A ROTATION OF WIENER INTEGRAL ASSOCIATED WITH BOUNDED OPERATORS ON ABSTRACT WIENER SPACES

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Abstract. In this paper, we establish a rotation theorem for the Wiener integral associated with bounded operator on abstract Wiener spaces. To do this, we introduce a class of the angle preserving operators defined on the abstract Wiener space B.

1. Introduction

Let $C_0[0,T]$ denote the one-parameter Wiener space [15], that is, the space of all real-valued continuous functions x on the time interval $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m_{w} denote the Wiener measure. Then, as is well-known, $(C_0[0,T], \mathcal{M}, m_{\mathrm{w}})$ is a complete probability space, and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x) dm_{\mathrm{w}}(x).$$

The function space $C_0[0,T]$ can be considered as a space of all continuous sample paths of a Brownian motion $\{B_t : t \in [0,T]\}$ on a probability space Ω, see [7].

The motivation of this work will be introduced by two folds: In [1], Bearman provided a rotation theorem for the Wiener measure on the product Wiener space $C_0^2[0,T] \equiv C_0[0,T] \times C_0[0,T]$, as follows.

THEOREM 1.1. *Let* $\mathbf{F}$ *be a* $m_{\mathrm{w}} \times m_{\mathrm{w}}$ *-integrable functional on the product Wiener space* $C_0^2[0,T]$. *Given a function* θ *of bounded variation on* $[0,T]$, *let* $R_\theta : C_0^2[0,T] \to C_0^2[0,T]$ *be the transformation defined by* $R_\theta(w,z) = (x,y)$ *with*

$$\begin{cases} x(t) = \int_0^t \cos\theta(s) dw(s) - \int_0^t \sin\theta(s) dz(s), \\ y(t) = \int_0^t \sin\theta(s) dw(s) + \int_0^t \cos\theta(s) dz(s). \end{cases}$$

Then the transform R_θ *is measure preserving and*

$$\int_{C_0^2[0,T]} \mathbf{F}(R_\theta(w,z)) d(m_{\mathrm{w}} \times m_{\mathrm{w}})(w,z) = \int_{C_0^2[0,T]} \mathbf{F}(x,y) d(m_{\mathrm{w}} \times m_{\mathrm{w}})(x,y). \qquad (1.1)$$

As a special case of (1.1), one can see that for any Wiener integrable functional F on $C_0[0,T]$ and all $\theta \in \mathbb{R}$,

$$\int_{C_0^2[0,T]} F(w\cos\theta + z\sin\theta)d(m_{\mathrm{w}} \times m_{\mathrm{w}})(w,z) = \int_{C_0[0,T]} F(x)dm_{\mathrm{w}}(x). \qquad (1.2)$$

Using equation (1.2), one can also verify the following equation: for all real numbers α and β, it follows that

$$\int_{C_0^2[0,T]} F(\alpha w + \beta z)d(m_{\mathrm{w}} \times m_{\mathrm{w}})(w,z) = \int_{C_0[0,T]} F(\sqrt{\alpha^2 + \beta^2}x)dm_{\mathrm{w}}(x). \qquad (1.3)$$

In view of equation (1.1), we see that the Wiener measure $m_{\mathrm{w}} \times m_{\mathrm{w}}$ is rotation invariant. Equations (1.2) and (1.3) have been played an important role in various areas in mathematics and physics concerned with Wiener integration theory. Equation (1.2) was further developed by Cameron and Storvick [2] and by Johnson and Skoug [9] in their studies of Wiener integral equations. In [12], Lee extended equation (1.3) on the complexification of abstract Wiener space to study the solutions of the differential equation which is called a Cauchy problem. Furthermore, in [3], equation (1.3) was developed for the functionals in nonstationary Gaussian processes on the Wiener space $C_0[0,T]$.

The next illustration is the second motivation of the topic of this article. Equation (1.1) extended in [3] for cylinder functionals F on $C_0[0,T]$ defined by

$$F(Z_h(x,\cdot)) = f\left(\int_0^T \alpha_1(t)dZ_h(x,t),\ldots,\int_0^T \alpha_n(t)dZ_h(x,t)\right),$$

where $\{\alpha_1,\ldots,\alpha_n\}$ is an orthonormal set of functions from $L_2[0,T]$, $\int_0^T \alpha(t)dZ_h(x,t)$ and $Z_h(x,t) = \int_0^t h(s)dx(s)$ denote the stochastic integrals, and the paths x are in the Wiener space $C_0[0,T]$. In particular, the stochastic integral $Z_h(x,t) = \int_0^t h(s)dx(s)$ can be interpreted as the Itô integral of deterministic functions h in $L_2[0,T]$, see [13] and the references cited therein. Given a function h in $L_2[0,T]$, the integral process $Z_h(x,t) = \int_0^t h(s)dx(s)$ on $C_0[0,T] \times [0,T]$ is Gaussian with mean zero and covariance function

$$\mathrm{Cov}(Z_h(x,s),Z_h(x,t)) = \int_0^{\min\{s,t\}} h^2(u)du.$$

For more details, see [7, p. 157]. Also, if h is a function of bounded variation on $[0,T]$, Z_h is a continuous process on $C_0[0,T] \times [0,T]$.

Furthermore, for a function h in $L_\infty[0,T]$, the Itô integral $\int_0^t \alpha(s)dZ_h(x,s)$ of the deterministic function h has the kernel exchange properties as follows: given functions $\alpha \in L_2[0,T]$ and $h_1,h_2 \in L_\infty[0,T]$,

$$\int_0^t \alpha(s)dZ_h(x,s) = \int_0^t \alpha(s)h(s)dx(s) \qquad (1.4)$$

and

$$\int_0^t \alpha(s)dZ_{h_2}(Z_{h_1}(x,\cdot),s) = \int_0^t \alpha(s)h_1(s)h_2(s)dx(s) = \int_0^t \alpha(s)dZ_{h_1 h_2}(x,s). \qquad (1.5)$$

Let

$$C_0'[0,T] = \left\{ w \in C_0[0,T] : w(t) = \int_0^t v(s)dx(s) \text{ for some } v \in L_2[0,T] \right\}$$

and let

$$C_0^*[0,T] = \{ \beta \in C_0'[0,T] : D\beta \text{ is of bounded variation} \}$$

where $D\beta \equiv \frac{d}{dt}\beta$. Then it is well known in the abstract Wiener space theory that the space $C_0'[0,T]$ forms the Cameron–Martin space of the Wiener space $C_0[0,T]$. The inner product $\langle \cdot, \cdot \rangle$ is given by the formula $\langle w_1, w_2 \rangle = \int_0^T Dw_1(t)Dw_2(t)dt$. Given a function k in $C_0^*[0,T]$, we denote the stochastic integral $Z_{Dk}(x,T) = \int_0^T Dk(t)dx(t)$ by the symbol (k,x). Then the symbol $(\cdot,\cdot)$ on $C_0^*[0,T] \times C_0[0,T]$ is a bilinear form. In this case, equations (1.4) and (1.5) with t replaced with T, respectively, can be characterized by

$$(g, Z_{Dk}(x, \cdot)) = (k \odot g, x) \tag{1.6}$$

and

$$(g, Z_{Dk_2}(Z_{Dk_1}(x, \cdot), \cdot)) = (k_1 \odot k_2 \odot g, x) \tag{1.7}$$

for all g in $C_0'[0,T]$ and all $k, k_1, k_2 \in C_0^*[0,T]$, where $\odot$ is the operation between $C_0^*[0,T]$ and $C_0'[0,T]$ defined by $(k \odot g)(t) = \int_0^t Dg(s)Dk(s)ds$. In fact, these kernel exchange properties of the bilinear form associated with the processes Z_{Dk} are developed in [7] extensively. In [7, Chapter 5], these rotation transformations with the processes Z_{Dk} replaced with the operators on abstract Wiener spaces and rigorous structures of the rotation operators are introduced via white noise setting on nuclear spaces. For more information of those white noise analysis, we refer to the reference [8] and the references cited therein.

Based on those researches and the applications studied in [2, 3, 7, 8, 9, 12], it is worth-while to study another development of the rotation property of abstract Wiener measure. In this paper, we thus establish a rotation theorem for the Wiener integral associated with bounded operators on abstract Wiener spaces. The stochastic integrals in the product Wiener integral (1.1) will be replaced by bounded linear operators on abstract Wiener space B. To do this, we adopt a concept of the angle preserving operators on B (see Definition 2.5 below). The structures of the operators on abstract Wiener spaces and the Banach space adjoints involve the properties such as the kernel exchange properties appeared in (1.6) and (1.7).

2. Preliminaries

Let H be a real infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $|\cdot|$, and let B be a real separable Banach space with norm $\|\cdot\|$. It is assumed that H is continuously, linearly, and densely embedded in B. The natural injection (i.e., embedding) is denoted by $\iota : H \hookrightarrow B$. Let ν be a centered Gaussian probability measure on $(B, \mathscr{B}(B))$, where $\mathscr{B}(B)$ is the Borel σ-field of B. The triple (H, B, ν) is called an abstract Wiener space if

$$\int_B \exp\{i(\theta, x)\}d\nu(x) = \exp\left\{ -\frac{1}{2}|\iota^*(\theta)|^2 \right\} = \exp\left\{ -\frac{1}{2}|\theta|^2 \right\}$$

for any $\theta \in B^*$, where $(\cdot,\cdot)$ denotes the B^*-B paring, and $\iota^* : B^* \to H^*$ is the dual map to the natural injection $\iota : H \hookrightarrow B$, and where B^* and H^* are the topological duals of B and H, respectively. The space B^* is identified as a dense subspace of $H^* \approx H$ in the sense that, for all $\theta \in B^*$ and $x \in H$, $\langle \theta,x \rangle = (\theta,x)$. Thus we have the triple

$$B^* \subset H^* \approx H \subset B. \tag{2.1}$$

The Hilbert space H is called the Cameron–Martin space of the abstract Wiener space B. For more details, see [4, 6, 10, 11].

Given a Banach space X, let $\mathscr{L}(X) \equiv \mathscr{L}(X,X)$ denote the space of all bounded linear operators from X to X. Then $\mathscr{L}(B^*)$, $\mathscr{L}(H)$ and $\mathscr{L}(B)$ are Banach spaces. By the concept of the Banach space adjoint operator, given an operator $A \in \mathscr{L}(B)$, there exists a bounded linear operator $A^* : B^* \to B^*$ such that for all $\theta \in B^*$ and $x \in B$,

$$(A^*\theta)x = \theta(Ax). \tag{2.2}$$

By the structure of the B^*-B paring and the triple (2.1) (i.e., in the sense of Riesz representation theorem), equation (2.2) can be rewritten by

$$(A^*\theta,x) = (\theta,Ax).$$

For a finite subset $\mathscr{V} = \{\theta_1,\ldots,\theta_m\}$ of B^*, let $X_\mathscr{V} : B \to \mathbb{R}^m$ denote the random vector given by

$$X_\mathscr{V}(x) \equiv \big((\theta_1,x),\ldots,(\theta_m,x)\big). \tag{2.3}$$

A functional F is called a cylinder functional on B if there exists a linearly independent subset $\mathscr{V} = \{\theta_1,\ldots,\theta_m\}$ of B^* such that

$$F(x) = \psi\big(X_\mathscr{V}(x)\big) \equiv \psi\big((\theta_1,x),\ldots,(\theta_m,x)\big), \quad x \in B, \tag{2.4}$$

where ψ is a complex-valued Borel measurable function on $\mathbb{R}^m$. It is easy to show that for the cylinder functional F of the form (2.4), there exists an orthogonal subset $\mathscr{G} = \{g_1,\cdots,g_n\}$ of H whose elements are in B^* such that F is expressed as

$$F(x) = f\big(X_\mathscr{G}(x)\big) \equiv f\big((g_1,x),\cdots,(g_n,x)\big), \quad x \in B, \tag{2.5}$$

where f is a complex-valued Borel measurable function on $\mathbb{R}^n$. Thus, we loose no generality in assuming that every cylinder functional on B is of the form (2.5).

Proofs of the following lemma, corollaries and theorem are modifications of those in [5].

LEMMA 2.1. *Let (B,H,ν) be an abstract Wiener space. Let $\mathscr{V} = \{\theta_1,\ldots,\theta_m\}$ be a set of vectors in B^*, let $X_\mathscr{V}$ be given by (2.3), and let $\psi : \mathbb{R}^m \to \mathbb{C}$ be a Borel measurable function. Let V be the covariance matrix of the Gaussian random variables $\{(g_1,x),\ldots,(g_n,x)\}$ and suppose V is nonsingular (i.e., $\det V \neq 0$). Then it follows that*

$$\int_B \psi\big(X_\mathscr{G}(x)\big)d\nu(x) \overset{*}{=} \big((2\pi)^m \det V\big)^{-1/2} \int_{\mathbb{R}^m} \psi(\vec{u})\exp\left\{ -\frac{1}{2}(\vec{u}V^{-1})\cdot\vec{u} \right\}d\vec{u}, \tag{2.6}$$

in the sense that if either side exists, then both sides exist and equality holds, where $\vec{u}\cdot\vec{v}$ denotes the standard inner product of $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^m$.

The matrix V in the previous lemma is known to be positive definite in the case that the random variables (g_j, x), $j \in \{1, \ldots, n\}$, are non-degenerate and linearly independent.

COROLLARY 2.2. *Let (B, H, ν) be an abstract Wiener space. Let $\mathcal{V} = \{\theta_1, \ldots, \theta_m\}$ be a linearly independent set of vectors in B^*, and let $X_{\mathcal{V}}$, ψ, and V be as in Lemma 2.1. Then the covariance matrix V is nonsingular and equation (2.6) holds true.*

REMARK 2.3. Let $\mathcal{V} = \{\theta_1, \ldots, \theta_m\}$ be a linearly independent set of vectors in B^*. Then the covariance matrix V is given by

$$V = \begin{pmatrix} (\theta_1, \theta_1) & (\theta_1, \theta_2) & \cdots & (\theta_1, \theta_m) \\ (\theta_2, \theta_1) & (\theta_2, \theta_2) & \cdots & (\theta_2, \theta_m) \\ \vdots & \vdots & \ddots & \vdots \\ (\theta_m, \theta_1) & (\theta_m, \theta_2) & \cdots & (\theta_m, \theta_m) \end{pmatrix}.$$

COROLLARY 2.4. *Let (B, H, ν) be an abstract Wiener space. Let $\mathcal{G} = \{g_1, \ldots, g_n\}$ be an orthogonal subset of H whose elements are in B^*, let $X_{\mathcal{G}} : B \to \mathbb{R}^n$ denote the random vector given by $X_{\mathcal{G}}(x) = ((g_1, x), \ldots, (g_m, x))$, and let f be Borel measurable function on $\mathbb{R}^n$. Then it follows that*

$$\int_B f(X_{\mathcal{G}}(x)) d\nu(x) \overset{*}{=} \left((2\pi)^n \prod_{j=1}^n |g_j|^2 \right)^{-1/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{ -\frac{1}{2} \sum_{j=1}^n \frac{u_j^2}{|g_j|^2} \right\} d\vec{u}.$$

Given an orthogonal set $\mathcal{G} = \{g_1, \ldots, g_n\}$ of elements in B^*, let a functional F on B be given by (2.5) above. Then, for any operator A in $\mathcal{L}(B)$,

$$F(Ax) = f((g_1, Ax), \ldots, (g_n, Ax)) = f((A^* g_1, x), \ldots, (A^* g_n, x)).$$

Even though the set $\mathcal{G} = \{g_1, \ldots, g_n\}$ of vectors in B^* is orthogonal in H, the set $A^*(\mathcal{G}) \equiv \{A^* g : g \in \mathcal{G}\}$ of B^* might not be orthogonal in H.

DEFINITION 2.5. Given an orthogonal set $\mathcal{G} = \{g_1, \ldots, g_n\}$ of vectors in H such that each of whose elements is in B^*, let $\mathrm{AP}_B(\mathcal{G})$ be the class of all operators A in $\mathcal{L}(B)$ such that

$$(g_j, g_k) = \langle g_j, g_k \rangle = \langle A^* g_j, A^* g_k \rangle = (A^* g_j, A^* g_k)$$

for all $j, k \in \{1, \ldots, n\}$.

REMARK 2.6. Let A be an operator in $\mathcal{L}(B)$. If $A(H) \subseteq H$ and A^* is angle preserving as an operator in $\mathcal{L}(H)$, then $A \in \mathrm{AP}_B(\mathcal{G})$. In particular, if A is orthogonal (or unitary) on H, then A is in $\mathrm{AP}_B(\mathcal{G})$.

THEOREM 2.7. *Let $\mathscr{G} = \{g_1, \ldots, g_n\}$, $X_{\mathscr{G}}$ and f be as in Corollary 2.4. Then it follows that for any operator A in $\mathsf{AP}_B(\mathscr{G})$,*

$$\int_B f\big(X_{\mathscr{G}}(Ax)\big)d\nu(x) \equiv \int_B f\big((A^*g_1, x), \ldots, (A^*g_n, x)\big)d\nu(x)$$

$$\stackrel{*}{=} \left((2\pi)^n \prod_{j=1}^n (g_j, AA^*g_j)\right)^{-1/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{u_j^2}{(g_j, AA^*g_j)}\right\}d\vec{u}. \tag{2.7}$$

3. A rotation via the operators on abstract Wiener spaces

In order to obtain our rotation theorem for the abstract Wiener integral associated with bounded operators we provide the following structure for bounded operators on B.

(O1) Let A be an operator in $\mathscr{L}(B)$ such that $A(H) \subseteq H$. Then A is an element of $\mathscr{L}(H)$ (precisely speaking, the restriction, $A|_H$ is in $\mathscr{L}(H)$). We will denote the class of all operators satisfying the condition "$A \in \mathscr{L}(B)$ and $A(H) \subseteq H$" by $\mathscr{L}(B) \cap \mathscr{L}(H)$. Notice that $\mathscr{L}(B) \cap \mathscr{L}(H)$ is a linear space. For any operator A in $\mathscr{L}(B) \cap \mathscr{L}(H)$, AA^* is positive definite (as an operator on H) and so, by the square root lemma [14], there exists a positive operator $|A|$ such that $|A| = \sqrt{AA^*}$. We notice that every bounded positive definite operator on a Hilbert space is self-adjoint.

(O2) Given two bounded operators A_1 and A_2 in $\mathscr{L}(B) \cap \mathscr{L}(H)$, the operator $A_1A_1^* + A_2A_2^*$ is positive definite on H. Thus, by the square root lemma, there exists an operator $\sqrt{A_1A_1^* + A_2A_2^*}$, uniquely, in $\mathscr{L}(H)$. It is clear that the operator $\sqrt{A_1A_1^* + A_2A_2^*}$ is in $\mathscr{L}(B) \cap \mathscr{L}(H)$.

In order to identify these operators, we consider the relation $\sim$ on $\mathscr{L}(B) \cap \mathscr{L}(H)$ given by

$$A_1 \sim A_2 \iff A_1A_1^* = A_2A_2^*.$$

Then $\sim$ is an equivalence relation. For each A in $\mathscr{L}(B) \cap \mathscr{L}(H)$, let $[A]$ denote the equivalence class of A. In view of the observation (O1), it follows that there exists a positive definite operator $\mathfrak{S}(A)$ such that $A \sim \mathfrak{S}(A)$.

In this paper, given two bounded operators A_1 and A_2 in $\mathscr{L}(B) \cap \mathscr{L}(H)$, we will use the symbol '$\mathfrak{S}(A_1, A_2)$' to denote the representative of the equivalence class

$$\left[\mathfrak{S}(A_1, A_2)\right] = \left\{\mathfrak{S} \in \mathscr{L}(B) \cap \mathscr{L}(H) : \mathfrak{S} \sim \sqrt{A_1A_1^* + A_2A_2^*}\right\}.$$

Then, in view of the observation (O2), it follows that for any operators $\mathfrak{S}$ in $[\mathfrak{S}(A_1, A_2)]$ and all $g \in B^*$,

$$|\mathfrak{S}^*g|^2 = (\mathfrak{S}^*g, \mathfrak{S}^*g) = (g, \mathfrak{S}\mathfrak{S}^*g) = (g, (A_1A_1^* + A_2A_2^*)g).$$

Throughout the rest of this paper, for convenience, we will regard

$$[\mathfrak{S}(A_1, A_2)] \equiv \mathfrak{S}(A_1, A_2)$$

as an operator in $\mathscr{L}(B) \cap \mathscr{L}(H)$. Then $\mathfrak{S}(A_1, A_2)\mathfrak{S}(A_1, A_2)^* = A_1A_1^* + A_2A_2^*$.

Given an orthogonal set $\mathscr{G}$ in H whose elements are in B^*, let $\mathfrak{C}_{\mathscr{G}}(B^2)$ be the class of all $\nu \times \nu$-integrable functionals, $\mathbf{F} : B \times B \to \mathbb{C}$, given by

$$
\begin{aligned}
\mathbf{F}(x,y) &= f((g_1,x),\ldots,(g_n,x);(g_1,y),\ldots,(g_n,y)) \\
&\equiv f\big(X_{\mathscr{G}}(x);X_{\mathscr{G}}(y)\big)
\end{aligned}
\tag{3.1}
$$

for $\nu \times \nu$-a.e. $(x,y) \in B \times B$, where $f : \mathbb{R}^{2n} \to \mathbb{C}$ is a Borel measurable function. For notational convenience we define the class:

$$
\mathsf{AP}_{H,B}(\mathscr{G}) \equiv \mathsf{AP}_B(\mathscr{G}) \cap (\mathscr{L}(B) \cap \mathscr{L}(H)).
$$

We are now ready to provide our main assertion.

THEOREM 3.1. *Let (B,H,ν) be an abstract Wiener space. Let $\mathscr{G} = \{g_1,\ldots,g_n\}$ be an orthogonal subset of H whose elements are in B^*, let $\mathbf{F} \in \mathfrak{C}_{\mathscr{G}}(B^2)$ be given by* (3.1), *and let A_1 and A_2 be operators in $\mathsf{AP}_{H,B}(\mathscr{G})$. Suppose that* $\det \begin{pmatrix} A_1 & A_2 \\ A_1^* & A_2^* \end{pmatrix} = A_1 A_2^* - A_2 A_1^* = 0$ *on B^*. Then it follows that*

$$
\begin{aligned}
&\int_{B^2} \mathbf{F}\big(A_1 w - A_2 z, A_2 w + A_1 z\big) d(\nu \times \nu)(w,z) \\
&= \int_{B^2} \mathbf{F}\big(\mathfrak{S}(A_1,A_2)x, \mathfrak{S}(A_1,A_2)y\big) d(\nu \times \nu)(x,y).
\end{aligned}
\tag{3.2}
$$

It will be helpful to establish the following two lemmas before giving the proof of Theorem 3.1.

LEMMA 3.2. *Let $\mathscr{G} = \{g_1,\ldots,g_n\}$, A_1 and A_2 be as in Theorem* 3.1. *For each $j \in \{1,\ldots,n\}$, let $S_j, T_j : B \times B \to \mathbb{R}$ be given by*

$$
S_j(w,z) = (g_j,A_1 w) - (g_j,A_2 z) = (A_1^* g_j, w) - (A_2^* g_j, z)
\tag{3.3}
$$

and

$$
T_j(w,z) = (g_j,A_2 w) + (g_j,A_1 z) = (A_2^* g_j, w) + (A_1^* g_j, z),
\tag{3.4}
$$

respectively. Then

$$
\mathscr{R} = \{S_1,\ldots,S_n,T_1,\ldots,T_n\}
\tag{3.5}
$$

is a set of independent Gaussian random variables. For each $j \in \{1,\ldots,n\}$,

$$
S_j \sim N\big(0, (g_j,A_1 A_1^* g_j) + (g_j,A_2 A_2^* g_j)\big)
$$

and

$$
T_j \sim N\big(0, (g_j,A_1 A_1^* g_j) + (g_j,A_2 A_2^* g_j)\big).
$$

Proof. We note that for each $\theta \in B^*$, (θ, x), as a functional of x in B, has a Gaussian distribution with mean zero and variance $|\theta|^2 = \langle \theta, \theta \rangle = (\theta, \theta)$. Using this fact, we observe that for $\theta_1, \theta_2 \in B^*$,

$$\int_B (\theta_1, x)(\theta_2, x) d\nu(x) = (\theta_1, \theta_2). \tag{3.6}$$

However, using (3.6) and Fubini's theorem, we have that for all $j, l \in \{1, \ldots, n\}$ with $j \neq l$,

$$\int_{B^2} S_j(w, z) S_l(w, z) d(\nu \times \nu)(w, z) = (A_1^* g_j, A_1^* g_l) + (A_2^* g_j, A_2^* g_l) = 0$$

and

$$\int_{B^2} T_j(w, z) T_l(w, z) d(\nu \times \nu)(w, z) = (A_2^* g_j, A_2^* g_l) + (A_1^* g_j, A_1^* g_l) = 0.$$

Also, we have that for all $j, l \in \{1, \ldots, n\}$,

$$\int_{B^2} S_j(w, z) T_l(w, z) d(\nu \times \nu)(w, z) = (A_1^* g_j, A_2^* g_l) - (A_2^* g_j, A_1^* g_l) = 0,$$

because

$$(A_2^* g_j, A_1^* g_l) = (g_j, A_2 A_1^* g_l) = (g_j, A_1 A_2^* g_l) = (A_1^* g_j, A_2^* g_l).$$

From these facts above, we observe that for any $X, Y \in \mathscr{R}$ with $X \neq Y$, $\mathrm{Cov}(X, Y) = 0$. This completes the proof of this lemma. $\square$

The following lemma follows from Lemma 3.2.

LEMMA 3.3. *Let $\mathscr{G}$, A_1 and A_2 be as in Theorem 3.1. Then the Gaussian random vectors*

$$R^1_{A_1, A_2} : B \times B \to \mathbb{R}^n, \quad R^1_{A_1, A_2}(w, z) = X_{\mathscr{G}}(A_1 w) - X_{\mathscr{G}}(A_2 z)$$

and

$$R^2_{A_1, A_2} : B \times B \to \mathbb{R}^n, \quad R^2_{A_1, A_2}(w, z) = X_{\mathscr{G}}(A_2 w) + X_{\mathscr{G}}(A_1 z)$$

are independent. Furthermore, the covariance matrix of $R^1_{A_1, A_2}$ and $R^2_{A_1, A_2}$ is given by

$$(\mathrm{Cov}(X, Y))_{X, Y \in \mathscr{R}},$$

where $\mathscr{R}$ is given by (3.5) above.

We note that the determinant of the matrix $(\mathrm{Cov}(X, Y))_{X, Y \in \mathscr{R}}$ is given by

$$\det(\mathrm{Cov}(X, Y))_{X, Y \in \mathscr{R}} = \mathrm{trace}(\mathrm{Cov}(X, Y))_{X, Y \in \mathscr{R}}$$
$$= \sum_{j=1}^{n} \left((g_j, A_1 A_1^* g_j) + (g_j, A_2 A_2^* g_j) \right).$$

We are now finally ready to provide the proof of Theorem 3.1.

Proof. [Proof of Theorem 3.1] Using equations (3.3) and (3.4), we observe that for $(w,z) \in B \times B$,

$$X_{\mathscr{G}}(A_1 w) - X_{\mathscr{G}}(A_2 z) = (S_1(w,z), \ldots, S_n(w,z))$$

and

$$X_{\mathscr{G}}(A_2 w) + X_{\mathscr{G}}(A_1 z) = (T_1(w,z), \ldots, T_n(w,z)).$$

Thus, using these, (3.1), Fubini's theorem, and (2.7), and applying Lemmas 3.2 and 3.3, we obtain

$$
\begin{aligned}
&\int_{B \times B} \mathbf{F}(A_1 w - A_2 z, A_2 w + A_1 z) d(\nu \times \nu)(w,z) \\
&= \int_{B \times B} f(S_1(w,z), \ldots, S_n(w,z); T_1(w,z), \ldots, T_n(w,z)) d(\nu \times \nu)(w,z) \\
&= \left(\prod_{j=1}^{n} 2\pi \left((g_j, A_1 A_1^* g_j) + (g_j, A_2 A_2^* g_j) \right) \right)^{-1/2} \\
&\quad \times \int_{\mathbb{R}^n} \left[\int_B f(\vec{u}; A_2 w + A_1 z) d\nu(x) \right] \\
&\quad \times \exp \left\{ -\sum_{j=1}^{n} \frac{u_j^2}{2((g_j, A_1 A_1^* g_j) + (g_j, A_2 A_2^* g_j))} \right\} d\vec{u} \\
&= \left(\prod_{j=1}^{n} 2\pi \left((g_j, A_1 A_1^* g_j) + (g_j, A_2 A_2^* g_j) \right) \right)^{-1} \\
&\quad \times \int_{\mathbb{R}^{2n}} f(\vec{u}; \vec{v}) \exp \left\{ -\sum_{j=1}^{n} \frac{u_j^2 + v_j^2}{2((g_j, A_1 A_1^* g_j) + (g_j, A_2 A_2^* g_j))} \right\} d\vec{u} d\vec{v}.
\end{aligned}
\tag{3.7}
$$

Now, let $\beta_j = \mathfrak{S}(A_1, A_2)^* g_j$ for each $j \in \{1, \ldots, n\}$. Then we have that for all $j, l \in \{1, \ldots, n\}$ with $j \neq l$,

$$
\begin{aligned}
(\beta_j, \beta_l) &= (g_j, \mathfrak{S}(A_1, A_2) \mathfrak{S}(A_1, A_2)^* g_l) \\
&= (g_j, (A_1 A_1^* + A_2 A_2^*) g_l) \\
&= (g_j, A_1 A_1^* g_l) + (g_j, A_2 A_2^* g_l) \\
&= (A_1^* g_j, A_1^* g_l) + (A_2^* g_j, A_2^* g_l) = 0
\end{aligned}
\tag{3.8}
$$

and that for each $j \in \{1, \ldots, n\}$,

$$
\begin{aligned}
(\beta_j, \beta_j) &= (g_j, \mathfrak{S}(A_1, A_2) \mathfrak{S}(A_1, A_2)^* g_j) \\
&= (g_j, (A_1 A_1^* + A_1 A_2^*) g_j) \\
&= (A_1^* g_j, A_1^* g_j) + (A_2^* g_2, A_2^* g_j).
\end{aligned}
\tag{3.9}
$$

Hence, from (3.8) and (3.9), we see that $\mathscr{B} = \{\beta_1, \ldots, \beta_n\}$ is an orthogonal set of elements in B^* and that the $B^* - B$ parings

$$(\beta_j, x) = (\mathfrak{S}(A_1, A_2)^* g_j, x) = (g_j, \mathfrak{S}(A_1, A_2) x), \ j \in \{1, \ldots, n\}$$

form a set of independent Gaussian random variables on B with mean zero and variance $(g_j, A_1 A_1^* g_j) + (g_j, A_2 A_2^* g_j)$, respectively, by checking their covariances. Also, using Fubini's theorem and (2.7), we obtain that

$$
\begin{aligned}
&\int_{B^2} \mathbf{F}\big(\mathfrak{S}(A_1, A_2)x, \mathfrak{S}(A_1, A_2)y\big) d(v \times v)(x, y) \\
&= \int_{B^2} f\big((\beta_1, x), \ldots, (\beta_n, x); (\beta_1, y), \ldots, (\beta_n, y)\big) d(v \times v)(x, y) \\
&= \int_B \int_B f\big(X_{\mathscr{B}}(x); X_{\mathscr{B}}(y)\big) dv(x) dv(y) \\
&= \left(\prod_{j=1}^n 2\pi\big((g_j, A_1 A_1^* g_j) + (g_j, A_2 A_2^* g_j)\big) \right)^{-1/2} \\
&\quad \times \int_{\mathbb{R}^n} \int_B f\big(\vec{u}; X_{\mathscr{B}}(y)\big) dv(y) \exp\left\{ -\sum_{j=1}^n \frac{u_j^2}{2((g_j, A_1 A_1^* g_j) + (g_j, A_2 A_2^* g_j))} \right\} d\vec{u} \\
&= \left(\prod_{j=1}^n 2\pi\big((g_j, A_1 A_1^* g_j) + (g_j, A_2 A_2^* g_j)\big) \right)^{-1} \\
&\quad \times \int_{\mathbb{R}^{2n}} f\big(\vec{u}; \vec{v}\big) \exp\left\{ -\sum_{j=1}^n \frac{u_j^2 + v_j^2}{2((g_j, A_1 A_1^* g_j) + (g_j, A_2 A_2^* g_j))} \right\} d\vec{u} d\vec{v}.
\end{aligned}
\tag{3.10}
$$

Equation (3.2) now follows from equations (3.7) and (3.10). $\quad\square$

The following corollaries are very simple consequences of Theorem 3.1.

COROLLARY 3.4. *Let $\mathscr{G}$, A_1 and A_2 be as in Theorem 3.1. Let F be given by (2.5) and let a functional $K : B \to \mathbb{C}$ be given by $K(x) = k(X_{\mathscr{G}}(x))$, where k is a complex-valued Borel measurable function on $\mathbb{R}^n$. Then*

$$
\begin{aligned}
&\int_{B^2} F\big(A_1 w - A_2 z\big) K\big(A_2 w + A_1 z\big) d(v \times v)(w, z) \\
&= \int_B F\big(\mathfrak{S}(A_1, A_2)x\big) dv(x) \int_B K\big(\mathfrak{S}(A_1, A_2)y\big) dv(y).
\end{aligned}
\tag{3.11}
$$

COROLLARY 3.5. *Let $\mathscr{G}$, A_1, A_2, and K be as in Corollary 3.4. Then it follows that*

$$
\int_{B^2} K\big(A_1 w + A_2 z\big) d(v \times v)(w, z) = \int_B K\big(\mathfrak{S}(A_1, A_2)x\big) dv(x).
$$

Proof. Simply choose $F \equiv 1$ in equation (3.11). $\quad\square$

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