BIDIAGONAL DECOMPOSITIONS AND TOTAL
POSITIVITY OF SOME SPECIAL MATRICES

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Abstract. The matrix $S = [1 + x_i y_j]_{i,j=1}^n$, $0 < x_1 < \cdots < x_n$, $0 < y_1 < \cdots < y_n$, has gained importance lately due to its role in powers preserving total nonnegativity. We give an explicit decomposition of S in terms of elementary bidiagonal matrices, which is analogous to the *Neville decomposition*. We give a bidiagonal decomposition of $S^{\circ m} = [(1 + x_i y_j)^m]$ for positive integers $1 \leqslant m \leqslant n - 1$. We also explore the total positivity of Hadamard powers of another important class of matrices called *mean matrices*.

1. Introduction

A matrix is called *totally nonnegative* (respectively *totally positive*) if all its minors are nonnegative (respectively positive) (see [8]). Totally nonnegative (respectively totally positive) matrices have also been called totally positive (respectively strictly totally positive), as can be seen in [12, 21, 27]. Let A be an $n \times n$ matrix with nonnegative entries. The matrix A^T denotes the *transpose* of A. We assume $1 \leqslant i, j \leqslant n$, unless otherwise stated. The (i, j)th entry of A is denoted by A_{ij}. Let $1 \leqslant k \leqslant n$. A matrix is called TN_k (respectively TP_k) if all its minors upto order k are nonnegative (respectively positive). If $r > 0$, then the rth *Hadamard power* of A is given by $A^{\circ r} = [A_{ij}^r]$. The matrix A is said to be *infinitely divisible* if $A^{\circ r}$ is positive semidefinite for every $r > 0$. We refer the reader to [2, 4, 14, 26] for many examples and results on infinitely divisible matrices.

It was shown in [11] that if A is positive semidefinite, then for $r \geqslant n - 2$, $A^{\circ r}$ is also positive semidefinite. The sharpness of the lower bound $n - 2$ was given by considering the positive semidefinite matrix $A_\varepsilon = [1 + \varepsilon i j]$, where $\varepsilon > 0$. It was shown that if $r < n - 2$ is any positive non integer, then $A_\varepsilon^{\circ r}$ fails to be positive semidefinite for sufficiently small $\varepsilon > 0$.

A Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ is called TN_k if given any increasing sequences $\{x_m\}_{m \geqslant 1}$ and $\{y_m\}_{m \geqslant 1}$ of real numbers, the matrix $[f(x_i - y_j)]$ is TN_k. Let $W : \mathbb{R} \to \mathbb{R}$ be defined as

$$W(x) = \begin{cases} \cos x & \text{if } x \in (-\pi/2, \pi/2), \\ 0 & \text{otherwise.} \end{cases}$$

Schoenberg [30] showed that if $r \geqslant 0$ and k is an integer greater than or equal to 2, then $W(x)^r$ is TN_k if and only if $r \geqslant k-2$. (See also [23, Remark 6.2].)

A function $f : \mathbb{R} \to \mathbb{R}$ is called a *Pólya frequency function* if f is Lebesgue integrable on $\mathbb{R}$, does not vanish at at least two points, and for $n \in \mathbb{N}$ and real numbers $x_1 < \cdots < x_n, y_1 < \cdots < y_n$, the matrix $[f(x_i - y_j)]$ is totally nonnegative. Consider the Pólya frequency function $\Omega : \mathbb{R} \to \mathbb{R}$ defined as

$$\Omega(x) := \begin{cases} xe^{-x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Karlin [21] proved that for any integer $k \geqslant 2$ and any real number $r \geqslant 0$, the function $\Omega(x)^r$ is TN_k if r is a non negative integer or $r \geqslant k-2$, see also [22, p. 211]. Recently, Khare [23] showed its converse by proving that for any integer $k \geqslant 2$ and $r \in (0, k-2) \setminus \mathbb{Z}$, $\Omega(x)^r$ is not TN_k.

The characterizations of total nonnegativity of Hadamard powers of a matrix have also recently appeared in [7, 9]. Let $x_1, \ldots, x_n$ be distinct positive real numbers. Let $X = [1 + x_i x_j]$. Jain [17, Theorem 1.1] proved that the matrix $X^{\circ r}$ is positive semidefinite if and only if r is a nonnegative integer or $r > n-2$. So the matrix X serves as a stronger example than the matrix A_ε for proving the sharpness of the lower bound $n-2$, in the sense that it works for every positive non integer $r < n-2$. She also proved that if $0 < x_1 < \cdots < x_n$, then the matrix $X^{\circ r}$ is totally positive for $r > n-2$ (see [17, Theorem 2.4]). For any real numbers $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$ such that $1 + x_i y_j > 0$, let

$$S = [1 + x_i y_j].$$

Recently, Khare [23, Theorem C] showed that the matrix $S^{\circ r}$ is totally positive if $r > n-2$, and totally nonnegative if and only if $r = 0, 1, \ldots, n-2$. Jain [18, Corollary 5] showed that for $r = 0, 1, \ldots, n-2$, $\mathrm{rank}(S^{\circ r}) = r+1$, and therefore, $S^{\circ r}$ is not totally positive. Thus $S^{\circ r}$ is totally positive if and only if $r > n-2$ and $S^{\circ r}$ is totally nonnegative if and only if either $r > n-2$ or $r = 0, 1, \ldots, n-2$. The matrix S was used to prove the converse of Karlin's result, see [23, Theorem 1.7].

A matrix A is called *lower* (respectively *upper*) *bidiagonal* if $A_{ij} = 0$ for $i - j \neq 0, 1$ (respectively for $i - j \neq 0, -1$). For any real number s and positive integer $2 \leqslant i \leqslant n$, let $L_i(s)$ (respectively $U_i(s)$) be the matrix whose diagonal entries are one, $(i, i-1)$th (respectively $(i-1, i)$th) entry is s and the remaining entries are zero. These particular bidiagonal matrices are called *elementary bidiagonal matrices*. Cryer [5] showed that any $n \times n$ totally nonnegative matrix A can be written as

$$A = \prod_k L^{(k)} \prod_\ell U^{(\ell)}, \tag{1}$$

where $L^{(k)}$ and $U^{(\ell)}$ are, respectively, lower and upper elementary bidiagonal matrices (see also [6]). Careful analyses of the relationships between totally nonnegative matrices and bidiagonal decompositions have been done in [10, 12]. For more results on bidiagonal decompositions of matrices, see [1, 15, 16, 19]. In particular, in the case of invertible totally nonnegative matrices, the unicity of the bidiagonal decomposition under certain conditions was assured in [12]. Finding explicit decompositions

like (1) is a non trivial task as there may not be obvious patterns to guess the factors. One of the main aims of this paper is to give the decomposition (1) for the matrix S, which is similar to what appears in the *successive elementary bidiagonal decomposition* (also called *Neville decomposition*) for invertible totally nonnegative matrices, see [8, Theorem 2.2.2]. However, the matrix S is not invertible for $n \geqslant 3$. To find this decomposition, we also give an LU decomposition of S. We also give another interesting decomposition for S in terms of bidiagonal matrices. The difference with the earlier one is that here the lower and upper bidiagonal matrices appear in a mixed pattern; however, a major advantage is that this decomposition can be generalized to Hadamard integer powers of S.

Another important class of matrices is that of the *mean matrices*. For a discussion on infinite divisibility of these matrices, see [4]. Some important examples of means on positive real numbers are the *arithmetic mean* $\mathscr{A}(a,b) = \dfrac{a+b}{2}$, the *harmonic mean* $H(a,b) = \dfrac{2ab}{a+b}$, the *Heinz mean* $\mathscr{H}_\nu(a,b) = \dfrac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}$ for $0 \leqslant \nu \leqslant 1$ and the *binomial mean* $\mathscr{B}_\alpha(a,b) = \left(\dfrac{a^\alpha + b^\alpha}{2} \right)^{1/\alpha}$ for $-\infty \leqslant \alpha \leqslant \infty$, where it is understood that $\mathscr{B}_0(a,b) = \sqrt{ab}$, $\mathscr{B}_\infty(a,b) = \max(a,b)$ and $\mathscr{B}_{-\infty}(a,b) = \min(a,b)$. Let $0 < \lambda_1 < \cdots < \lambda_n$. In [4], it is shown that $\left[\dfrac{1}{\mathscr{A}(\lambda_i, \lambda_j)} \right]$, $[H(\lambda_i, \lambda_j)]$, $\left[\dfrac{1}{\mathscr{H}_\nu(\lambda_i, \lambda_j)} \right]$ and $\left[\dfrac{1}{\mathscr{B}_\alpha(\lambda_i, \lambda_j)} \right]$ $(\alpha \geqslant 0)$ are infinitely divisible. Since the *Cauchy matrix* $C = \left[\dfrac{1}{\lambda_i + \lambda_j} \right]$ is totally positive (see [27]), so are $\left[\dfrac{1}{\mathscr{A}(\lambda_i, \lambda_j)} \right]$, $[H(\lambda_i, \lambda_j)]$ and $\left[\dfrac{1}{\mathscr{H}_\nu(\lambda_i, \lambda_j)} \right]$ $\left(\nu \neq \tfrac{1}{2} \right)$.
In particular, they are TP_3. By [20, Theorem 4.2] (or [9, Theorem 5.2]), we have that for $r \geqslant 1$, their rth Hadamard powers are also TP_3. We give a simple proof to show that the Hadamard powers of these matrices are in fact totally positive.

In Section 2, we state and prove our results for the decompositions of S. In Section 3, we give the results for mean matrices.

2. Bidiagonal decompositions for $S = [1 + x_i y_j]$ and its Hadamard powers

We begin by giving an LU decomposition for S.

PROPOSITION 2.1. *Let* $0 < x_1 < \cdots < x_n$ *and* $0 < y_1 < \cdots < y_n$. *Then the matrix* $S = [1 + x_i y_j]$ *can be written as* LU, *where* L *and* U *are the lower and upper triangular matrices, respectively, given by*

$$L_{ij} = \begin{cases} \dfrac{1 + y_1 x_i}{\sqrt{1 + x_1 y_1}} & \text{if } j = 1, \\[3mm] \dfrac{(x_i - x_1)\sqrt{y_2 - y_1}}{\sqrt{x_2 - x_1}\sqrt{1 + x_1 y_1}} & \text{if } j = 2, \\[3mm] 0 & \text{otherwise} \end{cases}$$

and

$$
U_{ij} = \begin{cases} \dfrac{1+x_1 y_j}{\sqrt{1+x_1 y_1}} & \text{if } i = 1, \\[3ex] \dfrac{(y_j - y_1)\sqrt{x_2 - x_1}}{\sqrt{y_2 - y_1}\sqrt{1+x_1 y_1}} & \text{if } i = 2, \\[3ex] 0 & \text{otherwise}. \end{cases}
$$

Proof. This can be proved by checking the (i,j) th entry of LU :

$$
\begin{aligned}
(LU)_{ij} &= \sum_{k=1}^{2} L_{ik} U_{kj} \\
&= \left(\frac{1+y_1 x_i}{\sqrt{1+x_1 y_1}} \right) \left(\frac{1+x_1 y_j}{\sqrt{1+x_1 y_1}} \right) + \\
&\quad \left(\frac{(x_i - x_1)\sqrt{y_2 - y_1}}{\sqrt{x_2 - x_1}\sqrt{1+x_1 y_1}} \right) \left(\frac{(y_j - y_1)\sqrt{x_2 - x_1}}{\sqrt{y_2 - y_1}\sqrt{1+x_1 y_1}} \right) \\
&= \frac{1}{(1+x_1 y_1)} \left[(1+y_1 x_i)(1+x_1 y_j) + (x_i - x_1)(y_j - y_1) \right] \\
&= 1 + x_i y_j. \quad \square
\end{aligned}
$$

Let $\text{diag}[d_i]$ denote the diagonal matrix with diagonal entries $d_1,\dots,d_n$. Now, we give the decomposition (1) for S.

THEOREM 2.2. *Let* $n \geqslant 2$. *Let* $0 < x_1 < \cdots < x_n$ *and* $0 < y_1 < \cdots < y_n$. *For* $2 \leqslant i \leqslant n$, *let* $\alpha_i = \dfrac{1+y_1 x_i}{1+y_1 x_{i-1}}$ *and* $\alpha_i' = \dfrac{1+x_1 y_i}{1+x_1 y_{i-1}}$. *For* $3 \leqslant j \leqslant n$, *let*

$$
\beta_j = \frac{(x_j - x_{j-1})(1+y_1 x_{j-2})}{(x_{j-1} - x_{j-2})(1+y_1 x_{j-1})}
$$

and

$$
\beta_j' = \frac{(y_j - y_{j-1})(1+x_1 y_{j-2})}{(y_{j-1} - y_{j-2})(1+x_1 y_{j-1})}.
$$

Let

$$
D = \begin{bmatrix} 1+x_1 y_1 & 0 & 0 \cdots 0 \\[1ex] 0 & \dfrac{(x_2 - x_1)(y_2 - y_1)}{1+x_1 y_1} & 0 \cdots 0 \\[2ex] 0 & 0 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \ddots \vdots \\ 0 & 0 & 0 \cdots 0 \end{bmatrix}.
$$

Then

$$S = (L_n(\alpha_n) \cdots L_2(\alpha_2)) (L_n(\beta_n) \cdots L_3(\beta_3)) D \left(U_3(\beta_3') \cdots U_n(\beta_n') \right)$$
$$\left(U_2(\alpha_2') \cdots U_n(\alpha_n') \right). \tag{2}$$

Proof. Let the lower triangular matrices $M = [M_{ij}], M' = [M_{ij}'], N = [N_{ij}], N' = [N_{ij}'], Y_1 = [(Y_1)_{ij}]$ and $Y_2 = [(Y_2)_{ij}]$ be defined as follows:

$$M_{ij} = \begin{cases} \dfrac{1+y_1 x_i}{1+y_1 x_j} & \text{if } i \geqslant j, \\ 0 & \text{otherwise,} \end{cases}$$

$$M_{ij}' = \begin{cases} \dfrac{1+x_1 y_i}{1+x_1 y_j} & \text{if } i \geqslant j, \\ 0 & \text{otherwise,} \end{cases}$$

$$N_{ij} = \begin{cases} 1 & \text{if } i = j = 1, \\ \dfrac{(x_i - x_{i-1})(1+y_1 x_{j-1})}{(x_j - x_{j-1})(1+y_1 x_{i-1})} & \text{if } i \geqslant j \geqslant 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$N_{ij}' = \begin{cases} 1 & \text{if } i = j = 1, \\ \dfrac{(y_i - y_{i-1})(1+x_1 y_{j-1})}{(y_j - y_{j-1})(1+x_1 y_{i-1})} & \text{if } i \geqslant j \geqslant 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$(Y_1)_{ij} = \begin{cases} \dfrac{1+y_1 x_i}{1+y_1 x_1} & \text{if } j = 1, \\ \dfrac{x_i - x_{j-1}}{x_j - x_{j-1}} & \text{if } i \geqslant j \geqslant 2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(Y_2)_{ij} = \begin{cases} \dfrac{1+x_1 y_i}{1+x_1 y_1} & \text{if } j = 1, \\ \dfrac{y_i - y_{j-1}}{y_j - y_{j-1}} & \text{if } i \geqslant j \geqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $P_i = L_i(\alpha_i), P_i' = L_i(\alpha_i'), Q_j = L_j(\beta_j)$ and $Q_j' = L_j(\beta_j')$ for $2 \leqslant i \leqslant n$ and $3 \leqslant j \leqslant n$. To prove the theorem, we show the following:

(a) $P_n P_{n-1} \cdots P_2 = M$ and $P'_n P'_{n-1} \cdots P'_2 = M'$.

(b) $Q_n Q_{n-1} \cdots Q_3 = N$ and $Q'_n Q'_{n-1} \cdots Q'_3 = N'$.

(c) $MN = Y_1$ and $M'N' = Y_2$.

(d) $S = Y_1 D Y_2^T$.

We shall give a proof for the first part of each of (a), (b) and (c). The proofs of their second parts are analogous. So we omit them.

Note that for $i \geqslant 2$, multiplying any matrix by $L_i(s)$ on the left is equivalent to changing its ith row to the one obtained by adding s times the $(i-1)$th row to it. Let $\delta_{ij} = 1$ for $i = j$, and 0 otherwise. To prove (a), we show that for $2 \leqslant k \leqslant n$,

$$(P_k \cdots P_2)_{ij} = \begin{cases} M_{ij} & \text{if } i \leqslant k, \\ \delta_{ij} & \text{if } i > k. \end{cases} \tag{3}$$

Let I be the identity matrix of order n. Then for $i \neq 2$,

$$(P_2)_{ij} = (P_2 I)_{ij} = \delta_{ij}.$$

Note that $M_{1j} = \delta_{1j}$. We also have

$$(P_2)_{2j} = (P_2 I)_{2j}$$

$$= I_{2j} + \left(\frac{1 + y_1 x_2}{1 + y_1 x_1} \right) I_{1j}$$

$$= \delta_{2j} + \left(\frac{1 + y_1 x_2}{1 + y_1 x_1} \right) \delta_{1j}$$

$$= \begin{cases} \dfrac{1 + y_1 x_2}{1 + y_1 x_1} & \text{if } j = 1, \\ 1 & \text{if } j = 2, \\ 0 & \text{otherwise} \end{cases}$$

$$= M_{2j}.$$

Hence (3) holds for $k = 2$. Let it hold for $k = m$, where $2 \leqslant m \leqslant n - 1$. Then

$$(P_m \cdots P_2)_{ij} = \begin{cases} M_{ij} & \text{if } i \leqslant m, \\ \delta_{ij} & \text{if } i > m. \end{cases}$$

So for $i \leqslant m$, $(P_{m+1} \cdots P_2)_{ij} = (P_m \cdots P_2)_{ij} = M_{ij}$. For $i > m + 1$,

$$(P_{m+1} \cdots P_2)_{ij} = (P_m \cdots P_2)_{ij} = \delta_{ij}.$$

Also,

$$(P_{m+1} \cdots P_2)_{m+1,j} = (P_m \cdots P_2)_{m+1,j} + \left(\frac{1 + y_1 x_{m+1}}{1 + y_1 x_m} \right) (P_m \cdots P_2)_{mj}$$

$$= \delta_{m+1,j} + \left(\frac{1 + y_1 x_{m+1}}{1 + y_1 x_m} \right) M_{mj}$$

$$= M_{m+1,j}.$$

Hence

$$(P_{m+1} \cdots P_2)_{ij} = \begin{cases} M_{ij} & \text{if } i \leqslant m+1, \\ \delta_{ij} & \text{if } i > m+1. \end{cases}$$

Thus (3) holds for $k = m+1$. Hence (3) is true for every $2 \leqslant k \leqslant n$. Putting $k = n$ proves the first part of (a). To prove (b), we show that for $3 \leqslant k \leqslant n$,

$$(Q_k \cdots Q_3)_{ij} = \begin{cases} N_{ij} & \text{if } i \leqslant k, \\ \delta_{ij} & \text{if } i > k. \end{cases} \tag{4}$$

We prove this in a similar manner as above. Since $Q_3 = Q_3 I$, $(Q_3)_{ij} = (Q_3 I)_{ij} = \delta_{ij}$ for $i \neq 3$. Note that for $i = 1$ and $i = 2$, $N_{ij} = \delta_{ij}$. Now

$$(Q_3)_{3j} = I_{3j} + \left(\frac{(x_3 - x_2)(1 + y_1 x_1)}{(x_2 - x_1)(1 + y_1 x_2)} \right) I_{2j}$$

$$= \delta_{3j} + \left(\frac{(x_3 - x_2)(1 + y_1 x_1)}{(x_2 - x_1)(1 + y_1 x_2)} \right) \delta_{2j}$$

$$= N_{3j}.$$

Suppose (4) holds for $k = m$, where $3 \leqslant m \leqslant n-1$. Then

$$(Q_m \cdots Q_3)_{ij} = \begin{cases} N_{ij} & \text{if } i \leqslant m, \\ \delta_{ij} & \text{if } i > m. \end{cases}$$

So for $i \leqslant m$,

$$(Q_{m+1} \cdots Q_3)_{ij} = (Q_m \cdots Q_3)_{ij} = N_{ij}.$$

For $i > m+1$,

$$(Q_{m+1} \cdots Q_3)_{ij} = (Q_m \cdots Q_3)_{ij} = \delta_{ij}.$$

Also,

$$(Q_{m+1} \cdots Q_3)_{m+1,j} = (Q_m \cdots Q_3)_{m+1,j}$$

$$+ \left(\frac{(x_{m+1} - x_m)(1 + y_1 x_{m-1})}{(x_m - x_{m-1})(1 + y_1 x_m)} \right) (Q_m \cdots Q_3)_{mj}$$

$$= \delta_{m+1,j} + \left(\frac{(x_{m+1} - x_m)(1 + y_1 x_{m-1})}{(x_m - x_{m-1})(1 + y_1 x_m)} \right) N_{mj}$$

$$= N_{m+1,j}.$$

Therefore

$$(Q_{m+1} \cdots Q_3)_{ij} = \begin{cases} N_{ij} & \text{if } i \leqslant m+1, \\ \delta_{ij} & \text{if } i > m+1. \end{cases}$$

So (4) is true for every $3 \leqslant k \leqslant n$. Putting $k = n$ proves the first part of (b). Now for each i,

$$(MN)_{i1} = M_{i1}N_{11} = \left(\frac{1+y_1x_i}{1+y_1x_1}\right) = (Y_1)_{i1}.$$

For $i < j$,

$$(MN)_{ij} = 0 = (Y_1)_{ij}.$$

For $1 < j \leqslant i$,

$$\begin{aligned}
(MN)_{ij} &= \sum_{k=1}^{n} M_{ik}N_{kj} \\
&= \sum_{i \geqslant k \geqslant j} \left(\frac{1+y_1x_i}{1+y_1x_k}\right)\left(\frac{(x_k - x_{k-1})(1+y_1x_{j-1})}{(x_j - x_{j-1})(1+y_1x_{k-1})}\right) \\
&= \frac{(1+y_1x_i)(1+y_1x_{j-1})}{y_1(x_j - x_{j-1})} \sum_{i \geqslant k \geqslant j} \left[\frac{1}{(1+y_1x_{k-1})} - \frac{1}{(1+y_1x_k)}\right] \\
&= \frac{(1+y_1x_i)(1+y_1x_{j-1})}{y_1(x_j - x_{j-1})} \left[\frac{1}{(1+y_1x_{j-1})} - \frac{1}{(1+y_1x_i)}\right] \\
&= \frac{x_i - x_{j-1}}{x_j - x_{j-1}} \\
&= (Y_1)_{ij}.
\end{aligned}$$

This proves the first part of (c).

Let $\sqrt{D} = \text{diag}[\sqrt{D_{ii}}]$. Let L and U be the lower and upper triangular matrices, respectively, in Proposition 2.1. To prove (d), we note that $Y_1\sqrt{D} = L$ and $\sqrt{D}Y_2^T = U$. Since $LU = S$, we have $Y_1DY_2^T = (Y_1\sqrt{D})(\sqrt{D}Y_2^T) = S$. This completes our proof. □

In particular, if $x_i = y_i = i$ for $1 \leqslant i \leqslant n$, then we have the following corollary.

COROLLARY 2.3. *We have*

$$X = [1 + ij] = ZDZ^T, \tag{5}$$

where

$$Z = \left[L_n\left(\frac{n+1}{n}\right)L_{n-1}\left(\frac{n}{n-1}\right)\cdots L_2\left(\frac{3}{2}\right)\right]\left[L_n\left(\frac{n-1}{n}\right)L_{n-1}\left(\frac{n-2}{n-1}\right)\cdots L_3\left(\frac{2}{3}\right)\right]$$

and

$$D = \begin{bmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix}.$$

In the next theorem, we give a bidiagonal decomposition of the mth Hadamard powers of S for $m \in \{1, 2, \ldots, n-1\}$. For distinct real numbers $x_1, \ldots, x_n$ and $1 \leqslant k \leqslant n-1$, let the lower bidiagonal matrices $L^{\mathbf{x}(k)}$ and upper bidiagonal matrices $U^{\mathbf{x}(k)}$ be defined as

$$
(L^{\mathbf{x}(k)})_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } i = j+1, i = n-k+1, \\ \displaystyle\prod_{t=0}^{k-n+i-2} \frac{x_i - x_{i-1-t}}{x_{i-1} - x_{i-2-t}} & \text{if } i = j+1, i > n-k+1, \\ 0 & \text{otherwise,} \end{cases}
$$

and

$$
(U^{\mathbf{x}(k)})_{ij} = \begin{cases} 1 & \text{if } i = j, i \leqslant n-k, \\ x_i - x_{n-k} & \text{if } i = j, i > n-k, \\ x_1 & \text{if } i = j-1, i = n-k, \\ \displaystyle x_{k-n+i+1} \prod_{t=1}^{k-n+i} \frac{x_i - x_{i-t}}{x_{i+1} - x_{i+1-t}} & \text{if } i = j-1, i > n-k, \\ 0 & \text{otherwise.} \end{cases}
$$

THEOREM 2.4. *Let $n \geqslant 2$. Let $m \in \{1, \ldots, n-1\}$. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be distinct real numbers. Let*

$$
D_m = \begin{bmatrix} 1 & & & & & & \\ & \binom{m}{1} & & & & & \\ & & \binom{m}{2} & & & & \\ & & & \ddots & & & \\ & & & & \binom{m}{m-1} & & \\ & & & & & 1 & \\ & & & & & & \mathbf{0}_{n-m-1} \end{bmatrix},
$$

where $\binom{m}{i}$ denotes the binomial coefficient and $\mathbf{0}_{n-m-1}$ is the zero matrix of order $n-m-1$. Then

$$
S^{\circ m} = \left(L^{\mathbf{x}(1)} \cdots L^{\mathbf{x}(n-1)} U^{\mathbf{x}(n-1)} \cdots U^{\mathbf{x}(1)} \right) D_m \left(L^{\mathbf{y}(1)} \cdots L^{\mathbf{y}(n-1)} U^{\mathbf{y}(n-1)} \cdots U^{\mathbf{y}(1)} \right)^T.
$$

Proof. Let $V_{\mathbf{x}}$ be the Vandermonde matrix given by

$$
V_{\mathbf{x}} = \left[x_i^{j-1} \right] = \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}
$$

and let $V_{\mathbf{y}}$ be defined analogously. We note that

$$(V_{\mathbf{x}}D_mV_{\mathbf{y}}^T)_{ij} = \sum_{k=1}^{n}(V_{\mathbf{x}})_{ik}(D_mV_{\mathbf{y}}^T)_{kj}$$
$$= \sum_{k=1}^{n}x_i^{k-1}\binom{m}{k-1}y_j^{k-1}$$
$$= \sum_{k=0}^{m}\binom{m}{k}x_i^ky_j^k$$
$$= (1+x_iy_j)^m.$$

Thus

$$S^{\circ m} = V_{\mathbf{x}}D_mV_{\mathbf{y}}^T. \tag{6}$$

Further, by [25, Theorem 3.1], the Vandermonde matrices V_x and V_y can be factorized as follows:

$$V_{\mathbf{x}} = L^{\mathbf{x}(1)}L^{\mathbf{x}(2)}\cdots L^{\mathbf{x}(n-1)}U^{\mathbf{x}(n-1)}U^{\mathbf{x}(n-2)}\cdots U^{\mathbf{x}(1)} \tag{7}$$

and

$$V_{\mathbf{y}} = L^{\mathbf{y}(1)}L^{\mathbf{y}(2)}\cdots L^{\mathbf{y}(n-1)}U^{\mathbf{y}(n-1)}U^{\mathbf{y}(n-2)}\cdots U^{\mathbf{y}(1)}. \tag{8}$$

Substituting (7) and (8) in (6), we get the desired result.   □

3. Mean matrices

We first note the below easy proposition about the Cauchy matrix $C = \left[\dfrac{1}{\lambda_i + \lambda_j}\right]$, where $0 < \lambda_1 < \cdots < \lambda_n$ are positive real numbers. The matrix C is known to be infinitely divisible (see [2]).

PROPOSITION 3.1. *For $r > 0$, $C^{\circ r}$ is totally positive.*

Proof. Let $r > 0$. Every minor of $C^{\circ r}$ is of the form $\det\left(\left[\dfrac{1}{(p_i+q_j)^r}\right]\right)$, where $0 < p_1 < \cdots < p_n$ and $0 < q_1 < \cdots < q_n$. We have $\dfrac{1}{(p_i+q_j)^r} = \dfrac{1}{p_i^r}\dfrac{1}{(1+(q_j/p_i))^r}$. Since $\left[\dfrac{1}{(1+(q_j/p_i))^r}\right]$ is nonsingular (see [18, Corollary 5]), so is $\left[\dfrac{1}{(p_i+q_j)^r}\right]$. Therefore, $\det\left(\left[\dfrac{1}{(p_i+q_j)^r}\right]\right) \neq 0$ for every $r > 0$ and for every $0 < p_1 < \cdots < p_n, 0 < q_1 < \cdots < q_n$. The map $(p_1,\ldots,p_n,q_1,\ldots,q_n,r) \mapsto \det\left(\left[\dfrac{1}{(p_i+q_j)^r}\right]\right)$ is a continuous function of its variables. So by the intermediate value theorem, it retains its sign for all choices of $0 < p_1 < \cdots < p_n, 0 < q_1 < \cdots < q_n$ and $r > 0$. For $r = 1, p_i = i$

and $q_j = j$, we have, $\det\left(\left[\dfrac{1}{i+j}\right]\right) > 0$ (see [27, p. 92]). Thus $\det\left(\left[\dfrac{1}{(p_i+q_j)^r}\right]\right) > 0$. So $C^{\circ r}$ is totally positive. $\square$

We remark that the total positivity of Hadamard powers of Pascal matrices is shown in [13, Remark 2.2].

The main theorem of this section is as below. The proof is similar to [4], where their infinite divisibility is discussed.

THEOREM 3.2. *Let $r > 0$. The matrices* $\left[\dfrac{1}{\mathscr{A}(\lambda_i,\lambda_j)^r}\right]$, $[H(\lambda_i,\lambda_j)^r]$, $\left[\dfrac{1}{\mathscr{H}_\nu(\lambda_i,\lambda_j)^r}\right]$ *$(\nu \neq \frac{1}{2})$ and* $\left[\dfrac{1}{\mathscr{B}_\alpha(\lambda_i,\lambda_j)^r}\right]$ *$(0 < \alpha < \infty)$ are totally positive. The matrices* $\left[\dfrac{1}{\mathscr{H}_{\frac{1}{2}}(\lambda_i,\lambda_j)^r}\right]$ *$\left(= \left[\dfrac{1}{\mathscr{B}_0(\lambda_i,\lambda_j)^r}\right]\right)$ and* $\left[\dfrac{1}{\mathscr{B}_\infty(\lambda_i,\lambda_j)^r}\right]$ *are totally nonnegative.*

Proof. Since $\left[\dfrac{1}{\mathscr{A}(\lambda_i,\lambda_j)}\right]$ is a Cauchy matrix, the total positivity of its Hadamard powers follows from Proposition 3.1. The total positivity and total nonnegativity of a matrix are preserved under multiplication by a diagonal matrix with positive diagonal entries. Note that

$$[H(\lambda_i,\lambda_j)] = \operatorname{diag}\left[\sqrt{2}\lambda_i\right]\left[\dfrac{1}{\lambda_i+\lambda_j}\right]\operatorname{diag}\left[\sqrt{2}\lambda_i\right]$$

and

$$\left[\dfrac{1}{\mathscr{B}_\alpha(\lambda_i,\lambda_j)}\right] = \operatorname{diag}\left[2^{1/\alpha}\right]\left[\dfrac{1}{(\lambda_i^\alpha+\lambda_j^\alpha)^{1/\alpha}}\right].$$

Thus by Proposition 3.1, $[H(\lambda_i,\lambda_j)^r]$ is totally positive and so is $\left[\dfrac{1}{\mathscr{B}_\alpha(\lambda_i,\lambda_j)^r}\right]$ for $0 < \alpha < \infty$. Similarly, since we have

$$\left[\dfrac{1}{\mathscr{H}_\nu(\lambda_i,\lambda_j)}\right] = \operatorname{diag}\left[\dfrac{1}{\lambda_i^\nu}\right]\left[\dfrac{2}{\lambda_i^{1-2\nu}+\lambda_j^{1-2\nu}}\right]\operatorname{diag}\left[\dfrac{1}{\lambda_i^\nu}\right],$$

we get that $\left[\dfrac{1}{\mathscr{H}_\nu(\lambda_i,\lambda_j)^r}\right]$ is totally positive for $0 \leqslant \nu < \dfrac{1}{2}$. Since $\mathscr{H}_\nu$ is symmetric about $\nu = \dfrac{1}{2}$, the Hadamard powers of $\left[\dfrac{1}{\mathscr{H}_\nu(\lambda_i,\lambda_j)}\right]$ are also totally positive for $\dfrac{1}{2} < \nu \leqslant 1$. Also,

$$\left[\dfrac{1}{\mathscr{H}_{\frac{1}{2}}(\lambda_i,\lambda_j)^r}\right] = \left[\dfrac{1}{\mathscr{B}_0(\lambda_i,\lambda_j)^r}\right] = \operatorname{diag}\left[\dfrac{1}{(\sqrt{\lambda_i})^r}\right]F\operatorname{diag}\left[\dfrac{1}{(\sqrt{\lambda_i})^r}\right],$$

where F is the *flat matrix* with all its entries equal to 1. Thus it is totally nonnegative. The matrix

$$\left[\frac{1}{\mathscr{B}_\infty(\lambda_i,\lambda_j)^r}\right] = \left[\frac{1}{\max(\lambda_i^r,\lambda_j^r)}\right] = \left[\min\left(\frac{1}{\lambda_i^r},\frac{1}{\lambda_j^r}\right)\right].$$

So the (i,j)th entry of $\left[\frac{1}{\mathscr{B}_\infty(\lambda_i,\lambda_j)^r}\right]$ is the $(n+1-i,n+1-j)$th entry of $\left[\min\left(\frac{1}{\lambda_{n+1-i}^r},\frac{1}{\lambda_{n+1-j}^r}\right)\right]$. In view of Proposition 1.3 of [27], it is enough to show that if $0 < \mu_1 < \cdots < \mu_n$, then $[\min(\mu_i,\mu_j)]$ is totally nonnegative. To see this, let $L' = [L'_{ij}]$ and $U' = [U'_{ij}]$ be defined as

$$L'_{ij} = \begin{cases} \mu_1 & \text{if } i \geqslant j = 1, \\ (\mu_j - \mu_{j-1}) & \text{if } i \geqslant j \geqslant 2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$U'_{ij} = \begin{cases} 1 & \text{if } i \leqslant j, \\ 0 & \text{otherwise.} \end{cases}$$

Then L' and U' are totally nonnegative lower and upper triangular matrices, respectively, and

$$[\min(\mu_i,\mu_j)] = L'U'.$$

(For $\mu_i = i$, this decomposition is given in [2].) Thus, $[\min(\mu_i,\mu_j)]$ is totally nonnegative. $\square$

4. Remarks

1. Note that for $0 < \alpha < \infty$, $\mathscr{B}_{-\alpha}(a,b) = \dfrac{ab}{\mathscr{B}_\alpha(a,b)}$. Thus $[\mathscr{B}_{-\alpha}(\lambda_i,\lambda_j)^r]$ is totally positive for $0 < \alpha < \infty$. Also, $[\mathscr{B}_{-\infty}(\lambda_i,\lambda_j)^r] = [\min(\lambda_i^r,\lambda_j^r)]$ is totally nonnegative.

2. In Theorem 2.2, the decomposition holds for all real numbers $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ such that $1 + x_1 y_j, 1 + x_i y_1$ are nonzero for $1 \leqslant i,j \leqslant n$, and $x_i - x_{i-1}, y_i - y_{i-1}$ are nonzero for $2 \leqslant i \leqslant n$. In particular, it holds for all distinct positive real numbers.

REFERENCES

[1] A. BARRERAS, J. M. PEÑA, *Bidiagonal decompositions, minors and applications*, Electron. J. Linear Algebra. 2012, 25: 60–71.

[2] R. BHATIA, *Infinitely divisible matrices*, Am. Math. Mon. 2006, 113: 221–235.

[3] R. BHATIA, S. FRIEDLAND, T. JAIN, *Inertia of Loewner matrices*, Indiana Univ. Math. 2016, 65: 1251–1261.

[4] R. BHATIA, H. KOSAKI, *Mean matrices and infinite divisibility*, Linear Algebra Appl. 2007, 424: 36–54.

[5] C. W. CRYER, *Some properties of totally positive matrices*, Linear Algebra Appl. 1976, 15: 1–25.

[6] S. M. FALLAT, *Bidiagonal factorizations of totally nonnegative matrices*, Am. Math. Mon. 2001, 108: 697–712.

[7] S. M. FALLAT, C. R. JOHNSON, *Hadamard powers and totally positive matrices*, Linear Algebra Appl. 2007, 423: 420–427.

[8] S. M. FALLAT, C. R. JOHNSON, *Totally Nonnegative Matrices*, New Jersey (NJ): Princeton University Press, 2011.

[9] S. M. FALLAT, C. R. JOHNSON, *Total positivity of sums, Hadamard products and Hadamard powers: Results and counterexamples*, Linear Algebra Appl. 2017, 520: 242–259.

[10] M. FIEDLER, T. L. MARKHAM, *Consecutive-column and-row Properties of matrices and the Loewner-Neville factorization*, Linear Algebra Appl. 1997 Nov 15, 266: 243–59.

[11] C. H. FITZGERALD, R. A. HORN, *On fractional Hadamard powers of positive definite matrices*, J. Math. Anal. Appl. 1977, 61: 633–642.

[12] M. GASCA, J. M. PEÑA, *On factorizations of totally positive matrices*, In Total positivity and its applications (pp. 109–130), Springer, Dordrecht, 1996.

[13] P. GROVER, V. S. PANWAR, A. S. REDDY, *Positivity of some special matrices*, Linear Algebra Appl. 2020, 596: 203–215.

[14] R. A. HORN, *The theory of infinitely divisible matrices and kernels*, Trans. Am. Math. Soc. 1969, 136: 269–286.

[15] R. HUANG, J. LIU, *Bidiagonal factorizations with some parameters equal to zero*, Linear Algebra Appl. 2011, 434 (3): 730–40.

[16] R. HUANG, *A test and bidiagonal factorization for certain sign regular matrices*, Linear Algebra Appl. 2013, 438 (3): 1240–51.

[17] T. JAIN, *Hadamard powers of some positive matrices*, Linear Algebra Appl. 2017, 528: 147–158.

[18] T. JAIN, *Hadamard powers of rank two, doubly nonnegative matrices*, Adv. Oper. Theory. 2020, 5: 839–849.

[19] C. R. JOHNSON, D. D. OLESKY, P. VAN DEN DRIESSCHE, *Elementary bidiagonal factorizations*, Linear Algebra Appl. 1999, 292 (1–3): 233–244.

[20] C. R. JOHNSON, O. WALCH, *Critical exponents: old and new*, Electron. J. Linear Algebra. 2012, 25: 72–83.

[21] S. KARLIN, *Total positivity, absorption probabilities and applications*, Trans. Am. Math. Soc. 1964, 111: 33–107.

[22] S. KARLIN, *Total Positivity*, Vol. 1. Stanford: Stanford University Press, 1968.

[23] A. KHARE, *Multiply positive functions, critical exponent phenomena, and the Jain-Karlin-Schoenberg kernel*, arXiv preprint arXiv:2008.05121.2020.

[24] M. K. KWONG, *Some results on matrix monotone functions*, Linear Algebra Appl. 1989, 118: 129–153.

[25] H. ORUÇ, G. M. PHILLIPS, *Explicit factorization of the Vandermonde matrix*, Linear Algebra Appl. 2000, 315 (1–3): 113–123.

[26] V. S. PANWAR, A. S. REDDY, *Positivity of Hadamard powers of a few band matrices*, Electron. J. Linear Algebra 2022, 38: 85–90.

[27] A. PINKUS, *Totally Positive Matrices*, Cambridge (UK): Cambridge University Press, 2010.

[28] T. SANO, S. TACHIBANA, *On Loewner and Kwong matrices*, Sci. Math. Jpn. 2012, 75: 335–338.

[29] I. J. SCHOENBERG, *On Pólya frequency functions. I. The totally positive functions and their Laplace transforms*, J. d'Analyse Math. 1951, 1: 331–374.

[30] I. J. SCHOENBERG, *On the zeros of the generating functions of multiply positive sequences and functions*, Ann. Math. 1955, 62: 447–471.

[31] T. SIMON, *Total positivity of a Cauchy kernel*, J. Approx. Theory. 2014, 184: 238–58.

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