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SOLUTION ALGORITHM OF THE INVERSE SPECTRAL PROBLEM FOR DIRAC OPERATOR WITH A SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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Abstract. We consider an inverse problem for Dirac system in case where one of nonseparated boundary conditions involves a linear function of spectral parameter. We prove the uniqueness theorem for the solution of this problem and then, based on this theorem, we construct a solution algorithm for the considered problem.

1. Introduction

The problem of reconstructing the operators from some spectral data is called an inverse problem of spectral analysis. Theory of inverse problems is playing an important role in the study of spectrum of differential operators. Theory of inverse problems has wide applications in different fields of science and technics (in mechanics, physics, geophysics, electronics, meteorology, etc.). With the advance of quantum mechanics, recent years saw increased interest in inverse spectral problems for various differential operators. That's why the various versions of inverse problems became one of the actively developing fields of modern mathematics.

Dirac equation is one of the most extensively studied differential equations. It is a relativistic wave equation introduced by the English physicist Paul Dirac in 1928. Describing electron behaviour, this equation combines relativistic and quantum theories. It holds an important place in various fields of modern physics and mathematics. Thus, a system of Dirac equations is widely used in relativistic quantum theory. One-dimensional stationary Dirac system has the following form:

$$BY'(x)+Q(x)Y(x)=\lambda Y(x), \tag{1}$$

where

$$B=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x)=\begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad Y(x)=\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

Its canonical form has been obtained by M. G. Gasymov and B. M. Levitan in [12].

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Denote by $W_2^1[0,\pi]$ the Sobolev space of absolutely continuous functions whose derivatives are square summable in the interval $[0,\pi]$ (i.e. belong to $L_2[0,\pi]$). Assume that the elements $p(x)$ and $q(x)$ of the matrix function $Q(x)$ of the equation (1) belong to the space $W_2^1[0,\pi]$.

Consider in $[0,\pi]$ the boundary value problem generated by the canonical Dirac equation (1) and the general boundary conditions

$$A_0Y(0)+A_1Y(\pi)=0, \tag{2}$$

where

$$A_0=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix},\; A_1=\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24}\end{pmatrix},$$

$a_{ir}\left(i=1,2;\; r=\overline{1,4}\right)$ are arbitrary complex numbers. If

$$A_0=\begin{pmatrix} \alpha\lambda+\beta & 1 \\ -\overline{\omega} & 0\end{pmatrix},\; A_1=\begin{pmatrix} \omega & 0 \\ \gamma & 1\end{pmatrix},$$

then the boundary conditions (2) become

$$\begin{aligned} &y_2(0)+(\alpha\lambda+\beta)y_1(0)+\omega y_1(\pi)=0,\\ &y_2(\pi)+\gamma y_1(\pi)-\bar{\omega}y_1(0)=0, \end{aligned} \tag{3}$$

where λ is a spectral parameter, α,β,γ are real numbers, and ω is a complex number. We will denote the boundary value problem (1), (3) by $D(\omega,\alpha,\beta,\gamma)$.

Spectral properties of the Dirac operator have been studied in [6, 7, 9, 13, 15, 16, 18, 24, 26, 27] and other works. Inverse problems for the Dirac operator with separated boundary conditions $(\omega=0)$ have been treated in [3, 11, 19, 25] and other papers. Direct and inverse problems of spectral analysis for a Dirac system with nonseparated boundary conditions (such as periodic, antiperiodic, quasiperiodic, generalized periodic conditions) have been considered in [1, 8, 20, 21, 22]. In case where the nonseparated boundary conditions involve a spectral parameter, the spectral analysis inverse problems for the reconstruction of Dirac system have been treated in [2, 4, 5] with the proof of uniqueness theorem.

In this work, we consider an inverse spectral problem for the boundary value problem $D(\omega,\alpha,\beta,\gamma)$ in case $\alpha\omega\neq 0$, i.e. in case where one of the nonseparated boundary conditions involves a linear function of spectral parameter. Using spectral data, we prove the uniqueness theorem for the solution of inverse problem. Then, based on spectral data and the uniqueness theorem, we construct an algorithm for the reconstruction of Dirac system. In the case of separated boundary conditions, two spectra are usually sufficient to determine the potential. However, if the boundary condition is nonseparated, it is necessary to provide additional spectral data, as two spectra are not enough [23]. It is clear from the course of the proof of the theorem that this is a definite sequence of signs. Note that the other properties of the spectrum of the boundary value problem $D(\omega,\alpha,\beta,\gamma)$ have been studied in [10].

2. Inverse problem

In this section, we consider the problem of reconstructing the boundary value problem $D(\omega,\alpha,\beta,\gamma)$ from spectral data. Before proceeding to the inverse problem, let's consider the spectral data of the problem $D(\omega,\alpha,\beta,\gamma)$.

Denote by $C(x,\lambda)=\begin{pmatrix} c_1(x,\lambda) \\ c_2(x,\lambda)\end{pmatrix}$ and $S(x,\lambda)=\begin{pmatrix} s_1(x,\lambda) \\ s_2(x,\lambda)\end{pmatrix}$ the solutions of the equation (1) which satisfy the initial conditions

$$C(0,\lambda)=\begin{pmatrix} 1 \\ 0\end{pmatrix},\ S(0,\lambda)=\begin{pmatrix} 0 \\ 1\end{pmatrix}. \tag{4}$$

The Wronskian of these solutions is identically equal to 1, i.e.

$$c_1(x,\lambda)s_2(x,\lambda)-c_2(x,\lambda)s_1(x,\lambda)\equiv 1. \tag{5}$$

The general solution of the equation (1) has the form

$$Y(x,\lambda)=M_1C(x,\lambda)+M_2S(x,\lambda),$$

where M_1 and M_2 are arbitrary constants. Considering this solution in the boundary conditions (3), then the characteristic function of the boundary value problem $D(\omega,\alpha,\beta,\gamma)$ has the following form:

$$d(\lambda)=\det\left(A_0+A_1e(\pi,\lambda)\right),$$

where $e(\pi,\lambda)=\begin{pmatrix} c_1(\pi,\lambda) & s_1(\pi,\lambda) \\ c_2(\pi,\lambda) & s_2(\pi,\lambda)\end{pmatrix}$. Using the identity (5) here, we easily obtain the following equation for the characteristic function of the boundary value problem $D(\omega,\alpha,\beta,\gamma)$:

$$\begin{aligned} d(\lambda)=2Re\omega-c_2(\pi,\lambda)-\gamma c_1(\pi,\lambda)+|\omega|^2 s_1(\pi,\lambda)+ \\ +(\alpha\lambda+\beta)\left[s_2(\pi,\lambda)+\gamma s_1(\pi,\lambda)\right]. \end{aligned} \tag{6}$$

The zeros of this function are eigenvalues of problem $D(\omega,\alpha,\beta,\gamma)$. Using the representations given in [10, Lemma 2.1, 20, p. 108] for the functions $c_1(\pi,\lambda)$, $c_2(\pi,\lambda)$, $s_1(\pi,\lambda)$ and $s_2(\pi,\lambda)$ we can write the characteristic function $d(\lambda)$ as follows:

$$\begin{aligned} d(\lambda)=2Re\omega+\alpha\lambda\left(\cos\lambda\pi-\gamma\sin\lambda\pi\right)+\left(\alpha\gamma A_1+\alpha B_2+\beta-\gamma\right)\cos\lambda\pi+ \\ +\left(\alpha A_2+\alpha\gamma B_1-1-|\omega|^2-\beta\gamma\right)\sin\lambda\pi+\psi(\lambda), \end{aligned} \tag{7}$$

where

$$A_1=A+Q_1,\ A_2=A+Q_2,\ B_1=-\frac{p(0)+p(\pi)}{2},\ B_2=\frac{p(0)-p(\pi)}{2},$$

$$A=\frac{1}{2}\int_0^{\pi}\left[p^2(x)+q^2(x)\right]dx,\ Q_1=\frac{q(\pi)-q(0)}{2},\ Q_2=-\frac{q(0)+q(\pi)}{2},$$

$$\psi(\lambda)=\int_{-\pi}^{\pi}\tilde{\psi}(t)\,e^{i\lambda t}dt,\tilde{\psi}\in L_2[-\pi,\pi].$$

The following asymptotic formula is true for the eigenvalues γ_k $(k=\pm 0,\pm 1,\pm 2,\dots)$ of the boundary value problem $D(\omega,\alpha,\beta,\gamma)$ [10]:

$$\gamma_k=k+a+\frac{A}{\pi k}+$$

$$+\frac{4(-1)^k Re\omega b+\alpha q(\pi)(\gamma^2-1)-\alpha b^2 q(0)-2\alpha\gamma p(\pi)-2b^2-2|\omega|^2}{2\pi\alpha b^2 k}+\frac{\xi_k}{k}, \tag{8}$$

where $a=\frac{1}{\pi}\text{arcctg}\gamma,\ b=\sqrt{1+\gamma^2},\ \{\xi_k\}\in l_2.$

Consider the boundary conditions

$$\begin{aligned} &y_2(0)+(\alpha_j\lambda+\beta)y_1(0)+\omega y_1(\pi)=0,\\ &y_2(\pi)+\gamma y_1(\pi)-\bar{\omega}y_1(0)=0, \end{aligned} \tag{9}$$

$$j=1,2,$$

which are similar to (3). Denote the problem (1), (9) by $D(\omega,\alpha_j,\beta,\gamma)$, and the eigenvalues of this problem by $\left\{\gamma_k^{(j)}\right\}$ $(k=0,\pm 1,\pm 2,\dots)$. By the characteristic function (6), the eigenvalues of the problem $D(\omega,\alpha_j,\beta,\gamma)$ are the zeros of the characteristic function

$$d_j(\lambda)=2Re\omega+U_+(\lambda)+(\alpha_j\lambda+\beta)\sigma(\lambda), \tag{10}$$

where

$$U_+(\lambda)=|\omega|^2 s_1(\pi,\lambda)-c_2(\pi,\lambda)-\gamma c_1(\pi,\lambda), \tag{11}$$

$$\sigma(\lambda)=s_2(\pi,\lambda)+\gamma s_1(\pi,\lambda). \tag{12}$$

According to relation (8), we obtain the following asymptotic formula for the eigenvalues of this problem:

$$\gamma_k^{(j)}=k+a+\frac{A}{\pi k}+$$

$$+\frac{4(-1)^k Re\omega b+\alpha_j q(\pi)(\gamma^2-1)-\alpha_j b^2 q(0)-2\alpha_j\gamma p(\pi)-2b^2-2|\omega|^2}{2\pi\alpha_j b^2 k}+$$

$$+\frac{\xi_k^{(j)}}{k},\left\{\xi_k^{(j)}\right\}\in l_2. \tag{13}$$

Also, according to (7), for the characteristic function $d_j(\lambda)$ of the problem $D(\omega,\alpha_j,\beta,\gamma)$ we have

$$d_j(\lambda)=2Re\omega+\alpha_j\lambda(\cos\lambda\pi-\gamma\sin\lambda\pi)+(\alpha_j\gamma A_1+\alpha_j B_2+\beta-\gamma)\cos\lambda\pi+$$

$$+\left(\alpha_j A_2+\alpha_j\gamma B_1-1-|\omega|^2-\beta\gamma\right)\sin\lambda\pi+\psi_j(\lambda), \tag{14}$$

where $\psi_j(\lambda) = \int\limits_{-\pi}^{\pi} \tilde{\psi}_j(t)\, e^{i\lambda t} dt$, $\tilde{\psi}_j \in L_2[-\pi,\pi]$.

In what follows, an important role will also be played by the eigenvalues of the boundary value problems generated by the equation (1) and the boundary conditions

$$y_1(0) = y_1(\pi) = 0, \tag{15}$$

$$y_1(0) = y_2(\pi) + \gamma y_1(\pi) = 0. \tag{16}$$

We denote the sequence of eigenvalues of problem (1), (15) by $\{\lambda_k\}$ $(k = 0, \pm 1, \pm 2, \ldots)$ and the sequence of eigenvalues of problem (1), (16) by $\{\mu_k\}$ $(k = 0, \pm 1, \pm 2, \ldots)$. The following asymptotic formula is true for the sequence $\{\lambda_k\}$:

$$\lambda_k = k + r_k, \ [20]$$

where $\sum\limits_{k=-\infty}^{\infty} r_k^2 < \infty$. Using the characteristic equation $\sigma(\lambda) = 0$, [20] and Rouche's theorem, we can easily show that the asymptotic formula

$$\mu_k = k + \frac{1}{\pi}\operatorname{arcctg}\gamma + m_k \tag{17}$$

is true for the eigenvalues $\{\mu_k\}$ of the boundary value problem (1), (16), where $\sum\limits_{k=-\infty}^{\infty} m_k^2 < \infty$.

Let's state the inverse problem as follows:

INVERSE PROBLEM. *Given the spectra* $\left\{\gamma_k^{(1)}\right\}$, $\left\{\gamma_k^{(2)}\right\}$, *the sign sequence* $\delta_k = sign\ (1 - |\omega s_1(\pi, \mu_k)|)$ $(k = 0, \pm 1, \pm 2, \ldots)$ *and the number* ω, *recover the matrix coefficient function* $Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$ *of the equation* (1) *and the coefficients* $\alpha_1, \alpha_2, \beta, \gamma$ *of the boundary value problems* $D(\omega, \alpha_1, \beta, \gamma)$, $D(\omega, \alpha_2, \beta, \gamma)$.

3. Uniqueness theorem for the reconstruction of boundary value problem

To uniquely recover the boundary value problems $D(\omega, \alpha_1, \beta, \gamma)$ and $D(\omega, \alpha_2, \beta, \gamma)$, the knowledge of their spectra $\left\{\gamma_k^{(1)}\right\}$, $\left\{\gamma_k^{(2)}\right\}$ is not enough. So we have to prove the following theorem.

THEOREM 1. *The spectra* $\left\{\gamma_k^{(1)}\right\}$, $\left\{\gamma_k^{(2)}\right\}$, *the sign sequence*

$$\delta_k = sign\ (1 - |\omega s_1(\pi, \mu_k)|) \quad (k = 0, \pm 1, \pm 2, \ldots)$$

and the number ω *uniquely recovered the boundary value problems* $D(\omega, \alpha_1, \beta, \gamma)$ *and* $D(\omega, \alpha_2, \beta, \gamma)$.

Proof. From [14, Lemma 1.3] it follows that the characteristic function $d_j(\lambda)$ can be uniquely recovered by means of the sequence $\left\{\gamma_k^{(j)}\right\}$ in the form of infinite product

$$d_j(\lambda) = -\pi\alpha_j\sqrt{1+\gamma^2}\left(\gamma_{-0}^{(j)}-\lambda\right)\left(\gamma_{+0}^{(j)}-\lambda\right)\prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{\gamma_k^{(j)}-\lambda}{k}, \tag{18}$$

where $\alpha_j\sqrt{1+\gamma^2} = \frac{2Re\omega}{\pi \lim\limits_{k\to\infty} k\left(\gamma_{2k}^{(j)}-\gamma_{2k+1}^{(j)}+1\right)}$.

Using the asymptotic formula (13), we can determine γ as follows:

$$\gamma = \lim_{\mathrm{k}\to\infty}\operatorname{ctg}\pi\left(\gamma_k^{(j)}-k\right). \tag{19}$$

By (14),

$$d_j(2k) = 2Re\omega + 2k\alpha_j + \alpha_j\gamma A_1 + \alpha_j B_2 + \beta - \gamma + \psi_j(2k). \tag{20}$$

Using this equality, we can recover the parameters α_1 and α_2 as follows:

$$\alpha_j = \frac{1}{2}\lim_{k\to\infty}\frac{d_j(2k)}{k}. \tag{21}$$

After having determined the parameters γ and a_j, we can easily recover β. As $j=1,2$ in the equality (20), we have

$$d_1(2k) = 2Re\omega + 2k\alpha_1 + \alpha_1\gamma A_1 + \alpha_1 B_2 + \beta - \gamma + \psi_1(2k),$$

$$d_2(2k) = 2Re\omega + 2k\alpha_2 + \alpha_2\gamma A_1 + \alpha_2 B_2 + \beta - \gamma + \psi_2(2k).$$

On multiplying the first of the last two equalities by α_2, and the second one by α_1, and then subracting one from another, we easily obtain

$$\beta = -2Re\omega + \gamma + \frac{\alpha_2 d_1(2k) - \alpha_1 d_2(2k)}{\alpha_2 - \alpha_1} - \psi_3(2k),$$

where $\psi_3(\lambda) = \int\limits_{-\pi}^{\pi}\tilde{\psi}_3(t)\,e^{i\lambda t}dt$, $\tilde{\psi}_3 \in L_2[-\pi,\pi]$. Passing to the limit as $k\to\infty$, we obtain the following formula for the recovery of parameter β:

$$\beta = -2Re\omega + \gamma + \frac{1}{\alpha_1-\alpha_2}\lim_{k\to\infty}\left[\alpha_1 d_2(2k) - \alpha_2 d_1(2k)\right]. \tag{22}$$

Using the equality (10), we can determine the function $\sigma(\lambda)$ as follows:

$$\sigma(\lambda) = \frac{d_1(\lambda)-d_2(\lambda)}{(\alpha_1-\alpha_2)\lambda}. \tag{23}$$

Also, by means of (10), the function $U_+(\lambda)$ can be recovered as follows:

$$U_+(\lambda) = \frac{\alpha_2 d_1(\lambda) - \alpha_1 d_2(\lambda)}{\alpha_2-\alpha_1} - \beta\sigma(\lambda) - 2Re\omega. \tag{24}$$

Now let's consider the function

$$U_{-}(\lambda) = -|\omega|^2 s_1(\pi,\lambda) - c_2(\pi,\lambda) - \gamma c_1(\pi,\lambda). \tag{25}$$

From (11), (25) and

$$c_1(\pi,\lambda)\,\sigma(\lambda) - s_1(\pi,\lambda)\,[c_2(\pi,\lambda) + \gamma c_1(\pi,\lambda)] = 1 \tag{26}$$

it easily follows that

$$U_{-}^2(\lambda) - U_{+}^2(\lambda) = 4|\omega|^2 (c_1(\pi,\lambda)\,\sigma(\lambda) - 1).$$

Taking into account in the last equality that the eigenvalues μ_k are the zeros of the function $\sigma(\lambda)$, we obtain

$$U_{-}^2(\mu_k) - U_{+}^2(\mu_k) = -4|\omega|^2.$$

Then

$$U_{-}(\mu_k) = sign\, U_{-}(\mu_k)\sqrt{U_{+}^2(\mu_k) - 4|\omega|^2}. \tag{27}$$

From (25) and (26) it follows

$$U_{-}(\mu_k) = \frac{1}{s_1(\pi,\mu_k)} - |\omega|^2 s_1(\pi,\mu_k) = \frac{1-|\omega|^2 s_1^2(\pi,\mu_k)}{s_1(\pi,\mu_k)}. \tag{28}$$

Using the intermittency of the zeros of the functions $s_1(\pi,\lambda)$ and $\sigma(\lambda)$ (this fact is easily shown by the method given in [3]), we can obtain $sign\, s_1(\pi,\mu_k) = (-1)^{k+1}$. By the relations (27) and (28), we have

$$U_{-}(\mu_k) = (-1)^{k+1}\,\delta_k\sqrt{U_{+}^2(\mu_k) - 4|\omega|^2}. \tag{29}$$

Let

$$\varphi(\lambda) = U_{+}(\lambda) - U_{-}(\lambda) + 2|\omega|^2 \sin\lambda\pi. \tag{30}$$

It is known [10, 20] that

$$s_1(\pi,\lambda) = -\sin\lambda\pi + \varphi_1(\lambda),$$
$$c_1(\pi,\lambda) = \cos\lambda\pi + \varphi_2(\lambda),$$
$$c_2(\pi,\lambda) = \sin\lambda\pi + \varphi_3(\lambda),$$

where

$$\varphi_p(\lambda) = \int_{-\pi}^{\pi} \tilde{\varphi}_p(t)\,e^{i\lambda t}dt,\ \tilde{\varphi}_p(t) \in L_2[-\pi,\pi],\ p = 1,2,3.$$

Taking into account the last relations, the asymptotics μ_k and [20, Lemma 2], we obtain $\sum\limits_{k=-\infty}^{\infty} |\varphi(\mu_k)|^2 < \infty$. Then, from [17, Theorem 28], the function $\varphi(\lambda)$ satisfies the interpolation formula

$$\varphi(\lambda) = \sigma(\lambda)\sum_{k=-\infty}^{\infty} \frac{\varphi(\mu_k)}{(\lambda-\mu_k)\,\sigma'(\mu_k)}, \tag{31}$$

where

$$\varphi(\mu_k) = U_+(\mu_k) + (-1)^k \delta_k \sqrt{U_+^2(\mu_k) - 4|\omega|^2} + 2|\omega|^2 \sin\mu_k\pi. \tag{32}$$

Hence we see that in order to determine the function $U_-(\lambda)$ from (30), it suffices to determine, in addition to the spectra $\left\{\gamma_k^{(1)}\right\}$ and $\left\{\gamma_k^{(2)}\right\}$, the sequence $\{\delta_k\}$.

Then we can recover the function $s_1(\pi,\lambda)$ by means of the functions $U_\pm(\lambda)$ from the formula

$$s_1(\pi,\lambda) = \frac{1}{2|\omega|^2}\left[U_+(\lambda) - U_-(\lambda)\right]. \tag{33}$$

The zeros λ_k of this function are the eigenvalues of the boundary value problem generated by the equation (1) and the boundary conditions (15).

As $\sigma(\lambda)$, $s_1(\pi,\lambda)$ and γ are determined, the function $s_2(\pi,\lambda)$ can be recovered by means of the equality

$$s_2(\pi,\lambda) = \sigma(\lambda) - \gamma s_1(\pi,\lambda), \tag{34}$$

which follows from (12). The zeros θ_k $(k = 0, \pm 1, \pm 2, \ldots)$ of this function are the eigenvalues of the boundary value problem generated by the equation (1) and the boundary conditions $y_1(0) = y_2(\pi) = 0$.

As is known [11, 20], the sequences $\{\mu_k\}$ and $\{\lambda_k\}$ determine uniquely the matrix coefficient function $Q(x)$.

So, the spectra of the boundary value problems $D(\omega,\alpha_1,\beta,\gamma)$, $D(\omega,\alpha_2,\beta,\gamma)$, the sign sequence $\{\delta_k\}$ and the number ω uniquely recover also the coefficient $Q(x)$ of the Dirac equation and the parameters $\alpha_1,\alpha_2,\beta,\gamma$ of the boundary conditions. □

The theorem is proved.

4. An algorithm for the reconstruction of boundary value problem

Based on the uniqueness theorem proved above, let's construct the following algorithm for the recovery of boundary value problems $D(\omega,\alpha_1,\beta,\gamma)$ and $D(\omega,\alpha_2,\beta,\gamma)$.

ALGORITHM. Let the spectral data $\left\{\gamma_k^{(1)}\right\}$, $\left\{\gamma_k^{(2)}\right\}$ of the boundary value problems $D(\omega,\alpha_1,\beta,\gamma)$ and $D(\omega,\alpha_2,\beta,\gamma)$, the sign sequence $\{\delta_k\}$ and the number ω be given.

Step 1. Recover the characteristic function $d_j(\lambda)$ from the infinite product (18) by means of the sequence $\left\{\gamma_k^{(j)}\right\}$.

Step 2. Determine the parameters $\gamma, \alpha_j\ (j = 1,2), \beta$ of the boundary conditions (9) by the formulas (19), (21), (22), respectively.

Step 3. Recover the function $\sigma(\lambda)$ from the formula (23) and find the zeros μ_k of this function from (17).

Step 4. Recover the function $U_+(\lambda)$ from (24).

Step 5. Using (27) and (29), calculate the values of the function $U_-(\lambda)$ at the points $\{\mu_k\}$.

Step 6. Recover the function $\varphi(\lambda)$ by the interpolation formula (31), where the sequence $\varphi(\mu_k)$, $(k = 0, \pm 1, \pm 2, \ldots)$ is calculated by (32).

Step 7. Using the interpolation formula (31) and the function $U_+(\lambda)$ construct the function $U_-(\lambda)$ given by (25) from the equality (30).

Step 8. Using the functions $U_\pm(\lambda)$, find the characteristic function $s_1(\pi, \lambda)$ of the boundary value problem (1), (15) from the formula (33).

Step 9. Using the functions $\sigma(\lambda)$, $s_1(\pi, \lambda)$ and parameter γ, find the characteristic function $s_2(\pi, \lambda)$ of the boundary value problem (1), $y_1(0) = y_2(\pi) = 0$ from the formula (34).

Step 10. The recovery of coefficient $Q(x)$ of the Dirac equation (1) is performed by means of the sequences $\{\lambda_k\}$ and $\{\mu_k\}$ in accordance with the procedure described in [11, 20].

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