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ON MULTIPLICITIES OF EIGENVALUES OF A SPECTRAL PROBLEM ON A PROLATE TREE

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Abstract. It is known that there exists an ordered vector majorizing any ordered vector of possible multiplicities of eigenvalues of the star-patterned matrix. The same situation occurs in the problem of small transverse vibrations of a star graph of Stieltjes strings. We show that for a certain class of prolate trees of Stieltjes strings there exists an ordered vector majorizing any ordered vectors of possible eigenvalue multiplicities of spectral problems on such trees.

1. Introduction

This paper is devoted to spectral problems associated with small transverse vibrations of tree graphs, edges of which are Stieltjes strings [4, 9]. Such vibrations are described by second-order difference equations. The same equations appear in various fields of physics such as synthesis of electrical circuits [6, p. 129], and longitudinal vibrations of point masses connected by springs [11]. Eliminating time dependence in these equations, we arrive at a spectral problem for a tree patterned matrix (see [8]). Such spectral problems can have multiple eigenvalues. The paper deals with estimating multiplicities of these eigenvalues.

Massless elastic thread-bearing beads (point masses) are called *Stieltjes string* [4]. Small transverse vibrations of such a string are described by the equations

$$\frac{v_k(t)-v_{k+1}(t)}{l_k}+\frac{v_k(t)-v_{k-1}(t)}{l_{k-1}}+m_k v_k''(t)=0 \quad (k=1,2,\cdots,n),$$

where $v_k(t)$ is the transverse displacement of the kth bead of mass m_k and l_k is the distance between the beads of masses m_{k-1} and m_k. We assume the number n of beads to be finite.

Separating of variables $v_k(t)=u_k e^{i\lambda t}$ and $z=\lambda^2$ lead to

$$\frac{u_k-u_{k+1}}{l_k}+\frac{u_k-u_{k-1}}{l_{k-1}}-zm_k u_k=0 \quad (k=1,2,\cdots,n), \tag{1}$$

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where z is the spectral parameter. It is known (see e.g. [4, Supplement II] or [18, p. 55]) that the eigenvalues of the spectral problem which consists of (1) and the Dirichlet boundary conditions

$$u_0 = u_{n+1} = 0 \tag{2}$$

are simple, and for any sequence of distinct positive numbers $\{z_k\}_{k=1}^n$ and any positive number l there exist sequences $\{m_k\}_{k=1}^n$, $\{l_k\}_{k=0}^n$ of positive numbers such that $\sum_{k=0}^n l_k = l$ and $\{z_k\}_{k=1}^n$ is the spectrum of problem (1)–(2). A problem (1), (2) on an interval we call Dirichlet problem.

Equations (1) on the edges of metric trees were considered in [5, 8, 10, 12]. The corresponding spectral problems may have multiple eigenvalues. The number of multiple eigenvalues and their possible multiplicities depend on the form of the tree and the number of beads on the edges.

If the form of a tree is given together with the numbers of the beads on the edges, then it implies restrictions on possible multiplicities of the eigenvalues. For a star graph, such restrictions are known [24]: if $\{p_i^{\downarrow}\}_{i=1}^r$ is the vector of multiplicities in decreasing order, and N_j is the number of edges for which the number of masses is $\geqslant j$ $(j = 1, 2, \cdots, n_1)$. Then the vector $\{N_1, N_2, \cdots, N_{n_1}\}$ Hardy-Littlewood majorizes $\{p_1^{\downarrow}, p_2^{\downarrow}, \cdots, p_r^{\downarrow}\}$. It should be mentioned that those restrictions on the multiplicities of eigenvalues are similar to the ones obtained in [3, 16] for the so-called tree-patterned matrices if the corresponding tree is a generalized star graph. The difference is that if the beads in [24] are considered as the vertices of the corresponding generalized star graph then we have to require a bead to be placed at the central vertex to have the same situation as in [3, 16].

Unfortunately, there is no general answer about restrictions on eigenvalue multiplicities for a spectral problem on an arbitrary tree as well as for a spectral problem on an arbitrary tree-patterned matrix despite many particular results (for generalized star graphs, for double generalized stars) which have been established for tree-patterned matrices (see [15, 16, 17].)

The objects of investigation in this paper are spectral problems on a particular case of trees of Stieltjes strings (so-called prolate weighted trees). Since our tree is a metric we measure lengths in meters while *combinatorial length* as usually means the number of edges in a path. However, it is important to characterize a path by the number of beads in it. So we consider trees of Stieltjes strings as weighted graphs meaning the number of the beads on an edge to be the weight of the edge. We call this number of beads the *weight of the path*. A *prolate tree of Stieltjes strings* T is rooted at the beginning of the path $P := v_0 \to v_1 \to \cdots \to v_\tau$ of the maximal weight. If a subtree T_j is joined to P at a vertex v_i, then the maximal weight of the paths in the subtree T_j does not exceed the weight of any of the edges $v_{i-1}v_i$ and v_iv_{i+1} (see Definition 3 below).

The organization of this paper is as follows. In Section 2 we start with auxiliary results on spectral problems on arbitrary trees. In Section 3, we consider prolate weighted trees and we show that there exists an ordered vector majorizing the ordered vector of multiplicities of eigenvalues of the spectral problem on such a tree and propose how to find this majorizing vector.

2. Auxiliary results

In this section, we state some auxiliary results which will be used subsequently. Let T be a plane metric tree with q edges. We denote by v_i the vertices, by $d(v_i)$ their degrees, by e_j the edges, and by l_j their lengths. In this section, an arbitrary pendant vertex v_0 is chosen to be the root. Local coordinates for edges identify the edge e_j with the interval $[0,l_j]$ so that the local coordinate increases as the distance to the root increases. Each edge e_j is divided into n_j+1 subintervals of the lengths $l_0^j, l_1^j, \cdots, l_{n_j}^j$ by beads of masses $m_1^j, m_2^j, \cdots, m_{n_j}^j$ with $l_k^j>0$, $m_k^j>0$ and $l_j=\sum_{k=0}^{n_j} l_k^j$. The beads and the subintervals are enumerated such that the upper index corresponds to the index of the edge. Each pendant vertex, if it is not the root, is located at the end of a subinterval of the length $l_{n_j}^j$ where j is the index of the edge. The root is at the beginning of a subinterval of the lengths l_0^r on the edges e_r incident with v_0. Each interior vertex v_i has one incoming edge e_j ending with a subinterval of the length $l_{n_j}^j$, while each outgoing edge e_r starts with a subinterval of length l_0^r. It is assumed that the tree is stretched and the pendant vertices are fixed. The tree can vibrate in the direction orthogonal to the equilibrium position of the strings. We denote by $v_k^j(t)$ the transverse displacement of the bead of mass m_k^j at time t. If an edge e_j is incoming into an interior vertex v_i, then the displacement of the incoming end of the edge is denoted by $v_{n_j+1}^j(t)$, while if an edge e_r is outgoing from the vertex v_i then the displacement of the outgoing end of the edge is denoted by $v_0^r(t)$. In virtue of the above notations, vibrations of such a graph can be described by the system of equations

$$\frac{v_k^j(t)-v_{k+1}^j(t)}{l_k^j}+\frac{v_k^j(t)-v_{k-1}^j(t)}{l_{k-1}^j}+m_k^j\frac{\partial^2 v_k^j}{\partial t^2}(t)=0, \tag{3}$$

where $k=1,2,\cdots,n_j$; $n_j \geqslant 1$, $j=1,2,\cdots,q$. For each interior vertex with an incoming edge e_j and outgoing edges e_r, we impose the continuity conditions

$$v_0^r(t)=v_{n_j+1}^j(t) \tag{4}$$

for all r corresponding to outgoing edges. Balance of forces at such a vertex implies

$$\sum_r \frac{v_1^r(t)-v_0^r(t)}{l_0^r}=\frac{v_{n_j+1}^j(t)-v_{n_j}^j(t)}{l_{n_j}^j}, \tag{5}$$

where the sum in the left-hand side is taken over all the outgoing edges. We impose Dirichlet boundary condition

$$v_{n_j+1}^j(t)=0 \tag{6}$$

at each edge e_j incident with a pendant vertex (except for the root).

At the root we impose the Dirichlet condition

$$v_0^r(t)=0. \tag{7}$$

Substituting $v_k^j(t) = u_k^j e^{i\lambda t}$ into (3)–(7) and denoting $z = \lambda^2$, we obtain

$$\frac{u_k^j - u_{k+1}^j}{l_k^j} + \frac{u_k^j - u_{k-1}^j}{l_{k-1}^j} + m_k^j z u_k^j = 0, \tag{8}$$

where $k = 1,2,\cdots,n_j$ and $j = 1,2,\cdots,q$, for each interior vertex with an incoming edge e_j and outgoing edges e_r, we infer

$$u_0^r = u_{n_j+1}^j, \tag{9}$$

$$\sum_r \frac{u_1^r - u_0^r}{l_0^r} = \frac{u_{n_j+1}^j - u_{n_j}^j}{l_{n_j}^j}, \tag{10}$$

for each edge e_j incident with a pendant vertex (except for the root):

$$u_{n_j+1}^j = 0, \tag{11}$$

and at the root:

$$u_0^r = 0. \tag{12}$$

The problem (8)–(12) possesses a non-trivial solution for a discrete set of values of z (see, e.g. [21], or Sec. 4.1. in [19]). These values ν_i are called eigenvalues. The corresponding solutions are called eigenvectors.

Now we consider attaching edges to a tree. We attach an edge which we denote e_0 at a pendant vertex v_1 of the initial tree T and obtain a new tree T' (see Fig. 1).

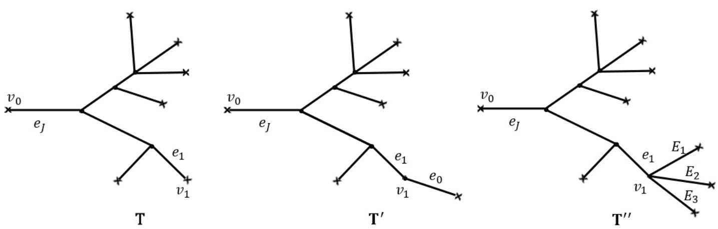

Figure 1: *Trees T, T' and T''.*

LEMMA 1. *Let ν be an eigenvalue of multiplicity p of problem* (8)–(12) *on a tree T. Among the eigenvectors corresponding to ν let one of them be such that its projection $\{u_1^1, u_2^1, \cdots, u_{n_1}^1\}$ on the edge e_1 incident with v_1 is not identically zero.*

1. If ν is also an eigenvalue of the Dirichlet problem

$$\frac{u_k^0 - u_{k+1}^0}{l_k^0} + \frac{u_k^0 - u_{k-1}^0}{l_{k-1}^0} - m_k^0 z u_k^0, \quad k = 1,2,\cdots,n_0, \tag{13}$$

$$u_0^0 = u_{n_0+1}^0 = 0, \tag{14}$$

then ν is an eigenvalue of problem (8)–(12) on T' of multiplicity p and the corresponding eigenspace can be chosen such that all the linearly independent eigenvectors attain zero value at the vertex v_1 and all the eigenvectors have nonzero components on the edge e_0.

2. If ν is not an eigenvalue of problem (13)–(14), then ν is an eigenvalue of problem (8)–(12) on T' of multiplicity of $p-1$.

Proof. Statement 1 was proved in [23, Lemma 2.1].

If ν is not an eigenvalue of problem (13)–(14), then all the eigenvectors of problem (8)–(12) on T' corresponding to ν have identically zero component on e_1 and on e_0. The number of linearly independent eigenvectors of problem (8)–(12) on T' with zero component on e_1 and on e_0 equals the number of linearly independent eigenvectors of (8)–(12) on T with zero component on e_1. Since all the eigenvectors except for this one can be continued by 0 onto e_0, the number of eigenvectors of problem (8)–(12) on T' is $p-1$. $\square$

REMARK 1. It follows from [4] that for any sequence of positive numbers $0 < \nu_1^0 < \nu_2^0 < \cdots < \nu_{n_0}^0$ and any $l > 0$, there exist sequences $\{m_k^0\}_{k=1}^{n_0}$ and $\{l_k^0\}_{k=0}^{n_0}$ such that $\sum_{k=0}^{n_0} l_k^0 = l$ and $\{\nu_s^0\}_{s=1}^{n_0}$ coincides with the spectrum of problem (13)–(14).

In what follows, we need the notion of vector majorization which goes back to Muirhead [7] for the case of vectors of integers and was generalized to vectors of nonnegative numbers by Hardy, Littlewood, and Pólya [13] (also see [20]).

DEFINITION 1. Let $x = \{x_j\}_{j=1}^s$ and $y = \{y_j\}_{j=1}^t$ be two vectors with nonnegative entries ordered nonincreasingly, i.e., $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_s \geqslant 0$, $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_t \geqslant 0$. If $s = t$, then x is said to majorize y, written $x \succ y$, if

$$\sum_{j=1}^{t} x_j = \sum_{j=1}^{t} y_j, \quad \sum_{j=1}^{\tau} x_j \geqslant \sum_{j=1}^{\tau} y_j \quad (\tau = 1, 2, \cdots, t-1).$$

If $s \neq t$, we fill up the shorter vector with zeros, i.e., we set $\tilde{x} := \{x_j\}_{j=1}^{\max\{s,t\}}$ and $\tilde{y} := \{y_j\}_{j=1}^{\max\{s,t\}}$ with $x_j = 0$ for $j = s+1, \cdots, \max\{s,t\}$ and $y_j = 0$ for $j = t+1, \cdots, \max\{s,t\}$. Then x is said to majorize y, $x \succ y$, if $\tilde{x}$ majorizes $\tilde{y}$, $\tilde{x} \succ \tilde{y}$.

NOTATION 1. 1. For a vector $x = \{x_j\}_{j=1}^s \in \mathbb{R}^s$ we denote by $x^{\downarrow} = \{x_j^{\downarrow}\}_{j=1}^s \in \mathbb{R}^s$ the vector with the same entries but ordered nonincreasingly, i.e.,

$$x_1^{\downarrow} \geqslant x_2^{\downarrow} \geqslant \cdots \geqslant x_s^{\downarrow}, \quad x_j^{\downarrow} = x_{\pi(j)}, \quad j = 1, 2, \cdots, s,$$

for some permutation π of $\{1, 2, \cdots, s\}$.

For example, by Definition 1 and Notation 1 we infer that if $x = \{3, 1, 2, 5, 2, 4\}$ then $x^{\downarrow} = \{5, 4, 3, 2, 2, 1\}$.

Now we consider the tree T'' consisting of the tree T with the edges $E_1, E_2, \cdots, E_\tau$ attached at the vertex v_1 (see Fig. 1). The masses of the beads on the edge E_j with $j = 1,2,\cdots,\tau$ we denote by $\tilde{m}_k^j$ $(k = 1,2,\cdots,\tilde{n}_j)$ and the lengths of the subintervals by $\tilde{l}_k^j$ $(k = 0,1,\cdots,\tilde{n}_j)$. We assume that $\tilde{n}_j \leqslant \tilde{n}_0$ for all j where $\tilde{n}_0$ is the number of beads on the edge e_0 of T'.

LEMMA 2. [23, Lemma 2.4] *Let $\{p_1^{\downarrow}, p_2^{\downarrow}, \cdots, p_{r'}^{\downarrow}\}$ be the ordered vector of multiplicities of eigenvalues of (8)–(12) on the tree $T' = T \cup e_0$ and let $\nu_k^{\downarrow}$ be an eigenvalue of multiplicity $p_k^{\downarrow}$ for all $k \in \{1,2,\cdots,r'\}$. Let $\nu_k^0 = \nu_k^{\downarrow}$ for $k = 1,2,\cdots,n_0$, $n_0 \leqslant r'$, where $\{\nu_k^0\}_{k=1}^{n_0}$ is the spectrum of Dirichlet problem (13)–(14) on the edge e_0.*

Then there exists a tree $T'' = T \cup_{j=1}^{\tau} E_j$ such that the spectrum of problem (8)–(12) on the tree T'' consists of the eigenvalues $\{\nu_1^{\downarrow}, \nu_2^{\downarrow}, \cdots, \nu_{r'}^{\downarrow}\}$ counted with multiplicities $\{\tilde{p}_1^{\downarrow}, \tilde{p}_2^{\downarrow}, \cdots, \tilde{p}_{r'}^{\downarrow}\}$ where $\tilde{p}_k^{\downarrow} = p_k^{\downarrow} + \max\{\tilde{N}_k - 1, 0\}$, and $\tilde{N}_k := \#\{j \in \{1,2,\cdots,\tau\} : \tilde{n}_j \geqslant k\}$. Here $\#\{\}$ means the number of elements in the set $\{\}$.

REMARK 2. From the proof of Lemma 2.5 and its proof in [23] it is clear that for any of E_j and any eigenvalue ν_k^0 of problem (8)–(12) on the tree T'', there is such an eigenvector that its projection on E_j is not identically zero. This eigenvector is among the linearly independent eigenvectors corresponding to the eigenvalue ν_k^0.

REMARK 3. The condition 'the spectrum of problem (8)–(12) on the tree T'' consists of the eigenvalues $\{\nu_1^{\downarrow}, \nu_2^{\downarrow}, \cdots, \nu_{r'}^{\downarrow}\}$' in Lemma 2 implies that any of the numbers of the beads on the edges E_j do not exceed the number of beads on e_0. This follows from the proof of Lemma 2.4 in [23] and from Theorem in [22].

3. Multiplicities of eigenvalues for a prolate weighted tree

In this section, we prove that there exists an ordered vector majorizing any ordered vectors of possible eigenvalue multiplicities of the spectral problem on a so-called prolate weighted tree and we show how to find this majorizing vector. For convenience, we slightly change the enumeration of edges and introduce the notion of the height of a tree in the following:

DEFINITION 2. Let T be a tree of Stieltjes strings. We call the *height* of the tree the maximal weight of the paths in this tree. We choose one of the two pendant vertices of the path of maximal weight (or one of them if it is not unique) as the root and denote it by v_0. We direct the edges away from the root.

DEFINITION 3. The simplest directed weighted tree is a path and we attribute this tree to the class of *prolate weighted* trees. Let $P = v_0 \to v_1 \to \cdots \to v_m$ be the maximal weight path of the tree T rooted at v_0. Let $T_{i,j}$ $(j = 3,4,\cdots,d(v_i))$ be the subtrees rooted at $v_i \in P$ $(i = 1,2,\cdots,v_{m-1})$ such that $P \cup \cup_{i=1,2,\cdots,m-1}(\cup_{j=3,4,\cdots,d(v_j)} T_{i,j}) = T$ and $T_{i,j} \cap T_{i,j'} = \{v_i\}$ for $j \neq j'$, $T_{i,j} \cap T_{i',j'} = 0$ for $i \neq i'$.

Let for each i the root v_i of the subtrees $T_{i,j}$ be the starting vertex of the path of the maximal weight in $T_{i,j}$. Denote by $L_{i,j}$ the height of $T_{i,j}$. If all $T_{i,j}$ are prolate and the numbers of beads on each of the edges $v_{i-1}v_i$ and v_iv_{i+1} are not less than any of $L_{i,j}$ $(j = 3,4,\cdots,d(v_i))$ then T is said to be prolate (an example of a prolate tree T and its subtrees $T_{i,j}$ is given at Fig. 2 where the beads are denoted by bullets).

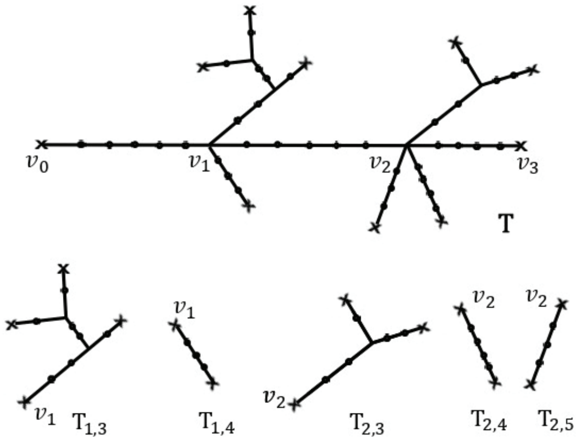

Figure 2: *Prolate tree T and its subtrees $T_{i,j}$.*

REMARK 4. In the case of a snowflake graph considered in [23], the condition of 'massive core' ($n_1 < n_0$) guarantee that the snowflake graph is a prolate tree.

Using Definitions 2 and 3, we give the main result of this paper, we show that there exists an ordered vector majorizing the ordered vectors of possible multiplicities of eigenvalues of the spectral problem for an arbitrary prolate weighted tree.

THEOREM 1. *Let T be a prolate weighted tree rooted at v_0 the beginning of the maximal weight path P with the side trees $T_{i,j}$ $(j = 3,4,\cdots,d(v_i);\ i = 1,2,\cdots,m-1)$. Let $\{M_1^{i,j}, M_2^{i,j}, \cdots, M_{L_{i,j}}^{i,j}\}$ be the majorizing vector for multiplicities of eigenvalues of problem (8)–(12) on $T_{i,j}$ and $L_{i,j}$ be the height of $T_{i,j}$. Then*

1. The ordered vector $\{\tilde{M}_1, \tilde{M}_2, \cdots, \tilde{M}_L\}$ majorizes $\{p_1^{\downarrow}, p_2^{\downarrow}, \cdots, p_r^{\downarrow}\}$ which is any possible ordered vector for multiplicities of eigenvalues of problem (8)–(12) on T. Here

$$\tilde{M}_k := \begin{cases} 1 + \sum\limits_{i=1}^{m-1} \sum\limits_{j=3}^{d(v_i)} M_k^{i,j}, & \text{if } k \leqslant \max\limits_{j=3,4,\cdots,d(v_i),i=1,2,\cdots,m-1} L_{i,j}, \\ 1, & \text{if } \max\limits_{j=3,4,\cdots,d(v_i),i=1,2,\cdots,m-1} L_{i,j} < k \leqslant L, \\ 0, & \text{if } k > L, \end{cases}$$

and L is the weight of P.

2. There exists a distribution of masses such that the number r of distinct eigenvalues equals the height of the tree T and the corresponding ordered vector of multiplicities

$$\{\check{p}_1^{\downarrow}, \check{p}_2^{\downarrow}, \cdots, \check{p}_r^{\downarrow}\} = \{\tilde{M}_1, \tilde{M}_2, \cdots, \tilde{M}_L\}.$$

3. In case of $\{\check{p}_1^{\downarrow}, \check{p}_2^{\downarrow}, \cdots, \check{p}_r^{\downarrow}\} = \{\tilde{M}_1, \tilde{M}_2, \cdots, \tilde{M}_L\}$, for each eigenvalue there is an eigenvector with the component on the edge incident with the root of T not zero identically.

Proof. Let us start with the maximal weight path $P = v_0v_1, v_1v_2, \cdots, v_{m-1}v_m$. Denote by n_i the number of beads on the edge $v_{i-1}v_i$ and define the number J by the equation $n_J = \max_{i\in\{1,2,\cdots,m\}} n_i$ and choose any positive numbers $\nu_1^{(J)}, \nu_2^{(J)}, \cdots, \nu_{n_J}^{(J)}$. According to Remark 1, we may choose masses of the beads on the edge $v_{J-1}v_J$ such that the numbers $\nu_1^{(J)}, \nu_2^{(J)}, \cdots, \nu_{n_J}^{(J)}$ are eigenvalues of the Dirichlet problem on the edge $v_{J-1}v_J$. Also we may choose the masses of the beads and the lengths of the subintervals on the edges $v_{i-1}v_i$ with $i \in \{1,2,\cdots,m\}\backslash\{J\}$ such that $\nu_1^{(i)} = \nu_1^{(J)}, \nu_2^{(i)} = \nu_2^{(J)}, \cdots, \nu_{n_i}^{(i)} = \nu_{n_i}^{(J)}$ for all $i = 1,2,\cdots,m$, where $\{\nu_k^{(i)}\}_{k=1}^{n_i}$ are eigenvalues of the Dirichlet problem on the edge $v_{i-1}v_i$.

Now let us fix positive numbers j and i and attach the maximal weight path $P_{i,j}$: $v_i =: v_0^{(i,j)} \to v_1^{(i,j)} \to \cdots \to v_{m_{i,j}}^{(i,j)}$ of the subtree $T_{i,j}$ at the vertex v_i. Denote by $n_s^{(i,j)}$ the number of beads on the edge $v_{s-1}^{(i,j)}v_s^{(i,j)}$. Define the number $J_{i,j}$ by the equation $n_{J_{i,j}} = \max_{s\in\{1,2,\cdots,m_{i,j}\}} n_s^{(i,j)}$, where $m_{i,j}$ is the number of edges in $P_{i,j}$. Denote by $\{\nu_k^{(i,j,s)}\}_{k=1}^{n_s^{(i,j)}}$ the eigenvalues of the Dirichlet problem on the edge $v_{s-1}^{(i,j)}v_s^{(i,j)}$.

Since the path $P_{i,j}$ is bead-shorter than $v_{i-1}v_i$ and v_iv_{i+1}, in terms of Lemma 1 we choose masses of the beads and the lengths of the subintervals on the edges of $P_{i,j}$ such that $\nu_1^{(i,j,s)} = \nu_1^{(J)}, \nu_2^{(i,j,s)} = \nu_2^{(J)}, \cdots, \nu_{n_s^{(i,j)}}^{(i,j,s)} = \nu_{n_s^{(i,j)}}^{(J)}$ are eigenvalues of the Dirichlet problem on the edge $v_{s-1}^{(i,j)}v_s^{(i,j)}$.

Applying Lemma 2 to the tree which consists of the paths P and $P_{i,j}$, we conclude that the ordered vector of multiplicities of eigenvalues of problem (8)–(12) can be $\{\underbrace{2,2,\cdots,2}_{L_{i,j}},\underbrace{1,1,\cdots,1}_{L-L_{i,j}}\}$. Since all the eigenvalues of Dirichlet problem of $P_{i,j}$ are distinct and the total number of eigenvalues of problem (8)–(12) on $P \cup P_{i,j}$ is $L + L_{i,j}$, we conclude that $\{\underbrace{2,2,\cdots,2}_{L_{i,j}},\underbrace{1,1,\cdots,1}_{L-L_{i,j}}\}$ majorizes any ordered vector of multiplicities of eigenvalues of problem (8)–(12) on $P \cup P_{i,j}$. If we attach all the paths $P_{i,j}$ $(j \in \{3,4,\cdots,d(v_i)\};\ i \in \{1,2,\cdots,m-1\})$ to P, then we obtain a caterpillar graph (see [14] for the definition) $P \cup \cup_{i=1,2,\cdots,m-1}(\cup_{j=3,4,\cdots,d(v_j)} P_{i,j})$. Consequently, in the same way as above we arrive at the following results:

1. The ordered vector $\{M_1, M_2, \cdots, M_L\}$ majorizes $\{p_1^{\downarrow}, p_2^{\downarrow}, \cdots, p_r^{\downarrow}\}$ that is any possible ordered vector of multiplicities of eigenvalues of the problem (8)–(12) on $P \cup$

$\cup_{i=1,2,\cdots,m-1}\left(\cup_{j=3,4,\cdots,d(v_j)}P_{i,j}\right)$. Here r is the number of distinct eigenvalues of the problem (8)–(12) on the above caterpillar tree and

$$M_k := \begin{cases} 1+\sum\limits_{i=1}^{m-1} N_k(i), & \text{if } k \leqslant \max\limits_{j=3,4,\cdots,d(v_i),i=1,2,\cdots,m-1} n_{i,j}, \\ 1, & \text{if } \max\limits_{j=3,4,\cdots,d(v_i),i=1,2,\cdots,m-1} L_{i,j} < k \leqslant L, \\ 0, & \text{if } k > L. \end{cases}$$

2. There exists a distribution of masses such that the number r of distinct eigenvalues equals the height L of the tree $P\cup\cup_{i=1,2,\cdots,m-1}(\cup_{j=3,4,\cdots,d(v_j)}P_{i,j})$ and the corresponding ordered vector of multiplicities of eigenvalues satisfies

$$\{\hat{p}_1^{\downarrow},\hat{p}_2^{\downarrow},\cdots,\hat{p}_L^{\downarrow}\} = \{M_1,M_2,\cdots,M_L\}. \tag{15}$$

By the definition of [14], the graph of Fig. 3 obtained from the graph of Fig. 2 is a caterpillar graph.

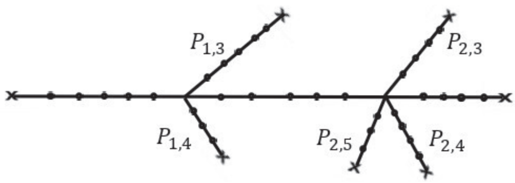

Figure 3: *Prolate caterpillar subtree for T.*

Now we apply the above procedure to the subtrees $T_{i,j}$ and then to their subtrees and so on. Finally we arrive at Statements 1 and 2 of our theorem. Statement 3 holds clearly from construction of the tree T with $\{\check{p}_1^{\downarrow},\check{p}_2^{\downarrow},\cdots,\check{p}_L^{\downarrow}\} = \{\tilde{M}_1,\tilde{M}_2,\cdots,\tilde{M}_L\}$. $\square$

EXAMPLE 1. It is prolate weighted graph in Fig. 2 because $n_1 = n_2 = n_{1,3} = 5 \geqslant n_{1,4}$ and $n_2 > n_3 = 4 = n_{2,3} = n_{2,4} > n_{2,5} = 3$. Applying Theorem 1 to the tree of Fig. 2, we arrive at the following majorizing vector

$$\{9,6,6,4,2,1,1,1,1,1,1,1,1,1\} \succ \{p_1^{\downarrow},p_2^{\downarrow},\cdots,p_r^{\downarrow}\}.$$

Here $\{p_1^{\downarrow},p_2^{\downarrow},\cdots,p_r^{\downarrow}\}$ is any possible ordered vector of multiplicities of eigenvalues of the tree in Fig. 2 and r is the number of distinct eigenvalues of the problem on the tree.

REMARK 5. Since all $n_i > 0$, it follows from Theorem 1 that $\tilde{M}_1 = p_{\text{pen}} - 1$, where p_{pen} is the number of pendant vertices which agrees with the result in [2].

It was proved in [22] that the number of distinct eigenvalues is not less than the length of the maximal weight path in the tree. From Statement 3 of Theorem 1, we obtain the following remark:

REMARK 6. According to Statement 2 of Theorem 1, there exists a distribution of masses on the prolate weighted tree T such that the number of distinct eigenvalues of problem (8)–(12) can be equal to the length L of maximal weight path (height) of the tree T. If the tree of Stieltjes strings is not prolate then this statement is not true. For example the height of the tree described on Fig. 4 is 2 but the minimal number of distinct eigenvalues is 3. Similar situation is described in [1] for the case of acyclic matrices.

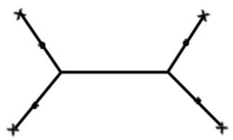

Figure 4: *Example of nonprolate tree.*

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REFERENCES

[1] F. BARIOLI AND S. M. FALLAT, *On two conjectures regarding an inverse eigenvalue problem for acyclic symmetric matrices*, Electron. J. Linear Algebra, **11** (2004), 41–50.

[2] O. BOYKO, O. MARTYNYUK AND V. PIVOVARCHIK, *On maximal multiplicity of eigenvalues of finite-dimensional spectral problem on a graph*, Methods Funct. Anal. Topology, **25** (2019), no. 2, 104–117.

[3] A. L. DUARTE, *Construction of acyclic matrices from spectral data*, Linear Algebra Appl., **113** (1989), 173–182.

[4] F. R. GANTMAKHER AND M. G. KREIN, *Oscillating Matrices and Kernels and Vibrations of Mechanical Systems* (revised edition), AMS, Providence RI, Chelsea, 2002.

[5] G. GLADWELL, *Inverse Problems in Vibration*, Kluwer Academic Publishers, Dordrecht, 2004.

[6] E. A. GUILLEMIN, *Synthesis of Passive Networks. Theory and Methods Appropriate to the Realization and Approximation Problems*, John Wiley and Sons, Inc., New York, 1958.

[7] R. F. MUIRHEAD, *Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters*, Proc. Edinburgh Math. Soc., **21** (1903), 144–157.

[8] J. GENIN AND J. S. MAYBEE, *Mechanical vibrations trees*, J. Math. Anal. Appl., **45** (1974), 746–763.

[9] M. G. KREIN, *On a generalization of investigations of Stieltjes* (Russian), Doklady Akad. Nauk SSSR, **87** (1952), 881–884.

[10] C.-K. LAW, V. PIVOVARCHIK AND W. C. WANG, *A polynomial identity and its applications to inverse spectral problems in Stieltjes strings*, Oper. Matrices, **7** (2013), no. 3, 603–616.

[11] V. A. MARCHENKO, *Introduction to the Theory of Inverse Problems of Spectral Analysis* (Russian), Acta, Kharkov, 2005.

[12] G. GLADWELL, *Matrix inverse eigenvalue problems. Dynamical Inverse Problems: Theory and Applications*, 1–28, CISM Courses and Lect., 529, Springer/Wien/New York, Vienna, 2011.

[13] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities* (second edition), University Press, Cambridge, 1952.

[14] F. HARARY AND A. SCHWENK, *The number of caterpillars*, Discrete Math., **6** (1973), 359–365.

[15] C. R. JOHNSON AND A. LEAL DUARTE, *On the possible multiplicities of the eigenvalues of a Hermitian matrix whose graph is a tree*, Linear Algebra Appl., **348** (2002), 7–21.

[16] C. R. JOHNSON, A. LEAL DUARTE AND C. M. SAIAGO, *Inverse eigenvalue problems and list of multiplicities of eigenvalues for matrices whose graph is a tree: the case of generalized. stars and double generalized stars*, Linear Algebra Appl., **373** (2003), 311–330.

[17] C. R. JOHNSON, B. KROSCHEL AND M. OMLADIČ, *Eigenvalue multiplicities in principal submatrices*, Linear Algebra Appl., **390** (2004), 111–120.

[18] M. MÖLLER AND V. PIVOVARCHIK, *Spectral Theory of Operator Pencils, Hermite-Biehler Functions, and their Applications*, Birkhäuser, Basel, 2015.

[19] M. MÖLLER AND V. PIVOVARCHIK, *Direct and Inverse Finite-Dimensional Spectral Problems on Graphs*, Birkhäuser, Basel, 2020.

[20] A. MARSHALL, I. OLKIN AND B. ARNOLD, *Inequalities: Theory of Majorization and Its Applications* (second edition), Springer, New York, 2011.

[21] V. PIVOVARCHIK, *Existence of a tree of Stieltjes strings corresponding to two given spectra*, J. Phys. A: Math. Theor., **42** (2009), no. 37, 375213, 16 pp.

[22] V. PIVOVARCHIK, *On the minimum number of distinct eigenvalues in a problem on a tree of Stieltjes strings*, Ukrainian Math. J., **72** (2020), no. 1, 149–156.

[23] V. PIVOVARCHIK, *On multiplicities of eigenvalues of a boundary value problem on a snowfake graph*, Linear Algebra Appl., **571** (2019), 78–91.

[24] V. PIVOVARCHIK AND C. TRETTER, *Location and multiplicities of eigenvalues for a star graph of Stieltjes strings*, J. Difference Equ. Appl., **21** (2015), no. 5, 383–402.

[25] C.-W. TSAO AND C.-K. LAW, *The Stieltjes string and its associated nodal points*, Oper. Matrices, **13** (2019), no. 2, 363–373.

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