Operators and Matrices
Volume 15, Number 3 (2021), 1073–1087
doi:10.7153/oam-2021-15-67

APPROXIMATION OF THE NUMERICAL RANGE OF POLYNOMIAL OPERATOR MATRICES

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(*Communicated by N.-C. Wong*)

Abstract. A linear operator on a Hilbert space may be approximated by finite matrices choosing an orthonormal basis of the Hilbert space. In this paper we establish an approximation of the *q-numerical range* of a bounded and an unbounded polynomial operator by variational methods. Applications to Hain-Lüst operator and Stokes operator are also given.

1. Introduction

In a complex Hilbert space $\mathscr{H}$ $(1 \leqslant \dim \mathscr{H} \leqslant \infty)$ with inner product $\langle \cdot,\cdot \rangle$, we consider a *polynomial operator* of degree m

$$Q(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0, \ \ (\lambda \in \mathbb{C}) \tag{1}$$

where A_j are bounded or (possibly unbounded) operators for $j = 0,1,\ldots,m$, with $A_m \neq 0$. In the special case $A_m \equiv I$, where I denotes the identity operator, the polynomial operator (1) is called *monic*. At the classical case of finite dimension $\dim \mathscr{H} = n < \infty$, A_j $(j = 0,\ldots,m)$ are $n \times n$ matrices and the polynomial operator (1) is called *matrix polynomial*. This is an important subject in spectral theory and its applications (see, [8], [9], [11], [13], [15], [23], [24]).

Let $S = \{x \in H : \|x\| = \sqrt{\langle x,x \rangle} = 1\}$ be the unit sphere in $\mathscr{H}$. A complex number λ is said to be a *root* of $Q(\lambda)$ if there exists $x \in S$ such that $\langle Q(\lambda)x,x \rangle = 0$, namely

$$\sum_{j=0}^{m} \langle A_j x,x \rangle \lambda^j = 0.$$

The set of all roots of the polynomial operator $Q(\lambda)$ is the well known *numerical range* of $Q(\lambda)$ denoted by $W(Q)$, [24]. The special case $Q(\lambda) = I\lambda - A$ yields the classical numerical range of operator A given by

$$W(A) = \{\langle Ax,x \rangle \text{ such that } x \in S\}.$$

Mathematics subject classification (2020): 47A58, 47A20, 47A12, 15A60.
Keywords and phrases: Polynomial operator, q-numerical range, projection method, Hain-Lüst operator;, Stokes operator.

Paper OaM-15-67

In this sense, the numerical range of a polynomial operator is a natural generalization of the numerical range of an operator. Generally, the numerical range of a polynomial operator is neither open nor closed.

During the recent decades, the numerical range of a polynomial operator, as well as an operator, specialized to a finite dimensional Hilbert space has been extensively studied by many researchers (see, [2], [14], [17], [19] and the references therein). In 1954 P. H. Muller [18] first introduced the notation for the classical numerical range of a matrix polynomial $Q(\lambda)$ in the finite dimensional Hilbert space $H=\mathbb{C}^n$ as

$$W(Q)=\{\lambda\in\mathbb{C}\ :\ x^*Q(\lambda)x=0 \text{ for any } x\in\mathbb{C}^n \text{ such that } x^*x=1\}, \tag{2}$$

where the inner product $\langle x,y\rangle = y^*x$ on the Hilbert space $\mathbb{C}^n$. Should we note that (only) in the finite dimensional case $W(Q)$ is always closed and contains the *spectrum* $\sigma(Q)=\{\lambda\in\mathbb{C}:\ \det Q(\lambda)=0\}$ of $Q(\lambda)$, that is, the set of all eigenvalues of $Q(\lambda)$. The notion of the numerical range of a matrix polynomial and its extensions are currently attracting attention based on several results still being published in the literature [5], [14], [19], [20].

For a given $q\in[0,1]$, a generalization of the classical numerical range (2), first mentioned by Psarrakos and Vlamos in [21], is the *q-numerical range* of an $n\times n$ matrix polynomial $Q(\lambda)$,

$$W_q(Q)=\{\lambda\in\mathbb{C}:\ \ y^*Q(\lambda)x=0,\ x,y\in\mathbb{C}^n \text{ with } x^*x=y^*y=1,\ y^*x=q\}. \tag{3}$$

A review of the properties of the latter set may be found in [21]. Here we simply note that $W_1(Q)\equiv W(Q)$, $W_q(Q)$ is always closed and it also contains $\sigma(Q)$.

Extending the definition in (3) for bounded or unbounded operators and a given real number $q\in[0,1]$, the *q-numerical range* of a polynomial operator $Q(\lambda)$ in (1) is defined by

$$W_q(Q)=\{\lambda\in\mathbb{C}:\ \ \langle Q(\lambda)x,y\rangle=0,\ x,y\in H \text{ with } \|x\|=\|y\|=1,\ \langle x,y\rangle=q\}. \tag{4}$$

The q-numerical range of a polynomial operator, like the numerical range of a polynomial operator, is closed only if $\dim\mathscr{H}<\infty$. The numerical computation of the boundary of q-numerical range of a matrix polynomial remains a challenging task, and up to date not entirely satisfactory numerical algorithm has been found. The situation becomes even worse when dealing with unbounded operators as the 'random vector' method proposed in [19] is then sampling from an infinite-dimensional space.

In this paper, we consider how to compute $W_q(Q)$ by projection methods, which reduce the problem to that of computing the q-numerical range of a (finite) matrix and a block matrix. Projection methods always yield a subset of the q-numerical range under hypotheses. It is necessary to make some extra assumptions only if one wishes to be sure of generating the whole of $W_q(Q)$. Our motivation for such a study comes from the fact that some problems arising in various research areas conclude in the study of unbounded linear operators. The approximation of $W_q(Q)$ of a polynomial operator $Q(\lambda)$ is approached by taking the q-numerical range of sufficiently many matrix polynomials of various size. The algorithms and procedures applied extend known results regarding the numerical range of a matrix polynomial in [17].

The paper is organized as follows. In Section 2, some theoretical results are investigated dealing with the approximation of the q-numerical range of bounded and unbounded polynomial operators using projection method.

In Section 3, we apply these results to compute the q-numerical range of a quadratic monic differential polynomial operator.

2. Convergence Theorems

In this section we will use finite matrices to approximate the numerical range of linear operators. However the idea of approximating linear operators by finite matrices is an obvious one that must happen again and again. Suppose that one wishes to compute the *q-numerical range* of polynomial operator $Q(\lambda)=\sum_{j=0}^{m}A_j\lambda^j$, by using the following projection method. Let $(V_k)_{k=1}^{\infty}$ be a nested family of spaces in $\mathscr{H}$ given by $V_k = span\{\phi_1,\phi_2,\ldots,\phi_k\}$, where $\{V_k : k\in\mathbb{N}\}$ is orthonormal basis of a Hilbert space $\mathscr{H}$ whose element lie in $\mathscr{D}(Q(\lambda))$ and suppose that the corresponding orthogonal projections $P_k:\mathscr{H}\to V_k$, converge strongly to the identity operator I.

For a fixed number $l\geqslant 2$, and $j=0,1,\ldots,m$, the $l\times l$ matrix is defined as

$$\mathbb{A}_j=\begin{pmatrix}\langle A_j\phi_1,\phi_1\rangle & \langle A_j\phi_1,\phi_2\rangle & \ldots & \langle A_j\phi_1,\phi_l\rangle\\ \langle A_j\phi_2,\phi_1\rangle & \langle A_j\phi_2,\phi_2\rangle & \ldots & \langle A_j\phi_2,\phi_l\rangle\\ \vdots & \vdots & & \vdots\\ \langle A_j\phi_l,\phi_1\rangle & \langle A_j\phi_l,\phi_2\rangle & \ldots & \langle A_j\phi_l,\phi_l\rangle\end{pmatrix}, \tag{5}$$

that is, the (p,r)-element of $\mathbb{A}_j$ matrix is equal to $\langle A_j\phi_p,\phi_r\rangle$, for $p,r=1,2,\ldots,l$. Moreover, the $l\times l$ matrix polynomial is formulated by

$$\mathbb{Q}_l(\lambda)=\sum_{j=0}^{m}\mathbb{A}_j\lambda^j, \tag{6}$$

where $\mathbb{A}_j$ is defined by (5) for $j=0,1,\cdots m$.

THEOREM 2.1. *Let $Q(\lambda)$ be a bounded polynomial operator on a Hilbert space H, and $\mathbb{Q}_\ell(\lambda)$ be as in Eq. (6). Let $(V_k)_{k=1}^{\infty}$ be a nested family of spaces in $\mathscr{H}$ given by $V_k=span\{\phi_1,\phi_2,\ldots,\phi_k\}$, where $(\phi_k)_{k=1}^{\infty}$ is an orthonormal. Then $W_q(\mathbb{Q}_k)\subseteq W_q(Q)$, for $q\in(0,1]$.*

Proof. Suppose that $\lambda_0\in W_q(\mathbb{Q}_\ell)$, then there exist two unit vectors $\alpha,\beta\in\mathbb{C}^\ell$ with $\langle\alpha,\beta\rangle=q$ such that $\langle\mathbb{Q}_\ell(\lambda_0)\alpha,\beta\rangle=0$. Define an isometry $\mathbf{i}:V_\ell\to\mathbb{C}^\ell$ by $\mathbf{i}(\alpha_1\phi_1+\alpha_2\phi_2+\cdots+\alpha_\ell\phi_\ell)=\alpha$, and $\mathbf{i}(\beta_1\phi_1+\beta_2\phi_2+\cdots+\beta_\ell\phi_\ell)=\beta$. Choose $x,y\in V_\ell$ such that $\mathbf{i}(x)=\alpha$, $\mathbf{i}(y)=\beta$, it is evident that $\|x\|=\|y\|=1$ with $\langle x,y\rangle=q$. A simple calculation shows that $\langle\mathbb{Q}_\ell(\lambda_0)\alpha,\beta\rangle=\langle Q(\lambda_0)x,y\rangle$ which, by (4), implies $\lambda_0\in W_q(Q)$, thus completing the proof. $\square$

The next inclusion, which will be used in the proof of Theorem 2.3, asserts that $\{W_q(\mathbb{Q}_\ell):\ell\geqslant 2\}$ forms an increasing sequence of sets.

LEMMA 2.2. *Let* $(V_\ell)_\ell^\infty$ *and* $\mathbb{Q}_\ell(\lambda)$ *be as in Theorem* 2.1. *Given* $q \in (0,1]$, *then* $W_q(\mathbb{Q}_\ell) \subseteq W_q(\mathbb{Q}_{\ell+r}),\ \ r = 1,2,\ldots$.

Proof. This is an immediate consequence of the fact that $\mathbb{C}^\ell$ is a subspace of $\mathbb{C}^{\ell+r}$. In detail, let $\lambda_0 \in W_q(\mathbb{Q}_\ell)$, then we can choose $\alpha, \beta \in \mathbb{C}^\ell$ such that $\|\alpha\| = \|\beta\| = 1$ and $\langle \alpha, \beta \rangle = q$, for which we have $\langle \mathbb{Q}_\ell(\lambda_0)\alpha, \beta \rangle = 0$. Both α and β can be extended to vectors in $\mathbb{C}^{\ell+r}$, say $\hat{\alpha}$ and $\hat{\beta}$, whose τ-th components are zero for each $\tau \geqslant \ell + r$. It is easy to see that $\langle \mathbb{Q}_\ell(\lambda_0)\alpha, \beta \rangle = \langle \mathbb{Q}_{\ell+r}(\lambda_0)\hat{\alpha}, \hat{\beta} \rangle$, which yields $\lambda_0 \in W_q(\mathbb{Q}_{\ell+r})$. $\square$

In the following theorem, the q-numerical range of bounded polynomial operator is presented as the infinity union of q-numerical ranges of the reduction polynomial operator in a matrix polynomial by finite dimension (smaller size of a firstly linear operator).

THEOREM 2.3. *Let* $Q(\lambda)$, $\mathbb{Q}_\ell(\lambda)$ *and* $(V_\ell)_\ell^\infty$ *be as in Theorem* 2.1. *Suppose that* $(\phi_k)_{k=1}^\infty$ *is orthnormal basis of* H. *then, for* $q \in (0,1]$ *holds* $\overline{W_q(Q)} = \overline{\bigcup_{\ell \in \mathbb{N}} W_q \mathbb{Q}_\ell(\lambda)}$.

Proof. In view of Theorem 2.1, it is sufficient to prove $W_q(Q) \subseteq \overline{\bigcup_{\ell=2}^{\infty} W_q(\mathbb{Q}_\ell)}$. Suppose $\lambda \in W_q(Q)$. Choose $x, y \in \mathscr{H}$ such that $\|x\| = \|y\| = 1$ and $\langle x, y \rangle = q$, for which holds $\langle Q(\lambda)x, y \rangle = 0$. Since $(\phi_k)_{k=1}^\infty$ is orthnormal basis of H, there exists a sequence $(x_k)_{k=1}^\infty$, with each $x_k \in span\{\phi_1, \phi_2, \ldots, \phi_{s_k}\}$ for some $s_k > 0$ such that $\|x - x_k\| \to 0$ and $\|Q(\lambda)x - Q(\lambda)x_k\| \to 0$. In a similar way, we may also find a sequence $(y_k)_{k=1}^\infty$, with each $y_k \in span\{\phi_1, \phi_2, \ldots, \phi_{s_k}\}$ for some $s_k > 0$, such that $\|y - y_k\| \to 0$ and $\|Q(\lambda)y - Q(\lambda)y_k\| \to 0$. by a simple calculation, we then obtain that $\langle Q_k(\lambda)x_k, y_k \rangle = \langle Q(\lambda)x, y \rangle$ with

$$\|x_k\| \to \|x\| = 1,\ \ \|y_k\| \to \|y\| = 1,\ \ \langle x_k, y_k \rangle \to \langle x, y \rangle = q,\ \text{ as }\ k \to \infty. \tag{7}$$

Now, consider V_{s_k} the closed subspace of $\mathscr{H}$ such that $V_{s_k} := span\{\phi_1, \phi_2, \ldots, \phi_{s_k}\}$ Fix $k > 0$. Let $\mathbf{i} : V_{s_k} \to \mathbb{C}^{s_k}$ be the standard isometries as in the proof of Theorem 2.1. Define $\tilde{\alpha}_k, \tilde{\beta}_k \in \mathbb{C}^{s_k}$ by $\tilde{\alpha}_k = i_{s_k}(x_k),\ \tilde{\beta}_k = i_{s_k}(y_k)$. Consider the $s_k \times s_k$ matrix polynomial

$$\mathbb{Q}_{s_k}(\lambda) = \sum_{j=0}^{m} \mathbb{A}_j \lambda^j, \tag{8}$$

where the (p,r)-element of the $s_k \times s_k$ matrix $\mathbb{A}_j$ is equal to $\langle A_j \phi_p, \phi_r \rangle$, for $p, r = 1, 2, \ldots, s_k$ and $j = 0, 1, \ldots, m$. A simple calculation shows that $\langle \mathbb{Q}_{s_k}(\lambda)\tilde{\alpha}_k, \tilde{\beta}_k \rangle = \langle Q_k(\lambda)x_k, y_k \rangle$ with

$$\|x_k\| = \|\tilde{\alpha}_{s_k}\|,\quad \|y_k\| = \|\tilde{\beta}_{s_k}\|\ \text{ and }\ \langle x_k, y_k \rangle = \langle \tilde{\alpha}_{s_k}, \tilde{\beta}_{s_k} \rangle. \tag{9}$$

Now, fix $\ell \geqslant 2$ and suppose that $\lambda_\ell \in W_q(Q)$, then there exist two unit vectors $x_\ell, y_\ell \in V_{s_\ell}$ with $\langle x_\ell, y_\ell \rangle \to q$ such that $\langle Q_k(\lambda_\ell)x_\ell, y_\ell \rangle = 0$, and which are related to $x, y \in H$.

From the above relation arises that $\langle \mathbb{Q}_{s_\ell}(\lambda_\ell)\tilde{\alpha}_{s_\ell}, \tilde{\beta}_{s_\ell}\rangle = 0$, which in turn implies $\lambda_\ell \in W_q(\mathbb{Q}_{s_\ell})$ due to (7). Therefore, the existence of a sequence $\{\lambda_\ell : \ell \in \mathbb{N}\} \subseteq W_q(\mathbb{Q}_{s_\ell})$ is guaranteed by the sequence $\{\lambda_\ell : \ell \in \mathbb{N}\} \subseteq W_q(Q)$. Moreover, the closure of $W_q(Q)$ for the sequence $\{\lambda_\ell : \ell \in \mathbb{N}\} \subseteq W_q(Q)$ yields $\underline{\lambda_\ell \to \lambda \text{ as } \ell \to +\infty}$. Thus, there exists $\{\lambda_\ell : \ell \in \mathbb{N}\} \subseteq W_q(\mathbb{Q}_{s_\ell})$. It is obvious that $\lambda \in \bigcup_{\ell=2}^{\infty} W_q(\mathbb{Q}_{s_\ell})$ and in view of Lemma 2.2, immediately gives $\lambda \in \overline{\bigcup_{\ell=2}^{\infty} W_q(\mathbb{Q}_{s_\ell})} \subseteq \overline{\bigcup_{\ell=2}^{\infty} W_q(\mathbb{Q}_\ell)}$. $\square$

REMARK 1. Let $Q(\lambda)$ and $(V_\ell)_\ell^\infty$ be as in Theorem 2.1. Let P_k denote orthogonal projection onto V_k. If $Q(\lambda)$ be a bounded polynomial operator on a Hilbert space H, then the hypotheses that $(\phi_k)_{k=1}^\infty$ is orthnormal basis of H, is equivalent to the statements that P_k converge strongly to the identity operator I as $k \to +\infty$.

REMARK 2. We assume readers familiar with basic notions and results about linear unbounded operators, as well as matrices of non necessarily bounded operators. useful references are [6] [7] and [24]. We call a few definitions though: A linear operator A with a domain $\mathscr{D}(A)$ contained in a Hilbert space $\mathscr{H}$ is said to be densely defined if $\overline{\mathscr{D}(A)} = \mathscr{H}$. Say that a linear operator A is closed if its graph Γ_A is closed in $\mathscr{H} \oplus \mathscr{H}$. A linear operator A is called closable if, the closure $\overline{\Gamma_A}$ of its graph is the graph of some operator. A subspace $\mathscr{D} \subset \mathscr{D}(A)$ is called a core of a closable operator A if $A\mid_{\mathscr{D}}$ is closable with closure $\overline{A}$.

We consider an unbounded polynomial operator

$$Q(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0, \ \ (\lambda \in \mathbb{C}) \tag{10}$$

in a Hilbert space H, where A_j are closable operators for $j = 0, 1, \ldots, m$, with $A_m \neq 0$, with dense domain $\mathscr{D}(Q(\lambda))$ and is always equal to $\mathscr{D}(Q(\lambda)) = \cap_{j=1}^m \mathscr{D}(A_j)$ where $\cap_{j=1}^m \mathscr{D}(A_j)$ is also densely defined for $j = 0, 1, \ldots, m$, and the domain $\mathscr{D}(Q(\lambda))$ do not dependent on λ.

The definition of the *q-numerical range* of a bounded polynomial operator $Q(\lambda)$ in Eq. (4) generalizes as follows to unbounded polynomial operator $Q(\lambda)$ with dense domain $\mathscr{D}(Q(\lambda))$.

DEFINITION 2.4. For a polynomial operator $Q(\lambda)$ with domain $\mathscr{D}(Q(\lambda)) \subset \mathscr{H}$ we define the *q-numerical range* of $Q(\lambda)$ for $0 \leqslant q \leqslant 1$ by

$$W_q(Q) = \{\lambda \in \mathbb{C} : \ \langle Q(\lambda)x, y\rangle = 0, \ x, y \in \mathscr{D}(Q(\lambda)) \text{ with } \|x\| = \|y\| = 1, \ \langle x, y\rangle = q\}. \tag{11}$$

In the following result we describe that the closure of the range $W_q(Q(\lambda))$ is approximated by $W_q(\mathbb{Q}_\ell)$ under the assumption that the linear span of $\{\phi_1, \phi_2, \ldots,\}$ is a core of $Q(\lambda)$.

THEOREM 2.5. *Let $Q(\lambda)$ be an unbounded polynomial operator on a Hilbert space H, and $\mathbb{Q}_\ell(\lambda)$ be as in Eq. (6). Let $(V_k)_{k=1}^{\infty}$ be a nested family of spaces in $\mathscr{D}(Q(\lambda))$ given by $V_k = span\{\phi_1,\phi_2,\ldots,\phi_k\}$, where $(\phi_k)_{k=1}^{\infty}$ is an orthonormal. Then $W_q(\mathbb{Q}_k) \subseteq W_q(Q)$, for $q \in (0,1]$.*

Proof. Define an isometry $\pi : V_\ell \to \mathbb{C}^\ell$ by $\pi(\zeta_1\phi_1 + \zeta_2\phi_2 + \cdots + \zeta_\ell\phi_\ell) = \zeta$, and $\pi(\gamma_1\phi_1 + \gamma_2\phi_2 + \cdots + \gamma_\ell\phi_\ell) = \gamma$. Suppose that $\lambda_0 \in W_q(\mathbb{Q}_\ell)$, then there exist two unit vectors $\zeta,\gamma \in \mathbb{C}^\ell$ with $\langle \zeta,\gamma\rangle = q$ such that $\langle \mathbb{Q}_\ell(\lambda_0)\zeta,\gamma\rangle = 0$. Choose $x,y \in V_\ell$ such that $\pi(x) = \zeta$, $\pi(y) = \gamma$, it is evident that $\|x\| = \|y\| = 1$ with $\langle x,y\rangle = q$. A simple calculation shows that $\langle \mathbb{Q}_\ell(\lambda_0)\zeta,\gamma\rangle = \langle Q(\lambda_0)x,y\rangle$ which, by (11), implies $\lambda_0 \in W_q(Q)$, thus completing the proof. $\square$

The following Lemma can be proof in a similar fashion as Lemma 2.2:

LEMMA 2.6. *Let $(V_k)_{k=1}^{\infty}$ be a nested family of spaces in $\mathscr{D}(Q(\lambda))$ given by $V_k = span\{\phi_1,\phi_2,\ldots,\phi_k\}$, where $(\phi_k)_{k=1}^{\infty}$ is an orthonormal. Let $\mathbb{Q}_\ell(\lambda)$ be as in Theorem 2.1. Given $q \in (0,1]$, then $W_q(\mathbb{Q}_\ell) \subseteq W_q(\mathbb{Q}_{\ell+r})$, $r = 1,2,\ldots$.*

THEOREM 2.7. *Let $Q(\lambda) = \sum_{j=0}^{m} A_j\lambda^j$, be an unbounded polynomial operators in $\mathscr{H}$, for which the domain $\mathscr{D}(Q(\lambda))$ do not dependent on λ, and $\mathbb{Q}_\ell(\lambda)$ denotes as in Theorem 2.1. Let $(V_k)_{k=1}^{\infty}$ be a nested family of spaces in $\mathscr{D}(Q(\lambda))$ given by $V_k = span\{\phi_1,\phi_2,\ldots,\phi_k\}$, where $(\phi_k)_{k=1}^{\infty}$ is an orthonormal. Suppose that A_j are closable operators with dense domains for $j = 0,1,\ldots,m$, and $\mathscr{C} = span\{\phi_1,\phi_2,\ldots\}$ is a core of $Q(\lambda)$. Given $q \in (0,1]$, then $\overline{W_q(Q)} = \overline{\bigcup_{\ell=2}^{\infty} W_q(\mathbb{Q}_\ell)}$.*

Proof. It is sufficient to prove $W_q(Q) \subseteq \overline{\bigcup_{\ell=2}^{\infty} W_q(\mathbb{Q}_\ell)}$. Suppose $\lambda \in W_q(Q)$. Choose $x,y \in \mathscr{D}(Q(\lambda))$ such that $\|x\| = \|y\| = 1$ and $\langle x,y\rangle = q$, for which holds $\langle Q(\lambda)x,y\rangle = 0$. Since $\mathscr{C}$ is a core of $\mathscr{D}(Q(\lambda))$, there exists a sequence $(x_k)_{k=1}^{\infty}$, with each $x_k \in span\{\phi_1,\phi_2,$
$\ldots,\phi_{t_k}\}$ for some $t_k > 0$ such that $\|x - x_k\| \to 0$ and $\|Q(\lambda)x - Q(\lambda)x_k\| \to 0$. In a similar way, we may also find a sequence $(y_k)_{k=1}^{\infty}$, with each $y_k \in span\{\phi_1,\phi_2,\ldots,\phi_{t_k}\}$ for some $t_k > 0$, such that $\|y - y_k\| \to 0$ and $\|Q(\lambda)y - Q(\lambda)y_k\| \to 0$. by a simple calculation, we then obtain that $\langle Q_k(\lambda)x_k,y_k\rangle = \langle Q(\lambda)x,y\rangle$ with

$$\|x_k\| \to \|x\| = 1,\ \ \|y_k\| \to \|y\| = 1,\ \ \langle x_k,y_k\rangle \to \langle x,y\rangle = q,\ \text{ as }\ k \to \infty. \tag{12}$$

Fix $k > 0$. Let $\pi : span\{\phi_1,\phi_2,\ldots,\phi_{t_k}\} \to \mathbb{C}^{t_k}$ be the standard isometries as in the proof of Theorem 2.1. Define $\tilde{\alpha}_k, \tilde{\beta}_k \in \mathbb{C}^{t_k}$ by $\tilde{\alpha}_k = \pi_{t_k}(x_k)$, $\tilde{\beta}_k = \pi_{t_k}(y_k)$. Consider the $t_k \times t_k$ matrix polynomial

$$\mathbb{Q}_{t_k}(\lambda) = \sum_{j=0}^{m} \mathbb{A}_j\lambda^j, \tag{13}$$

where the (p,r)-element of the $t_k \times t_k$ matrix $\mathbb{A}_j$ is equal to $\langle A_j\phi_p, \phi_r\rangle$, for $p,r = 1,2,\dots,t_k$ and $j = 0,1,\dots,m$. A simple calculation shows that $\langle \mathbb{Q}_{t_k}(\lambda)\tilde{\alpha}_k, \tilde{\beta}_k\rangle = \langle Q_k(\lambda)x_k, y_k\rangle$ with

$$\|x_k\| = \|\tilde{\alpha}_{t_k}\|, \quad \|y_k\| = \|\tilde{\beta}_{t_k}\| \text{ and } \langle x_k, y_k\rangle = \langle \tilde{\alpha}_{t_k}, \tilde{\beta}_{t_k}\rangle. \tag{14}$$

Now, fix $\ell \geqslant 2$ and suppose that $\lambda_\ell \in W_q(Q)$, then there exist two unit vectors $x_\ell, y_\ell \in V_{t_\ell}$ with $\langle x_\ell, y_\ell\rangle \to q$ such that $\langle Q(\lambda_\ell)x_\ell, y_\ell\rangle = 0$, and which are related to $x,y \in H$. From the above relation arises that $\langle \mathbb{Q}_{t_\ell}(\lambda_\ell)\tilde{\alpha}_{t_\ell}, \tilde{\beta}_{t_\ell}\rangle = 0$, which in turn implies $\lambda_\ell \in W_q(\mathbb{Q}_{t_\ell})$ due to (14). Therefore, the existence of a sequence $\{\lambda_\ell : \ell \in \mathbb{N}\} \subseteq W_q(\mathbb{Q}_{t_\ell})$ is guaranteed by the sequence $\{\lambda_\ell : \ell \in \mathbb{N}\} \subseteq W_q(Q)$. Moreover, the closure of $W_q(Q)$ for the sequence $\{\lambda_\ell : \ell \in \mathbb{N}\} \subseteq W_q(Q)$ yields $\lambda_\ell \to \lambda$ as $\ell \to +\infty$. Thus, there exists $\{\lambda_\ell : \ell \in \mathbb{N}\} \subseteq W_q(\mathbb{Q}_{t_\ell})$. It is obvious that $\lambda \in \overline{\bigcup_{\ell=2}^{\infty} W_q(\mathbb{Q}_{t_\ell})}$ and in view of Lemma 2.2, immediately gives $\lambda \in \overline{\bigcup_{\ell=2}^{\infty} W_q(\mathbb{Q}_{t_\ell})} \subseteq \overline{\bigcup_{\ell=2}^{\infty} W_q(\mathbb{Q}_\ell)}$. $\square$

3. Numerical experiments on a quadratic polynomial differential operator

In this section we study some concrete examples and demonstrate that, in spite of the results obtained in the previous section, practical computation of the q-numerical range of a quadratic polynomial differential operator is very far from being straightforward. We define the inner product $\langle u,v\rangle$ to be linear in the first parameter and conjugate linear in the second parameter, and we consider the space of square-integrable functions, $L^2(\Omega, dx)$, where Ω is an interval in $\mathbb{R}$, a Hilbert space with inner product

$$\langle u,v\rangle = \int_\Omega u\bar{v}dx. \tag{15}$$

The computations were performed in Matlab.

3.1. Application to Hain-Lüst operator and Stokes operator.

Assume that $w : [0,1] \to [0,\infty)$, $\widetilde{w} : [0,1] \to [0,\infty)$, and $u : [0,1] \to \mathbb{C}$ are such that $w(x) = 1$, $\widetilde{w}(x) = 1$, $u(x) = 18e^{2\pi ix} - 20$, for each $x \in [0,1]$. We introduce the differential expression

$$\tau_{\widetilde{A}} := -\tfrac{d^2}{dx^2}, \ \tau_{\widetilde{B}} := w(x), \tag{16}$$

$$\tau_{\widetilde{C}} := \widetilde{w}(x), \ \tau_{\widetilde{D}} := u(x). \tag{17}$$

Let A, B, C, D be the operators in the Hilbert space $L^2(0,1)$ induced by the differential expressions $\tau_{\widetilde{A}}$, $\tau_{\widetilde{B}}$, $\tau_{\widetilde{C}}$, $\tau_{\widetilde{D}}$ with domain

$$\mathscr{D}(A) := H^2(0,1) \cap H_0^1(0,1), \quad \mathscr{D}(B) = \mathscr{D}(C) = \mathscr{D}(D) := L^2(0,1).$$

In the Hilbert space $L^2_2(0,1) := L^2(0,1) \oplus L^2(0,1)$, we introduce the matrix differential operator

$$\mathscr{A}_1 := \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -\frac{d^2}{dx^2} & w(x) \\ \widetilde{w}(x) & u(x) \end{pmatrix}, \tag{18}$$

on the domain

$$\mathscr{D}(\mathscr{A}_1) := (H^2(0,1) \cap H^1_0(0,1)) \oplus L^2(0,1). \tag{19}$$

This operator was introduced by K. Hain and R. Lüst in application to problems of magnetohydrodynamics [10] and the problem of this type has been studied in [1], [3], [12], [13] and [16].

REMARK 3.

(i) It is clear that the eigenvalues and normalized eigenfunctions for the operator A in $L_2[0,1]$ are

$$\lambda_j = j^2\pi^2, \quad \phi_j(x) = \sqrt{2}\sin(j\pi x), \quad j = 1,2,3,\dots \tag{20}$$

(ii) By [4, Corollary VII.2.7], the operator $A := -\frac{d^2}{dx^2}$ with domain $\mathscr{D}(A) = H^2(0,1) \cap H^1_0(0,1)$ is closed and because $\mathscr{D}(A) \subset \mathscr{D}(C)$, then the operator C is A-bounded with relative bound 0. This follows that, there is a $\gamma > 0$ such that, for every $\varepsilon > 0$

$$\|Cf\|^2 \leqslant \gamma|\langle Af, f\rangle| \leqslant \gamma(\varepsilon\|Af\|^2 + \varepsilon^{-1}\|f\|^2) \text{ for every } f \in \mathscr{D}(A). \tag{21}$$

On the other hand $\mathscr{D}(D) \subset \mathscr{D}(B)$, then the operator B is D-bounded we conclude that the operator matrix $\mathscr{A}_1$ is diagonally dominant of order 0; it is closed by [24, Corollary 2.2.9 i)].

(iii) The linear span $\mathscr{C}_A = \operatorname{span}\{\phi_1, \phi_2, \dots\}$ is a core of A, where $\{\phi_k : k \in \mathbb{N}\}$ is an orthonormal basis of $L^2(0,1)$ also the linear span $\mathscr{C}_D = \operatorname{span}\{\phi_1, \phi_2, \dots\}$ is dense set in $L_2(0,1)$. In order to show that $\mathscr{C} := \mathscr{C}_A \oplus \mathscr{C}_D$ is a core of $\mathscr{A}_1$. Suppose $y = (y_1, y_2) \in \mathscr{D}(\mathscr{A}_1)$, it is sufficient to find a sequence $\{y^{(m)} : m \in \mathbb{N}\} \subset \mathscr{C}$ such that $y^{(m)} \to y$ and $\mathscr{A}_1 y^{(m)} \to \mathscr{A}_1 y$ as m tends to infinity. Since $\mathscr{C}_1$ is a core of A and $\mathscr{C}_2$ is dense set in $L_2(0,1)$, then there exist sequences $\{y_1^{(m)} : m \in \mathbb{N}\} \subset \mathscr{C}_1$, $\{y_2^{(m)} : m \in \mathbb{N}\} \subset \mathscr{C}_2$ with $y_1^{(m)} \to y_1$, $Ay_1^{(m)} \to Ay_1$ and $y_2^{(m)} \to y_2$, $Dy_2^{(m)} \to Dy_2$ as m tends to infinity. Moreover, $\mathscr{A}_1$ is diagonally dominant (that is, C is A-bounded and B is D-bounded), there exist constants $a_1, b_1, a_2, b_2 \geqslant 0$ such that

$$\|Cy_1^{(m)} - Cy_1\| \leqslant a_1\|y_1^{(m)} - y_1\| + b_1\|Ay_1^{(m)} - Ay_1\| \to 0 \text{ as } m \to +\infty,$$
$$\|By_2^{(m)} - By_2\| \leqslant a_2\|y_2^{(m)} - y_2\| + b_2\|Dy_2^{(m)} - Dy_2\| \to 0 \text{ as } m \to +\infty.$$

The above convergences imply $y^{(m)} \to y$, $\mathscr{A} y^{(m)} \to \mathscr{A} y$ as m tends to infinity. Hence the subspace $\mathscr{C} := \mathscr{C}_A \oplus \mathscr{C}_D \subset \mathscr{D}(\mathscr{A}_1) = (\mathscr{D}(A) \cap \mathscr{D}(C)) \oplus (\mathscr{D}(B) \cap \mathscr{D}(D))$ is a core of $\mathscr{A}_1$.

(iv) We may use these eigenfunctions in Eq. (20) as basis elements for a discretization of the type discussed in Section 2, forming the matrix elements $\langle A\phi_k,\phi_j\rangle$, $\langle w\phi_k,\phi_j\rangle$, $\langle \widetilde{w}\phi_k,\phi_j\rangle$, $\langle u\phi_k,\phi_j\rangle$, with respect to the inner product in Eq. (15) and considering the infinite block matrix

$$\mathscr{Q}_1 := \begin{pmatrix} \langle A\phi_k,\phi_j\rangle & \langle w\phi_k,\phi_j\rangle \\ \\ \langle \widetilde{w}\phi_k,\phi_j\rangle & \langle u\phi_k,\phi_j\rangle \end{pmatrix}. \tag{22}$$

The matrix $\mathbb{A}_1$ defined in Eq. (5) is obtained by taking the leading sub-matrices of the block $\mathscr{Q}_1$, in Eq. (22) with appropriate dimensions.

Observe that $\langle A\phi_k,\phi_j\rangle = \operatorname{diag}\{\pi^2,4\pi^2,9\pi^2,\ldots\}$, $\langle w\phi_k,\phi_j\rangle = \operatorname{diag}\{1,1,1,\ldots\}$, $\langle \widetilde{w}\phi_k,\phi_j\rangle = \operatorname{diag}\{1,1,1,\ldots\}$, and $\langle u\phi_k,\phi_j\rangle = 36\int_0^1 e^{2\pi ix}\sin(k\pi x)\sin(j\pi x)dx$ $-20\delta_{k,j}$, which can be evaluated explicitly.

If the operator A included a potential, for instance, then its eigenfunctions would not generally be explicitly computable. We could still use the functions ϕ_j in Eq. (20) as basis functions, but the matrix elements $\langle A\phi_k,\phi_j\rangle$ would have to be computed by quadrature and the corresponding matrix would no longer diagonal.

On the other hand, let $\mathscr{A}_0$ be a 2×2 Stokes type system of ordinary differential equations subject to the Dirichlet boundary conditions on $[0,1]$, and also in this case, the underlying Hilbert space is $\mathscr{H} := L^2(0,1)\times L^2(0,1)$ and the operator is

$$\mathscr{A}_0 := \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} = \begin{pmatrix} -\frac{d^2}{dx^2} & -\frac{d}{dx} \\ \frac{d}{dx} & -\frac{3}{2} \end{pmatrix}, \tag{23}$$

The domain of $\mathscr{A}_0$ is given by

$$\mathscr{D}(\mathscr{A}_0) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u(0)=0=u(1), u\in H_0^1(0,1)\cap H^2(0,1) \text{ and } v\in H^1(0,1) \right\}.$$

REMARK 4.

(i) The operator $\mathscr{A}_0$ is not closed operator, however it is closable and its closure is self-adjoint. (see PhD thesis [1]).

(ii) It is not difficult to show that the subspace $\mathscr{C}_0 := \mathscr{C}_{A_0}\oplus\mathscr{C}_{B_0}$ is a core of $\mathscr{A}_0$, where The linear span $\mathscr{C}_{A_0} = \operatorname{span}\{\phi_1,\phi_2,\ldots\}$ is a core of A_0 where $\{\phi_k : k\in\mathbb{N}\}$ is an orthonormal basis of $L^2(0,1)$ also the linear span $\mathscr{C}_{B_0} = \operatorname{span}\{\phi_1,\phi_2,\ldots\}$ is a core of B_0. Now for $x=(x_1,x_2)\in\mathscr{D}(\mathscr{A}_0)$, it is sufficient to find a sequence $\{x^{(m)} : m\in\mathbb{N}\}\subset\mathscr{C}_0$ such that $x^{(m)}\to x$ and $\mathscr{A}_0x^{(m)}\to\mathscr{A}_0x$ as m tends to infinity. Since $\mathscr{C}_{A_0}$ is a core of A_0 and $\mathscr{C}_{B_0} = \operatorname{span}\{\phi_1,\phi_2,\ldots\}$ is a core of B_0., there exist sequences $\{x_1^{(m)} : m\in\mathbb{N}\}\subset\mathscr{C}_{A_0}, \{x_2^{(m)} : m\in\mathbb{N}\}\subset\mathscr{C}_{B_0}$ with

$$x_1^{(m)}\to x_1, \quad A_0x_1^{(m)}\to A_0x_1, \qquad x_2^{(m)}\to x_2, \quad B_0x_2^{(m)}\to B_0x_2$$

as m tends to infinity. Since $\mathscr{A}_0$ is upper dominant, (that is C is A-bounded and D is B-bounded), there exist constants $a_1, b_1, a_2, b_2 \geqslant 0$ such that

$$||C_0x_1^{(m)} - C_0x_1|| \leqslant a_1||x_1^{(m)} - x_1|| + b_1||A_0x_1^{(m)} - A_0x_1|| \to 0 \text{ as} m \to +\infty,$$

$$||D_0x_2^{(m)} - D_0x_2|| \leqslant a_2||x_2^{(m)} - x_2|| + b_2||B_0x_2^{(m)} - B_0x_2|| \to 0 \text{ as} m \to +\infty.$$

The above convergence imply $x^{(m)} \to x$, $\mathscr{A}_0x^{(m)} \to \mathscr{A}_0x$ as m tends to infinity. Hence the subspace $\mathscr{C}_0 = \mathscr{C}_{A_0} \oplus \mathscr{C}_{B_0} \subset \mathscr{D}(\mathscr{A}_0) = (\mathscr{D}(A_0) \cap \mathscr{D}(C_0)) \oplus (\mathscr{D}(B_0) \cap \mathscr{D}(D_0)) = (H^2(0,1) \cap H_0^1(0,1)) \oplus H^1(0,1)$ is a core of $\mathscr{A}_0$.

(iii) Now form the matrix elements $\langle -\phi_k'', \phi_j\rangle$, $\langle -\phi_k', \phi_j\rangle$, $\langle \phi_k', \phi_j\rangle$, $\langle -\frac{3}{2}\phi_k, \phi_j\rangle$ with respect to the orthonormal basis in Eq. (20) and consider the (infinite) block operator matrix

$$\mathscr{Q}_0 := \begin{pmatrix} \langle -\phi_k'', \phi_j\rangle & \langle -\phi_k', \phi_j\rangle \\ \\ \langle \phi_k', \phi_j\rangle & \langle -\frac{3}{2}\phi_k, \phi_j\rangle \end{pmatrix}. \tag{24}$$

The matrix $\mathbb{A}_0$ defined in Eq. (5) is obtained by taking the leading sub-matrices of the block $\mathscr{Q}_0$, with appropriate dimensions.

Observe that $\langle -\phi_k'', \phi_j\rangle = \operatorname{diag}\{\pi^2, 4\pi^2, 9\pi^2, \ldots\}$,

$$\langle -\phi_k', \phi_j\rangle = \begin{cases} 0, & \text{if } k = j; \\ -2k\pi \int_0^1 \cos(k\pi x)\sin(j\pi x)dx, & \text{if } k \neq j; \end{cases}$$

$$\langle \phi_k', \phi_j\rangle = \begin{cases} 0, & \text{if } k = j; \\ 2k\pi \int_0^1 \cos(k\pi x)\sin(j\pi x)dx, & \text{if } k \neq j; \end{cases}$$

and $\langle -\frac{3}{2}\phi_k, \phi_j\rangle = -\frac{3}{2}\,\delta_{k,j}$, which can be evaluated explicitly.

3.2. The quadratic differential block polynomial operator

Furthermore, consider the quadratic monic differential polynomial operator

$$Q(\lambda) = I\lambda^2 + \mathscr{A}_1\lambda + \mathscr{A}_0 = I\lambda^2 + \begin{pmatrix} -\frac{d^2}{dx^2} & w(x) \\ \widetilde{w}(x) & u(x) \end{pmatrix}\lambda + \begin{pmatrix} -\frac{d^2}{dx^2} & -\frac{d}{dx} \\ \frac{d}{dx} & -\frac{3}{2} \end{pmatrix}, \tag{25}$$

in a Hilbert space $\mathscr{H}$, for which the domains $\mathscr{D}(Q(\lambda))$ do not depend on λ, and are always $\mathscr{D}(Q(\lambda)) := \mathscr{D}(\mathscr{A}_1) \cap \mathscr{D}(\mathscr{A}_0)$, where the block infinite matrices $\mathscr{A}_1$, $\mathscr{A}_0$ have been formulated in (18),(23), respectively.

REMARK 5. According to Remark 3 (i) and Remark 4 (ii), it is not difficult to see that the subspace $\mathscr{C} = \mathscr{C}_1 \oplus \mathscr{C}_0 \subset \mathscr{D}(\mathscr{A}_1) \cap \mathscr{D}(\mathscr{A}_0)$ is a core of $Q(\lambda)$ in (25), so in this case the main Theorem 2.7 is applicable to this example.

The differential block polynomial operator in Eq. (25) allows us to define the quadratic monic polynomial operator

$$\begin{aligned} Q(\lambda) &= I\lambda^2 + \mathscr{Q}_1\lambda + \mathscr{Q}_0 \\ &= I\lambda^2 + \begin{pmatrix} \langle A\phi_k, \phi_j\rangle & \langle w\phi_k, \phi_j\rangle \\ \langle \widetilde{w}\phi_k, \phi_j\rangle & \langle u\phi_k, \phi_j\rangle \end{pmatrix}\lambda + \begin{pmatrix} \langle -\phi_k'', \phi_j\rangle & \langle -\phi_k', \phi_j\rangle \\ \langle \phi_k', \phi_j\rangle & \langle -\frac{3}{2}\phi_k, \phi_j\rangle \end{pmatrix}, \end{aligned} \tag{26}$$

where the block infinite matrices $\mathscr{Q}_1, \mathscr{Q}_0$ have been formulated in Eq. (22) and Eq. (24), respectively.

Now, the projection method reduces the quadratic monic polynomial operator in (26) to the $\ell \times \ell$ quadratic monic matrix polynomial with coefficients the $\ell \times \ell$ matrices $\mathbb{A}_1$ and $\mathbb{A}_0$ as defined in Eq. (5), derived by taking the leading sub-matrices of the matrices $\mathscr{Q}_1, \mathscr{Q}_0$ in Eq. (22) and Eq. (24) with appropriate dimensions. Hence fix an $\ell \geqslant 2$, the $\ell \times \ell$ quadratic monic matrix polynomial is given by

$$\mathbb{Q}_\ell(\lambda) = \mathbb{I}\lambda^2 + \mathbb{A}_1\lambda + \mathbb{A}_0. \tag{27}$$

we consider the monic block polynomial operator $Q(\lambda)$ determined by Eq. (25). Based on the analysis described in Subsection 3.2 and carrying out analogous steps as in Subsection 3.1, we attempt to estimate its q-numerical range by sketching the q-numerical range $W_q(\mathbb{Q}_\ell)$ of the reduced $\ell \times \ell$ monic matrix polynomial $\mathbb{Q}_\ell(\lambda)$ in Eq. (27) for some values of ℓ ($\ell = 4, 6$). The following figures have been produced by two different ways. The first one uses a simple Matlab code, which can plot the roots of a sufficient number of polynomials of the form $y^*\mathbb{Q}_\ell(\lambda)x$, where $x, y \in \mathbb{C}^\ell$ are unit vectors satisfying $y^*x = q \in [0,1]$. On the other hand, Algorithm 3 in [19] is used to draw an estimation of the boundary of $W_q(\mathbb{Q}_\ell)$. Alternatively, the boundary of $W_1(\mathbb{Q}_\ell) \equiv W(\mathbb{Q}_\ell)$ may be drawn by Psarrakos' Matlab code in [22].

3.3. Analytical estimates for Figure 1

Figure 1 illustrates the 1-numerical range $W_1(\mathbb{Q}_6) \equiv W(\mathbb{Q}_6)$ of the 6×6 monic matrix polynomial $\mathbb{Q}_6(\lambda)$ in (27). On the left-hand side of figure 1, 40000 random points are sketched, the roots of $x^*\mathbb{Q}_6(\lambda)x$, with $\|x\| = 1$, whilst On the right-hand side, its boundary is estimated by using the algorithm in [22]. The spectrum is $\sigma(\mathbb{Q}_6) = \{-87.8243 + 0.0001i, -38.4685 - 0.0003i, 33.1825 + 0.0000i, 18.0067 - 13.8987i, 18.0085 + 13.8941i, -8.7706 - 0.0080i, -1.1014 + 0.0128i, -0.9998 + 0.0065i, -0.9992 - 0.0051i, -0.0720 - 0.0548i, -0.0721 + 0.0530i, -0.0644 + 0.0002i\}$ and the eigenvalues are marked with red 'x' in both pictures. Observe that $W(\mathbb{Q}_6)$ lies in the open rectangle $(-100, 50) \times (-20i, 20i)$; there is a clear picture only for the central area of $W(\mathbb{Q}_6)$ and the eigenvalues $-87.8243 + 0.0001i, 33.1825, 18.0067 - 13.8987i, 18.0085 + 13.8941i$ appear to lie out of $W(\mathbb{Q}_6)$. The above rectangle has been used as the range in Psarrakos' code with partition equal to $hx = hy = 0.1$. As referred to [19] and verified in the right part of figure, the existence of the non-differentiable points on the boundary of $W(\mathbb{Q}_6)$ affects the accuracy of the code and it requires further investigation.

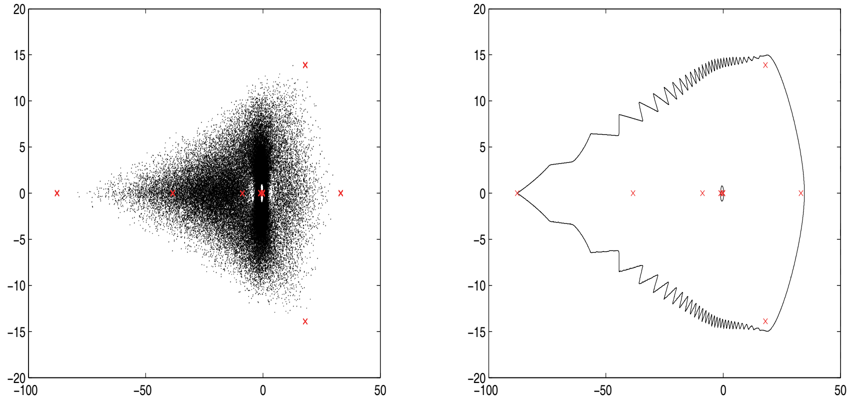


Figure 1: *The 1-numerical range,* $W_1(\mathbb{Q}_6) \equiv W(\mathbb{Q}_6)$*, of the monic matrix polynomial* $\mathbb{Q}_6$.

3.4. Analytical estimates for Figure 2

For $q = 0.7$, the q-numerical range $W_{0.7}(\mathbb{Q}_6)$ is sketched in Figure 2 by the roots of 40000 randomly generated polynomials $y^*\mathbb{Q}_6(\lambda)x$, where $\|x\| = \|y\| = 1$, with $y^*x = 0.7$. Observe that $W_{0.7}(\mathbb{Q}_6)$ has only one connected component, which seems to lie inside the open rectangle $(-100, 50) \times (-50i, 50i)$. Using this rectangle as grid in Algorithm 3 in [19], an approximation of $W_{0.7}(\mathbb{Q}_6)$ is drawn in the right part of the figure (white areas in the dark rectangle). Here, one can see that the spectrum $\sigma(\mathbb{Q}_6)$ lies in the interior of at least three connected components, which is a contradiction (see [21, Theorem 2.1]).

As it has been already noticed in [19], Figures 1 and 2 demonstrate that the re-

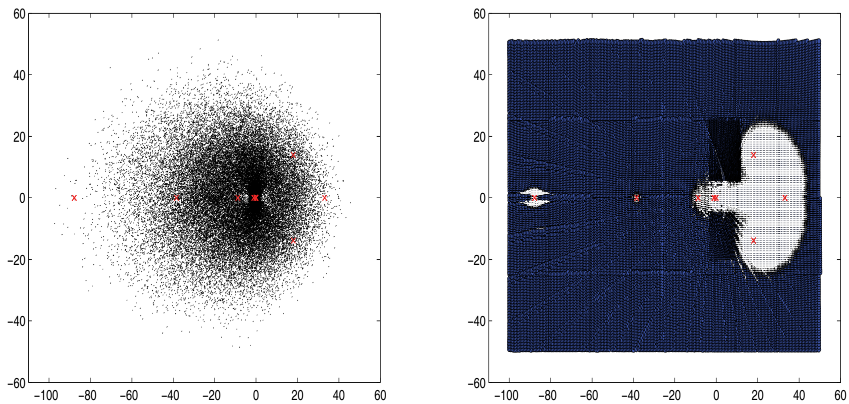
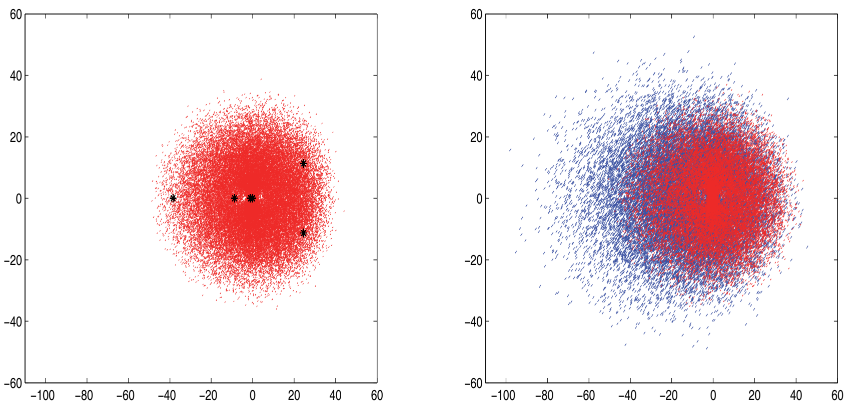


Figure 2: *A connected component of* $W_{0.7}(\mathbb{Q}_6)$.

liable design of the q-numerical range of a monic matrix polynomial depends on the existence of real intervals and/or non-differentiable points on its boundary. This is still an open and challenging problem. Furthermore, if we apply the existing algorithms for plotting the q-numerical range of a matrix polynomial, whose design area extends to a large scale with respect to the axes, then the procedure is burdened with large computation time, some hundreds of thousands, due to the required partitions. This is another problem for further study and optimization.

3.5. Analytical estimates for Figure 3

At this point, take the 4×4 monic matrix polynomial $\mathbb{Q}_4(\lambda)$ in (27). Figure 3(a) approximates $W_{0.7}(\mathbb{Q}_4)$ by 45000 red points, the roots of $y^*\mathbb{Q}_4(\lambda)x$ such that $\|x\| = \|y\| = 1$ and $y^*x = 0.7$. The spectrum is $\sigma(\mathbb{Q}_4) = \{-0.0677+0.0242i, -0.0689-0.0253i, -1.0208 - 0.0029i, -1.0908 + 0.0126i, -8.7705 - 0.0070i, -38.4687 + 0.0004i, 24.5706+11.3325i, 24.5689-11.3346i\}$ marked with black '*'. Observe that $W_{0.7}(\mathbb{Q}_4)$ has one connected component like $W_{0.7}(\mathbb{Q}_6)$. This component seems to lie inside the open rectangle $(-50,50)\times(-40i,40i)$, which is subset of the rectangle containing $W_{0.7}(\mathbb{Q}_6)$. In Figure 3(b), $W_{0.7}(\mathbb{Q}_4)$ and $W_{0.7}(\mathbb{Q}_6)$ are depicted at the same graph by red and blue points, respectively, confirming their inclusion stated in Lemma 2.2.



(a) (b)

Figure 3: *The connected component of $W_{0.7}(\mathbb{Q}_4)$ is illustrated in (a) and the inclusion $W_{0.7}(\mathbb{Q}_4) \subseteq W_{0.7}(\mathbb{Q}_6)$ in (b).*

3.6. Analytical estimates for Figure 4

The sub-figures of Figure 4 present $W_q(\mathbb{Q}_4)$ for some values of $q \in [0,1]$. In particular, we present $W_{0.2}(\mathbb{Q}_4)$ (black points), $W_{0.7}(\mathbb{Q}_4)$ (red points) and $W_1(\mathbb{Q}_4)$ (blue points) in sub-figures 4(a), 4(b) and 4(c), respectively, by drawing the roots

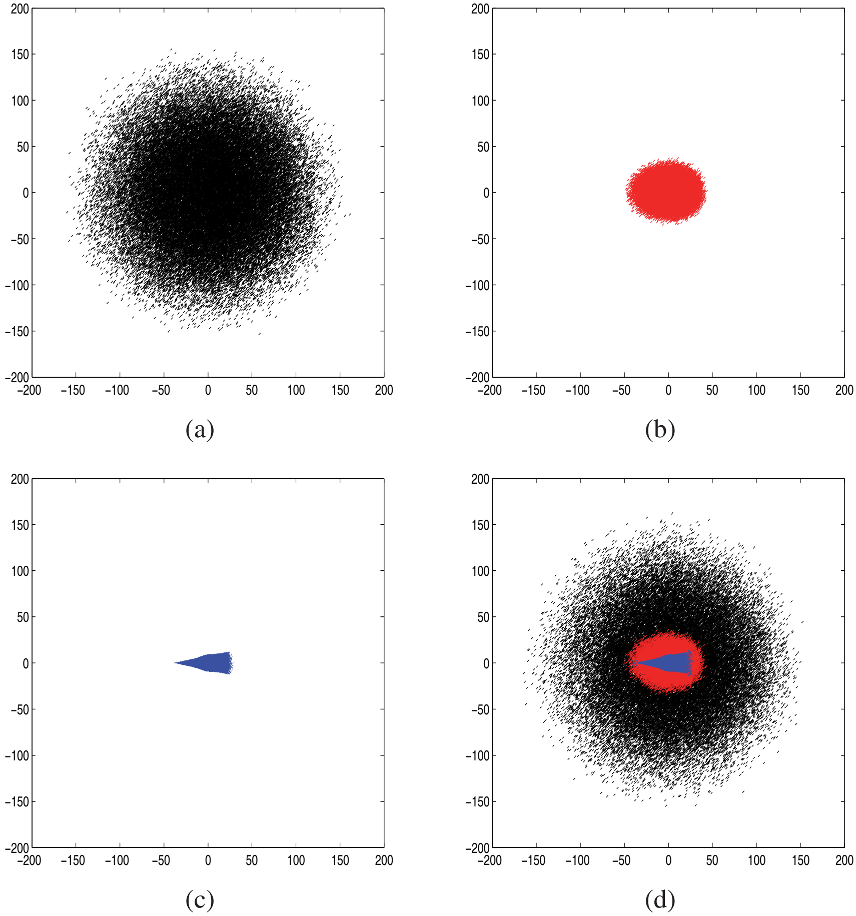


Figure 4: *The subfigures (a), (b), (c) illustrate* $W_{0.2}(\mathbb{Q}_4)$, $W_{0.7}(\mathbb{Q}_4)$ *and* $W(\mathbb{Q}_4)$, *respectively, and* $W(\mathbb{Q}_4) \subseteq W_{0.7}(\mathbb{Q}_4) \subseteq W_{0.2}(\mathbb{Q}_4)$ *appears in (d).*

of a few thousands randomly generated polynomials $y^*\mathbb{Q}_4(\lambda)x$, where $x, y \in \mathbb{C}^4$ are unit vectors such that $y^*x = q$. The last sub-figure 4(d) demonstrates their inclusion $W(\mathbb{Q}_4) \subseteq W_{0.7}(\mathbb{Q}_4) \subseteq W_{0.2}(\mathbb{Q}_4)$, confirming the well known relation

$$W_1(\mathbb{Q}_4) \subseteq W_{q_2}(\mathbb{Q}_4) \subseteq W_{q_1}(\mathbb{Q}_4), \quad 0 < q_1 \leqslant q_2 \leqslant 1.$$

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(Received May 27, 2020)

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