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OPERATOR SPLITTING FOR ABSTRACT CAUCHY PROBLEMS WITH DYNAMICAL BOUNDARY CONDITIONS

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Abstract. In this work we study operator splitting methods for a certain class of coupled abstract Cauchy problems, where the coupling is such that one of the sub-problems prescribes a "boundary type" extra condition for the other one. The theory of one-sided coupled operator matrices provides an excellent framework to study the well-posedness of such problems. We show that with this machinery even operator splitting methods can be treated conveniently and rather efficiently. We consider three specific examples: the Lie (sequential), the Strang, and the weighted splitting, and prove the convergence of these methods along with error bounds under fairly general assumptions. Simple numerical examples show that the obtained theoretical bounds can be computationally realised.

1. Introduction

Operator splitting procedures provide an efficient way of solving time-dependent differential equations which describe the combined effect of several processes. In this case the operator describing the time-evolution is the sum of certain sub-operators corresponding to the different processes. The main idea of operator splitting is that one solves the sub-problems corresponding to the sub-operators separately, and constructs the solution of the original problem from the sub-solutions.

Depending on how the sub-solutions define the solution itself, one can distinguish several operator splitting procedures, such as sequential (proposed by Bagrinovskii and Godunov in [3]), Strang (proposed by Strang and Marchuk in [53] and [48]), or weighted ones (see e.g. in Csomós et al. [14]). An application of sequential splitting, for instance, results in the subsequent solution of the sub-problems using the previously obtained sub-solution as initial condition for the next sub-problem.

Although operator splitting procedures enable the numerical treatment of complicated differential equations, their application leads to an approximate solution which usually differs from the exact one. The accuracy can be increased by considering the sequence of the sub-problems on short time intervals in a cycle, which will in turn increase the computational effort. However, the analysis of the error, caused by the use of operator splitting, stands in the main focus of related research. For general

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overviews on splitting methods we refer the interested reader to the vast literature. For instance, Bjørhus analysed the consistency of sequential (Lie) splitting in an abstract framework in [9], Sportisse considered the stiff case in [52], Hansen and Ostermann also treated the abstract case in [28], while Bátkai et al. applied the splitting methods for non-autonomous evolution equations in [7]. Error bounds in the abstract setting were proved by Jahnke and Lubich in [36] for the Strang splitting. While Hansen and Ostermann in [29] have treated higher order splitting methods. A survey can be found in [22] by Geiser.

Another challenging issue is what kinds of processes of the sub-operators describe. They can e.g. correspond to various physical, chemical, biological, financial, etc. phenomena. Hundsdorfer and Verwer analysed the splitting of advection–diffusion–reaction equations in [35, Chapter IV], Dimov et al. solved air pollution transport models in [15], Jacobsen et al. considered the Hamilton–Jacobi equations in [38], Holden et al. partial differential equations with Burgers nonlinearity in [34], while in [12] Csomós and Nickel and in [5, 6] Bátkai et al. applied splitting methods for delay equations. Splitting methods for Schrödinger equations are treated, e.g., in Hochbruck et al. [37], Caliari et al. [10].

The sub-operators can also correspond to the change (derivative) with respect to various spatial coordinates or other variables, as Hansen and Ostermann have studied in [28], [30]; or for the case of Maxwell equations, see, e.g., Jahnke et al. [33] or Eilinghoff and Schnaubelt [16]. Furthermore, the sub-problems may originate from other (mathematical) properties of the problem itself, as in the present case of dynamical boundary problems.

We emphasise that the analysis and the numerical treatment of dynamical boundary problems has been attracting the attention of several researchers recently, cf. the work of Hipp [31, 32] for wave-type equations or Knopf et al. [40, 39, 41] on the Cahn–Hilliard equation or Kovács et al. [42] and Kovács, Lubich [43] on parabolic equations. The literature is extensive, and we mention some very recent papers by Altmann [1], Epshteyn, Xia [20], Fukao et al. [21], Langa, Pierre [44], and refer to the references therein.

In the present work we focus on the *abstract setting* of coupled Cauchy problems, where one of the subproblems provides an extra condition, of boundary type, to the other. We consider equations of the form:

$$\begin{cases} \dot{u}(t) = A_m u(t) & \text{for } t \geqslant 0, \quad u(0) = u_0 \in E, \\ \dot{v}(t) = Bv(t) & \text{for } t \geqslant 0, \quad v(0) = v_0 \in F, \\ Lu(t) = v(t) & \text{for } t \geqslant 0, \end{cases} \tag{1.1}$$

where E and F are Banach spaces over the complex field $\mathbb{C}$, A and B are (unbounded) linear operators on E and F, respectively. The coupling of the two problems involves the unbounded linear operator L acting between E and F. Moreover, this coupling is of "boundary type", i.e., as concrete examples we have in mind problems of the following form:

$$\dot{u}(t) = \Delta_\Omega u(t), \qquad u(0) = u_0 \in \mathrm{L}^2(\Omega), \tag{1.2}$$

$$\begin{aligned} \dot{v}(t) &= \Delta_{\partial\Omega} v(t), \qquad v(0) = v_0 \in \mathrm{L}^2(\partial\Omega), \\ u(t)|_{\partial\Omega} &= v(t), \end{aligned} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with sufficiently regular boundary and $A_m = \Delta_\Omega$, $B = \Delta_{\partial\Omega}$ are the (maximal) distributional Laplace and Laplace–Beltrami operators restricted to the respective L^2-space. In this example L denotes the trace operator (the precise ingredients will be discussed in Example 2.7 below.)

It is a natural idea for the numerical treatment of (1.2)–(1.3) to apply operator splitting methods, i.e. to treat the first and second equations separately, see also in [43]. The purpose of this work is to investigate such possibilities, and as a splitting strategy we propose the following steps:

1. Choose a time step $\tau > 0$.
2. Solve the second equation (1.3) with the initial condition $v(0) = u_0|_{\partial\Omega} = v_0$, set $v_1 := v(\tau)$.
3. Solve the first equation (1.2) on $[0,\tau]$ with the inhomogeneous boundary condition $u(t)|_{\partial\Omega} = v_1$ and the initial condition $u(0) = \widetilde{u}_0$. The method determines how $\widetilde{u}_0$ is calculated from u_0 (and v_0), and in general $\widetilde{u}_0$ does not need to equal u_0. Set $u_1 = u(\tau)$.
4. The new initial condition for equation (1.3) is then $u_1|_{\partial\Omega} = v_1$.
5. Iterate this procedure for $n \in \mathbb{N}$ time steps.

The aim of this paper is to formulate this splitting method as *an operator splitting* in an abstract *operator semigroup theoretic* framework and investigate its convergence properties. The method then becomes applicable for a wider class of equations than in (1.1). Our choice for the auxiliary, modified initial value $\widetilde{u}_0$, in Step 3 above, is motivated by this approach. Indeed, the abstract theory will immediately yield the convergence of the method as an instance of the Lie–Trotter formula. However, we shall briefly touch upon other possible choices for $\widetilde{u}_0$ as well.

As a matter of fact our proposed methods, at a first sight, will be slightly different in that we decompose the system in not two but three sub-problems. This idea is nicely illustrated in the above example of diffusion: We separate the dynamics in the domain and assume homogeneous boundary conditions, the dynamics on the boundary, and as the third component the interaction between the two dynamics, i.e., how the boundary dynamics is fed into the domain. In fact, this decomposition is responsible for the modified form $\widetilde{u}_0$ of the initial condition. This approach will also have the advantage that the internal and boundary dynamics are completely separated. Hence well-established methods can be used for solving each of the subproblems. We also note that the splitting approach here gives a way to parallelisation of the solution to the subproblems.

This work is organised as follows. In Section 2 we recall the necessary operator theoretic background for this programme and in Section 3 we introduce the different splitting approaches for the dynamical boundary conditions: the Lie splitting, the Strang splitting and the weighted splitting. We also prove the convergence of these methods under fairly general assumptions. Section 4 contains error bounds for the above mentioned splitting methods. Finally, in Section 5 we illustrate the proposed methods by numerical examples, and show that the analytically proved error bounds are realised computationally, too.

2. Abstract dynamical boundary conditions

Before discussing splitting methods in more detail let us briefly recall a possible approach for treating such abstract dynamical boundary value problems. The abstract treatment of boundary perturbations, i.e., techniques for altering the domain of the generator of a C_0-semigroup goes back to the work of Greiner [26]. Many results have been building on his theory, and our main sources for describing the abstract setting will be the works by Casarino, Engel, Nagel, and Nickel [11] and Engel [17, 18]. In [11] the following set of conditions were posed for treating the well-posedness of the problem (1.1).

HYPOTHESIS 2.1. The $\mathbb{C}$-vector spaces E and F are Banach spaces.

(i) The operators $A_m : \mathrm{dom}(A_m) \subseteq E \to E$ and $B : \mathrm{dom}(B) \subseteq F \to F$ are linear.

(ii) The linear operator $L : \mathrm{dom}(A_m) \to F$ is surjective and bounded with respect to the graph norm of A_m on $\mathrm{dom}(A_m)$.

(iii) The restriction A_0 of A_m to $\ker(L)$ generates a strongly continuous semigroup $\big(T_0(t)\big)_{t\geqslant 0}$ on E.

(iv) The operator B generates a strongly continuous semigroup $\big(S(t)\big)_{t\geqslant 0}$ on F.

(v) The operator (matrix) $\binom{A_m}{L} : \mathrm{dom}(A_m) \to E \times F$ is closed.

REMARK 2.2. Consider the following conditions.

(i') The operator $A_m : \mathrm{dom}(A_m) \subseteq E \to E$ is linear.

(ii') The linear operator $L : \mathrm{dom}(A_m) \to F$ is surjective.

(iii') $L : \mathrm{dom}(A_m) \to F$ has a bounded right-inverse $R : F \to E$ with $\mathrm{rg}(R) \subseteq \ker(A_m)$.

(iv') The restriction A_0 of A_m to $\mathrm{dom}(A_0) := \ker(L)$ is (boundedly) invertible (i.e., $0 \in \rho(A_0)$; in general, it is sufficient to assume that the resolvent set $\rho(A_0)$ is non-empty).

Under these assumptions $\binom{A_m}{L} : \mathrm{dom}(A_m) \to E \times F$ is closed. To see this, we first recall from the proof of Lemma 2.2 in [11] that in this case

$$\mathrm{dom}(A_m) = \mathrm{dom}(A_0) \oplus \ker(A_m).$$

We also repeat the quick argument for this, taken from [11]: For $x \in \mathrm{dom}(A_m)$ we have

$$x = A_0^{-1}A_m x + (x - A_0^{-1}A_m x) \quad \text{with} \quad A_0^{-1}A_m x \in \mathrm{dom}(A_0),\, x - A_0^{-1}A_m x \in \ker(A_m).$$

Furthermore, if $x \in \mathrm{dom}(A_0) \cap \ker(A_m)$, then $A_0 x = A_m x = 0$, and $x = 0$ follows by $0 \in \rho(A_0)$.

Now, let $x_n \in \mathrm{dom}(A_m)$ be with $x_n \to x$, $A_m x_n \to y$ in E and $Lx_n \to z$ in F as $n \to \infty$. We need to show $x \in \mathrm{dom}(A_m)$, $A_m x = y$, $Lx = z$. For each $n \in \mathbb{N}$ write $x_n = x_n^0 + x_n^1$ with $x_n^0 \in \mathrm{dom}(A_0)$ and $x_n^1 \in \ker(A_m)$. Then $A_m x_n = A_m x_n^0 + A_m x_n^1 = A_m x_n^0 = A_0 x_n^0$, thus $x_n^0 \to A_0^{-1} y$ in E as $n \to \infty$. On the other hand $Lx_n = Lx_n^0 + Lx_n^1 = Lx_n^1 \to z$ in F as $n \to \infty$. It follows that

$$RLx_n = RLx_n^1 \to Rz \in \mathrm{dom}(A_m).$$

Moreover, from $L(x_n^1 - RLx_n^1) = 0$ we conclude $x_n^1 - RLx_n^1 \in \mathrm{dom}(A_0)$ and $A_0(x_n^1 - RLx_n^1) = A_m x_n^1 - A_m RLx_n^1 = 0$, so that $x_n^1 = RLx_n^1$ follows. This implies $x - A_0^{-1} y = Rz$, $x \in \mathrm{dom}(A_m)$, $A_m x = y$ and $Lx = LRz = z$.

In this paper we make the following technical assumption to simplify the things a bit.

HYPOTHESIS 2.3. The operators A_0 and B are invertible.

However, let us note that for the splitting procedures this makes *no theoretical difference*, since (for semigroup generators) one always finds sufficiently large $\lambda > 0$ such that $A_0 - \lambda$ and $B - \lambda$ become invertible. Then the numerical schemes can be applied in this rescaled situation.

Next, we recall the following definition from [11, Lemma 2.2], and note that under the previous assumption the following operator is bounded

$$D_0 := L|_{\ker(A_m)}^{-1} : F \to \ker(A_m) \subseteq E. \tag{2.1}$$

The operator D_0 is called the *abstract Dirichlet operator*; the operator $L|_{\ker(A_m)}$ is indeed invertible, see the mentioned lemma in [11]. We remark that the existence of this Dirichlet operator $D_0 =: R$, a continuous right-inverse to L as in Remark 2.2, is therefore equivalent to the closedness of $\binom{A_m}{L}$ (under the assumption that $0 \in \rho(A_0)$).

REMARK 2.4. (a) The operator $D_0 B : \mathrm{dom}(B) \to E$ is bounded if $\mathrm{dom}(B)$ is supplied with the graph-norm $\|\cdot\|_B$.

(b) We have $\mathrm{rg}(D_0) \cap \mathrm{dom}(A_0) = \{0\}$.

Following [11] we introduce the product space $E \times F$ and the operator $\mathscr{A}$ acting on it as

$$\mathscr{A} := \begin{pmatrix} A_m & 0 \\ 0 & B \end{pmatrix} \text{ with } \mathrm{dom}(\mathscr{A}) := \left\{ \tbinom{x}{y} \in \mathrm{dom}(A_m) \times \mathrm{dom}(B) : Lx = y \right\}. \tag{2.2}$$

Section 1.1 in [51] relates the well-posedness of (1.1) to the generation property of $\mathscr{A}$, see also [50].

The first thing to be settled is therefore, whether the abstract Cauchy problem

$$\dot{\boldsymbol{u}}(t) = \mathscr{A}u(t), \quad \text{for } t \geqslant 0, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 = (u_0, v_0)^\top,$$

is well-posed in the sense of C_0-semigroups, see [19, Section II.6]. In this case the solution satisfies $\boldsymbol{u}(t) = \mathscr{T}(t)\boldsymbol{u}_0$, where $(\mathscr{T}(t))_{t\geqslant 0}$ is the semigroup generated by $\mathscr{A}$. The problem of well-posedness is solved in [11]. We briefly recall here the following results from Theorem 2.7 in [11] and from its proof.

THEOREM 2.5. *Let the operators $\mathscr{A}$, D_0 be as defined in (2.2) and (2.1) and assume Hypotheses 2.1 and 2.3. For $y \in \operatorname{dom}(B)$ define*

$$Q(t)y = D_0S(t)y - T_0(t)D_0y - \int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s. \tag{2.3}$$

Operator $\mathscr{A}$ is the generator of a C_0-semigroup if and only if for each $t \geqslant 0$ the operator (extends to)

$$Q(t) \in \mathscr{L}(F,E) \quad \text{and} \quad \limsup_{t\downarrow 0} \|Q(t)\| < \infty. \tag{2.4}$$

In this case the semigroup $\big(\mathscr{T}(t)\big)_{t\geqslant 0}$ generated by $\mathscr{A}$ is given by

$$\mathscr{T}(t) = \begin{pmatrix} T_0(t) & Q(t) \\ 0 & S(t) \end{pmatrix}. \tag{2.5}$$

The next condition will be important throughout the paper.

HYPOTHESIS 2.6. The operator A_0 generates a bounded analytic semigroup (see [47, 27] for details about analytic semigroups).

If Hypotheses 2.1, 2.3 and 2.6 are fulfilled and also B is a generator of an analytic semigroup, then Theorem 2.5 applies and assures that the semigroup $(\mathscr{T}(t))_{t\geqslant 0}$ generated by $\mathscr{A}$ is analytic too, see [11, Corollary 2.8].

The motivating example from the introduction is discussed in [11, Section 3] in detail. We recall here the ingredients, to illustrate that our proposed methods will be applicable also for this equation.

EXAMPLE 2.7. (Laplace and Laplace–Beltrami operators) Let Ω be a bounded domain in $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^2.

- $E := \mathrm{L}^2(\Omega)$, $F := \mathrm{L}^2(\partial\Omega)$ are the L^2-spaces with respect to the Lebesgue and the surface measure, respectively.
- Δ_Ω and $\Delta_{\partial\Omega}$ are the (maximal) distributional Laplace and Laplace–Beltrami operators, respectively.

- $A_m := \Delta_\Omega$ with domain

$$\mathrm{dom}(A_m) := \{f : f \in \mathrm{H}^{1/2}(\Omega) \text{ with } \Delta_\Omega f \in \mathrm{L}^2(\Omega)\}.$$

- $Lf = f|_{\partial\Omega}$ the trace of $f \in \mathrm{dom}(A_m)$ on $\partial\Omega$.
- $B = \Delta_{\partial\Omega}$ with domain

$$\mathrm{dom}(B) = \{g : g \in \mathrm{L}^2(\partial\Omega) \text{ with } \Delta_{\partial\Omega} g \in \mathrm{L}^2(\Omega\}.$$

Hypotheses 2.1, 2.3, 2.6 are satisfied for these choices. In particular, $\mathscr{A}$ generates an analytic semigroup on $E \times F$, see [11, Section 3]. We also have the following:

- The Dirichlet operator $D_0 : \mathrm{L}^2(\partial\Omega) \to \mathrm{H}^{1/2}(\Omega)$ assigns to a prescribed boundary value g a function f with $f|_{\partial\Omega} = g$ (in the sense of traces) and $\Delta_\Omega f = 0$.
- $A_0 = \Delta_\mathrm{D}$ is the Laplace operator with (homogeneous) Dirichlet boundary condition, generating the Dirichlet heat semigroup $(T_0(t))_{t\geqslant 0}$ on $\mathrm{L}^2(\Omega)$.
- The semigroup $(S(t))_{t\geqslant 0}$ is the heat semigroup on $\mathrm{L}^2(\partial\Omega)$.

EXAMPLE 2.8. (Bounded Lipschitz domains) In this example we indicate that one can relax the smoothness condition on the boundary of the domain from Example 2.7. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$.

(a) Consider the following operators:

- $A_m = \Delta_\Omega$ with domain

$$\mathrm{dom}(A_m) := \{f : f \in \mathrm{H}^{1/2}(\Omega) \text{ with } \Delta_\Omega f \in \mathrm{L}^2(\Omega)\}.$$

- $Lf = f|_{\partial\Omega}$ the trace of $f \in \mathrm{dom}(A_m)$ on $\partial\Omega$ (see, e.g., [49, pp. 89–106]).

Then L is surjective and actually has a bounded right-inverse

$$R : \mathrm{L}^2(\partial\Omega) \to \ker(A_m),$$

where $\mathrm{dom}(A_m)$ is endowed with the norm $u \mapsto \|u\|_{\mathrm{H}^{1/2}} + \|\Delta u\|_2$, see Theorem 3.6 (i) in [8] for precisely this statement (or [25, Lemma 3.1, Theorem 5.3], [24], [49, Theorem 3.37]). The restriction A_0 of A_m to

$$\ker(L) = \{f : \mathrm{H}^{1/2}(\Omega), \Delta_\Omega f \in \mathrm{L}^2(\Omega), Lf = 0\}$$

is strictly positive, self-adjoint, in particular A_0 is invertible and generates a bounded analytic semigroup, [25, Theorem 5.1], [24, Theorem 2.11] (see also [8, Theorem 3.6 (v)]). By invoking Remark 2.2 we obtain that $\binom{A_m}{L} : \mathrm{dom}(A_m) \to \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\partial\Omega)$ is closed, and altogether that A_m and L satisfy the relevant conditions from Hypothesis 2.1.

(b) One can also consider the Laplace–Beltrami operator $B := \Delta_{\partial\Omega}$ on $\mathrm{L}^2(\partial\Omega)$, which (with an appropriate domain) is also a strictly positive, self-adjoint operator, see [25, Theorem 2.5] or [23] for details.

Summing up, we see that the abstract framework of [11], hence of this paper, covers also some interesting cases of dynamical boundary value problems on bounded Lipschitz domains.

The decisive tool, based on the theory of coupled operator matrices [17, 18], is to bring the *formally diagonal* operator $\mathscr{A}$ with a *non-diagonal domain* into an upper triangular form with the state space transformations

$$\mathscr{R}_0 = \begin{pmatrix} I & -D_0 \\ 0 & I \end{pmatrix}, \qquad \mathscr{R}_0^{-1} = \begin{pmatrix} I & D_0 \\ 0 & I \end{pmatrix}.$$

Accordingly, we obtain the following representation:

$$\mathscr{A} = \mathscr{R}_0^{-1}\mathscr{A}_0\mathscr{R}_0, \tag{2.6}$$

where

$$\mathscr{A}_0 = \begin{pmatrix} A_0 & -D_0 B \\ 0 & B \end{pmatrix} \quad \text{with} \quad \mathrm{dom}(\mathscr{A}_0) = \mathrm{dom}(A_0) \times \mathrm{dom}(B),$$

see [11, Lemma 2.6 and the proof of Corollary 2.8].

3. Operator splitting methods for dynamical boundary conditions problems

Since the form of the semigroup $(\mathscr{T}(t))_{t\geqslant 0}$ can be rarely determined in practice, our aim is to determine an approximation to it, and denote at time $t = k\tau$ the approximation of $\boldsymbol{u}(k\tau)$ by $\boldsymbol{u}_k(\tau)$ for all $k \in \mathbb{N}$. The natural requirement is that the approximate value should converge to the exact one when refining the temporal resolution (letting $\tau \to 0$). We recall the following definition from [45] due to Lax and Richtmyer.

DEFINITION 3.1. (Convergence) The approximation $\boldsymbol{u}_k$ is called convergent to the solution $\boldsymbol{u}$ of problem (1.1) on $[0, t_{\max}]$ (for given $t_{\max} > 0$) if $\boldsymbol{u}(t) = \lim\limits_{n\to\infty} \boldsymbol{u}_n(\frac{t}{n})$ holds uniformly for all $t \in [0, t_{\max}]$.

Starting from the representation (2.6), we construct approximations of the form

$$\boldsymbol{u}_k(\tau) := \mathscr{R}_0^{-1}\mathbb{T}(\tau)^k\mathscr{R}_0\binom{u_0}{v_0}, \tag{3.1}$$

where the operator $\mathbb{T}(\tau)\colon E \times \mathrm{dom}(B) \to E \times \mathrm{dom}(B)$, $\tau \geqslant 0$ describes the actual numerical method, and $\boldsymbol{u}(0) = \boldsymbol{u}_0 = (u_0, v_0)^\top$. In order to specify the operator $\mathbb{T}(\tau)$, we remark that the operator $\mathscr{A}_0$ can be written as the sum

$$\mathscr{A}_0 =: \mathscr{A}_1 + \mathscr{A}_2 + \mathscr{A}_3,$$

where

$$\mathscr{A}_1 = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathscr{A}_2 = \begin{pmatrix} 0 & -D_0 B \\ 0 & 0 \end{pmatrix}, \quad \mathscr{A}_3 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix},$$

with

$$\mathrm{dom}(\mathscr{A}_1) = \mathrm{dom}(A_0) \times F,\ \mathrm{dom}(\mathscr{A}_2) = E \times \mathrm{dom}(B),\ \mathrm{dom}(\mathscr{A}_3) = E \times \mathrm{dom}(B).$$

We point out that $\mathscr{A}_1$ and $\mathscr{A}_3$ commute (in the sense of resolvents). From Hypothesis 2.1 and Remark 2.4 we immediately obtain the following proposition.

PROPOSITION 3.2. *The operator semigroups* $(\mathscr{T}_i(t))_{t\geqslant 0}$, $i = 1,2,3$ *given by*

$$\mathscr{T}_1(t) = \begin{pmatrix} T_0(t) & 0 \\ 0 & I \end{pmatrix}, \quad \mathscr{T}_2(t) = \begin{pmatrix} I & -tD_0 B \\ 0 & I \end{pmatrix}, \quad \mathscr{T}_3(t) = \begin{pmatrix} I & 0 \\ 0 & S(t) \end{pmatrix}$$

are strongly continuous on $E \times \mathrm{dom}(B)$ *with generator*

$$\mathscr{A}_1|_{E\times\mathrm{dom}(B)},\ \mathscr{A}_2 \text{ and } \mathscr{A}_3|_{E\times\mathrm{dom}(B)}, \text{ respectively.}$$

Here we consider the parts of the respective operators in the space $E \times \mathrm{dom}(B)$. *The semigroups* $(\mathscr{T}_1(t))_{t\geqslant 0}$ *and* $(\mathscr{T}_3(t))_{t\geqslant 0}$ *are even strongly continuous on* $E \times F$. *Their generators are* $\mathscr{A}_1$ *and* $\mathscr{A}_3$, *respectively.*

In this work we focus on methods (3.1) with the following choices for the operator $\mathbb{T}(\tau)$:

$$\mathbb{T}^{[\mathrm{Lie}]}(\tau) := \mathscr{T}_1(\tau)\mathscr{T}_2(\tau)\mathscr{T}_3(\tau) \tag{3.2}$$

for the *Lie (or sequential) splitting*;

$$\mathbb{T}^{[\mathrm{Str}]}(\tau) := \mathscr{T}_1(\tfrac{\tau}{2})\mathscr{T}_3(\tfrac{\tau}{2})\mathscr{T}_2(\tau)\mathscr{T}_3(\tfrac{\tau}{2})\mathscr{T}_1(\tfrac{\tau}{2}) \tag{3.3}$$

for the *Strang (or symmetrical) splitting*;

$$\mathbb{T}^{[\mathrm{wgh}]}(\tau) := \Theta\mathscr{T}_1(\tau)\mathscr{T}_2(\tau)\mathscr{T}_3(\tau) + (1-\Theta)\mathscr{T}_3(\tau)\mathscr{T}_2(\tau)\mathscr{T}_1(\tau) \tag{3.4}$$

for the *weighted splitting*, where the parameter $\Theta \in [0,1]$ is fixed. We note that the case $\Theta = 1$ corresponds to the Lie splitting, while $\Theta = 0$ gives the Lie splitting in the reverse order. Computing the composition of the operators leads to the common form

$$\mathbb{T}(\tau) = \begin{pmatrix} T_0(\tau) & V(\tau) \\ 0 & S(\tau) \end{pmatrix} \tag{3.5}$$

with the operators

$$\text{Lie splitting:} \quad V^{[\mathrm{Lie}]}(\tau) = -\tau T_0(\tau) D_0 B S(\tau), \tag{3.6}$$

$$\text{Strang splitting:} \quad V^{[\mathrm{Str}]}(\tau) = -\tau T_0(\tfrac{\tau}{2}) D_0 B S(\tfrac{\tau}{2}), \tag{3.7}$$

$$\text{weighted splitting:} \quad V^{[\mathrm{wgh}]}(\tau) = -\tau(\Theta T_0(\tau) D_0 B S(\tau) + (1-\Theta) D_0 B) \tag{3.8}$$

for all $\tau > 0$. The approximation (3.1) requires the powers of the operator $\mathbb{T}(\tau)$ to be computed next.

PROPOSITION 3.3. *For the operator family* $\mathbb{T}(\tau)\colon E\times\operatorname{dom}(B)\to E\times\operatorname{dom}(B)$, $\tau>0$, *from* (3.5) *we have the identity*

$$\mathbb{T}(\tau)^k=\begin{pmatrix}T_0(k\tau) & V_k(\tau)\\ 0 & S(k\tau)\end{pmatrix}$$

with

$$V_k(\tau)=\sum_{j=0}^{k-1}T_0\big((k-1-j)\tau\big)V(\tau)S(j\tau). \tag{3.9}$$

Proof. We show the assertion by induction. For $k=1$ we have formula (3.5) with $V_1(\tau)=V(\tau)$. If the assertion is valid for some $k\geqslant 1$, then

$$\mathbb{T}(\tau)^{k+1}=\begin{pmatrix}T_0(k\tau) & V_k(\tau)\\ 0 & S(k\tau)\end{pmatrix}\begin{pmatrix}T_0(\tau) & V(\tau)\\ 0 & S(\tau)\end{pmatrix}=\begin{pmatrix}T_0\big((k+1)\tau\big) & V_{k+1}(\tau)\\ 0 & S\big((k+1)\tau\big)\end{pmatrix}$$

holds with

$$\begin{aligned}V_{k+1}(\tau)&=T_0(k\tau)V(\tau)+V_k(\tau)S(\tau)\\&=T_0(k\tau)V(\tau)+\sum_{j=0}^{k-1}T_0\big((k-1-j)\tau\big)V(\tau)S(j\tau)S(\tau)\\&=T_0(k\tau)V(\tau)+\sum_{j=1}^{k}T_0\big((k-j)\tau\big)V(\tau)S\big((j-1)\tau\big)S(\tau)\\&=\sum_{j=0}^{k}T_0\big((k-j)\tau\big)V(\tau)S(j\tau).\end{aligned}$$

This proves the assertion for all $k\in\mathbb{N}$ by induction. $\square$

The convergence of the approximation relies on the following result.

PROPOSITION 3.4. *Under Hypotheses* 2.1, 2.3, (2.4) *and with the notation in* (3.5), *the approximation* (3.1) *is convergent for* $y\in\operatorname{dom}(B)$ *if the condition*

$$\lim_{n\to\infty}V_n(\tfrac{t}{n})y=-\int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s \tag{3.10}$$

holds uniformly for t *in compact intervals.*

Proof. From Proposition 3.3, the approximation has the form

$$\begin{aligned}\boldsymbol{u}_k(\tau)&=\begin{pmatrix}I & D_0\\ 0 & I\end{pmatrix}\begin{pmatrix}T_0(k\tau) & V_k(\tau)\\ 0 & S(k\tau)\end{pmatrix}\begin{pmatrix}I & -D_0\\ 0 & I\end{pmatrix}\boldsymbol{u}_0\\&=\begin{pmatrix}T_0(k\tau) & V_k(\tau)-T_0(k\tau)D_0+D_0S(k\tau)\\ 0 & S(k\tau)\end{pmatrix}\boldsymbol{u}_0.\end{aligned}\tag{3.11}$$

By comparing with formula (2.5) and using the relation (2.3), condition (3.10) implies the assertion. □

The convergence of the Riemann sums implies our next result concerning the approximation of the convolution in (3.10).

LEMMA 3.5. *Let $t_{\max} \geqslant 0$, let $f\colon [0,t_{\max}] \to \mathscr{L}(F,E)$ be strongly continuous, and let $g\colon [0,t_{\max}] \to F$ be continuous. For each $n \in \mathbb{N}$ and $t \in [0,t_{\max}]$ define the following expressions*

$$
\begin{aligned}
C_n^{[1]}(t) &:= \tfrac{t}{n} \sum_{j=0}^{n-1} f\big((n-j)\tfrac{t}{n}\big)\, g(j\tfrac{t}{n}),\\
C_n^{[2]}(t) &:= \tfrac{t}{n} \sum_{j=0}^{n-1} f\big((n-j-\tfrac{1}{2})\tfrac{t}{n}\big)\, g\big((j+\tfrac{1}{2})\tfrac{t}{n}\big).
\end{aligned}
$$

Then for $j = 1,2$ we have that

$$
\lim_{n\to\infty} C_n^{[j]}(t) = \int_0^t f(t-s)\, g(s)\, \mathrm{d}s
$$

holds uniformly for $t \in [0,t_{\max}]$.

We can now state the main result of this section concerning convergent approximations of the solution to problem (1.1).

PROPOSITION 3.6. *Under Hypotheses 2.1, 2.3, and (2.4) the approximations defined in (3.2), (3.3) and (3.4) are convergent for all $\boldsymbol{u}_0 \in E \times \operatorname{dom}(B)$.*

Proof. It suffices to prove that condition (3.10) holds for the operators $V(\tau)$ defined in (3.6), (3.7) and (3.8). By Proposition 3.3, we have the following identity for the Lie splitting:

$$
\begin{aligned}
V_k^{[\mathrm{Lie}]}(\tau)y &= -\tau \sum_{j=0}^{k-1} T_0\big((k-1-j)\tau\big) T_0(\tau) D_0 B S(\tau) S(j\tau) y\\
&= -\tau \sum_{j=0}^{k-1} T_0\big((k-j)\tau\big) D_0 B S\big((j+1)\tau\big) y,
\end{aligned}
$$

for the Strang splitting:

$$
\begin{aligned}
V_k^{[\mathrm{Str}]}(\tau)y &= -\tau \sum_{j=0}^{k-1} T_0\big((k-1-j)\tau\big) T_0(\tfrac{\tau}{2}) D_0 B S(\tfrac{\tau}{2}) S(j\tau) y \qquad (3.12)\\
&= -\tau \sum_{j=0}^{k-1} T_0\big((k-j-\tfrac{1}{2})\tau\big) D_0 B S\big((j+\tfrac{1}{2})\tau\big) y,
\end{aligned}
$$

and for the weighted splitting:

$$\begin{aligned}
V_k^{[\mathrm{wgh}]}(\tau)y &= -\tau\sum_{j=0}^{k-1} T_0\big((k-1-j)\tau\big)\big(\Theta T_0(\tau)D_0BS(\tau)+(1-\Theta)D_0B\big)S(j\tau)y\\
&= -\Theta\tau\sum_{j=0}^{k-1} T_0\big((k-j)\tau\big)D_0BS\big((j+1)\tau\big)y\\
&\quad -(1-\Theta)\tau\sum_{j=0}^{k-1} T_0\big((k-j-1)\tau\big)D_0BS(j\tau)y
\end{aligned}$$

for all $y\in\mathrm{dom}(B)$, $\tau>0$, and $\Theta\in[0,1]$. Since B and the semigroup operators $S(t)$ commute on $\mathrm{dom}(B)$, we have

$$\begin{aligned}
V_n^{[\mathrm{Lie}]}(\tfrac{t}{n})y &= -\tau\sum_{j=0}^{n-1} T_0((n-j)\tfrac{t}{n})D_0S((j+1)\tfrac{t}{n})By,\\
V_n^{[\mathrm{Str}]}(\tfrac{t}{n})y &= -\tau\sum_{j=0}^{n-1} T_0((n-j-\tfrac{1}{2})\tfrac{t}{n})D_0S((j+\tfrac{1}{2})\tfrac{t}{n})By,\\
V_n^{[\mathrm{wgh}]}(\tfrac{t}{n})y &= -\Theta\tau\sum_{j=0}^{n-1} T_0((n-j)\tfrac{t}{n})D_0S((j+1)\tfrac{t}{n})By\\
&\quad -(1-\Theta)\tau\sum_{j=0}^{n-1} T_0((n-j-1)\tfrac{t}{n})D_0S(j\tfrac{t}{n})By.
\end{aligned}$$

Now, Lemma 3.5 yields the convergence to the convolution in (3.10) for each of these cases. $\square$

REMARK 3.7. The stability of splitting methods for triangular operator matrices has been studied in [4]. If we write

$$\mathscr{A}_0=\begin{pmatrix}A_0 & 0\\ 0 & B\end{pmatrix}+\begin{pmatrix}0 & -D_0B\\ 0 & 0\end{pmatrix}=\mathscr{B}+\mathscr{A}_2,$$

then $\mathscr{B}$ with $\mathrm{dom}(\mathscr{B})=E\times\mathrm{dom}(B^2)$ generates the strongly continuous semigroup

$$\mathscr{S}(t)=\begin{pmatrix}T_0(t) & 0\\ 0 & S(t)\end{pmatrix}$$

on $E\times\mathrm{dom}(B)$. Since $\mathscr{A}_2$ is bounded on this space, by [4, Prop. 2.4] we obtain that for some $M\geqslant 0$ and $\omega\in\mathbb{R}$

$$\big\|\big(\mathscr{S}(\tfrac{t}{n})\mathscr{T}_2(\tfrac{t}{n})\big)^n\big\|_{\mathscr{L}(E\times\mathrm{dom}(B))}\leqslant M\mathrm{e}^{\omega}\quad\text{for every } t\geqslant 0.$$

Thus we immediately obtain the convergence of the corresponding Lie splitting procedure on $E\times\mathrm{dom}(B)$ by the Lie–Trotter product formula, see [19, Section III.5], or [54]. As a matter of fact, in this way we obtain also the generator property of $\mathscr{A}_0$ on $E\times\mathrm{dom}(B)$ directly, without recurring to [11].

REMARK 3.8. Let us comment on the relation between the previously proposed Lie splitting and the one from the introduction. Given $u_0 \in \mathrm{H}^{1/2}(\Omega)$ such that $v_0 = u_0|_{\partial\Omega}$ belongs to $\mathrm{dom}(B) = \mathrm{H}^2(\partial\Omega)$, we have that the Lie splitting corresponds to the choices $v_1 = S(\tau)v_0 \in \mathrm{dom}(B)$ and

$$\begin{aligned}\widetilde{u}_0 &= u_0 - D_0v_0 + D_0v_1 - \tau D_0Bv_1 = u_0 + D_0\Big(\int_0^\tau S(r)Bv_0\,\mathrm{d}r - \tau D_0Bv_1\Big)\\ &= u_0 + D_0\int_0^\tau \big(S(r) - S(\tau)\big)Bv_0\,\mathrm{d}r.\end{aligned}$$

If $v_0 \in \mathrm{dom}(B^2)$, we obtain $\widetilde{u}_0 = u_0 + \mathrm{O}(\tau^2)$, where $\mathrm{O}(\tau^2)$ denotes a term with norm less than or equal to $\tau^2C\|B^2v_0\|$.

It can be proved that if a method (more precisely the choice of $\widetilde{u}_0$) satisfies $\widetilde{u}_0 = u_0 + \mathrm{O}(\tau^2)$, then the corresponding splitting method (e.g. the one in the introduction with $\widetilde{u}_0 = u_0$) is convergent. In addition, its convergence order is the same as for the Lie splitting, cf. the next section.

4. Order of convergence

In this section we will investigate the order of convergence of the proposed splitting schemes. We begin with recalling a standard definition, see, e.g., [2].

DEFINITION 4.1. (Order of convergence) The approximation $\boldsymbol{u}_n$ to $\boldsymbol{u}$ is called *convergent of order* $p > 0$ on $[0, t_{\max}]$ (for some fixed $t_{\max} > 0$) if there exists a constant $C \geqslant 0$ such that $\|\boldsymbol{u}(t) - \boldsymbol{u}_n(\frac{t}{n})\| \leqslant Cn^{-p}$ for every $t \in [0, t_{\max}]$ and $n \in \mathbb{N} \setminus \{0\}$.

The rest of this paper is devoted to the investigation of such estimates for the approximations given in (3.1).

REMARK 4.2. Jahnke and Lubich [36] studied the convergence order of the Strang splitting for generators of bounded analytic semigroups under certain commutator conditions (for the Lie splitting, see [13, Chapter 10]). If we split

$$\mathscr{A}_0 = \begin{pmatrix} A_0 & 0\\ 0 & B\end{pmatrix} + \begin{pmatrix} 0 & -D_0B\\ 0 & 0\end{pmatrix} = \mathscr{B} + \mathscr{A}_2,$$

and assume that A_0, B are generators of bounded analytic semigroups, then in order to apply their result we need to calculate the commutator $[\mathscr{B}, \mathscr{A}_2]$. We have

$$\begin{aligned}[\mathscr{B}, \mathscr{A}_2] &= \begin{pmatrix} A_0 & 0\\ 0 & B\end{pmatrix}\begin{pmatrix} 0 & -D_0B\\ 0 & 0\end{pmatrix} - \begin{pmatrix} 0 & -D_0B\\ 0 & 0\end{pmatrix}\begin{pmatrix} A_0 & 0\\ 0 & B\end{pmatrix}\\ &= \begin{pmatrix} 0 & -A_0D_0B\\ 0 & 0\end{pmatrix} - \begin{pmatrix} 0 & D_0B^2\\ 0 & 0\end{pmatrix} = -\begin{pmatrix} 0 & D_0B^2\\ 0 & 0\end{pmatrix}\end{aligned}$$

with the domain

$$\mathrm{dom}([\mathscr{B}, \mathscr{A}_2]) = \mathrm{dom}(A_0) \times \{0\},$$

by Remark 2.4. This renders the direct application of the Jahnke–Lubich result impossible. Moreover, in contrast to [36] we do not need to require that the operator B is also an analytic generator, only the well-posedness of (1.1) (or equivalently (2.4)). The price to be paid for this simplification is the requirement of increased regularity conditions for the initial value.

Before proceeding to the error estimates we start with an important observation, whose proof is a small modification of the one of Lemma 3.4, cf. (3.11).

PROPOSITION 4.3. *Let $V(\tau)$ be as in (3.5) and let $D \subseteq F$ be a subspace with a given norm $\|\cdot\|_D$. Let $r \geqslant 0$, let $t_{\max} > 0$ and $C \geqslant 0$ such that for every $y \in D$ and for every $t \in [0,t_{\max}]$*

$$\Big\|V_n(\tfrac{t}{n})y + \int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s\Big\| \leqslant \frac{Ct^r\log(n)}{n^r}\|y\|_D. \tag{4.1}$$

Then

$$\Big\|\mathscr{R}_0^{-1}\mathbb{T}^n(\tfrac{t}{n})\mathscr{R}_0\tbinom{x}{y} - \mathscr{T}(t)\tbinom{x}{y}\Big\| \leqslant \frac{Ct^r\log(n)}{n^r}\|y\|_D$$

for every $x \in E$, $y \in D$ and $t \in [0,t_{\max}]$. In particular, the approximation $\boldsymbol{u}_k$ defined in (3.1) is convergent of order p for any $p \in (0,r)$ and every initial value $\boldsymbol{u}_0 \in E \times D$.

From now on we will focus on the error estimates concerning the approximation $V_n(\frac{t}{n})$, where the corresponding V is either given in (3.6), or (3.7) or (3.8) (but note that many other choices for V are possible, cf. Remark 3.8.)

LEMMA 4.4. (Local error of splittings I) *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\geqslant 0}$ and $(S(t))_{t\geqslant 0}$, respectively, and suppose Hypotheses 2.1, 2.3, (2.4), 2.6. For every $t_{\max} > 0$ there is $C \geqslant 0$ such that for every $h \in [0,t_{\max}]$, for every $s_0, s_1 \in [0,h]$, and for every $y \in \operatorname{dom}(B^2)$ we have*

$$\Big\|\int_0^h T_0(h-s)A_0^{-1}D_0S(s)By\,\mathrm{d}s - hT_0(h-s_0)A_0^{-1}D_0S(s_1)By\Big\| \leqslant Ch^2(\|By\| + \|B^2y\|).$$

Proof. For each $y \in \operatorname{dom}(B^2)$ we can write

$$\begin{aligned}
&\int_0^h T_0(h-s)A_0^{-1}D_0S(s)By\,\mathrm{d}s - hT_0(h-s_0)A_0^{-1}D_0S(s_1)By\\
&= \int_0^h \big(T_0(h-s)A_0^{-1}D_0S(s)By - T_0(h-s_0)A_0^{-1}D_0S(s_1)By\big)\,\mathrm{d}s\\
&= \int_0^h (T_0(h-s) - T_0(h-s_0))A_0^{-1}D_0S(s)By\,\mathrm{d}s\\
&\quad + \int_0^h T_0(h-s_0)A_0^{-1}D_0(S(s)-S(s_1))By\,\mathrm{d}s = I_1 + I_2,
\end{aligned}$$

where I_1, I_2 denote the occurring integrals on the right-hand side in the order of appearance. The first term I_1 can be estimated as

$$\begin{aligned}\|I_1\| &= \Big\|\int_0^h (T_0(h-s)-T_0(h-s_0))A_0^{-1}D_0S(s)By\,\mathrm{d}s\Big\| \\ &\leqslant \int_0^h \|(T_0(h-s)-T_0(h-s_0))A_0^{-1}\|\cdot\|D_0\|\cdot\|S(s)By\|\,\mathrm{d}s \\ &\leqslant C_1\int_0^h |s-s_0|\,\mathrm{d}s\cdot\|By\| = C_2h^2\|By\|.\end{aligned}$$

For the second term I_2 we obtain the estimate:

$$\begin{aligned}\|I_2\| &= \Big\|\int_0^h T_0(h-s_0)A_0^{-1}D_0(S(s)-S(s_1))By\,\mathrm{d}s\Big\| \\ &\leqslant C_3\int_0^h \|(S(s)-S(s_1))By\|\,\mathrm{d}s = C_3\int_0^h\Big\|\int_{s_1}^s S(r)B^2y\,\mathrm{d}r\Big\|\,\mathrm{d}s \\ &\leqslant C_4h^2\|B^2y\|.\end{aligned}$$

Putting these estimates together finishes the proof of the lemma. □

The validity of the following condition makes it possible to prove convergence rates for the other types of splittings.

HYPOTHESIS 4.5. (We suppose as in Hypothesis 2.6 that A_0 generates a bounded analytic semigroup.) The number $\gamma \in [0,1]$ is such that $\operatorname{rg}(D_0) \subseteq \operatorname{dom}((-A_0)^\gamma)$.

We refer to [27, Chapter 3], [47, Chapter 4], [19, Chapter II.5] or [13, Chapter 9] for details concerning fractional powers of sectorial operators. In particular, at this point it is important to recall the following result.

REMARK 4.6. If $\alpha \in [0,1]$ and A_0 is the generator of the bounded analytic semigroup $(T_0(t))_{t\geqslant 0}$, then

$$\sup_{t>0}\|t^\alpha(-A_0)^\alpha T_0(t)\| < \infty,$$

and

$$\sup_{t>0}\|t^{-\alpha}(T_0(t)-I)(-A_0)^{-\alpha}\| < \infty.$$

REMARK 4.7. (a) For $\gamma = 0$ the condition in Hypothesis 4.5 is always trivially satisfied, and this choice will suffice for the Lie splitting. The requirement $\gamma > 0$ is only needed for the cases of the Strang and the weighted splittings.

(b) Hypothesis 4.5 is fulfilled in the setting of Example 2.7 for the Dirichlet Laplace operator Δ_{D} with $\gamma \in [0,1/4)$. Indeed, we have $\operatorname{rg}(D_0) \subseteq \mathrm{H}^{1/2}(\Omega)$. For $\gamma \in [0,1/4)$ we have by [46, Theorem 11.1] that

$$\mathrm{H}^{2\gamma}(\Omega) = \mathrm{H}_0^{2\gamma}(\Omega),$$

and then by complex interpolation, [46, Theorem 11.6], we can write

$$\left[\mathrm{H}^2_0(\Omega), \mathrm{L}^2(\Omega)\right]_\gamma = \mathrm{H}^{2\gamma}(\Omega).$$

Moreover, since

$$\mathrm{H}^2_0(\Omega) \subseteq \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) = \mathrm{dom}(\Delta_\mathrm{D})$$

with continuous inclusion, we obtain (see, e.g., [47, Chapter 4]) that

$$\mathrm{H}^{2\gamma}(\Omega) = \left[\mathrm{H}^2_0(\Omega), \mathrm{L}^2(\Omega)\right]_\gamma \subseteq \left[\mathrm{dom}(\Delta_\mathrm{D}), \mathrm{L}^2(\Omega)\right]_\gamma \subseteq \mathrm{dom}\big((-\Delta_\mathrm{D})^\gamma\big).$$

Finally, this yields

$$\mathrm{rg}(D_0) \subseteq \mathrm{H}^{1/2}(\Omega) \subseteq \mathrm{H}^{2\gamma}(\Omega) \subseteq \mathrm{dom}\big((-\Delta_\mathrm{D})^\gamma\big).$$

(c) It is important to note that if for some $\gamma \geqslant 0$ Hypothesis 4.5 is satisfied, then $(-A_0)^\gamma D_0 : F \to E$ is a closed, and hence bounded, linear operator.

LEMMA 4.8. (Local error of splittings II) *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\geqslant 0}$ and $(S(t))_{t\geqslant 0}$, respectively. Suppose Hypotheses 2.1, 2.3, (2.4), 2.6 and also 4.5, i.e., that $\mathrm{rg}(D_0) \subseteq \mathrm{dom}((-A_0)^\gamma)$ for some $\gamma \in [0,1]$. For every $t_{\max} > 0$ there is $C \geqslant 0$ such that for every $h \in [0, t_{\max}]$, for every $s_0, s_1 \in [0,h]$ and for every $y \in \mathrm{dom}(B^2)$ we have*

$$\left\| \int_0^h T_0(h-s) D_0 S(s) By \,\mathrm{d}s - hT_0(h-s_0) D_0 S(s_1) By \right\| \leqslant Ch^{1+\gamma}(\|By\| + \|B^2 y\|).$$

Proof. For any $y \in \mathrm{dom}(B^2)$ we can write

$$\begin{aligned}
&\left\| \int_0^h T_0(h-s) D_0 S(s) By \,\mathrm{d}s - hT_0(h-s_0) D_0 S(s_1) By \right\| \\
&\quad = \left\| \int_0^h \big(T_0(h-s) D_0 S(s) By - T_0(h-s_0) D_0 S(s_1) By \big) \,\mathrm{d}s \right\| \\
&\quad \leqslant \left\| \int_0^h \big(T_0(h-s) - T_0(h-s_0) \big) D_0 S(s) By \,\mathrm{d}s \right\| \\
&\qquad + \left\| \int_0^h T_0(h-s_0) D_0 \big(S(s) - S(s_1) \big) By \,\mathrm{d}s \right\|.
\end{aligned}$$

The second term can be further estimated as follows:

$$\begin{aligned}
&\left\| \int_0^h T_0(h-s_0) D_0 \big(S(s) - S(s_1) \big) By \,\mathrm{d}s \right\| \\
&\quad \leqslant C_1 \int_0^h \|\big(S(s) - S(s_1)\big) By\| \,\mathrm{d}s = C_1 \int_0^h \left\| \int_{s_1}^s S(r) B^2 y \,\mathrm{d}r \right\| \mathrm{d}s \qquad (4.2) \\
&\quad \leqslant C_2 h^2 \|B^2 y\|.
\end{aligned}$$

It remains to estimate the first term. Since $(-A_0)^{\gamma}D_0$ is closed and everywhere defined, it is bounded (see Remark 4.7) and hence we can write

$$
\begin{aligned}
&\Big\|\int_0^h \big(T_0(h-s)-T_0(h-s_0)\big)D_0S(s)By\,\mathrm{d}s\Big\| \\
&\qquad\leqslant \int_0^h \|(-A_0)^{-\gamma}\big(T_0(h-s)-T_0(h-s_0)\big)(-A_0)^{\gamma}D_0S(s)By\|\,\mathrm{d}s \\
&\qquad\leqslant \int_0^h \|(-A_0)^{-\gamma}\big(T_0(h-s)-T_0(h-s_0)\big)\|\cdot\|(-A_0)^{\gamma}D_0S(s)By\|\,\mathrm{d}s \\
&\qquad\leqslant C_3\|By\|\int_0^h \|(-A_0)^{-\gamma}\big(T_0(h-s)-T_0(h-s_0)\big)\|\,\mathrm{d}s.
\end{aligned}
$$

Now, by Remark 4.6 we have

$$
\|(-A_0)^{-\gamma}\big(T_0(h-s)-T_0(h-s_0)\big)\| \leqslant C_4|s-s_0|^{\gamma}.
$$

Inserting this back into the previous inequality and integrating with respect to s we finally obtain the statement. □

The next result yields that the order of Lie splitting is (at most 1 but) as near to 1 as we wish, provided the initial data is smooth enough.

THEOREM 4.9. (Convergence of the Lie splitting) *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\geqslant 0}$ and $(S(t))_{t\geqslant 0}$, respectively. Suppose Hypotheses 2.1, 2.3, (2.4), 2.6. For each $t_{\max}>0$ there is $C\geqslant 0$ such that for every $n\in\mathbb{N}$, $y\in\operatorname{dom}(B^2)$ and $t\in[0,t_{\max}]$ we have*

$$
\Big\|V_n^{[\mathrm{Lie}]}(\tfrac{t}{n})y+\int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s\Big\| \leqslant C\frac{t\log(n+1)}{n}\big(\|By\|+\|B^2y\|\big).
$$

Proof. With $\tau=\frac{t}{n}$ we have

$$
\begin{aligned}
V_n^{[\mathrm{Lie}]}(\tau)y&+\int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s \\
&= -\tau\sum_{j=0}^{n-1} T_0((n-1-j)\tau)T_0(\tau)D_0BS(\tau)S(j\tau)y \\
&\qquad + \int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s \\
&= -\sum_{j=0}^{n-1}\Big(\tau T_0((n-1-j)\tau)T_0(\tau)D_0BS(\tau)S(j\tau)y \\
&\qquad -\int_{j\tau}^{(j+1)\tau} T_0(t-s)D_0S(s)By\,\mathrm{d}s\Big).
\end{aligned}
$$

Notice that for $j \in \{0,\dots,n-1\}$ we have

$$\begin{aligned}
&\int_{j\tau}^{(j+1)\tau} T_0(t-s)D_0S(s)By\,\mathrm{d}s - \tau T_0((n-1-j)\tau)T_0(\tau)D_0BS(\tau)S(j\tau)y \\
&= T_0(t-(j+1)\tau)\int_{j\tau}^{(j+1)\tau} T_0((j+1)\tau-s)D_0S(s)By\,\mathrm{d}s \\
&\quad - \tau T_0(t-(j+1)\tau)T_0(\tau)D_0S((j+1)\tau)By.
\end{aligned}$$

If $j \in \{0,\dots,n-2\}$ then by Lemma 4.4 we conclude that

$$\begin{aligned}
&\Big\|\int_{j\tau}^{(j+1)\tau} T_0(t-s)D_0S(s)By\,\mathrm{d}s - \tau T_0((n-1-j)\tau)T_0(\tau)D_0BS(\tau)S(j\tau)y\Big\| \\
&\leqslant \|A_0T_0(t-(j+1)\tau)\|\cdot\Big\|\int_{j\tau}^{(j+1)\tau} T_0((j+1)\tau-s)A_0^{-1}D_0S(s)By\,\mathrm{d}s \\
&\qquad - \tau T_0(\tau)A_0^{-1}D_0S((j+1)\tau)By\Big\| \\
&\leqslant C_1\frac{1}{t-(j+1)\tau}\Big\|\int_0^{\tau} T_0(\tau-s)A_0^{-1}D_0S(s+j\tau)By\,\mathrm{d}s \\
&\qquad - \tau T_0(\tau)A_0^{-1}D_0S(\tau)S(j\tau)By\Big\| \\
&\leqslant C_2\frac{1}{t-(j+1)\tau}\tau^2(\|BS(j\tau)y\|+\|B^2S(j\tau)y\|) \\
&\leqslant C_3\frac{t}{n(n-(j+1))}(\|By\|+\|B^2y\|).
\end{aligned}$$

Whereas for $j=n-1$ we have by Lemma 4.8 (with $\gamma=0$, $h=\tau$, $s_0=s_1=\tau$) that

$$\begin{aligned}
&\Big\|\int_{j\tau}^{(j+1)\tau} T_0(t-s)D_0S(s)By\,\mathrm{d}s - \tau T_0((n-1-j)\tau)T_0(\tau)D_0BS(\tau)S(j\tau)y\Big\| \\
&= \Big\|\int_0^{\tau} T_0(\tau-s)D_0S((n-1)\tau+s)By\,\mathrm{d}s - \tau T_0(\tau)D_0S(\tau)S((n-1)\tau)By\Big\| \\
&\leqslant C_4\frac{t}{n}(\|By\|+\|B^2y\|).
\end{aligned}$$

Summing these terms for $j=0,\dots,n-1$ we obtain that

$$\begin{aligned}
&\Big\|V_n^{[\mathrm{Lie}]}(\tau)y+\int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s\Big\| \\
&\leqslant \sum_{j=0}^{n-2} C_3\frac{t}{n(n-(j+1))}(\|By\|+\|B^2y\|)+C_4\frac{t}{n}(\|By\|+\|B^2y\|) \\
&\leqslant C\frac{t\log(n)}{n}(\|By\|+\|B^2y\|),
\end{aligned}$$

as asserted. $\square$

LEMMA 4.10. (Local error of the Strang splitting) *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\geqslant 0}$ and $(S(t))_{t\geqslant 0}$, respectively. Suppose Hypotheses 2.1, 2.3, (2.4), 2.6 and also 4.5, i.e., that* $\mathrm{rg}(D_0)\subseteq \mathrm{dom}((-A_0)^{\gamma})$ *for some $\gamma\in[0,1]$. For every $t_{\max}>0$ there is $C\geqslant 0$ such that for every $h\in[0,t_{\max}]$ and for every $y\in \mathrm{dom}(B^3)$ we have*

$$\Big\|\int_0^h T_0(h-s)A_0^{-1}D_0S(s)By\,\mathrm{d}s - hT_0(\tfrac{h}{2})A_0^{-1}D_0S(\tfrac{h}{2})By\Big\| \leqslant Ch^{2+\gamma}(\|By\|+\|B^2y\|+\|B^3y\|).$$

Proof. We have

$$\begin{aligned}
&\int_0^h T_0(h-s)A_0^{-1}D_0S(s)By\,\mathrm{d}s - hT_0(\tfrac{h}{2})A_0^{-1}D_0S(\tfrac{h}{2})By\\
&=\int_0^h T_0(h-s)A_0^{-1}D_0S(s)By - T_0(\tfrac{h}{2})A_0^{-1}D_0S(\tfrac{h}{2})By\,\mathrm{d}s\\
&=\int_0^h T_0(h-s)A_0^{-1}D_0\big(S(s)-S(\tfrac{h}{2})\big)By\,\mathrm{d}s\\
&\quad+\int_0^h \big(T_0(h-s)-T_0(\tfrac{h}{2})\big)A_0^{-1}D_0S(\tfrac{h}{2})By\,\mathrm{d}s\\
&=\int_0^h \Big(T_0(h-s)-T_0(\tfrac{h}{2})\Big)A_0^{-1}D_0\big(S(s)-S(\tfrac{h}{2})\big)By\,\mathrm{d}s\\
&\quad+\int_0^h T_0(\tfrac{h}{2})A_0^{-1}D_0\big(S(s)-S(\tfrac{h}{2})\big)By\,\mathrm{d}s\\
&\quad+\int_0^h \big(T_0(h-s)-T_0(\tfrac{h}{2})\big)A_0^{-1}D_0S(\tfrac{h}{2})By\,\mathrm{d}s = I_1+I_2+I_3,
\end{aligned}$$

where I_1,I_2,I_3 denote the integrals on the right-hand side in the respective order of appearance.

We start with the estimation of I_1. Inserting the Taylor remainder

$$(S(s)-S(\tfrac{h}{2}))By=\int_{\frac{h}{2}}^s S(r)B^2y\,\mathrm{d}r,$$

and the analogous formula for $T_0(h-s)-T_0(\frac{h}{2})$, in the definition of I_1 yields that

$$\begin{aligned}
I_1&=\int_0^h \Big(T_0(h-s)-T_0(\tfrac{h}{2})\Big)A_0^{-1}D_0\big(S(s)-S(\tfrac{h}{2})\big)By\,\mathrm{d}s\\
&=\int_0^h\int_{\frac{h}{2}}^{h-s} T_0(t)A_0A_0^{-1}D_0\int_{\frac{h}{2}}^s S(r)B^2y\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s\\
&=\int_0^h\int_{\frac{h}{2}}^{h-s} T_0(t)D_0\int_{\frac{h}{2}}^s S(r)B^2y\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

Whence we conclude

$$
\begin{aligned}
\|I_1\| &\leqslant \int_0^h \Big| \int_{\frac{h}{2}}^{h-s} \|T_0(t)D_0\| \Big| \int_{\frac{h}{2}}^{s} \|S(r)\|\|B^2y\|\,\mathrm{d}r \Big| \mathrm{d}t \Big| \,\mathrm{d}s \qquad (4.3)\\
&\leqslant C_1\|D_0\| \int_0^h |h-s-\tfrac{h}{2}|\cdot|\tfrac{h}{2}-s|\,\mathrm{d}s\cdot\|B^2y\|\\
&\leqslant C_2h^3\|B^2y\|,
\end{aligned}
$$

where C_1 and C_2 depend only on the growth bounds of $(T_0(t))_{t\geqslant}$ and $(S(t))_{t\geqslant 0}$ and on $t_{\max}$ and $\|D_0\|$.

The next is the estimation of the integral I_2. Now instead of inserting a first order Taylor approximation for $S(s)$ in the definition of I_2 we make use of the special structure of the Strang splitting and recall the following Taylor formula

$$
S(s)By = S(\tfrac{h}{2})By + (s-\tfrac{h}{2})S(\tfrac{h}{2})B^2y + \int_0^{s-\frac{h}{2}} (s-\tfrac{h}{2}-r)S(\tfrac{h}{2}+r)B^3y\,\mathrm{d}r.
$$

If we substitute this into the definition of I_2, we arrive at

$$
\begin{aligned}
I_2 &= \int_0^h T_0(\tfrac{h}{2})A_0^{-1}D_0\big(S(s)-S(\tfrac{h}{2})\big)By\,\mathrm{d}s\\
&= T_0(\tfrac{h}{2})A_0^{-1}D_0 \int_0^h \big(S(s)-S(\tfrac{h}{2})\big)By\,\mathrm{d}s\\
&= T_0(\tfrac{h}{2})A_0^{-1}D_0 \int_0^h \Big(S(\tfrac{h}{2})By + (s-\tfrac{h}{2})S(\tfrac{h}{2})B^2y\\
&\qquad + \int_0^{s-\frac{h}{2}} (s-\tfrac{h}{2}-r)S(r)S(\tfrac{h}{2})B^3y\,\mathrm{d}r - S(\tfrac{h}{2})By\Big)\,\mathrm{d}s\\
&= T_0(\tfrac{h}{2})A_0^{-1}D_0 \Big(\int_0^h (s-\tfrac{h}{2})S(\tfrac{h}{2})B^2y\,\mathrm{d}s\\
&\qquad + \int_0^h\int_0^{s-\frac{h}{2}} (s-\tfrac{h}{2}-r)S(r)S(\tfrac{h}{2})B^3y\,\mathrm{d}r\,\mathrm{d}s\Big)\\
&= T_0(\tfrac{h}{2})A_0^{-1}D_0 \int_0^h\int_0^{s-\frac{h}{2}} (s-\tfrac{h}{2}-r)S(r)S(\tfrac{h}{2})B^3y\,\mathrm{d}r\,\mathrm{d}s,
\end{aligned}
$$

the last equality being true since the first integral on the right-hand side on the line before is 0. This immediately implies the desired norm-estimate for I_2:

$$
\begin{aligned}
\|I_2\| &\leqslant \|T_0(\tfrac{h}{2})A_0^{-1}D_0\|\cdot\Big\|\int_0^h\int_0^{s-\frac{h}{2}} (s-\tfrac{h}{2}-r)S(r)S(\tfrac{h}{2})B^3y\,\mathrm{d}r\,\mathrm{d}s\Big\|\\
&\leqslant C_3h^3\|B^3y\|,
\end{aligned}
$$

where C_3 is an appropriate constant independent of y and $h\in[0,t_{\max}]$.

We finally turn to the estimation of the term I_3, and this is only where the order reduction by $1-\gamma$ occurs. If we abbreviate $z = D_0 S(\frac{h}{2})By$, then

$$I_3 = \int_0^h \big(T_0(h-s) - T_0(\tfrac{h}{2})\big)A_0^{-1}z.$$

By analyticity we have $T_0(\frac{h}{2})z \in \operatorname{dom}(A_0^2)$ so, similarly to the case of the term I_2, we can use the Taylor expansion for $0 \leqslant s < h$

$$T_0(h-s)A_0^{-1}z = T_0(\tfrac{h}{2})A_0^{-1}z + (\tfrac{h}{2}-s)A_0 T_0(\tfrac{h}{2})A_0^{-1}z \\ + A_0 \int_0^{\frac{h}{2}-s} (\tfrac{h}{2}-s-r)A_0 T_0(\tfrac{h}{2}+r)A_0^{-1}z\,\mathrm{d}r.$$

Whence we conclude

$$\begin{aligned} I_3 &= \int_0^h \big(T_0(h-s) - T_0(\tfrac{h}{2})\big)A_0^{-1}z\,\mathrm{d}s \\ &= \int_0^h \Big((\tfrac{h}{2}-s)T_0(\tfrac{h}{2})z \\ &\quad + A_0 \int_0^{\frac{h}{2}-s} (\tfrac{h}{2}-s-r)T_0(\tfrac{h}{2})T_0(r)z\,\mathrm{d}r\Big)\,\mathrm{d}s, \end{aligned}$$

since the first integral here is 0, we arrive at

$$I_3 = A_0 \int_0^h \int_0^{\frac{h}{2}-s} (\tfrac{h}{2}-s-r)T_0(\tfrac{h}{2})T_0(r)D_0S(\tfrac{h}{2})By\,\mathrm{d}r\,\mathrm{d}s.$$

We take the norm here and, using Remark 4.6, estimate trivially:

$$\begin{aligned} \|I_3\| &= \Big\|\int_0^h (-A_0)^{1-\gamma} \int_0^{\frac{h}{2}-s} (\tfrac{h}{2}-s-r)T_0(\tfrac{h}{2})T_0(r)(-A_0)^{\gamma}D_0S(\tfrac{h}{2})By\,\mathrm{d}r\,\mathrm{d}s\Big\| \qquad (4.4) \\ &\leqslant C_4\|(-A_0)^{1-\gamma}T_0(\tfrac{h}{2})\| \cdot \|(-A_0)^{\gamma}D_0\| \cdot \|By\| \int_0^h \Big|\int_0^{\frac{h}{2}-s} |\tfrac{h}{2}-s-r|\,\mathrm{d}r\Big|\,\mathrm{d}s \\ &= C_5 \frac{h^3}{h^{1-\gamma}}\|By\| = C_5 h^{2+\gamma}\|By\|. \end{aligned}$$

The proof of the lemma is now finished by putting together the estimates for I_1, I_2 and I_3. □

THEOREM 4.11. (Convergence of the Strang splitting) *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\geqslant 0}$ and $(S(t))_{t\geqslant 0}$, respectively. Suppose Hypotheses 2.1, 2.3, (2.4), 2.6 and also 4.5, i.e., that* $\operatorname{rg}(D_0) \subseteq \operatorname{dom}((-A_0)^{\gamma})$ *for some $\gamma \in [0,1]$. For each $t_{\max} > 0$ there is $C \geqslant 0$ such that for every $n \in \mathbb{N}$, $y \in \operatorname{dom}(B^3)$ and $t \in [0, t_{\max}]$ we have*

$$\Big\|V_n^{[\mathrm{Str}]}(\tfrac{t}{n})y + \int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s\Big\| \leqslant C\frac{t^{1+\gamma}\log(n+1)}{n^{1+\gamma}}(\|By\| + \|B^2y\| + \|B^3y\|).$$

Proof. Set $\tau := \frac{t}{n}$. Recall from (3.12) that for $y \in \operatorname{dom}(B)$ we have

$$V_n^{[\mathrm{Str}]}(\tau)y = -\tau \sum_{j=0}^{n-1} T_0((n-j-\tfrac{1}{2})\tau)D_0S((j+\tfrac{1}{2})\tau)By,$$

so that

$$\begin{aligned}
&\int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s + V_n^{[\mathrm{Str}]}(\tau)y \\
&\quad= \sum_{j=0}^{n-1}\int_{j\tau}^{(j+1)\tau} T_0(t-s)D_0S(s)By - T_0((n-j-\tfrac{1}{2})\tau)D_0S((j+\tfrac{1}{2})\tau)By\,\mathrm{d}s \\
&\quad= \sum_{j=0}^{n-1} T_0((n-j-1)\tau)\int_0^\tau T_0(\tau-s)D_0S(s)S(j\tau)By \qquad (4.5)\\
&\qquad\qquad\qquad\qquad - T_0(\tfrac{\tau}{2})D_0S(\tfrac{\tau}{2})S(j\tau)By\,\mathrm{d}s.
\end{aligned}$$

We first consider the term for $j = n-1$. By Lemma 4.8, with $h = \tau$ and $s_0 = s_1 = \frac{\tau}{2}$ we have that

$$\begin{aligned}
&\Big\|\int_0^\tau T_0(\tau-s)D_0S(s)S((n-1)\tau)By - T_0(\tfrac{\tau}{2})D_0S(\tfrac{\tau}{2})S((n-1)\tau)By\,\mathrm{d}s\Big\| \\
&\quad\leqslant C_1\tau^{1+\gamma}(\|BS((n-1)\tau)y\| + \|B^2S((n-1)\tau)y\|) \\
&\quad\leqslant C_2\tau^{1+\gamma}(\|By\| + \|B^2y\|)
\end{aligned}$$

for $t \in [0, t_{\max}]$. Next we consider the summands in (4.5) for $j \in \{0, \dots, n-2\}$. In these cases we can write

$$\begin{aligned}
&\Big\|T_0\big((n-j-1)\tau\big)\int_0^\tau T_0(\tau-s)D_0S(s)S(j\tau)By - T_0(\tfrac{\tau}{2})D_0S(\tfrac{\tau}{2})S(j\tau)By\,\mathrm{d}s\Big\| \\
&\quad= \|A_0T_0\big((n-j-1)\tau\big)\int_0^\tau T_0(\tau-s)A_0^{-1}D_0S(s)S(j\tau)By \\
&\qquad\quad - T_0(\tfrac{\tau}{2})A_0^{-1}D_0S(\tfrac{\tau}{2})S(j\tau)By\,\mathrm{d}s\Big\| \\
&\quad\leqslant \|A_0T_0\big((n-j-1)\tau\big)\| \cdot \Big\|\int_0^\tau T_0(\tau-s)A_0^{-1}D_0S(s)S(j\tau)By \\
&\qquad\quad - T_0(\tfrac{\tau}{2})A_0^{-1}D_0S(\tfrac{\tau}{2})S(j\tau)By\,\mathrm{d}s\Big\|,
\end{aligned}$$

and by Lemma 4.10 and Remark 4.6 we can continue as follows:

$$\begin{aligned}
&\leqslant \frac{C_3}{(n-j-1)\tau}\tau^{2+\gamma}\big(\|BS(j\tau)y\| + \|B^2S(j\tau)y\| + \|B^3S(j\tau)y\|\big) \\
&\leqslant \frac{C_4}{(n-j-1)\tau}\tau^{2+\gamma}\big(\|By\| + \|B^2y\| + \|B^3y\|\big)
\end{aligned}$$

for constants C_3, C_4 independent of y, n and $t \in [0, t_{\max}]$. Summing up these estimates we arrive at

$$\begin{aligned}
&\Big\| \int_0^t T_0(t-s) D_0 S(s) By \,\mathrm{d}s + V_n^{[\mathrm{Str}]}(\tau) y \Big\| \\
&\quad \leqslant C_2 \tau^{1+\gamma} \big(\|By\| + \|B^2 y\| \big) + \sum_{j=0}^{n-2} \frac{C_4}{(n-j-1)\tau} \tau^{2+\gamma} \big(\|By\| + \|B^2 y\| + \|B^3 y\| \big) \\
&\quad \leqslant \Big(C_2 \frac{t^{1+\gamma}}{n^{1+\gamma}} + \frac{C_4 t^{2+\gamma}}{n^{2+\gamma}} \sum_{k=1}^{n-1} \frac{n}{tk} \Big) \cdot \big(\|By\| + \|B^2 y\| + \|B^3 y\| \big) \\
&\quad \leqslant C \frac{t^{1+\gamma} \log(n)}{n^{1+\gamma}} \big(\|By\| + \|B^2 y\| + \|B^3 y\| \big),
\end{aligned}$$

with an appropriate constant $C \geqslant 0$. The proof is complete. $\square$

Finally, let us turn to the weighted splittings. For any $\Theta \in [0,1]$ the weighted splitting possess at least the convergence properties as the Lie splitting. For the case $\Theta = 1/2$ one can prove even more.

LEMMA 4.12. (Local error of the symmetrically weighted Splitting) *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t \geqslant 0}$ and $(S(t))_{t \geqslant 0}$, respectively. Suppose Hypotheses 2.1, 2.3, (2.4), 2.6 and also 4.5, i.e., that* $\mathrm{rg}(D_0) \subseteq \mathrm{dom}((-A_0)^\gamma)$ *for some $\gamma \in [0,1]$. For every $t_{\max} > 0$ there is $C \geqslant 0$ such that for every $h \in [0, t_{\max}]$ and for every $y \in \mathrm{dom}(B^3)$ we have*

$$\begin{aligned}
\Big\| \int_0^h T_0(h-s) A_0^{-1} D_0 S(s) By \,\mathrm{d}s - \frac{1}{2} \Big(h T_0(h) A_0^{-1} D_0 B S(h) y + h A_0^{-1} D_0 By \Big) \Big\| \\
\leqslant C h^{2+\gamma} (\|By\| + \|B^2 y\| + \|B^3 y\|).
\end{aligned}$$

Proof. We have that

$$\begin{aligned}
2 \int_0^h & T_0(h-s) A_0^{-1} D_0 S(s) By \,\mathrm{d}s - \Big(h T_0(h) A_0^{-1} D_0 B S(h) y + h A_0^{-1} D_0 By \Big) = \\
&= \int_0^h T_0(h-s) A_0^{-1} D_0 S(s) By - T_0(h) A_0^{-1} D_0 B S(h) y \,\mathrm{d}s \\
&\quad + \int_0^h T_0(h-s) A_0^{-1} D_0 S(s) By - A_0^{-1} D_0 By \,\mathrm{d}s \\
&= \int_0^h T_0(h) A_0^{-1} D_0 (2S(s) - S(h) - \mathrm{I}) By \,\mathrm{d}s \\
&\quad + \int_0^h (2T_0(h-s) - T_0(h) - \mathrm{I}) A_0^{-1} D_0 S(h) By \,\mathrm{d}s \\
&\quad + \int_0^h (2T_0(h-s) - T_0(h) - \mathrm{I}) A_0^{-1} D_0 (S(s) - S(h)) By \,\mathrm{d}s \\
&\quad + \int_0^h (\mathrm{I} - T_0(h)) A_0^{-1} D_0 (S(s) - \mathrm{I}) By \,\mathrm{d}s = I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where I_j denotes the respective term on the right-hand side in order of occurrence. We first consider the term I_1. Since $y \in \mathrm{dom}(B^3)$ for any $t > 0$ we have the Taylor expansion

$$S(t)By = By + tB^2y + \int_0^t (t-r)S(r)B^3y\,\mathrm{d}r.$$

Substituting this into the formula for I_1 we obtain that

$$\begin{aligned} I_1 &= T_0(h)A_0^{-1}D_0\int_0^h \Big(2By + 2sB^2y + 2\int_0^s (s-r)S(r)B^3y\,\mathrm{d}r - By - hB^2y \\ &\qquad - \int_0^h (h-r)S(r)B^3y\,\mathrm{d}r - By\Big)\,\mathrm{d}s \\ &= T_0(h)A_0^{-1}D_0\int_0^h \Big[(2s-h)B^2y + 2\int_0^s (s-r)S(r)B^3y\,\mathrm{d}r \\ &\qquad - \int_0^h (h-r)S(r)B^3y\,\mathrm{d}r\Big]\,\mathrm{d}s \\ &= T_0(h)A_0^{-1}D_0\int_0^h \Big[2\int_0^s (s-r)S(r)B^3y\,\mathrm{d}r - \int_0^h (h-r)S(r)B^3y\,\mathrm{d}r\Big]\,\mathrm{d}s, \end{aligned}$$

since the integral of the first term is 0. Whence we conclude

$$\|I_1\| \leqslant C_3h^3\|B^3y\|.$$

The term I_2 can be estimated similarly, and we obtain

$$\|I_2\| \leqslant C_1h^{2+\gamma}(\|By\| + \|B^2y\|),$$

cf. (4.4) in the estimation of the term I_3 in Lemma 4.10. Finally, for the terms I_3 and I_4 we have

$$\|I_3\|,\ \|I_4\| \leqslant C_2h^3\|B^2y\|,$$

cf. (4.3) in the estimation of the term I_1 in Lemma 4.10. Putting the estimates for the terms I_1, I_2, I_3, I_4 together finishes the proof of the lemma. □

Based on Lemma 4.12 we immediately obtain the following error estimate for the symmetrically weighted splitting, the proof is analogous to the one of Theorem 4.11.

THEOREM 4.13. (Convergence of the symmetrically weighted splitting) *Let A_0 and B be the generator of the strongly continuous semigroups $(T_0(t))_{t\geqslant 0}$ and $(S(t))_{t\geqslant 0}$, respectively. Suppose Hypotheses 2.1, 2.3, (2.4), 2.6 and also 4.5, i.e., that $\mathrm{rg}(D_0) \subseteq \mathrm{dom}((-A_0)^\gamma)$ for some $\gamma \in [0,1]$. For each $t_{\max} > 0$ there is $C \geqslant 0$ such that for every $n \in \mathbb{N}$, $y \in \mathrm{dom}(B^3)$ and $t \in [0,t_{\max}]$ we have*

$$\Big\|V_n^{[\mathrm{wgh}]}(\tfrac{t}{n})y + \int_0^t T_0(t-s)D_0S(s)By\,\mathrm{d}s\Big\| \leqslant C\frac{t^{1+\gamma}\log(n+1)}{n^{1+\gamma}}(\|By\| + \|B^2y\| + \|B^3y\|).$$

5. Numerical examples

We present two examples to illustrate the theoretical results concerning the accuracy of the splitting procedures applied to problem (1.1). In both cases, we analyse the order of the global error by solving the problems using various values of the (splitting) time step $\tau > 0$. Fitting a straight line to these data in the logarithmic scale, the resulting slope yields the computational (numerical) order of the method. This is then to be compared to the theoretical order obtained in Theorems 4.9, 4.11, and 4.13 for the Lie (3.2), Strang (3.3), and weighted splittings (3.4), respectively.

For such illustration purposes the simplest non-trivial example serves as a good basis. In fact, we consider the heat equation on the domain $(0,\beta)$, $\beta > 0$ and a system of ordinary differential equations on the boundary for the unknown function $w\colon [0,\infty)\times[0,\beta] \to \mathbb{R}$:

$$\begin{cases} \partial_t w(t,x) = c\partial_{xx} w(t,x) & \text{for } t>0,\ x\in(0,\beta), \\ w(0,x) = w_0(x) & \text{for } x\in(0,\beta), \\ \dot{w}(t,0) = b_{11}w(t,0) + b_{12}w(t,\beta) & \text{for } t\geqslant 0, \\ \dot{w}(t,\beta) = b_{21}w(t,0) + b_{22}w(t,\beta) & \text{for } t\geqslant 0. \end{cases} \tag{5.1}$$

As explained in Section 2 this equation can be casted in the form of (1.1). The occurring spaces and operators are as follows (cf. Example 2.7):

- $E = \mathrm{L}^2(0,\beta)$, $F = \mathbb{C}^2$.
- $A_m = c\Delta$, and Δ is the distributional Laplace operator on $\mathrm{dom}(A_m) = \{f \in \mathrm{H}^1(0,\beta) : f'' \in \mathrm{L}^2(0,\beta)\}$.
- L is the Dirichlet trace on $\{0,\beta\}$.
- A_0 is the scalar multiple of the Dirichlet Laplace operator generating the Dirichlet heat semigroup $(T_0(t))_{t\geqslant 0}$.
- $D_0 : \mathbb{C}^2 \to H^{1/2}(0,\beta)$, $D_0(a,b)(r) = br/\beta + a(\beta - r)/\beta$.
- The operator B is given by

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

  generating the semigroup $(S(t))_{t\geqslant 0} = (\mathrm{e}^{tB})_{t\geqslant 0}$.

Implementation

Many ingredients, needed for the splitting methods, can be calculated explicitly, such as D_0, $(S(t))_{t\geqslant 0}$. For the implementation we used the corresponding built-in MATLAB functions. For the solution of the heat equation with *homogeneous* Dirichlet boundary condition, i.e., for determining $y(t) = T_0(t)y(0)$, it is plausible to apply an appropriate spectral method, in this case the Fourier method and use expansion with respect to the orthogonal basis of eigenfunctions of A_0. This is implemented by using MATLAB's built in `fft` function (Fast Fourier Transform). To this end we choose an integer N_x and split the space interval $[0,\beta]$ into N_x pieces of equal length, and the Fourier (sine) series is also truncated at N_x. We compute the values $u(t_n)$ only at grid points $x_j = j\beta/N_x$ for all $j = 1,\dots,N_x$ and for all time levels, $t_n = n\tau \in [0,t_{\max}]$, $n = 1,\dots,N_t := t_{\max}/\tau$ for some $t_{\max} > 0$ (such that N_t is an integer). Similarly, the boundary values $v(t_n)$ are also computed only at the same time levels. For all $n = 1,\dots,N_t$, the numerical solution $\boldsymbol{u}_n^{N_x} \in \mathbb{R}^{N_x}$ has then the elements $(\boldsymbol{u}_n^{N_x})_j$ being the approximation to $(\boldsymbol{u}(t_n))(x_j)$ for all $j = 1,\dots,N_x$.

The relative global error of the splittings is then computed as

$$\varepsilon(\tau) := \frac{\|\boldsymbol{u}(t_{\max}) - \boldsymbol{u}_{N_t}^{N_x}\|_2}{\|\boldsymbol{u}(t_{\max})\|_2} \tag{5.2}$$

with the Euclidean vector norm. The results of Section 4 forecast the following behaviour (for fixed $t_{\max} > 0$ and initial value)

$$\begin{array}{rl} \text{for the sequential splitting} & \varepsilon(\tau) = O(\tau|\log(\tau)|) \\ \left.\begin{array}{l}\text{for the symmetrically weighted}\\ \text{and Strang splittings}\end{array}\right\} & \varepsilon(\tau) = O(\tau^{1+\gamma}), \end{array}$$

where $\gamma \in [0,1/4)$ is as near to $1/4$ as we want, see Remark 4.7. For the computational orders of these methods we therefore expect 1.0 and 1.25, respectively. In the next two examples we test the methods against these expectations.

Exponential growth and decay on the boundary

We consider the following specific problem:

$$\begin{cases} \partial_t w(t,x) = \partial_{xx} w(t,x) & \text{for } t > 0,\, x \in (0,\pi), \\ w(0,x) = \cos(\frac{x}{2}) + \sinh(x) & \text{for } x \in [0,\pi], \\ \dot{w}(t,0) = -\frac{1}{4} w(t,0) & \text{for } t \geqslant 0, \\ \dot{w}(t,\pi) = w(t,\pi) & \text{for } t \geqslant 0. \end{cases} \tag{5.3}$$

This corresponds to the choices $\beta = \pi$, $c = 1$ and

$$B = \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}.$$

For this test example, the exact solution

$$w(t,x) = \mathrm{e}^{-t/4}\cos(\tfrac{x}{2}) + \mathrm{e}^{t}\sinh(x)$$

is known for all $t \geqslant 0$ and $x \in [0,\pi]$, therefore, we have

$$(u(t))(x) = \mathrm{e}^{-t/4}\cos(\tfrac{x}{2}) + \mathrm{e}^{t}\sinh(x) \quad \text{for} \quad x \in (0,\pi), \tag{5.4}$$

$$v(t) = \begin{pmatrix} e^{-t/4} \\ e^{t}\sinh(\pi) \end{pmatrix} \tag{5.5}$$

for all $t \geqslant 0$. Based on this, Figure 1 shows the exact solution $u(t)$ at certain time levels, while in Figure 2 the time evolution of the boundary values $v(t)$ is presented.

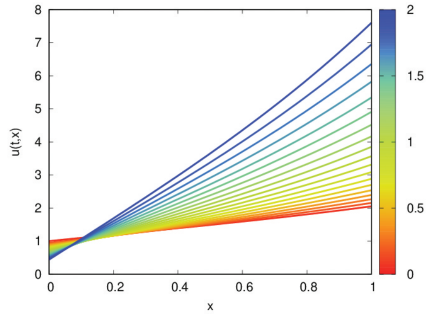

Figure 1: *Exact solution $u(t)$ of Example 1 for time levels $t_n = n\cdot 0.1$, $n = 1,\dots,20$. Red lines correspond to the beginning and blue ones to the end.*

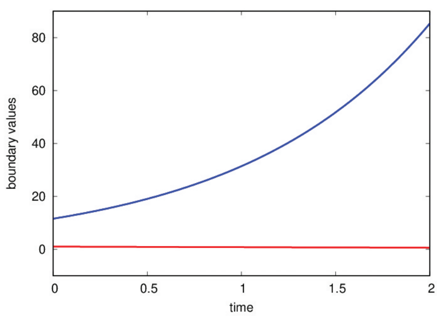

Figure 2: *Exact solution $v(t)$ of Example 1 for all $t \in [0,2]$. The red line corresponds to the first coordinate function (i.e., the left boundary value), while the blue line to the second one (right boundary value).*

In Figure 3, the order plots of the splittings are shown: the error values $\log(\varepsilon(\tau))$ via several values of $\log(\tau)$. The slope of the lines corresponds to the approximation of the order (called computational order later on). For comparison purposes, we also included the lines with slope 1 and 1.25.

One can see that for each splitting, there exists a value of $\log(\tau)$ below which the error no longer decreases. At that point the global error is dominated by the error of the spatial discretisation (the implementation/truncation error of the Fourier method in this case) and not the splitting error anymore. When computing the order, we fit a straight line only to the relevant data. The resulting values of the slopes are listed in Table 1.

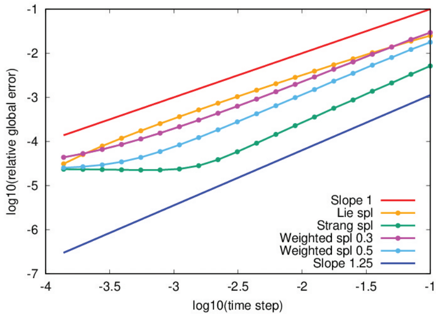

Figure 3: *Order plots of the various splitting procedures for problem* (5.3) *with* $N_x = 32768$.

Table 1: *Computational orders of the splitting procedures computed as the slope of the fitted line for* $N_x = 32768$ *in the case of problem* (5.3).

splitting	Lie	Strang	Weighted $\Theta = 0.3$	Weighted $\Theta = 0.5$
analytical order	~ 1	~ 1.25	~ 1	~ 1.25
computational order	1.0004	1.2405	1.0549	1.1646

Note that the asymmetrically weighted splitting is not studied in this paper analytically, but the numerical findings coincide with what one might expect: the same behaviour as for the Lie splitting. For each of the splitting methods, as one expects, the larger the N_x values are, the smaller becomes the threshold where the discretisation error starts dominating the splitting error.

Harmonic oscillation on the boundary

In the second example, we consider again the heat equation but this time we have a system of differential equations on the boundary describing the harmonic oscillator:

$$\begin{cases} \partial_t w(t,x) = c_1 \partial_{xx} w(t,x) & \text{for } t > 0,\ x \in (0,1), \\ w(0,x) = c_2 \mathrm{e}^{-c_3 (x-\frac{1}{2})^2} & \text{for } x \in (0,1), \\ \dot{w}(t,0) = w(t,1) & \text{for } t \geqslant 0 \text{ with } w(0,0) = 1, \\ \dot{w}(t,1) = -c_3 w(t,0) & \text{for } t \geqslant 0 \text{ with } w(0,1) = -c_4 \end{cases} \tag{5.6}$$

with some given constants $c_i > 0, i = 1,2,3,4$ and $\beta = 1$. We applied the same numerical method as described above. Note that the exact boundary function is

$$v(t) = \begin{pmatrix} \cos(\sqrt{c_3} t) - \frac{c_4}{\sqrt{c_3}} \sin(\sqrt{c_3} t) \\ -\sqrt{c_3} \sin(\sqrt{c_3} t) - c_4 \cos(\sqrt{c_3} t) \end{pmatrix}.$$

Since in this example the exact solution $u(t)$ is unknown, we computed a reference solution instead, by using the same numerical method but with much larger N_t value (but the same N_x value). This choice is justified by the fact that, at this point, we are interested in the order of the splitting method only and not in the error of the entire numerical method (including spatial and temporal discretisations). Figure 4 shows the reference solution at certain time levels, and in Figure 5 the time evolution of the boundary values are presented. For our numerical experiments, we chose the values $c_1 = 0.1, c_2 = 9, c_3 = 10, c_4 = 0.1$.

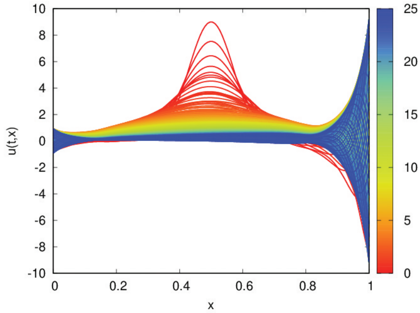

Figure 4: *Exact solution $u(t)$ of Example 2 for time levels $t_n = n \cdot 0.1$, $n = 1, \ldots, 250$ with $N_x = 128$. Red lines correspond to the beginning and blue ones to the end.*

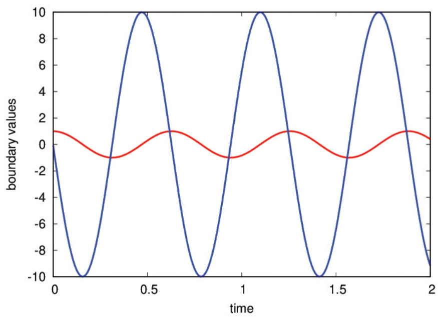

Figure 5: *Exact solution $v(t)$ of Example 2 for all $t \in [0, 2]$. The red line corresponds to the first coordinate function (i.e., the left boundary value), while the blue line to the second one (right boundary value).*

In Figure 6, the order plots of the splittings are presented for this example. As before, the slope of the lines corresponds to the approximation of the order. For comparison purposes, we included the lines with slope 1, 1.25 and 2. One can see that the classical first-order splittings (Lie and weighted with $\Theta = 0.3$) possess computational order 1 as well. The symmetrically weighted splitting ($\Theta = 0.5$) shows some oscillation after a while. The Strang splitting, however, behaves very well. For time step values with $\log(\tau)$ greater than approximately -2.5, its computational order is around 1.25 as expected. For smaller time steps its order becomes 2, which corresponds to the general order of the Strang splitting without the order reduction caused by the inhomogeneous boundary (the effect of the Dirichlet operator D_0).

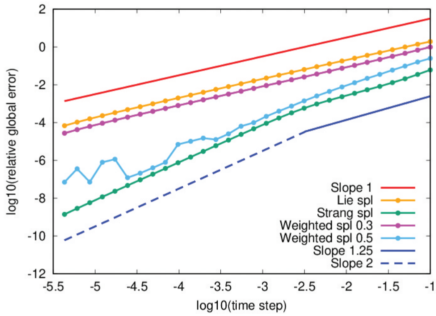

Figure 6: *Order plots of the various splitting procedures for problem* (5.3) *with* $N_x = 128$.

By considering larger N_x values (i.e., finer spatial resolution), the threshold where the break occurs can be pushed down and the numerical order gets closer to 1.25. Moreover, the oscillation for the weighted splittings in this case occurs also only for smaller τ values.

Table 2: *Computational orders of the splitting procedures computed as the slope of the fitted line for* $N_x = 128$ *in the case of problem* (5.6).

splitting	Lie	Strang	Weighted $\Theta = 0.3$	Weighted $\Theta = 0.5$
analytical order	~ 1	~ 1.25	~ 1	~ 1.25
computational order	1.0100	1.3056 if $\log(\tau) > -2.5$ 1.9812 if $\log(\tau) < -2.5$	1.0256	1.5765

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