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ON THE COMPACTNESS AND SPECTRA OF THE GENERALIZED DIFFERENCE OPERATOR ON THE SPACES ℓ^∞ AND bv

SAAD R. EL-SHABRAWY AND YOSHIHIRO SAWANO

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Abstract. In this paper, we consider the compactness and the spectrum of the generalized difference operator Δ_{ab} on the Banach sequence spaces ℓ^∞, of bounded sequences, and bv, of bounded variation sequences, which allows us to generalize and extend some existing results for the operator Δ_{ab}. Furthermore, the results for the operator Δ_{ab} on ℓ^∞ are new even in the case where Δ_{ab} is reduced to the difference operator Δ.

1. Introduction and the results

In this paper, we concern ourselves with the generalized difference operator Δ_{ab}, which is defined on a sequence space μ by

$$\Delta_{ab}x := (a_k x_k + b_{k-1}x_{k-1})_{k=0}^{\infty}, \qquad x := (x_k) = (x_k)_{k=0}^{\infty} \in \mu,$$

where $b_{-1} = x_{-1} = 0$. Here, (a_k) and (b_k) are sequences of real numbers [5]. If $a_k = 1$ and $b_k = -1$ for $k \geqslant 0$, then the operator Δ_{ab} is reduced to the difference operator Δ. Suppose that Δ acts on ℓ^∞. The spectrum of Δ is $\sigma(\Delta, \ell^\infty) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leqslant 1\}$; see [9, Theorem 2.12]. If $a_k = r$ and $b_k = s \neq 0$ for $k \geqslant 0$, where $r, s \in \mathbb{R}$, then the operator Δ_{ab} is reduced to the generalized difference operator $B(r,s)$; see [10]. Let us remark that $B(r,s) = -s\Delta + (r+s)\mathrm{I}$, where I is the identity operator. Then the spectra of $B(r,s)$ are obtained from the corresponding spectra of Δ by the application of the transformation $\lambda' = -s\lambda + r + s$. This is an obvious consequence of the fact, that the equations

$$(B(r,s) - \lambda'\mathrm{I})x = -sy, \qquad (\Delta - \lambda\mathrm{I})x = y$$

are equivalent when $\lambda' = -s\lambda + r + s$.

We are interested in the boundedness, the compactness and spectra of the generalized difference operator Δ_{ab} on the sequence space ℓ^∞, as well as on the sequence space bv. That is, we consider a small perturbation of $B(r,s) = -s\Delta + (r+s)\mathrm{I}$, where $r, s \in \mathbb{R}$, and observe what happens to the spectra in both compact and noncompact cases of the operator.

Among other Banach sequence spaces, we take up ℓ^∞ and bv. Recall that ℓ^∞ is the Banach space of bounded sequences of complex numbers with the well known

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ℓ^∞-norm. The Banach space bv is defined by $\mathrm{bv} := \{x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty\}$, with the norm

$$\|x\|_{\mathrm{bv}} := \sum_{k=0}^{\infty} |x_k - x_{k-1}|, \qquad (x_{-1} := 0).$$

By the triangle inequality, bv is continuously embedded into ℓ^∞. From the proof of this proposition, we feel that these two spaces are close to each other. Thus, one of the aims of this paper is to detect the difference between them in the study of spectra of Δ_{ab}.

The spectral problem of infinite matrices, in general, has deserved the attention of researchers, and some of them are motivated by the numerous applications of this scientific area. For example. Hilbert studied the eigenvalues of integral operators by viewing the operators as infinite matrices [30, *p*. 1063]. Further, it is known that infinite system of linear equations can be represented alternatively by infinite "coefficient" matrix. In [45] Shivakumar and Wong discussed infinite systems for algebraic equations, while Chew, Shivakumar and Williams [17] discussed systems of differential equations. In [44], Shivakumar, Williams and Rudraiah discussed eigenvalues of infinite matrices as operators acting on ℓ^1 and ℓ^∞. Further, Shivakumar and Williams [43] discussed the existence and uniqueness of solutions to infinite linear systems in ℓ^∞; they obtained necessary and sufficient conditions for the convergence of certain iteration scheme in terms of the spectral radius of the associated operator.

Spectral problems for infinite matrices arise frequently in mathematics and engineering. We find the theoretical and computational difficulties both in finite and infinite cases. For finite matrices, we have some technique both in theory and in computation. However, for a general infinite matrix, there is no known method for obtaining its spectrum. To the authors' knowledge, Brown, Halmos and Shields started the investigation of this problem in their paper [15] where they investigated and solved the problem in the case of the Cesàro matrix C_1 as an operator on the sequence space ℓ^2. More papers by different authors were devoted to the spectra of C_1 on the sequence spaces c [32], c_0 [32, 38], ℓ^{p} $(1 < p < \infty)$ [15, 18, 32], ℓ^∞ [32, 37], bv_0 [35], bv [36], and to the Bachelis space N^p $(1 < p < \infty)$ [19] and the weighted ℓ^{p} $(1 \leqslant p < \infty)$ spaces [1, 2]. Motivated by the paper [15], in [39], Rhoades started (see also the later papers of Leibowitz [33] and Rhoades and Sharma [40]) to deal with the above problems in the case of certain classes of Hausdorff matrices. Başar overviewed the extensive literature on the spectrum of matrix operators in [12]; see also [53] and [34, Chapter 5].

The spectra of the generalized difference operator Δ_{ab} in various sequence spaces have attracted a lot of attention. For example, we mention the works in ℓ^1 [7, 46, 47], ℓ^{p} $(1 \leqslant p < \infty)$ [8], c [5, 6], c_0 [22], bv_0 [23], h [23] and cs [42]. For the operator $B(r,s)$, the problem was studied in the Banach spaces c_0, c, ℓ^{p} $(1 \leqslant p < \infty)$, $\mathrm{bv_p}$ $(1 \leqslant p < \infty)$, cs, bv_0 and h; see [10, 13, 21, 23, 25]. The spectra of the difference operator Δ was also considered in [3, 4, 9]. In similar investigations [20, 24, 28, 29, 42, 52], the problem was studied for the upper triangular double-band matrix Δ^{ab} as an operator in the Banach spaces ℓ^{p} $(1 \leqslant p < \infty)$, c_0, c, cs and bv, where Δ^{ab} is the transpose of Δ_{ab}. We observed that the spectral problem of the generalized difference operator on ℓ^∞ is not completely settled down in the literature.

This work is a continuation of the works by El-Shabrawy and Abu-Janah [23] and

Sawano and El-Shabrawy [41, 42].

Below, we formally state and discuss our main results in full detail. The proofs of these results are given in Sections 3, 4 and 5.

The boundedness and the compactness of the operator Δ_{ab} are characterized in the following theorem, which is our starting point in this paper.

THEOREM 1.1.

(1) *The operator Δ_{ab} is bounded on ℓ^∞ if and only if*

$$\mathcal{M}_1 = \sup_{j\in\mathbb{N}_0} \left(|a_j| + |b_{j-1}|\right) < \infty.$$

If this is the case, the operator norm of Δ_{ab} equals to $\mathcal{M}_1$.

(2) *The operator Δ_{ab} is compact on ℓ^∞ if and only if*

$$\lim_{j\to\infty} a_j = \lim_{j\to\infty} b_j = 0.$$

(3) *The operator Δ_{ab} is bounded on* bv *if and only if*

$$\mathcal{M}_2 = \sup_{j\in\mathbb{N}_0} \left(|a_j| + |a_{j+1} - a_j + b_j| + \sum_{k=j+2}^{\infty} |a_k - a_{k-1} + b_{k-1} - b_{k-2}|\right) < \infty.$$

If this is the case, the operator norm of Δ_{ab} is less than or equal to $\mathcal{M}_2$.

(4) *The operator Δ_{ab} is compact on* bv *if and only if*

$$\sum_{k=0}^{\infty} |a_k - a_{k-1} + b_{k-1} - b_{k-2}| < \infty, \quad \lim_{j\to\infty} a_j = \lim_{j\to\infty} b_j = 0.$$

In Theorem 1.1, there is a gap between ℓ^∞ and bv as the following example shows:

EXAMPLE 1.1. The operator Δ_{ab} need not be bounded on the space bv even though (a_k) and (b_k) have finite limits as the example of $a_k = \frac{(-1)^k}{k+1}$ and $b_k = 0$ shows.

Before we discuss a more detailed description, we give further examples.

EXAMPLE 1.2. Suppose that $b_k \neq 0$ for all k and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$. If $\lambda = a_j$ for infinitely many j, then $\lambda = 0$ does not belong to the point spectrum; $\lambda = a_j = 0 \notin \sigma_{\mathrm{p}}(\Delta_{ab}, \ell^\infty)$.

It should be observed that if $b_k = 0$ for some k, then we will need to decompose the action of Δ_{ab} into the direct sum. So to avoid this situation, we assume that $b_k \neq 0$ for any k.

EXAMPLE 1.3. Suppose that $b_k \neq 0$ for all k and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$. If $\lambda = a_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$ for finitely many n_0, we may not be able to generally determine whether $0 \in \sigma_{\mathrm{p}}(\Delta_{ab}, \ell^\infty)$ or $0 \notin \sigma_{\mathrm{p}}(\Delta_{ab}, \ell^\infty)$. This will depend on the choice of (a_k) and (b_k).

To avoid the situation in Examples 1.2 and 1.3, it seems natural to assume that $a_k \neq 0$ for all k. Thus, the following assumptions are natural.

ASSUMPTION 1.1. Let (a_k) and (b_k) satisfy

$$a_k, b_k \neq 0 \quad (k \in \mathbb{N}_0), \quad \lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0. \tag{1}$$

ASSUMPTION 1.2. Let (a_k) and (b_k) satisfy

$$\sum_{k=0}^{\infty} |a_{k+1} - a_k + b_k - b_{k-1}| < \infty.$$

So, based on Assumption 1.1, we consider the case of the small perturbation of the zero operator.

We write

$$A := \{a_k : k = 0, 1, 2, \ldots\}. \tag{2}$$

Under Assumption 1.1, together with Assumption 1.2 in the case of bv, the operator Δ_{ab} will be compact in ℓ^∞ and bv as we will see in the next theorem.

THEOREM 1.2. *Suppose we have sequences (a_k) and (b_k) satisfying Assumption 1.1. Let $\mu \in \{\ell^\infty, \mathrm{bv}\}$ and assume in addition Assumption 1.2 if $\mu = \mathrm{bv}$. Then $\Delta_{ab} : \mu \mapsto \mu$ is a compact operator and the following hold:*

(1) $\sigma(\Delta_{ab}, \mu) = A \cup \{0\}$.

(2) $\sigma_{\mathrm{p}}(\Delta_{ab}, \mu) = A$.

(3) $\sigma_{\mathrm{r}}(\Delta_{ab}, \mu) = \{0\}$.

(4) $\sigma_{\mathrm{c}}(\Delta_{ab}, \mu) = \varnothing$.

(5) $\mathrm{III}_2\sigma(\Delta_{ab}, \mu) = \{0\}$.

(6) $\mathrm{III}_3\sigma(\Delta_{ab}, \mu) = A$.

No wonder that with different assumptions we obtain different results; it is a well-known fact that if assumptions are different, the results may (although they need not) be different. This will be demonstrated in determining the point spectrum by the following example.

EXAMPLE 1.4. Let (a_k) and (b_k) satisfying $\lim_{k\to\infty} b_k = 0$ and $a_0 = a_1 = a_2 = \cdots = 0$. Suppose that $(\Delta_{ab} - \lambda \mathrm{I})x = 0$, $x = (x_k) \in \ell_\infty$. Then

$$\lambda x_0 = 0, \qquad b_k x_k - \lambda x_{k+1} = 0, \qquad k \in \mathbb{N}_0.$$

It is clear that $\lambda \notin \sigma_{\mathrm{p}}(\Delta_{ab}, \ell_\infty)$ for all $\lambda \neq 0$. Now, $\lambda = 0 \notin \sigma_{\mathrm{p}}(\Delta_{ab}, \ell_\infty)$ whenever $b_k \neq 0$ for all $k \in \mathbb{N}_0$. But $\lambda = 0 \in \sigma_{\mathrm{p}}(\Delta_{ab}, \ell_\infty)$ whenever $b_k = 0$ for some $k \in \mathbb{N}_0$. Similar results can be obtained in bv.

Note that if $b_0 = b_1 = b_2 = \cdots = 0$ and $\lim\limits_{k\to\infty} a_k = 0$, then the results remain valid in Theorem 1.2.

Next, we consider another assumption.

ASSUMPTION 1.3. Let (a_k) and (b_k) have finite limits and

$$b_k \neq 0 \quad (k \in \mathbb{N}_0), \qquad \lim_{k\to\infty} a_k = a, \qquad \lim_{k\to\infty} b_k = b \neq 0.$$

In Assumption 1.3, we assumed $b_j \neq 0$ for any j. This is a reasonable assumption. In fact, $\lim\limits_{j\to\infty} b_j \neq 0$ implies b_j can be zero only for a finite number of j. Choosing the maximal j with $b_j = 0$, we can decompose the operator Δ_{ab} into direct sum of two operators. One is generated by a finitely supported (small) matrix and the other is an operator satisfying Assumption 1.3. To avoid the effect of this small matrix, we assume $b_j \neq 0$ for any j.

If $a_k = 1$ and $b_k = -1$ for all $k \in \mathbb{N}_0$, then Assumptions 1.2 and 1.3 are satisfied. In this case Δ_{ab} is nothing but the difference operator Δ.

Now, under Assumption 1.3, let $\lambda \in \partial\Delta(a, |b|) = \{\lambda \in \mathbb{C} : |\lambda - a| = |b|\}$. Then we have

$$\lim_{i\to\infty} |\lambda - a_i| = |\lambda - a| = |b| \neq 0.$$

Let $N(\lambda)$ denote the smallest $N \in \mathbb{N}$ for which $\lambda \neq a_j$ for all $j \geqslant N$. Based on this notation, consider the set

$$K_\infty := \left\{ \lambda \in A \cap \partial\Delta(a, |b|) : \sup_{k\geqslant 1} \prod\nolimits_{i=N(\lambda)}^{k+N(\lambda)} \left| \frac{b_{i-1}}{\lambda - a_i} \right| < \infty \right\}.$$

We have the following main result:

THEOREM 1.3. *Let the sequences (a_k) and (b_k) satisfy Assumption 1.3. Then, the operator $\Delta_{ab} : \ell^\infty \to \ell^\infty$ is a bounded operator with*

$$\|\Delta_{ab}\|_{\ell^\infty} = \sup_{j\in\mathbb{N}_0} \left\{ |a_j| + |b_{j-1}| \right\}.$$

Furthermore, the following hold:

(1) $\sigma(\Delta_{ab}, \ell^\infty) = \overline{\Delta}(a, |b|) \cup \left(A \cap \left(\mathbb{C} \setminus \overline{\Delta}(a, |b|) \right) \right)$.

(2) $\sigma_{\mathrm{p}}(\Delta_{ab},\ell^\infty) = \left(A \cap \left(\mathbb{C} \setminus \overline{\Delta}(a,|b|)\right)\right) \cup K_\infty$.

Some difficulties caused some problems in characterizing the residual and the continuous spectra of the operator $\Delta_{ab} : \ell^\infty \to \ell^\infty$ in general. However, we give specific results; see Theorems 1.4, 1.5, 1.6 and Corollary 1.1 below. We hope to have advances regarding these results in another publication.

THEOREM 1.4. *Assume that Assumption 1.3 holds and that* $\sum_{k=0}^\infty |a_k + b_k|$ *is convergent (clearly, in this case* $a = -b \neq 0$*). If* $0 \notin \sigma_{\mathrm{p}}(\Delta_{ab},\ell^\infty)$*, then* $0 \in \sigma_{\mathrm{r}}(\Delta_{ab},\ell^\infty)$.

We assert that the class of the operator Δ_{ab}, under the assumptions in Theorem 1.4, includes in particular the generalized difference operator Δ_υ investigated in [7, 46].

The following theorem gives an inclusion region for the residual spectrum.

THEOREM 1.5. *Under Assumption 1.3, the following hold:*

(1) $\sigma_{\mathrm{r}}(\Delta_{ab},\ell^\infty) \cup \sigma_{\mathrm{c}}(\Delta_{ab},\ell^\infty) = \overline{\Delta}(a,|b|) \setminus K_\infty$.

(2) $\{\lambda \in \mathbb{C} : \sum_{k=0}^\infty |\lambda - a_k - b_k| < \infty\} \setminus \sigma_{\mathrm{p}}(\Delta_{ab},\ell^\infty) \subseteq \sigma_{\mathrm{r}}(\Delta_{ab},\ell^\infty)$.

(3) $A \cap \left(\overline{\Delta}(a,|b|) \setminus K_\infty\right) \subseteq \sigma_{\mathrm{r}}(\Delta_{ab},\ell^\infty)$.

Note that $\{\lambda \in \mathbb{C} : \sum_{k=0}^\infty |\lambda - a_k - b_k| < \infty\}$ contains at most one element.

A complete characterization of the spectra of the difference operator Δ on ℓ^∞ is settled in the following main theorem of the present paper.

THEOREM 1.6. *The following hold:*

(1) $\sigma(\Delta,\ell^\infty) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leqslant 1\}$.

(2) $\sigma_{\mathrm{p}}(\Delta,\ell^\infty) = \varnothing$.

(3) $\sigma_{\mathrm{r}}(\Delta,\ell^\infty) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leqslant 1\}$.

(4) $\sigma_{\mathrm{c}}(\Delta,\ell^\infty) = \varnothing$.

(5) $\mathrm{III}_1\sigma(\Delta,\ell^\infty) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$.

(6) $\mathrm{III}_2\sigma(\Delta,\ell^\infty) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\}$.

Theorem 1.6 can be extended easily for the operator $B(r,s)$;

COROLLARY 1.1. *The following hold:*

(1) $\sigma(B(r,s),\ell^\infty) = \{\lambda \in \mathbb{C} : |\lambda - r| \leqslant |s|\}$.

(2) $\sigma_{\mathrm{p}}(B(r,s),\ell^\infty) = \varnothing$.

(3) $\sigma_{\mathrm{r}}(B(r,s),\ell^\infty) = \{\lambda \in \mathbb{C} : |\lambda - r| \leqslant |s|\}$.

(4) $\sigma_{\mathrm{c}}(B(r,s),\ell^\infty)=\varnothing$.

(5) $\mathrm{III}_1\sigma(B(r,s),\ell^\infty)=\{\lambda\in\mathbb{C}:|\lambda-r|<|s|\}$.

(6) $\mathrm{III}_2\sigma(B(r,s),\ell^\infty)=\{\lambda\in\mathbb{C}:|\lambda-r|=|s|\}$.

The reader can easily prove Corollary 1.1 by observing that the spectra of the operator $B(r,s)=-s\Delta+(r+s)\mathrm{I}$ are obtained from the corresponding spectra of Δ by using the transformation $\lambda'=-s\lambda+r+s$ to the λ-plane.

To state the next theorem, we adopt the following additional notation:

$$K_{\mathrm{b}}:=\left\{\lambda\in A\cap\partial\Delta(a,|b|):\left(\prod_{i=N(\lambda)}^{k+N(\lambda)}\frac{b_i}{\lambda-a_{i+1}}\right)\in\mathrm{bv}\right\},$$
$$H_{\mathrm{b}}:=\left\{\lambda\in\partial\Delta(a,|b|):\left(\prod_{i=0}^{k}\frac{\lambda-a_i}{b_i}\right)\in\mathrm{bs}\simeq\mathrm{bv}^*\right\}.$$

One can easily observe that $K_{\mathrm{b}}\subseteq K_\infty\subseteq H_{\mathrm{b}}$.

For any complex number $\lambda\in\mathbb{C}\setminus A$, let $\left(t_k^m\right)$ be a double sequence which is given by

$$t_k^m=t_k^m(\lambda)=\frac{1}{b_k}\sum_{j=0}^{m}\left(\prod_{i=j}^{k}\frac{b_i}{\lambda-a_i}\right),\qquad k,m\in\mathbb{N}_0.$$

and consider the set

$$F_{\mathrm{b}}=\left\{\lambda\in\mathbb{C}\setminus\left(\overline{\Delta}(a,|b|)\cup A\right):\sup_m\sum_{k=0}^{\infty}\left|t_k^m-t_{k+1}^m\right|=\infty\right\}.$$

Note that $F_{\mathrm{b}}=\varnothing$ in the special case where (a_k) and (b_k) are constant sequences.

THEOREM 1.7. *Let (a_k) and (b_k) satisfy Assumptions 1.2 and 1.3. Then*

(1) $\sigma(\Delta_{ab},\mathrm{bv})=\overline{\Delta}(a,|b|)\cup\left(A\cap\left(\mathbb{C}\setminus\overline{\Delta}(a,|b|)\right)\right)\cup F_{\mathrm{b}}$.

(2) $\sigma_{\mathrm{p}}(\Delta_{ab},\mathrm{bv})=\left(A\cap\left(\mathbb{C}\setminus\overline{\Delta}(a,|b|)\right)\right)\cup K_{\mathrm{b}}$.

(3) $\sigma_{\mathrm{r}}(\Delta_{ab},\mathrm{bv})=\Delta(a,|b|)\cup((H_{\mathrm{b}}\cup\{a+b\})\setminus K_{\mathrm{b}})$.

(4) $\sigma_{\mathrm{c}}(\Delta_{ab},\mathrm{bv})=[\partial\Delta(a,|b|)\setminus(H_{\mathrm{b}}\cup\{a+b\})]\cup F_{\mathrm{b}}$.

(5) $\mathrm{III}_{3\sigma}(\Delta_{ab},\mathrm{bv})=\left(A\cap\left(\mathbb{C}\setminus\overline{\Delta}(a,|b|)\right)\right)\cup K_{\mathrm{b}}$.

(6) $\mathrm{III}_{1\sigma}(\Delta_{ab},\mathrm{bv})\cup\mathrm{III}_{2\sigma}(\Delta_{ab},\mathrm{bv})=\Delta(a,|b|)\cup((H_{\mathrm{b}}\cup\{a+b\})\setminus K_{\mathrm{b}})$.

(7) $\mathrm{II}_{2\sigma}(\Delta_{ab},\mathrm{bv})=[\partial\Delta(a,|b|)\setminus(H_{\mathrm{b}}\cup\{a+b\})]\cup F_{\mathrm{b}}$.

Further subdivision of the spectrum will be given in Theorem 5.8.

We organize this paper as follows: Section 2 collects some preliminary facts and results. We prove Theorem 1.1 in Section 3. Section 4 is devoted to the proof of Theorem 1.2, while Section 5 proves Theorems 1.3–1.7.

Here, we explain the standard notation used in this paper.

- Denote the set of all natural numbers by $\mathbb{N}$ and the set of all nonnegative integers by $\mathbb{N}_0$.
- The set of complex numbers is denoted by $\mathbb{C}$.
- The set of all complex sequences is denoted by $\ell^0 = \ell^0(\mathbb{N}_0)$.
- We use the conventions that $\sum_{k=n}^{m} c_k = 0$ and $\prod_{k=n}^{m} c_k = 1$, for any $n,m \in \mathbb{N}_0$ with $n > m$. Also, any term with negative index is equal to zero.
- We write $\theta := (0,0,0,\dots)$.
- Let $r > 0$ and $z \in \mathbb{C}$. Then we write

$$\begin{aligned}
\Delta(z,r) &:= \{\lambda \in \mathbb{C} : |\lambda - z| < r\},\\
\Delta^*(z,r) &:= \{\lambda \in \mathbb{C} : 0 < |\lambda - z| < r\},\\
\overline{\Delta}(z,r) &:= \{\lambda \in \mathbb{C} : |\lambda - z| \leqslant r\},\\
\partial\Delta(z,r) &:= \{\lambda \in \mathbb{C} : |\lambda - z| = r\}.
\end{aligned}$$

2. Preliminary facts and results

To make this paper self-contained, we briefly overview notation used in this paper.

Let μ_1 and μ_2 be two Banach sequence spaces, and let $A = (a_{n,k})$ an infinite matrix. We identify the matrix A with the linear operator $A : \mu_1 \to \mu_2$ if for every sequence $x := (x_k) \in \mu_1$, the sequence

$$Ax = ((Ax)_n) := \left(\sum_{k=0}^{\infty} a_{n,k} x_k\right),$$

the A-transform of x, is in μ_2. We denote the class of all matrices A that maps μ_1 into μ_2 by $(\mu_1 : \mu_2)$. Thus we write $A \in (\mu_1 : \mu_2)$, called matrix transformation from μ_1 into μ_2, if and only if $Ax \in \mu_2$ for every $x \in \mu_1$.

By bs we denote the Banach space of all sequences $x = (x_k)$ for which

$$\|x\|_{\mathrm{bs}} = \sup_{n \in \mathbb{N}_0} \left|\sum_{k=0}^{n} x_k\right|$$

is finite. It is known that $\mathrm{bv} = \mathbb{C} \oplus \mathrm{bv}_0$, where $\mathrm{bv}_0 = \mathrm{bv} \cap c_0$ is a subspace of bv. Then, $\mathrm{bv}^* = \mathbb{C}^* \oplus \mathrm{bv}_0^* = \mathbb{C} \oplus \mathrm{bs} = \mathrm{bs}$. The last isomorphism is given by $(L,(x_1,x_2,x_3,\dots,)) \mapsto (L,x_1,x_2,x_3\dots)$, where $L = \lim_{k\to\infty} x_k$. (cf. [14, 7.5.11(b), p. 385]).

If $T : \mathrm{bv} \to \mathrm{bv}$ is a bounded linear operator with matrix $A = (a_{n,k})$, then its adjoint $T^* : \mathrm{bv}^* \simeq \mathbb{C} \oplus \mathrm{bs} \to \mathrm{bv}^* \simeq \mathbb{C} \oplus \mathrm{bs}$ has the matrix representation of the form (cf. [36])

$$T^* = \begin{pmatrix}
\overline{\psi} & \omega_0 - \overline{\psi} & \omega_1 - \overline{\psi} & \omega_2 - \overline{\psi} & \cdots\\
v_0 & r_{00} - v_0 & r_{10} - v_0 & r_{20} - v_0 & \cdots\\
v_1 & r_{01} - v_1 & r_{11} - v_1 & r_{21} - v_1 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

where $v_n = \lim_{m\to\infty} r_{m,n}$, $\omega_n = \sum_{m=0}^{\infty} r_{n,m}$ and $\overline{\psi} = \lim_{n\to\infty}\omega_n$.

The operator Δ_{ab} may be considered as a generalized summability method, so that it can be represented by the infinite matrix

$$\Delta_{ab} = (a_{nk}) = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \cdots \\ 0 & b_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

So, for the generalized difference operator Δ_{ab} acting on bv, we have

$$\Delta_{ab}^* = \begin{pmatrix} a+b & a_0-a-b & a_1+b_0-a-b & a_2+b_1-a-b & \cdots \\ 0 & a_0 & b_0 & 0 & \cdots \\ 0 & 0 & a_1 & b_1 & \cdots \\ 0 & 0 & 0 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We assume some familiarity with the basic concepts of spectral theory of bounded linear operators, principally the spectrum and the classical subdivisions of the spectrum and we refer to the books by Stone [49], Taylor and Lay [51] and Kubrusly [31] for basic definitions.

Let X be an infinite dimensional complex Banach space. The set of all bounded linear operators on X into itself is denoted by $\mathcal{B}(X)$. For $T \in \mathcal{B}(X)$, we use $\mathcal{R}(T)$ to denote the range of T. Write $T_\lambda = T - \lambda \mathrm{I}$, where λ is a complex number and I is the identity operator acting on X.

There are different types of classification of the *spectrum* of a bounded linear operator on a Banach space; all of them are based on the possible behaviors of $\mathcal{R}(T_\lambda)$ and T_λ^{-1}. The first classification provides a disjoint subdivision of the spectrum into the *point spectrum* $\sigma_{\mathrm{p}}(T,X)$, the *residual spectrum* $\sigma_{\mathrm{r}}(T,X)$ and the *continuous spectrum* $\sigma_{\mathrm{c}}(T,X)$. This is the customary classification of the spectrum [49].

Following Appell et al. [11], more subdivisions of the spectrum can be defined; the *approximate point spectrum* $\sigma_{\mathrm{ap}}(T,X)$, the *defect spectrum* $\sigma_\delta(T,X)$ and the *compression spectrum* $\sigma_{\mathrm{co}}(T,X)$.

Another classification is due to Taylor and Halberg [50]; we can classify the spectrum into seven disjoint sets $\mathrm{I}_2\sigma(T,X)$, $\mathrm{I}_3\sigma(T,X)$, $\mathrm{II}_2\sigma(T,X)$, $\mathrm{II}_3\sigma(T,X)$, $\mathrm{III}_1\sigma(T,X)$, $\mathrm{III}_2\sigma(T,X)$ and $\mathrm{III}_3\sigma(T,X)$.

For the sequel we need the following lemmas.

LEMMA 2.1. [26, 27] *Let T be a bounded linear operator on a complex Banach space X. Then $\mathrm{III}_1\sigma(T,X)$ and $\mathrm{I}_3\sigma(T,X)$ are open sets.*

LEMMA 2.2. [41, Lemma 2.2] *Let T be a linear operator on a Banach sequence space X that has a lower triangular matrix representation $A = (a_{n,k})$. Then the point spectrum of T on X is at most countable. More precisely, we have $\sigma_{\mathrm{p}}(T,X) \subseteq \{a_{n,n} : n \in \mathbb{N}_0\}$.*

The proof of Theorem 1.1 hinges on the following fact on matrix operators on ℓ^1 and ℓ^∞.

PROPOSITION 2.1. *Let $V=(v_{jk})_{j,k\in\mathbb{N}_0}$ be an infinite matrix with complex entries. The following holds:*

(1) *The matrix $V=(v_{jk})_{j,k\in\mathbb{N}_0}$ generates a bounded linear operator on ℓ^∞ into itself if and only if*

$$\mathcal{I}=\sup_{j\in\mathbb{N}_0}\left(\sum_{k=0}^{\infty}|v_{jk}|\right)<\infty.$$

If this is the case, the operator norm of V is equal to $\mathcal{I}$.

(2) *The matrix $V=(v_{jk})_{j,k\in\mathbb{N}_0}$ generates a compact linear operator on ℓ^∞ into itself if and only if*

$$\lim_{L\to\infty}\left\{\sup_{j\in\mathbb{N}_0}\left(\sum_{k=L}^{\infty}|v_{jk}|\right)\right\}=0.$$

(3) *The matrix $V=(v_{jk})_{j,k\in\mathbb{N}_0}$ generates a bounded linear operator on ℓ^1 into itself if and only if*

$$\mathcal{II}=\sup_{j\in\mathbb{N}_0}\left(\sum_{k=0}^{\infty}|v_{kj}|\right)<\infty.$$

If this is the case, the operator norm of V is equal to $\mathcal{II}$.

(4) *The matrix $V=(v_{jk})_{j,k\in\mathbb{N}_0}$ generates a compact linear operator on ℓ^1 into itself if and only if*

$$\lim_{L\to\infty}\left\{\sup_{j\in\mathbb{N}_0}\left(\sum_{k=L}^{\infty}|v_{kj}|\right)\right\}=0.$$

Proof.

1. Although (1) is recorded in [48, Formula (1)], we can quickly prove the assertion if we test the operator on $(e^{i\varphi_0},e^{i\varphi_1},\dots)$ for $\varphi_0,\varphi_1,\dots\in\mathbb{R}$.

2. The “if” part is clear since

$$\lim_{L\to\infty}\|V-V_L\|_{\ell^\infty\to\ell^\infty}=0,$$

where

$$V_L=(\chi_{[0,L]}(k)v_{jk})_{j,k\in\mathbb{N}_0}$$

is a finite rank operator for each $L\in\mathbb{N}$.

The converse, or the “only if” part is proved by contrapositive. Assume

$$\lim_{L\to\infty}\left\{\sup_{j\in\mathbb{N}_0}\left(\sum_{k=L}^{\infty}|v_{jk}|\right)\right\}\neq 0,$$

so that there exists a constant $\kappa > 0$ such that

$$\sup_{j\in\mathbb{N}_0}\left(\sum_{k=L}^{\infty}|v_{jk}|\right) > \kappa > 0$$

for each $L \in \mathbb{N}_0$. Thus we can find $j_L \in \mathbb{N}_0$ such that

$$\sum_{k=L}^{\infty}|v_{j_L,k}| > \kappa > 0.$$

Furthermore, we can also find $K_L > L$ such that

$$\sum_{k=L}^{K_L}|v_{j_L,k}| > \kappa > 0.$$

We define a sequence by the recurrence formula:

$$A_0 = 1, \quad A_{L+1} = K_{A_L} + 1 \quad (L \in \mathbb{N}_0).$$

Then for each L,

$$\sum_{k=A_L}^{A_{L+1}-1}|v_{j_L,k}| = \sum_{k=A_L}^{K_{A_L}}|v_{j_L,k}| > \kappa.$$

If we set

$$x_L = (0,\dots,0,\overline{\mathrm{sgn}}(v_{j_L,A_L}),\overline{\mathrm{sgn}}(v_{j_L,A_L+1}),\dots,\overline{\mathrm{sgn}}(v_{j_L,A_{L+1}-1}),0\dots),$$

then we have

$$\|Vx_L - Vx_{L'}\|_{\ell^\infty} \geqslant \|Vx_L\|_{\ell^\infty} \geqslant \sum_{k=A_L}^{K_{A_L}}|v_{j_L,k}| > \kappa \quad (1 \leqslant L < L' < \infty).$$

This means that $\{Vx_L\}_{L=1}^{\infty}$ is not a Cauchy sequence, while $\{x_L\}_{L=1}^{\infty}$ is bounded. So, V is not compact.

3. Simply take the transpose of (1) using the duality $\ell^1(\mathbb{N}_0)$–$\ell^\infty(\mathbb{N}_0)$ or resort to [48, Formula (77)].

4. Simply take the transpose of (2) using the duality $\ell^1(\mathbb{N}_0)$–$\ell^\infty(\mathbb{N}_0)$. □

3. Proof of Theorem 1.1: Boundedness and compactness of Δ_{ab}

Assertions (1) and (2) are consequences of Proposition 2.1(1)-(2).
We prove (3). We use the isomorphism

$$U : (x_k) \in \ell^1(\mathbb{N}_0) \mapsto \left(\sum_{j=0}^{k} x_j\right) \in \mathrm{bv}$$

and its inverse

$$(x_k) \in \mathrm{bv} \mapsto (x_k - x_{k-1}) \in \ell^1.$$

We consider the following commutative diagram.

$$\begin{array}{ccc}
\ell^1(\mathbb{N}_0) & \xrightarrow{U} & \mathrm{bv} \\
{\scriptstyle K}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_{ab}} \\
\ell^1(\mathbb{N}_0) & \xrightarrow{U} & \mathrm{bv}
\end{array}$$

We calculate

$$\Delta_{ab} \circ Ux = \left(a_k \sum_{j=0}^{k} x_j + b_{k-1} \sum_{j=0}^{k-1} x_j\right) \in \mathrm{bv}$$

and

$$\begin{aligned}
Kx &= U^{-1} \circ \Delta_{ab} \circ Ux \\
&= \left(a_k \sum_{j=0}^{k} x_j + b_{k-1} \sum_{j=0}^{k-1} x_j - a_{k-1} \sum_{j=0}^{k-1} x_j - b_{k-2} \sum_{j=0}^{k-2} x_j\right) \\
&= \left(a_k x_k + (a_k - a_{k-1} + b_{k-1} - b_{k-2}) \sum_{j=0}^{k-2} x_j + (a_k - a_{k-1} + b_{k-1}) x_{k-1}\right) \\
&\in \ell^1.
\end{aligned}$$

Here K is a linear operator given by

$$Kx = \left(a_j x_j + (a_j - a_{j-1} + b_{j-1}) x_{j-1} + (a_j - a_{j-1} + b_{j-1} - b_{j-2}) \sum_{k=0}^{j-2} x_k\right).$$

Using Proposition 2.1(3), we obtain the desired result.

Finally we prove (4). If the conditions in (4) are satisfied, then

$$\lim_{m \to \infty} \sup_{(x_j) \in \mathrm{bv} \setminus \{0\}} \frac{\|\Delta_{ab}(x_j) - \Delta_{ab}((\chi_{[0,m]}(j) x_j)_{j=0}^{\infty})\|_{\mathrm{bv}}}{\|(x_j)\|_{\mathrm{bv}}} = 0.$$

In fact,

$$\begin{aligned}
&\sup_{(x_j) \in \mathrm{bv} \setminus \{0\}} \frac{\|\Delta_{ab}(x_j) - \Delta_{ab}((\chi_{[0,m]}(j) x_j)_{j=0}^{\infty})\|_{\mathrm{bv}}}{\|(x_j)\|_{\mathrm{bv}}} \\
&\leqslant \sup_{j \in \mathbb{N}_0} |\chi_{[m+1,\infty)}(j) a_j| \\
&\quad + \sup_{j \in \mathbb{N}_0} |\chi_{[m+1,\infty)}(j+1) a_{j+1} - \chi_{[m+1,\infty)}(j) a_j + \chi_{[m+1,\infty)}(j) b_j| \\
&\quad + \sup_{j \in \mathbb{N}_0} \sum_{k=j+2}^{\infty} |\chi_{[m+1,\infty)}(k) a_k - \chi_{[m+2,\infty)}(k)(a_{k-1} - b_{k-1}) - \chi_{[m+3,\infty)}(k) b_{k-2}|.
\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}
&\sup_{(x_j)\in \mathrm{bv}\setminus\{0\}} \frac{\|\Delta_{ab}(x_j)-\Delta_{ab}((\chi_{[0,m]}(j)x_j)_{j=0}^{\infty})\|_{\mathrm{bv}}}{\|(x_j)\|_{\mathrm{bv}}}\\
&\leqslant \sup_{j\in\mathbb{N}_0\cap[m+1,\infty)} |a_j|+|a_{m+1}|+\sup_{j\in\mathbb{N}_0\cap[m+1,\infty)} |a_{j+1}-a_j+b_j|\\
&\quad +|a_m|+|a_{m-1}|+|b_m|+|b_{m-1}|+|a_{m-1}|+|a_{m-2}|+|b_{m-1}|+|b_{m-2}|\\
&\quad +\sup_{j\in\mathbb{N}_0\cap[m+1,\infty)} \sum_{k=j+2}^{\infty}|a_k-a_{k-1}+b_{k-1}-b_{k-2}|\\
&\to 0
\end{aligned}$$

as $m\to\infty$. So Δ_{ab} is a compact operator.

Conversely, assume that Δ_{ab} is a compact operator. Observe that the j-th elementary vector e_j is mapped to $a_je_j+b_je_{j+1}$ for each j. Thus, if $\lim\limits_{j\to\infty} a_j\neq 0$ or $\lim\limits_{j\to\infty} b_j\neq 0$ then $(a_je_j+b_je_{j+1})$ does not have any convergent subsequence. In fact, if this is the case, there exists $\kappa>0$ so that either $|a_j|>\kappa$ for infinitely many j or $|b_j|>\kappa$ for infinitely many j. Let us write $\mathcal{J}$ for the set of all such j. Then if $j,j'\in\mathcal{J}$ satisfies $j+2<j'$, then $\|a_je_j+b_je_{j+1}-a_{j'}e_{j'}-b_{j'}e_{j'+1}\|_{\mathrm{bv}}>\kappa$. Thus $(a_je_j+b_je_{j+1})$ does not have any convergent subsequence. So the conditions $\lim\limits_{j\to\infty} a_j=0$ and $\lim\limits_{j\to\infty} b_j=0$ are necessary for Δ_{ab} is compact. Since the compactness of Δ_{ab} implies that Δ_{ab} is bounded, $\sum_{j=0}^{\infty}|a_j-a_{j-1}+b_{j-1}-b_{j-2}|<\infty$ is necessary as well.

4. Proof of Theorem 1.2: Spectra of Δ_{ab} (compact case)

We specify the spectra of the generalized difference operator Δ_{ab} as a compact linear operator in the sequence space ℓ^∞, as well as in the sequence space bv. To the best of our knowledge, no contribution has appeared so far to study this problem in the space ℓ^∞ even in the case of the operator $B(r,s)$. Theorem 1.2 will be decomposed into Theorems 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7 and 4.8, which will be proved in the next two subsections.

4.1. The case of ℓ^∞

Throughout this subsection, we assume that Assumption 1.1 holds. It follows from Theorem 1.1(2) that the operator Δ_{ab} is compact on ℓ^∞.

THEOREM 4.1. $\sigma_{\mathrm{p}}(\Delta_{ab},\ell^\infty)=A$.

Proof. Applying Lemma 2.2, we learn that the only possible eigenvalues of Δ_{ab} are the diagonal elements; $\sigma_{\mathrm{p}}(\Delta_{ab},\ell^\infty)\subseteq A$. To prove the reverse inclusion, let $\lambda=a_{n_0}$, for some fixed $n_0\in\mathbb{N}_0$, and consider the relation $(\Delta_{ab}-\lambda\mathrm{I})x=\theta$ for some $x=(x_k)\in\ell^0(\mathbb{N}_0)$. Then, as a candidate of the solution x, we define $(x_j)_{j=n_0}^{\infty}$ which is subject to the following recurrence formula

$$x_{n_0}=1, \qquad b_kx_k+(a_{k+1}-\lambda)x_{k+1}=0, \qquad \text{for all } k\geqslant n_0,$$

where it is understood that n_0 is the largest k so that $\lambda = a_k = a_{n_0}$. Hence $a_k \neq a_{n_0}$ for all $k > n_0$. We also define $x_j = 0$ if $j < n_0$. Since $b_k \neq 0$ for all $k \in \mathbb{N}_0$, each x_k is well defined and satisfies $x_k \neq 0$ as long as $k \geqslant n_0$. We see that $x \in \ell^\infty$ since

$$\lim_{k\to\infty}\left|\frac{x_{k+1}}{x_k}\right| = \lim_{k\to\infty}\left|\frac{b_k}{\lambda - a_{k+1}}\right| = 0.$$

Consequently, we conclude $\lambda = a_{n_0} \in \sigma_{\mathrm{p}}(\Delta_{ab}, \ell^\infty)$. This completes the proof. □

Now, we arrive at one of the main theorems of this subsection.

THEOREM 4.2. $\sigma(\Delta_{ab}, \ell^\infty) = A \cup \{0\}$.

Proof. Since Δ_{ab} is compact, the spectrum consists of the point zero together with the non-zero eigenvalues. □

THEOREM 4.3. $\sigma_{\mathrm{r}}(\Delta_{ab}, \ell^\infty) = \{0\}$ *and* $\sigma_{\mathrm{c}}(\Delta_{ab}, \ell^\infty) = \varnothing$.

Proof. Using Theorems 4.1 and 4.2, we obtain $\sigma_{\mathrm{r}}(\Delta_{ab}, \ell^\infty) \cup \sigma_{\mathrm{c}}(\Delta_{ab}, \ell^\infty) = \{0\}$. So we need to specify which set does 0 belong to. But, the range of the operator Δ_{ab} is contained in c_0. In fact, let $x = (x_n) \in \ell^\infty$. Then $y = (y_n) = \Delta_{ab}x$, where $y_n = a_n x_n + b_{n-1}x_{n-1}$. Then

$$|y_n| \leqslant \sup_k |x_k| \left(|a_n| + |b_{n-1}|\right) \to 0 \qquad \text{as } n \to \infty.$$

That is, $\mathcal{R}(\Delta_{ab}) \subseteq c_0$. So, since c_0 is closed in ℓ^∞, $\mathcal{R}(\Delta_{ab})$ is not dense in ℓ^∞. Thus $0 \in \sigma_{\mathrm{r}}(\Delta_{ab}, \ell^\infty)$. The second statement follows directly from the fact that

$$\sigma_{\mathrm{c}}(\Delta_{ab}, \ell^\infty) = \sigma(\Delta_{ab}, \ell^\infty) \setminus \left(\sigma_{\mathrm{p}}(\Delta_{ab}, \ell^\infty) \cup \sigma_{\mathrm{r}}(\Delta_{ab}, \ell^\infty)\right).$$

This completes the proof. □

Furthermore, we have the following:

THEOREM 4.4. *The following hold:*

(1) $\mathrm{III}_2\sigma(\Delta_{ab}, \ell^\infty) = \{0\}$.

(2) $\mathrm{III}_3\sigma(\Delta_{ab}, \ell^\infty) = A$.

Proof.

(1) As in Lemma 2.1, $\mathrm{III}_1\sigma(\Delta_{ab}, \ell^\infty)$ is an open set. Meanwhile,

$$\mathrm{III}_1\sigma(\Delta_{ab}, \ell^\infty) \cup \mathrm{III}_2\sigma(\Delta_{ab}, \ell^\infty) = \sigma_{\mathrm{r}}(\Delta_{ab}, \ell^\infty) = \{0\}.$$

Thus, $\mathrm{III}_1\sigma(\Delta_{ab}, \ell^\infty)$ is empty and $\mathrm{III}_2\sigma(\Delta_{ab}, \ell^\infty) = \{0\}$.

(2) Let $\lambda = a_{n_0} \in \sigma_p(\Delta_{ab}, \ell^\infty)$ with $n_0 \in \mathbb{N}_0$. Then $\mathcal{R}(\Delta_{ab} - \lambda I)$ is contained in

$$\{y = (y_0, y_1, \dots, y_{n_0}, \dots) \in \ell^\infty : r_0 y_0 + r_1 y_1 + \dots + r_{n_0} y_{n_0} = 0\}.$$

for some $(r_0, \dots, r_{n_0}) \in \mathbb{R}^{n_0+1} \setminus \{0\}$. So, $\mathcal{R}(\Delta_{ab} - \lambda I)$ is not dense. Note that $I_3\sigma(\Delta_{ab}, \ell^\infty) = \varnothing$ by Lemma 2.1. Thus, $I_3\sigma(\Delta_{ab}, \ell^\infty) = II_3\sigma(\Delta_{ab}, \ell^\infty) = \varnothing$. It remains to use Theorem 4.1 and the fact that

$$\sigma_p(\Delta_{ab}, \ell^\infty) = I_3\sigma(\Delta_{ab}, \ell^\infty) \cup II_3\sigma(\Delta_{ab}, \ell^\infty) \cup III_3\sigma(\Delta_{ab}, \ell^\infty). \qquad \square$$

4.2. The case of bv

In this subsection, we assume that Assumption 1.1 holds, as well as Assumption 1.2. That is, the operator Δ_{ab} is compact on bv; see Theorem 1.1(4).

THEOREM 4.5. $\sigma_p(\Delta_{ab}, \mathrm{bv}) = A$.

Proof. The proof is similar to that of Theorem 4.1, and so is omitted. $\square$

LEMMA 4.1. $\sigma_p(\Delta_{ab}^*, \mathrm{bv}^*) = \sigma_p(\Delta_{ab}^*, \mathrm{bs}) = A \cup \{0\}$.

Proof. Suppose $\Delta_{ab}^* f = \lambda f$ for $f = (f_k) \neq \theta$ in $\mathrm{bs} \simeq \mathrm{bv}^*$. Then

$$\begin{aligned} -\lambda f_0 + \textstyle\sum_{k=0}^{\infty} (a_k + b_{k-1}) f_{k+1} &= 0, \\ (a_k - \lambda) f_{k+1} + b_k f_{k+2} &= 0, \qquad \text{for all } k \in \mathbb{N}_0. \end{aligned}$$

It is observed that $(1,0,0,\dots)$ is an eigenvector associated with the eigenvalue $\lambda = 0$. That is, $0 \in \sigma_p(\Delta_{ab}^*, \mathrm{bv}^*)$. Moreover, one can easily show that $A \subseteq \sigma_p(\Delta_{ab}^*, \mathrm{bv}^*)$. Furthermore, for all $\lambda \notin A \cup \{0\}$, we have

$$\lim_{k\to\infty} \left| \frac{f_{k+1}}{f_k} \right| = \lim_{k\to\infty} \left| \frac{\lambda - a_{k-1}}{b_{k-1}} \right| = \infty.$$

That is, $f = (f_k) \notin \mathrm{bs} \simeq \mathrm{bv}^*$. This completes the proof. $\square$

Similar to Theorem 4.2, we can specify $\sigma(\Delta_{ab}, \mathrm{bv})$:

THEOREM 4.6. $\sigma(\Delta_{ab}, \mathrm{bv}) = A \cup \{0\}$.

THEOREM 4.7. $\sigma_r(\Delta_{ab}, \mathrm{bv}) = \{0\}$ *and* $\sigma_c(\Delta_{ab}, \mathrm{bv}) = \varnothing$.

Proof. Simply observe that

$$\sigma_r(\Delta_{ab}, \mathrm{bv}) = \sigma_p(\Delta_{ab}^*, \mathrm{bv}^*) \setminus \sigma_p(\Delta_{ab}, \mathrm{bv}),$$

(cf. [11, Relation 1.56, Proposition 1.3(e)]) and then apply Theorem 4.5 and Lemma 4.1. The second result follows immediately;

$$\sigma_c(\Delta_{ab}, \mathrm{bv}) = \sigma(\Delta_{ab}, \mathrm{bv}) \setminus \left(\sigma_p(\Delta_{ab}, \mathrm{bv}) \cup \sigma_r(\Delta_{ab}, \mathrm{bv})\right).$$

Alternatively, the result follows from the fact that the range of the operator Δ_{ab} is contained in $c_0 \cap \mathrm{bv} =: \mathrm{bv}_0$, so that the range is not dense in bv. $\square$

Furthermore, the following theorem gives the spectra of Δ_{ab} on bv, where its proof is omitted since it is similar to that of Theorem 4.4.

THEOREM 4.8. *The following hold:*

(1) $\mathrm{III}_2\sigma(\Delta_{ab}, \mathrm{bv}) = \{0\}$.

(2) $\mathrm{III}_3\sigma(\Delta_{ab}, \mathrm{bv}) = A$.

REMARK 4.1. The reader can check that

$$\sigma_\delta(\Delta_{ab}, \mu) = \sigma_{\mathrm{co}}(\Delta_{ab}, \mu) = \sigma_{\mathrm{ap}}(\Delta_{ab}, \mu) = \sigma(\Delta_{ab}, \mu), \quad \mu \in \{\ell^\infty, \mathrm{bv}\}.$$

That is, no subdivision of the spectrum with respect to

$$\left\{\sigma_\delta(\Delta_{ab}, \mu), \sigma_{\mathrm{co}}(\Delta_{ab}, \mu), \sigma_{\mathrm{ap}}(\Delta_{ab}, \mu)\right\}$$

for each $\mu \in \{\ell^\infty, \mathrm{bv}\}$. However, under Assumptions 1.2 and 1.3, one will have a different situation, as presented in the next section.

5. Proof of Theorems 1.3-1.7: Spectra of Δ_{ab} (noncompact case)

Throughout this section we assume that Assumption 1.3 holds.

5.1. Proof of Theorem 1.3

COROLLARY 5.1. *The operator Δ_{ab} is in the class $(\ell^\infty : \ell^\infty)$ with*

$$\|\Delta_{ab}\| = \sup_{j \in \mathbb{N}_0} \left\{|a_j| + |b_{j-1}|\right\}.$$

Proof. The proof follows by a direct application of Theorem 1.1(1) $\square$

THEOREM 5.1. *The spectrum of the operator Δ_{ab} is*

$$\sigma(\Delta_{ab}, \ell^\infty) = \overline{\Delta}(a, |b|) \cup \left(A \cap \left(\mathbb{C} \setminus \overline{\Delta}(a, |b|)\right)\right).$$

Proof. From [6, Theorem 2] (see also [5, Theorem 2.2]), we have $\sigma(\Delta_{ab}, c) = \overline{\Delta}(a, |b|) \cup \left(A \cap \left(\mathbb{C} \setminus \overline{\Delta}(a, |b|)\right)\right)$. Now, the required result follows by applying [16, Lemma 4.8, *p.* 33]; $\sigma(\Delta_{ab}, c) = \sigma(\Delta_{ab}, \ell^\infty)$. $\square$

THEOREM 5.2. *The point spectrum of Δ_{ab} on ℓ^∞ is*

$$\sigma_{\mathrm{p}}(\Delta_{ab}, \ell^\infty) = \left(A \cap \left(\mathbb{C} \setminus \overline{\Delta}(a, |b|)\right)\right) \cup K_\infty.$$

Proof. From Lemma 2.2, we know that the set $A=\{a_k : k\in\mathbb{N}_0\}$ contains the eigenvalues; $\sigma_{\rm p}(\Delta_{ab},\ell^\infty)\subseteq A$. Now, let $\lambda\in A$. That is, $\lambda=a_{k_0}$ for some $k_0\in\mathbb{N}_0$. Consider the eigenvalue problem $\Delta_{ab}x=a_{k_0}x$ for $x=(x_k)$ in ℓ^∞. We obtain the system

$$b_{k-1}x_{k-1}+\left(a_k-a_{k_0}\right)x_k=0,\;\; k\in\mathbb{N}_0. \tag{3}$$

Then, we have two cases; $a_{k_0}=a$, or $a_{k_0}\neq a$.

If $a_{k_0}=a$, then $a_{k_0}\notin\sigma_{\rm p}(\Delta_{ab},\ell^\infty)$ since, in this case, any nonzero solution for (3) will not be an element in ℓ^∞. In fact, we obtain either $x=\theta$ or

$$\lim_{k\to\infty}\left|\frac{x_{k+1}}{x_k}\right|=\infty.$$

In the other case, $a_{k_0}\neq a$, we note that $a_{k_0}=a_k\neq a$ for finitely many k since $\lim_{j\to\infty}\left|a_j-a_{k_0}\right|=\left|a-a_{k_0}\right|\neq 0$. Put

$$k_1=\max\{k\in\mathbb{N}_0 : a_k=a_{k_0}\}.$$

If $x_{k_1}=0$, then $x=\theta$. Otherwise, from (3), we have

$$\lim_{k\to\infty}\left|\frac{x_{k+1}}{x_k}\right|=\left|\frac{b}{a_{k_0}-a}\right|.$$

Therefore, $a_{k_0}\notin\sigma_{\rm p}(\Delta_{ab},\ell^\infty)$ if $a_{k_0}\in\Delta^*(a,|b|)$, while $a_{k_0}\in\sigma_{\rm p}(\Delta_{ab},\ell^\infty)$ if $a_{k_0}\in A\cap\left(\mathbb{C}\diagdown\overline{\Delta}(a,|b|)\right)$. Finally, for $a_{k_0}\in A\cap\partial\Delta(a,|b|)$, we observe that any nonzero solution satisfies

$$x_k=x_{j_1}\prod\nolimits_{i=j_1}^{k-1}\frac{b_i}{a_{k_0}-a_{i+1}},$$

for $k>j_1$. Thus, $a_{k_0}\in\sigma_{\rm p}(\Delta_{ab},\ell^\infty)$ if $a_{k_0}\in K_\infty$. This completes the proof. $\square$

5.2. Proof of Theorem 1.4

To prove Theorem 1.4 we need to review some information about the space f of almost convergent sequences [14, p. 17]:

Let

$$f=\left\{x=(x_k)\mid \frac{1}{n+1}\sum\nolimits_{k=p}^{n+p}x_k \text{ converges } (as\ n\to\infty) \text{ uniformly over } p\right\}.$$

Then f is a sequence space. The notion of convergence generated in this way is called almost convergence, and the elements of f are called almost convergent sequences. Further, let

$$f_0:=\left\{x=(x_k)\mid \frac{1}{n+1}\sum\nolimits_{k=p}^{n+p}x_k\to 0\ (as\ n\to\infty) \text{ uniformly over } p\right\}.$$

Then f_0 is a sequence subspace of f.

Important properties of almost convergence are stated in the following lemma.

LEMMA 5.1. [14, Theorem 1.2.18, *p.* 17] *We have* $f_0 \subsetneqq f \subsetneqq \ell^\infty$.

LEMMA 5.2. [14, Lemma 2.4.11, *p.* 56, Exercise 2.9.11, *p.* 97]*We have* $\mathrm{bs} \subsetneqq f_0$. *Moreover, the space* bs *is a proper dense subspace in* $(f_0, \|.\|_{\ell^\infty})$.

REMARK 5.1. From Lemmas 5.1 and 5.2, we conclude that bs is not dense in ℓ^∞.

Proof of Theorem 1.4. It can be shown that the range of the operator Δ_{ab} is contained in bs. Indeed, let $x = (x_k) \in \ell^\infty$. Then $y = \Delta_{ab}x = (y_k)$, where

$$y_k = a_k x_k + b_{k-1}x_{k-1}.$$

Then

$$\sup_{n\in\mathbb{N}_0} \left|\sum_{k=0}^{n} y_k\right| \leqslant \sup_{n\in\mathbb{N}_0} |x_n| \left(\sum_{k=0}^{\infty} |a_k + b_k| + \sup_{n\in\mathbb{N}_0} |b_n|\right) < \infty.$$

Therefore, for all $x = (x_n) \in \ell^\infty$, we have $\Delta_{ab}x = y \in \mathrm{bs}$. That is,

$$\mathcal{R}(\Delta_{ab}) \subseteq \mathrm{bs}.$$

So

$$\overline{\mathcal{R}(\Delta_{ab})} \subseteq f_0 \neq \ell^\infty.$$

This implies that $\mathcal{R}(\Delta_{ab})$ is not dense in ℓ^∞. The result follows immediately. $\square$

5.3. Proof of Theorem 1.5

Proof.

(1) Theorems 5.1 and 5.2 directly imply that

$$\sigma_{\mathrm{r}}(\Delta_{ab}, \ell^\infty) \cup \sigma_{\mathrm{c}}(\Delta_{ab}, \ell^\infty) = \overline{\Delta}(a, |b|) \diagdown K_\infty.$$

(2) The proof follows by a similar argument as in the proof of Theorem 1.4.

(3) If $\lambda = a_{n_0} \in \overline{\Delta}(a, |b|) \diagdown K_\infty$ for some $n_0 \in \mathbb{N}_0$, then

$$\mathcal{R}(\Delta_{ab} - \lambda \mathrm{I}) \subseteq \{y = (y_0, y_1, \ldots, y_{n_0}, \ldots) \in \ell^\infty : y_0 + r_0 y_1 + \cdots + r_{n_0} y_{n_0} = 0\},$$

for some $(r_0, \ldots, r_{n_0}) \in \mathbb{R}^{n_0+1} \setminus \{0\}$. Therefore $\lambda = a_{n_0} \in \sigma_{\mathrm{r}}(\Delta_{ab}, \ell^\infty)$. $\square$

5.4. Proof of Theorem 1.6

Statements (1) and (2) follow immediately from Theorem 1.3 for $a_k = 1$ and $b_k = -1$, for all $k \in \mathbb{N}_0$.

Now, we prove Statement (3). The proof is via three steps:

Step 1 *The set* $\{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$ *is contained in* $\sigma_r(\Delta, \ell^\infty)$.

Indeed, for any $\lambda \in \Delta(1,1) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$, $\Delta - \lambda \mathrm{I}$ does not have dense range. In fact, if we let

$$y_0 = (1-\lambda)x_0, \qquad y_1 = -x_0 + (1-\lambda)x_1, \qquad y_2 = -x_1 + (1-\lambda)x_2, \ldots,$$

then we have

$$y_0 + (1-\lambda)y_1 + (1-\lambda)^2 y_2 + \cdots = 0.$$

So, the range of $\Delta - \lambda \mathrm{I}$ is included in a closed subspace whose codimension is 1. Thus, any element in $\Delta(1,1)$ is the residual spectrum of Δ.

Step 2 *The set* $\{\lambda \in \mathbb{C} : |\lambda - 1| = 1\}$ *is also contained in* $\sigma_r(\Delta, \ell^\infty)$.

Let $\lambda \in \partial\Delta(1,1) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\}$ and define $U(x_n) = (\rho^n x_n)$ with $\rho = (1-\lambda)^{-1}$. Then $U^{-1}(\Delta - \lambda \mathrm{I})U(x_n) = (1-\lambda)\Delta(x_n)$. Consequently, $\lambda \in \sigma_r(\Delta, \ell^\infty)$ if and only if $0 \in \sigma_r(\Delta, \ell^\infty)$. We will show that $0 \in \sigma_r(\Delta, \ell^\infty)$ by checking that the distance d between $(1,1,1,\ldots)$ and the range of Δ is more than or equal to 1, so that $\partial\Delta(1) \subset \sigma_r(\Delta, \ell^\infty)$.

Let $d' > d$ be fixed and $(y_n) \in \ell^\infty$ be chosen suitably. Then we have $|1 - y_n + y_{n-1}| < d'$, so that $1 - d' < \mathrm{Re}(y_n - y_{n-1}) < 1 + d'$. If $d < 1$, then by choosing $d' \in (d,1)$, we would have $\mathrm{Re}(y_n) \to \infty$. This contradicts $(y_n) \in \ell^\infty$.

Alternatively, from Theorem 1.4, it follows that $0 \in \sigma_r(\Delta, \ell^\infty)$.

It follows from Steps 1 and 2 that $\overline{\Delta}(1,1) \subseteq \sigma_r(\Delta, \ell^\infty)$.

Step 3 *The set* $\sigma_r(\Delta, \ell^\infty)$ is actually contained in $\overline{\Delta}(1,1)$.

It follows from the Neumann expansion that $\sigma(\Delta_{ab}, \ell^\infty) \subseteq \overline{\Delta}(1,1)$.

This completes the proof of Statement (3).

Statement (4) can be easily observed.

To prove Statements (5) and (6) of Theorem 1.6 we need to review some information about the *minimum modulus* of an operator. Let $T : X \longrightarrow X$ be a bounded linear operator, where X is a normed space. Define

$$\mu(T) = \inf_{\substack{x \in X \\ \|x\| = 1}} \|Tx\|.$$

We call $\mu(T)$ the *minimum modulus* of T [26]. It is non-negative real number. If λ is in the boundary of the spectrum $\sigma(T,X)$, then $\mu(T - \lambda \mathrm{I}) = 0$ [26, Theorem 3.2]. For each scalar $\lambda \in \mathbb{C}$, define

$$\Phi(\lambda) = \mu(T - \lambda \mathrm{I}).$$

It is clear that Φ is a continuous non-negative function. Further, since Φ is continuous, the set $\{\lambda \in \mathbb{C} : \Phi(\lambda) > 0\}$ is an open set. We have

$$\{\lambda \in \mathbb{C} : \Phi(\lambda) > 0\} = \rho(T,X) \cup \mathrm{III}_1\sigma(T,X);$$

see [26, page 19]. However, any connected open subset of the set $\{\lambda \in \mathbb{C} : \Phi(\lambda) > 0\}$ lies entirely in $\rho(T,X)$ or entirely in $\mathrm{III}_1\sigma(T,X)$ [26, Theorem 3.4].

PROPOSITION 5.1. [26, Theorem 3.8] *Let $T : X \longrightarrow X$ be a bounded linear operator on a normed space X. Suppose $\Phi(\lambda_0) > 0$. Let*

$$E = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r\},$$

where

$$r = \left\{\sup_n \left[\mu\left((T - \lambda_0 \mathrm{I})^n\right)\right]\right\}^{\frac{1}{n}}.$$

Then E lies either all in $\rho(T,X)$ or all in $\mathrm{III}_1\sigma(T,X)$.

Furthermore, let us recall the notion of the frame of the spectrum.

The set $\mathrm{II}_2\sigma(T,X) \cup \mathrm{II}_3\sigma(T,X) \cup \mathrm{III}_2\sigma(T,X) \cup \mathrm{III}_3\sigma(T,X)$ is referred to as the *frame* of the spectrum and denote it by $F\sigma(T,X)$.

LEMMA 5.3. [27, Corollary 2.4] *If T is a bounded linear operator on a complex Banach space X, then the frame, $F\sigma(T,X)$, is a nonvoid compact subset of the spectrum $\sigma(T,X)$, containing the boundary $\partial\sigma(T,X)$.*

Now, we are ready to prove Statements (5) and (6) of Theorem 1.6 as follows:

For the difference operator $\Delta : \ell^\infty \longrightarrow \ell^\infty$, we have

$$\Phi(1) = \mu\left((\Delta - I)^n\right) = r = 1.$$

One can easily observe that $1 \in \mathrm{III}_1\sigma(\Delta,\ell^\infty)$. Then $E = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\} \subseteq \mathrm{III}_1\sigma(\Delta,\ell^\infty)$. But

$$\mathrm{III}_1\sigma(\Delta,\ell^\infty) \subseteq \sigma_{\mathrm{r}}(\Delta,\ell^\infty) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leqslant 1\}.$$

Therefore, since $\mathrm{III}_1\sigma(\Delta,\ell^\infty)$ is open,

$$\mathrm{III}_1\sigma(\Delta,\ell^\infty) \subseteq \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}.$$

This completes the proof of statement (5). Further, since

$$\begin{aligned}\sigma_{\mathrm{r}}(\Delta,\ell^\infty) &= \{\lambda \in \mathbb{C} : |\lambda - 1| \leqslant 1\} \\ &= \mathrm{III}_1\sigma(\Delta,\ell^\infty) \cup \mathrm{III}_2\sigma(\Delta,\ell^\infty),\end{aligned}$$

then

$$\mathrm{III}_2\sigma(\Delta,\ell^\infty) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\}.$$

This completes the proof of Theorem 1.6.

The result in Theorem 1.6(5) asserts that $\mathrm{III}_1\sigma(\Delta,\ell^\infty)$ must be an open set; Lemma 2.1. Further, for the difference operator $\Delta:\ell^\infty\longrightarrow\ell^\infty$, we have

$$\mathrm{II}_2\sigma(\Delta,\ell^\infty)=\mathrm{II}_3\sigma(\Delta,\ell^\infty)=\mathrm{III}_3\sigma(\Delta,\ell^\infty)=\varnothing.$$

This implies that

$$F\sigma(\Delta,\ell^\infty)=\mathrm{III}_2\sigma(\Delta,\ell^\infty)=\{\lambda\in\mathbb{C}:|\lambda-1|=1\}.$$

This asserts the fact that $F\sigma(\Delta,\ell^\infty)$ must be compact and containing the boundary of the spectrum; $\partial\sigma(\Delta,\ell^\infty)$.

5.5. Proof of Theorem 1.7

In this subsection, we assume that Assumption 1.2 holds as well as Assumption 1.3. It follows directly from Theorem 1.1(3)-(4), that Δ_{ab} is bounded but not compact on bv. Theorem 1.7 will be decomposed into Theorems 5.3, 5.4, 5.5, 5.6 and 5.7, which will be presented as follows.

THEOREM 5.3. $\sigma_{\mathrm{p}}(\Delta_{ab},\mathrm{bv})=\big(A\cap\big(\mathbb{C}\setminus\overline{\Delta}(a,|b|)\big)\big)\cup K_{\mathrm{b}}$.

Proof. The proof is omitted since it is similar to that of Theorem 5.2. In fact, the proof can be easily adapted to the space bv. $\square$

LEMMA 5.4. $\sigma_{\mathrm{p}}(\Delta^*_{ab},\mathrm{bv}^*)=\Delta(a,|b|)\cup\big(A\cap\big(\mathbb{C}\setminus\overline{\Delta}(a,|b|)\big)\big)\cup H_{\mathrm{b}}\cup\{a+b\}$.

Proof. Suppose that $\Delta^*_{ab}f=\lambda f$ for $f\in\mathrm{bv}^*$, where $f=(f_k)\neq\theta$ and $\mathrm{bv}^*\cong\mathrm{bs}$. Then, we have

$$(a+b-\lambda)f_0+\sum_{n=0}^{\infty}(a_n+b_{n-1}-a-b)f_{n+1}=0,$$

and

$$f_{k+1}=\left(\frac{\lambda-a_{k-1}}{b_{k-1}}\right)f_k,\ \ k\geqslant 1.$$

There are five possibilities:

(1) Let $\lambda=a+b$. It follows that $f=(1,0,0,\dots)\in\mathrm{bs}$ is an eigenvector associated with the eigenvalue $\lambda=a+b$.

(2) Let $\lambda\in A$, so that $\lambda=a_{k_0}\neq a+b$ for some $k_0\in\mathbb{N}_0$. Then, $f=(f_0,f_1,f_2,\dots,f_{k_0+1}=1,0,0,\dots)\in\mathrm{bs}$ is an eigenvector associated with the eigenvalue $\lambda=a_{k_0}$. Thus $A=\{a_k:k\in\mathbb{N}\}\subseteq\sigma_{\mathrm{p}}(\Delta^*_{ab},\mathrm{bv}^*)$.

(3) Let $\lambda\in\Delta(a,|b|)\setminus(A\cup\{a+b\})$. Then

$$\lim_{k\to\infty}\left|\frac{f_{k+1}}{f_k}\right|=\left|\frac{\lambda-a}{b}\right|<1.$$

That is, $f=(f_k)\in\mathrm{cs}\subset\mathrm{bs}$. Thus $\Delta(a,|b|)\subseteq\sigma_{\mathrm{p}}(\Delta^*_{ab},\mathrm{bv}^*)$.

(4) Let $\lambda \in H_{\mathrm{b}} \setminus (A \cup \{a+b\})$. Then,

$$\sup_{n\in\mathbb{N}_0} \left|\sum_{k=2}^{n} f_k\right| = \sup_{n\in\mathbb{N}_0} \left|\sum_{k=2}^{n} \left(\prod_{i=0}^{k-2} \frac{\lambda - a_i}{b_i}\right) f_1\right| < \infty,$$

and so, $f = (f_k) \in \mathrm{bs}$ is the desired eigenvector. Thus $H_{\mathrm{b}} \subseteq \sigma_{\mathrm{p}}(\Delta_{ab}^*, \mathrm{bv}^*)$.

(5) It is now easy to observe that $\lambda \notin \sigma_{\mathrm{p}}(\Delta_{ab}^*, \mathrm{bv}^*)$ for all $\lambda \notin \Delta(a,|b|) \cup \left(A \cap \left(\mathbb{C} \setminus \overline{\Delta}(a,|b|)\right)\right) \cup H_{\mathrm{b}} \cup \{a+b\}$. $\square$

THEOREM 5.4. $\sigma_{\mathrm{r}}(\Delta_{ab}, \mathrm{bv}) = \Delta(a,|b|) \cup ((H_b \cup \{a+b\}) \setminus K_{\mathrm{b}})$.

Proof. This follows from the fact that $\sigma_{\mathrm{r}}(\Delta_{ab}, \mathrm{bv}) = \sigma_{\mathrm{p}}(\Delta_{ab}^*, \mathrm{bv}^*) \setminus \sigma_{\mathrm{p}}(\Delta_{ab}, \mathrm{bv})$. We apply Theorem 5.3 and Lemma 5.4. $\square$

Using Lemma 5.4, we deduce that

$$\overline{\Delta}(a,|b|) \cup \left(A \cap \left(\mathbb{C} \setminus \overline{\Delta}(a,|b|)\right)\right) \subseteq \sigma(\Delta_{ab}, \mathrm{bv}).$$

Now, we will give the result concerning the spectrum of the operator Δ_{ab} on bv.

THEOREM 5.5. $\sigma(\Delta_{ab}, \mathrm{bv}) = \overline{\Delta}(a,|b|) \cup \left(A \cap \left(\mathbb{C} \setminus \overline{\Delta}(a,|b|)\right)\right) \cup F_{\mathrm{b}}$.

Proof. Suppose that $\lambda \notin \overline{\Delta}(a,|b|) \cup \left(A \cap \left(\mathbb{C} \setminus \overline{\Delta}(a,|b|)\right)\right) \cup F_{\mathrm{b}}$, namely, $\left|\frac{b}{a-\lambda}\right| < 1$, $\lambda \notin F_{\mathrm{b}}$ and $\lambda \neq a_k$ for all $k \in \mathbb{N}_0$. Once we check that $(\Delta_{ab} - \lambda \mathrm{I})^{-1} \in B(\mathrm{bv})$, we will have $\lambda \notin \sigma(\Delta_{ab}, \mathrm{bv})$.

Since $\lambda \neq a_k$ for all $k \in \mathbb{N}_0$, it follows that $\Delta_{ab} - \lambda \mathrm{I}$ is lower triangular and has inverse at least formally, which is given by the matrix $(d_{n,k})$, where $d_{n,k} = 0$ for $k > n$ and

$$d_{n,k} = \frac{(-1)^{n+k}}{b_n} \prod_{j=k}^{n} \frac{b_j}{a_j - \lambda} = -\frac{1}{b_n} \prod_{j=k}^{n} \frac{b_j}{\lambda - a_j}$$

for $k \leqslant n$.

We need to apply the result in [48, Formula (99)] for the matrix $(d_{n,k})$. Firstly, since $(d_{n,k})$ is a lower triangular matrix, condition (1) of the result in [48, Formula (99)] is satisfied; $\sum_{k=0}^{\infty} d_{n,k}$ is trivially convergent for all $n \in \mathbb{N}_0$. Next, we aim to prove that $\sup_{m\in\mathbb{N}_0} C_m < \infty$, where

$$C_m = \sum_{n=1}^{\infty} \left|\sum_{k=0}^{m} (d_{n-1,k} - d_{n,k})\right|.$$

One can show that

$$C_m = \sum_{k=0}^{\infty} \left|t_k^m - t_{k+1}^m\right|,$$

where

$$t_k^m = \frac{1}{b_k} \sum_{j=0}^{m} \left(\prod_{i=j}^{k} \frac{b_i}{\lambda - a_i}\right), \qquad k,m \in \mathbb{N}_0.$$

Since $\lambda \notin F_{\mathrm{b}}$, then $\sup_m C_m = \sup_m \sum_{k=0}^{\infty} \left|t_k^m - t_{k+1}^m\right| < \infty$. Thus $\lambda \notin \sigma(\Delta_{ab}, \mathrm{bv})$. This completes the proof. $\square$

THEOREM 5.6. $\sigma_{\mathrm{c}}(\Delta_{ab},\mathrm{bv}) = [\partial\Delta(a,|b|)\setminus(H_{\mathrm{b}}\cup\{a+b\})]\cup F_{\mathrm{b}}$.

Proof. Simply observe that $\sigma_{\mathrm{c}}(\Delta_{ab},\mathrm{bv}) = \sigma(\Delta_{ab},\mathrm{bv})\setminus\big(\sigma_{\mathrm{p}}(\Delta_{ab},\mathrm{bv})\cup\sigma_{\mathrm{r}}(\Delta_{ab},\mathrm{bv})\big)$. It remains to apply Theorems 5.3, 5.4 and 5.5 □

Other spectra of the operator Δ_{ab} are given in the following theorems.

THEOREM 5.7. *The following hold:*

(1) $\mathrm{III}_{3\sigma}(\Delta_{ab},\mathrm{bv}) = \big(A\cap\big(\mathbb{C}\setminus\overline{\Delta}(a,|b|)\big)\big)\cup K_{\mathrm{b}}$,

(2) $\mathrm{III}_{1\sigma}(\Delta_{ab},\mathrm{bv})\cup\mathrm{III}_{2\sigma}(\Delta_{ab},\mathrm{bv}) = \Delta(a,|b|)\cup((H_{\mathrm{b}}\cup\{a+b\})\setminus K_{\mathrm{b}})$,

(3) $\mathrm{II}_{2\sigma}(\Delta_{ab},\mathrm{bv}) = [\partial\Delta(a,|b|)\setminus(H_{\mathrm{b}}\cup\{a+b\})]\cup F_{\mathrm{b}}$.

THEOREM 5.8. *The following hold:*

(1) $\sigma_{\mathrm{ap}}(\Delta_{ab},\mathrm{bv}) = \big(\overline{\Delta}(a,|b|)\setminus\mathrm{III}_{1\sigma}(\Delta_{ab},\mathrm{bv})\big)\cup\big(A\cap\big(\mathbb{C}\setminus\overline{\Delta}(a,|b|)\big)\big)\cup F_{\mathrm{b}}$,

(2) $\sigma_{\delta}(\Delta_{ab},\mathrm{bv}) = \overline{\Delta}(a,|b|)\cup\big(A\cap\big(\mathbb{C}\setminus\overline{\Delta}(a,|b|)\big)\big)\cup F_{\mathrm{b}}$,

(3) $\sigma_{\mathrm{co}}(\Delta_{ab},\mathrm{bv}) = \Delta(a,|b|)\cup\big(A\cap\big(\mathbb{C}\setminus\overline{\Delta}(a,|b|)\big)\big)\cup H_{\mathrm{b}}\cup\{a+b\}$.

REMARK 5.2. In comparison with Remark 4.1, it is noted that a subdivision (not necessarily disjoint) of the spectrum with respect to

$$\big\{\sigma_{\delta}(\Delta_{ab},\mathrm{bv}),\sigma_{\mathrm{co}}(\Delta_{ab},\mathrm{bv}),\sigma_{\mathrm{ap}}(\Delta_{ab},\mathrm{bv})\big\}$$

exists properly.

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REFERENCES

[1] A. A. ALBANESE, J. BONET, W. J. RICKER, *Spectrum and compactness of the Cesàro operator on weighted ℓ_p spaces*, J. Aust. Math. Soc. 99 (2015) 287–314.

[2] A. A. ALBANESE, J. BONET, W. J. RICKER, *The Cesàro operator in weighted ℓ^1 spaces*, Math. Nachr. 291 (2018) 1015–1048.

[3] A. M. AKHMEDOV, F. BAŞAR, *The fine spectra of the difference operator Δ over the sequence space ℓ_p, $(1\leqslant p<\infty)$*, Demonstratio Math. 39 (3) (2006), 585–595.

[4] A. M. AKHMEDOV, F. BAŞAR, *The fine spectra of the difference operator Δ over the sequence space bv_p, $(1\leqslant p<\infty)$*, Acta Math. Sin. (Engl. Ser.) 23 (10) (2007), 1757–1768.

[5] A. M. AKHMEDOV, S. R. EL-SHABRAWY, *On the fine spectrum of the operator Δ_{ab} over the sequence space c*, Comput. Math. Appl. 61 (10) (2011), 2994–3002.

[6] A. M. AKHMEDOV, S. R. EL-SHABRAWY, *Notes on the fine spectrum of the operator Δ_{ab} over the sequence space c*, J. Fract. Calc. Appl. 3 (S) (2012), 1–7 (Proc. of the 4th. Symb. of Fractional Calculus and Applications, July, 11, 2012, Alexandria, Egypt).

[7] A. M. AKHMEDOV, S. R. EL-SHABRAWY, *Notes on the spectrum of lower triangular double-band matrices*, Thai J. Math. 10 (2) (2012), 415–421.

[8] A. M. AKHMEDOV, S. R. EL-SHABRAWY, *Spectra and fine spectra of lower triangular double-band matrices as operators on ℓ_p, $(1 \leqslant p < \infty)$*, Math. Slovaca 65 (5) (2015), 1137–1152.

[9] B. ALTAY, F. BAŞAR, *On the fine spectrum of the difference operator Δ on c_0 and c*, Inform. Sci. 168 (2004), 217–224.

[10] B. ALTAY, F. BAŞAR, *On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c*, Int. J. Math. Math. Sci. 18 (2005), 3005–3013.

[11] J. APPELL, E. DE PASCALE, A. VIGNOLI, *Nonlinear Spectral Theory*, De Gruyter Series in Nonlinear Analysis and Applications 10, Walter de Gruyter, Berlin, 2004.

[12] F. BAŞAR, *Summability Theory and Its Applications*, Bentham Science Publishers, Istanbul (2012).

[13] H. BILGIÇ, H. FURKAN, *On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bv_p, $(1 < p < \infty)$*, Nonlinear Anal. 68 (2008), 499–506.

[14] J. BOOS, *Classical and Modern Methods in Summability*, Oxford University Press Inc., New York, 2000.

[15] A. BROWN, P. R. HALMOS, A. L. SHIELDS, *Cesàro operators*, Acta Sci. Math. (Szeged) 26 (1965) 125–137.

[16] J. P. CARTLIDGE, *Weighted mean matrices as operators on ℓ^p*, Ph. D. Dissertation, Indiana University (1978).

[17] K. H. CHEW, P. N. SHIVAKUMAR, J. J. WILLIAMS, *Error bounds for the truncation of infinite linear differential systems*, J. Inst. Maths. Appl. 25 (1980) 37–51.

[18] G. P. CURBERA, W. J. RICKER, *Spectrum of the Cesàro operator in ℓ^p*, Arch. Math. 100 (2013) 267–271.

[19] G. P. CURBERA, W. J. RICKER, *The Cesàro operator and unconditional Taylor series in Hardy spaces*, Integral Equations Operator Theory 83(2) (2015) 179–195.

[20] E. DÜNDAR, F. BAŞAR, *On the fine spectrum of the upper triangle double band matrix Δ^+ on the sequence space c_0*, Math. Commun. 18 (2013), 337–348.

[21] A. J. DUTTA, B. C. TRIPATHY, *Fine spectrum of the generalized difference operator $B(r,s)$ over the class of convergent series*, Int. J. Anal., 2013, Art. ID 630436, 4 pp.

[22] S. R. EL-SHABRAWY, *Spectra and fine spectra of certain lower triangular double-band matrices as operators on c_0*, J. Inequal. Appl. 241 (2014), 9 pp.

[23] S. R. EL-SHABRAWY, S. H. ABU-JANAH, *Spectra of the generalized difference operator on the sequence spaces bv_0 and h*, Linear Multilinear Algebra 66 (8) (2018), 1691–1708.

[24] J. FATHI, R. LASHKARIPOUR, *On the fine spectrum of generalized upper double-band matrices Δ^{uv} over the sequence space ℓ_1*, Math. Vesnik 65 (1) (2013), 64–73.

[25] H. FURKAN, H. BILGIÇ AND K. KAYADUMAN, *On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv*, Hokkaido Math. J. 35 (2006), 893–904.

[26] H. A. GINDLER, A. E. TAYLOR, *The minimum modulus of a linear operator and its use in spectral theory*, Studia Math. 22 (1962), 15–41.

[27] C. J. A. HALBERG, JR., A. SAMUELSSON, *On the fine structure of spectra*, Math. Scand. 29 (1971), 37–49.

[28] A. KARAISA, *Fine spectra of upper triangular double-band matrices over the sequence space ℓ_p, $(1 < p < \infty)$*, Discrete Dyn. Nat. Soc, 2012, Art. ID 381069, 19 pp.

[29] V. KARAKAYA, M. ALTUN, *Fine spectra of upper triangular double-band matrices*, J. Comput. Appl. Math. 234 (2010), 1387–1394.

[30] M. KLINE, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972.

[31] C. S. KUBRUSLY, *Spectral Theory of Operators on Hilbert Spaces*, Birkhäuser-Springer, New York, 2012.

[32] G. LEIBOWITZ, *Spectra of discrete Cesàro operators*, Tamkang J. Math. 3 (1972) 123–132.

[33] G. LEIBOWITZ, *Discrete Hausdorff transformations*, Proc. Amer. Math. Soc. 38 (1973), 541–544.

[34] M. MURSALEEN, F. BAŞAR, *Sequence Spaces: Topics in Modern Summability Theory*, CRC Press, Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton London New York, 2020.

[35] J. I. OKUTOYI, *On the spectrum of C_1 as an operator on* bv_0, J. Aust. Math. Soc. Ser. A 48 (1990) 79–86.
[36] J. T. OKUTOYI, *On the spectrum of C_1 as an operator on* bv, Commun. Fac. Sci. Univ. Ank. Series A_1 41 (1992), 197–207.
[37] L. PROUZA, *The spectrum of the discrete Cesàro operator*, Kybernetika 12 (4) (1976), 260–267.
[38] J. B. READE, *On the spectrum of the Cesàro operator*, Bull. Lond. Math. Soc. 17 (1985) 263–267.
[39] B. E. RHOADES, *Spectra of some Hausdorff operators*, Acta Sci. Math. (Szeged) 32 (1971), 91–100.
[40] B. E. RHOADES, N. K. SHARMA, *Spectral results for some Hausdorff matrices*, Acta Sci. Math. (Szeged) 44 (3–4) (1982), 359–364.
[41] Y. SAWANO, S. R. EL-SHABRAWY, *Fine spectra of the discrete generalized Cesàro operator on Banach sequence spaces*, Monatsh. Math. 192 (2020), 185–224.
[42] Y. SAWANO, S. R. EL-SHABRAWY, *The spectra of the generalized difference operators on the spaces of convergent series*, Linear Multilinear Algebra 69 (4) (2021), 732–751.
[43] P. N. SHIVAKUMAR, J. J. WILLIAMS, *An iterative method with truncation for infinite linear systems*, J. Comput. Appl. Math. 24 (1988), 199–207.
[44] P. N. SHIVAKUMAR, J. J. WILLIAMS AND N. RUDRAIAH, *Eigenvalues for infinite matrices*, Linear Algebra Appl. 96 (1987), 35–63.
[45] P. N. SHIVAKUMAR, R. WONG, *Linear equations in infinite matrices*, Linear Algebra Appl. 7 (1973), 53–62.
[46] P. D. SRIVASTAVA, S. KUMAR, *Fine spectrum of the generalized difference operator Δ_v on sequence space* ℓ_1, Thai J. Math. 8 (2) (2010), 221–233.
[47] P. D. SRIVASTAVA, S. KUMAR, *Fine spectrum of the generalized difference operator Δ_{uv} on sequence space* ℓ_1, Appl. Math. Comput. 218 (11) (2012), 6407–6414.
[48] M. STIEGLITZ, H. TIETZ, *Matrixtranformationen von Folgenräumen Eine Ergebnisübersicht*, Math Z. 154 (1977), 1–16.
[49] M. H. STONE, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, American Mathematical Society, New York, 1932.
[50] A. E. TAYLOR, C. J. A. HALBERG, JR., *General theorems about a bounded linear operator and its conjugate*, J. Reine Angew. Math. 198 (1957), 93–111.
[51] A. E. TAYLOR, D. C. LAY, *Introduction to Functional Analysis*, 2nd ed., Robert E. Krieger Publishing Co., Malabar, Florida, 1986.
[52] B. C. TRIPATHY, R. DAS, *Spectrum and fine spectrum of the upper triangular matrix $U(r,s)$ over the sequence space* cs, Proyecciones 34 (2) (2015), 107–125.
[53] M. YEŞILKAYAGIL, F. BAŞAR, *A survey for the spectrum of triangles over sequence spaces*, Numer. Funct. Anal. Optim. 40 (16) (2019), 1898–1917.

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Saad Rashad El-Shabrawy
Department of Mathematics
College of Science and Arts, Jouf University
Gurayyat, Al-Jouf, Saudi Arabia
and
Department of Mathematics
Faculty of Science, Damietta University
Damietta 34517, Egypt e-mail: srshabrawy@yahoo.com

Yoshihiro Sawano
Department of Mathematics
Chuo University
Kasuga 1-13-27, Bunkyo-ku, Tokyo, 112-8551, Japan
and
Department of Mathematics Analysis and the Theory of functions
Peoples' Friendship University of Russia
Moscow, Russia
e-mail: yoshihiro-sawano@celery.ocn.ne.jp