SOME RESULTS ON CAUCHY DUAL
OF CONDITIONAL TYPE OPERATORS

MORTEZA SOHRABI

(*Communicated by R. Curto*)

Abstract. In this paper, we discuss measure theoretic some characterizations of the notion of Cauchy dual for Lambert conditional operators in some operator classes on $L^2(\Sigma)$ such as, n-normal, n-quasi-normal, n-power †-normal, n-power †-quasi-normal. Moreover, the relations between these classes and some basic properties of these operators are studied. Finally, using the matrix representation, some examples are provided to illustrate the obtained results.

1. Introduction

Let (X, Σ, μ) be a complete σ-finite measure space. For any complete σ-finite subalgebra $\mathscr{A} \subseteq \Sigma$ the Hilbert space $L^2(X, \mathscr{A}, \mu_{|\mathscr{A}})$ is abbreviated to $L^2(\mathscr{A})$ where $\mu_{|\mathscr{A}}$ is the restriction of μ to $\mathscr{A}$. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$ and $L^0_+(\Sigma) = \{f \in L^0(\Sigma) : f \geqslant 0\}$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All sets and functions statements are to be interpreted as being valid almost everywhere with respect to μ. For each nonnegative $f \in L^0(\Sigma)$ or $f \in L^2(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathscr{A}$-measurable function $E^{\mathscr{A}}(f)$ such that

$$\int_A f d\mu = \int_A E^{\mathscr{A}}(f) d\mu,$$

where A is any $\mathscr{A}$-measurable set for which $\int_A f d\mu$ exists. Now associated with every complete σ-finite subalgebra $\mathscr{A} \subseteq \Sigma$, the mapping $E^{\mathscr{A}} : L^2(\Sigma) \to L^2(\mathscr{A})$ uniquely defined by the assignment $f \mapsto E^{\mathscr{A}}(f)$, is called the conditional expectation operator with respect to $\mathscr{A}$. Put $E = E^{\mathscr{A}}$. The mapping E is a linear orthogonal projection. Note that $\mathscr{D}(E)$, the domain of E, contains $L^2(\Sigma) \cup \{f \in L^0(\Sigma) : f \geqslant 0\}$. This operator will play a major role in our work. A detailed discussion and verification of most of the properties may be found in [15, 21, 26]. Those properties of E used in our discussion are summarized below.

- If f is an $\mathscr{A}$-measurable function, then $E(fg) = fE(g)$.
- If $f \geqslant 0$ then $E(f) \geqslant 0$. If $f > 0$ then $E(f) > 0$.
- $\sigma(E(|f|))$ is the smallest $\mathscr{A}$-measurable set containing $\sigma(f)$.
- $\sigma(f) \subseteq \sigma(E(f))$, for each nonnegative $f \in L^2(\Sigma)$.
- $E(|f|^2) = |E(f)|^2$ if and only if $f \in L(\mathscr{A})$.

The products of conditional expectation and multiplication operators appear more often in the service of the study of other operators rather than being the object of study in and of themselves. Weighted Lambert conditional operators in $L^2(\Sigma)$-spaces turn out to be interesting objects of measure and operator theory. The class of these operators includes multiplication operators, integral operators and their adjoints. Throughout the paper, we assume that the measure spaces under consideration are complete and that the corresponding Lambert conditional operators are densely defined. From now on we assume that $(u, w, uw) \in \mathscr{D}(E)$. Operators of the form $M_w E M_u(f) = wE(uf)$ acting in $L^2(\Sigma)$ with $\mathscr{D}(M_w E M_u) = \{fwE(uf) : wE(uf)wE(uf)\}$ are called Lambert conditional operator. Several aspects of this operator were studied in ([8, 11, 19]).

PROPOSITION 1. [17] *Let* $T : L^2(\Sigma) \to L^0(\Sigma)$ *defined by* $T = M_w E M_u$ *is a Lambert multiplication operator.*

(a) $T \in B(L^2(\Sigma))$ *if and only if* $E(|w|^2)E(|u|^2) \in L^\infty(\mathscr{A})$, *and in this case* $\|T\| = \|E(|w|^2)E(|u|^2)\|_\infty^{1/2}$.

(b) Let $T \in B(L^2(\Sigma))$, $0 \leqslant u \in L^0(\Sigma)$ *and* $\upsilon = u(E(|w|^2))^{\frac{1}{2}}$. *If* $E(\upsilon) \geqslant \delta$ *on* $\sigma(\upsilon)$, *then* T *has closed range.*

(c) Let $T \in B(L^2(\Sigma))$, $0 \leqslant u \in L^0(\Sigma)$ *and* $\upsilon = u(E(|w|^2))^{\frac{1}{2}}$. *If* $E(\upsilon) \geqslant \delta$ *on* $\sigma(\upsilon)$, *then* T *has closed range.*

Given a complex separable Hilbert space $\mathscr{H}$, let $B(\mathscr{H})$ denotes the linear space of all bounded linear operators on $\mathscr{H}$. $\mathscr{N}(T)$ and $\mathscr{R}(T)$ denote the null-space and range of an operator T, respectively. Recall that for $T \in B(\mathscr{H})$, there is a unique factorization $T = U|T|$, where $\mathscr{N}(T) = \mathscr{N}(U) = \mathscr{N}(|T|)$, U is a partial isometry; i.e. $UU^*U = U$ and $|T| = \sqrt{(T^*T)}$ is a positive operator. This factorization is called the polar decomposition of T. If $T = U|T|$ is the polar decomposition of $T \in B(\mathscr{H})$, then $\widetilde{T} = \sqrt{|T|}U\sqrt{|T|}$ is called the Aluthge transformation of T.

Let $B_C(\mathscr{H})$ be the set of all bounded linear operators on $\mathscr{H}$ with closed range. For $T \in B_C(\mathscr{H})$, the Moore-Penrose inverse of T, denoted by $T^\dagger$, is the unique bounded operator $T^\dagger$ that satisfies following:

$$TT^\dagger T = T, \ T^\dagger TT^\dagger = T^\dagger, \ (TT^\dagger)^* = TT^\dagger, \ (T^\dagger T)^* = T^\dagger T.$$

The Moore-Penrose inverse of an operator T may always be defined as a densely defined and closed operator. The condition $T \in B_C(\mathscr{H})$ guarantees that $T^\dagger$ is bounded. The Moore-Penrose inverse is designed as a measure for the invertibility of an operator. If $T = U|T|$ is invertible, then $T^{-1} = T^\dagger$, U is unitary and so $|T| = \sqrt{(T^*T)}$ is invertible. For other important properties of $T^\dagger$ see [2, 6].

From now on, we assume that $u, w \in L^0_+(\Sigma)$, $Q = \sigma(E(uw))$ and $K := S \cap G$, where $G = \sigma(E(w))$ and $S = \sigma(E(u))$.

PROPOSITION 2. [17] *Let* $T = M_w E M_u \in B_C(L^2(\Sigma))$ *with* $u, w \geqslant 0$. *Then*

$$T^\dagger = M_{\frac{\chi_K}{E(u^2)E(w^2)}} T^*.$$

The Cauchy dual of left-invertible operators is introduced in [25] as a powerful tool in the model theory of left-invertible operators. To be precise, if T is left invertible, it is easy to see that T^*T is invertible and the operator given by $L_T := (T^*T)^{-1}T^*$ is a canonical left inverse of T. The Cauchy dual of T is then defined as

$$\omega(T) := T(T^*T)^{-1} = L_T^*,$$

which is a right inverse of T^*. For more details on the properties of Cauchy dual see [1, 3, 12, 25, 27].

We introduce now the notion of Cauchy dual for Moore–Penrose inverse.

2. Cauchy dual and conditional type operators

DEFINITION 1. Let $T \in B_C(\mathscr{H})$. The Cauchy dual T is is defined as

$$\omega(T) = T(T^*T)^\dagger.$$

Conditional operators and the various types of generalized inverse have been widely used in practice. In this section, first we review some basic results on the Cauchy dual of Lambert conditional operator $M_w E M_u$ on $L^2(\Sigma)$. Moreover, we discuss the measure theoretic characterizations of the notion of Cauchy dual for Lambert conditional operators in some operator classes on $L^2(\Sigma)$. Finally, some examples are provided to illustrate the obtained results.

PROPOSITION 3. *Let* $T = M_w E M_u \in B_C(L^2(\Sigma))$. *We have*
(a) $\omega(T) = M_{\frac{\chi_K}{E(u^2)E(w^2)}} T, \ \omega(T)^\dagger = T^*$;
(b) $\omega(T^*) = M_{\frac{\chi_K}{E(u^2)E(w^2)}} T^* = \omega(T)^*$;
(c) $\omega(T^\dagger) = M_{\frac{\chi_K}{(E(u^2)E(w^2))^2}} T^* = M_{\frac{\chi_K}{E(u^2)E(w^2)}} \omega(T)^*$;
(d) $\omega(T)^* \omega(T) = M_{\frac{u\chi_K}{(E(u^2))^2 E(w^2)}} E M_u, \ \omega(T)\omega(T)^* = M_{\frac{w\chi_K}{E(u^2)(E(w^2))^2}} E M_w$;
(e) $\omega(T^\dagger)\omega(T) = M_{\frac{u\chi_K}{(E(u^2))^3(E(w^2))^2}} E M_u, \ \omega(T)\omega(T^\dagger) = M_{\frac{w\chi_K}{(E(u^2))^2(E(w^2))^3}} E M_w$;
(f) $\omega(\omega(T)) = M_{\frac{\chi_K}{(E(u^2)E(w^2))^2}} T$.

Proof. (a) Direct computations show that $(T^*T)f = M_{uE(w^2)} E M_u$. Then by Proposition 2 we get that $(T^*T)^\dagger = M_{\frac{u\chi_K}{(E(u^2))^2 E(w^2)}} E M_u$. Hence,

$$\omega(T) = T(T^*T)^\dagger = M_{\frac{w\chi_K}{E(u^2)E(w^2)}} E M_u = M_{\frac{\chi_K}{E(u^2)E(w^2)}} T.$$

Also by Proposition 2, it is easy to check that

$$\omega(T)^{\dagger} = T^*.$$

Parts (b) and (c) are obtained by similar calculations.

(d) We have

$$\omega(T)^*\omega(T) = M_{\frac{\chi_K}{(E(u^2)E(w^2))^2}} T^*T = M_{\frac{u\chi_K}{(E(u^2))^2 E(w^2)}} EM_u$$

and also,

$$\omega(T)\omega(T)^* = M_{\frac{\chi_K}{(E(u^2)E(w^2))^2}} TT^* = M_{\frac{w\chi_K}{E(u^2)(E(w^2))^2}} EM_w.$$

(e) This assertion Is similar to the previous part.

(f) With simple calculations, we get that

$$\omega(\omega(T)) = M_{\frac{\chi_K}{E(u^2)E(w^2)}} \omega(T) = M_{\frac{\chi_K}{(E(u^2)E(w^2))^2}} T. \quad \square$$

LEMMA 1. [16] *Let* $\omega \in L^0(\Sigma)$, $0 \leqslant v \in L^0(\mathscr{A})$ *and let* $A := M_{v\omega}EM_{\omega} \in B(L^2(\Sigma))$. *Then for each* $p \in (0,\infty)$ *and* $f \in L^2(\Sigma)$, $A^p(f) = v^p\omega E(\omega^2)^{p-1}E(\omega f)$.

PROPOSITION 4. *Let* $T = M_wEM_u \in \mathrm{B}_C(\mathscr{H})$ *and let* $\omega(T) = U|\omega(T)|$ *be the polar decomposition of* $\omega(T)$. *Then*

$$|\omega(T)| = M_{\frac{u\chi_K}{\sqrt{(E(u^2))^3 E(w^2)}}} EM_u;$$

$$U = M_{\frac{w\chi_K}{\sqrt{E(u^2)E(w^2)}}} EM_u.$$

Proof. Let $f \in L^2(\Sigma)$. Then $|\omega(T)|^2 = \omega(T)^*\omega(T) = M_{\frac{u\chi_K}{(E(u^2))^2 E(w^2)}} EM_u$.

Now $|\omega(T)|$ follows from Lemma 1. Moreover, it is easy to check that $\omega(T) = U|\omega(T)|$, $UU^*U = U$ and $\mathscr{N}(U) = \mathscr{N}(\omega(T)^*) = \mathscr{N}(|\omega(T)|)$. This completes the proof. $\square$

LEMMA 2. [18] *Let* $0 \leqslant u,w \in \mathscr{D}(E)$. $E(uw)^2 = E(u^2)E(w^2)$ *if and only if* $w = gu$ *for some* $g \in L^0(\mathscr{A})$.

DEFINITION 2. Let $T \in B(\mathscr{H})$. For $n \in \mathbb{N}$, T is said to be n-normal operator if $T^nT^* = T^*T^n$ and T is n-quasi-normal operator if $T^n(T^*T) = (T^*T)T^n$

PROPOSITION 5. *Let* $T = M_wEM_u \in B_C(L^2(\Sigma))$. *The following statements are equivalent*

    (a) $\omega(T)$ *is n-normal;*

    (b) $(E(uw))^2 = E(u^2)E(w^2)$, *on* Q;

    (c) $\omega(T)$ *is n-quasi-normal.*

Proof. (a)$\Leftrightarrow$ (b) Let $n \in \mathbb{N}$, $f \in L^2(\Sigma)$. Then by induction we obtain

$$(\omega(T))^n(f) = \frac{w(E(uw))^{n-1}}{(E(u^2))^n(E(w^2))^n}E(uf).$$

Then,

$$(\omega(T))^n\omega(T)^*(f) = \frac{wE(u^2)(E(uw))^{n-1}}{(E(u^2))^n(E(w^2))^{n+1}}E(wf).$$

Also,

$$\omega(T)^*(\omega(T))^n(f) = \frac{uE(w^2)(E(uw))^{n-1}}{(E(u^2))^n(E(w^2))^{n+1}}E(uf).$$

Thus, $\omega(T)$ is n-normal iff

$$\frac{wE(u^2)(E(uw))^{n-1}}{(E(u^2))^n(E(w^2))^{n+1}}E(wf) = \frac{uE(w^2)(E(uw))^{n-1}}{(E(u^2))^n(E(w^2))^{n+1}}E(uf). \tag{1}$$

Since $\mathscr{A}$ is σ−finite, there exists $\{A_n\} \subseteq \mathscr{A}$ such that $X = \cup A_n, A_n \subseteq A_{n+1}$ with $0 < \mu(A_n) < \infty$. In this case $\chi_{A_n} \nearrow \chi_X$. Then by simplify and substituting $f_n = w\sqrt{E(u^2)}\chi_{A_n}$ in (1) and taking limit on n, we obtain

$$wE(u^2)E(w^2)(E(uw))^{n-1}\sqrt{E(u^2)}$$
$$= uE(w^2)(E(uw))^n\sqrt{E(u^2)}, \quad \text{on } K.$$

Equivalently,

$$wE(u^2)\chi_Q = uE(uw). \tag{2}$$

Now, multiplying both sides of (2) by w and then taking E we get that

$$(E(uw))^2 = E(u^2)E(w^2).$$

Consequently, let $(E(uw))^2 = E(u^2)E(w^2)$, then the desired conclusion follows from Lemma 2.

(b)$\Leftrightarrow$ (c) Since

$$(\omega(T))^n(f) = \frac{w(E(uw))^{n-1}}{(E(u^2))^n(E(w^2))^n}E(uf).$$

Then, we get that

$$(\omega(T))^n\omega(T)^*\omega(T)(f) = \frac{wE(uw))^{n-1}}{(E(u^2))^{n+1}(E(w^2))^{n+1}}E(uf).$$

Also,

$$\omega(T)^*\omega(T)(\omega(T))^n(f) = \frac{u(E(uw))^n}{(E(u^2))^{n+2}(E(w^2))^{n+1}}E(uf).$$

Now let $E(uw) = E(u^2)E(w^2)$. Then by Lemma 2 and above relations $\omega(T)$ is n-quasi-normal. Conversely, let $\omega(T)$ is n-quasi-normal. Thus

$$\frac{wE(uw))^{n-1}}{(E(u^2))^{n+1}(E(w^2))^{n+1}}E(uf) = \frac{u(E(uw))^n}{(E(u^2))^{n+2}(E(w^2))^{n+1}}E(uf). \qquad (3)$$

Now by a similar argument used in (a), The proof is complete. $\square$

Recently, the authors M. Dana and R. Yousfi in [4] has introduced the some classes of operators such as, D-quasi-normal, n-power D-normal, n-power D-quasi-normal. Like this definitions we introduce the classes of n-power $\dagger$-normal and n-power $\dagger$-quasi-normal associated with a Moore-Penrose invertible operator using its Moore-Penrose inverse. Also, we discuss measure theoretic characterizations for Cauchy dual of Lambert conditional operators for these classes.

DEFINITION 3. Let $T \in B_C(\mathscr{H})$. For $n \in \mathbb{N}$, T is said to be n-power $\dagger$-normal if $(T^\dagger)^n T^* = T^*(T^\dagger)^n$ and T is n-power $\dagger$-quasi-normal if $(T^\dagger)^n(T^*T) = (T^*T)(T^\dagger)^n$

PROPOSITION 6. *Let* $T = M_w E M_u \in B_C(L^2(\Sigma))$. *Then the following assertions hold on* Q
   (a) $\omega(T)$ *is n-power $\dagger$-normal;*
   (b) $\omega(T)$ *is n-power $\dagger$-quasi-normal iff* $(E(uw))^2 = E(u^2)E(w^2)$.

Proof. (a) Since $(\omega(T)^\dagger)^n(f) = u(E(uw))^{n-1}E(wf)$. Then we have

$$(\omega(T)^\dagger)^n \omega(T)^*(f) = \frac{u(E(uw))^n}{E(u^2)E(w^2)}E(wf)$$
$$= \omega(T)^*(\omega(T)^\dagger)^n(f).$$

Therefore, the proof is compleat.
   (b) It is easy to check that

$$(\omega(T)^\dagger)^n \omega(T)^* \omega(T)(f) = \frac{uE(uw))^n}{(E(u^2))^2 E(w^2)}E(uf);$$

$$\omega(T)^* \omega(T)(\omega(T)^\dagger)^n(f) = \frac{u(E(uw))^{n-1}}{E(u^2)E(w^2)}E(wf).$$

Thus, $\omega(T)$ is n-power $\dagger$-quasi-normal if and only if

$$\frac{uE(uw))^n}{(E(u^2))^2 E(w^2)}E(uf) = \frac{u(E(uw))^{n-1}}{E(u^2)E(w^2)}E(wf). \qquad (4)$$

Put $f_n = w\sqrt{E(u^2)}\chi_{A_n}$. After substituting f_n in (4) and using the similar argument in Proposition 5, we obtain

$$(E(uw))^2 = E(u^2)E(w^2). \quad \square$$

PROPOSITION 7. *Let $T = M_w E M_u \in B_C(L^2(\Sigma))$. Then $\omega(T)^n = \omega(T)$ iff $E(uw) = E(u^2)E(w^2)$.*

Proof. Since for $f \in L^2(\Sigma)$, $n \in \mathbb{N}$, we have

$$(\omega(T))^n(f) = \frac{w(E(uw))^{n-1}}{(E(u^2))^n(E(w^2))^n} E(uf);$$

$$\omega(T)(f) = \frac{wE(uf)}{E(u^2)E(w^2)}.$$

Let $(\omega(T))^n = \omega(T)$. Put

$$\lambda = \left\{ \frac{(E(uw))^{n-1}}{(E(u^2))^n(E(w^2))^n} - \frac{1}{E(u^2)E(w^2)} \right\}.$$

Thus $M_\lambda T = 0$. Hence

$$\|M_\lambda T\| = \||\lambda| E(u^2)^{\frac{1}{2}} E(w^2)^{\frac{1}{2}}\|_\infty = 0.$$

Therefore, $\lambda = 0$, and so $E(uw) = E(u^2)E(w^2)$.

Now let $E(uw) = E(u^2)E(w^2)$. In this case clearly, $(\omega(T))^n = \omega(T)$. $\quad\square$

PROPOSITION 8. *Let $T = M_w E M_u \in B_C(L^2(\Sigma))$. Then $\omega(T)$ is a partial isometry iff $E(u^2)E(w^2) = 1$ on K.*

Proof. It is easy to check that $\omega(T)(\omega(T))^* \omega(T) = M_\alpha \omega(T)$, where $\alpha = \frac{\chi_K}{E(u^2)E(w^2)}$. Then, $\omega(T)$ is a partial isometry iff $M_{\alpha-1}\omega(T) = 0$ iff $\||\alpha - 1|\omega(T)\| = 0$. That is equivalent to $E(u^2)E(w^2) = 1$ on K. $\quad\square$

PROPOSITION 9. *Let $T = M_w E M_u \in B_C(L^2(\Sigma))$. Then the following assertions hold on Q.*
(a) $\omega(T^n) = \omega(T)^n$ iff $E(u^2)E(w^2) = 1$;
(b) $(\omega(T)^*\omega(T))^n = (\omega(T)^*)^n \omega(T)^n$ iff $(E(uw))^2 = E(u^2)E(w^2)$;
(c) $\omega(\omega(T)) = T$ iff $E(u^2)E(w^2) = 1$.

Proof. (a) We have

$$\omega(T)^n(f) = \frac{w(E(uw))^{n-1}}{(E(u^2))^n(E(w^2))^n} E(uf);$$

$$\omega(T^n)(f) = w(E(uw))^{n-1}E(uf).$$

Thus,

$$\omega(T)^n(f) = \omega(T^n)$$

$$\Leftrightarrow \frac{w(E(uw))^{n-1}}{(E(u^2))^n(E(w^2))^n} E(uf) = w(E(uw))^{n-1}E(uf).$$

By using the similar argument in Proposition 5, we get that $E(u^2)E(w^2) = 1$.

(b) Direct computations show that,

$$(\omega(T)^*\omega(T))^n(f) = \frac{u(E(u^2))^{n-1}}{(E(u^2))^{2n}(E(w^2))^n}E(uf);$$

$$(\omega(T)^*)^n\omega(T)^n(f) = \frac{u(E(uw))^{2n-2}E(w^2)}{(E(u^2))^{2n}(E(w^2))^{2n}}E(wf).$$

Thus we have,

$$(\omega(T)^*\omega(T))^n(f) = (\omega(T)^*)^n\omega(T)^n(f)$$

$$\Leftrightarrow \frac{u(E(u^2))^{n-1}}{(E(u^2))^{2n}(E(w^2))^n}E(uf) = \frac{u(E(uw))^{2n-2}E(w^2)}{(E(u^2))^{2n}(E(w^2))^{2n}}E(wf).$$

Again by similar argument in Proposition 5, the proof is complete.

(c) By using the Lemma 1 and Lemma 2, it is clear. □

We now turn to the computation of $\omega(\widetilde{T})$ and $\widetilde{\omega(T)}$.

PROPOSITION 10. *Let $T, \widetilde{T} \in B_C(L^2(\Sigma))$. Then*

(a) $\widetilde{\omega(T)} = M_{\frac{u(E(uw))\chi_K}{(E(u^2))^2 E(w^2)}} EM_u$;

(b) $\omega(\widetilde{T}) = M_{\frac{u\chi_K}{E(u^2)E(uw)}} EM_u$.

Proof. (a) We know that, if $T = M_w EM_u$, then $\widetilde{T} = M_v EM_u$, where $v = \frac{uE(uw)}{E(u^2)}$. It follows that

$$\widetilde{\omega(T)} = M_{\frac{u(E(uw))\chi_K}{(E(u^2))^2 E(w^2)}} EM_u.$$

(b) Knowing that $\widetilde{T} = M_{\frac{uE(uw)}{E(u^2)}} EM_u$, where $v = \frac{uE(uw)}{E(u^2)}$. We get that

$$\omega(\widetilde{T}) = M_{\frac{v\chi_K}{E(v^2)E(w^2)}} EM_u = M_{\frac{u\chi_K}{E(u^2)E(uw)}} EM_u.$$

This completes the proof. □

COROLLARY 1. *Let $T, \widetilde{T} \in B_C(L^2(\Sigma))$. Then $\widetilde{\omega(T)} = \omega(\widetilde{T}) = \omega(T)$ if and only if $(E(uw))^2 = E(u^2)E(w^2)$.*

COROLLARY 2. *Let $\widetilde{T} \in B_C(L^2(\Sigma))$. Then $\widetilde{\omega(\widetilde{T})} = \omega(\widetilde{T})$.*

In the following example by using the matrix representation, we show some applications of these results.

EXAMPLE 1. Let $X = \{1,2,3\}$, $\Sigma = 2^X$, $\mu\{n\} = \frac{1}{3}$ let $\mathscr{A}$ be the σ-algebra generated by the partition $\{\{1,2\},\{3\}\}$. Then $L^2(\Sigma) \cong \mathbb{C}^3$ and

$$E(f) = \left(\frac{1}{\mu(A_1)} \int_{A_1} f d\mu\right)\chi_{A_1} + \left(\frac{1}{\mu(A_2)} \int_{A_2} f d\mu\right)\chi_{A_2}$$
$$= \frac{f_1 + f_2}{2}\chi_{A_1} + f_3\chi_{A_2}.$$

Where $A_1 = \{1,2\}$ and $A_1 = \{3\}$. Then matrix representation of $E = E^{\mathscr{A}}$ with respect to the standard orthonormal basis is

$$E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $w = (w_1, w_2, w_3)$ and $u = (u_1, u_2, u_3)$ be nonzero elements of $\mathbb{C}^3$. Thus

$$T = M_w E M_u = \begin{bmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{w_1 u_1}{2} & \frac{w_1 u_2}{2} & 0 \\ \frac{w_2 u_1}{2} & \frac{w_2 u_2}{2} & 0 \\ 0 & 0 & w_3 u_3 \end{bmatrix}.$$

Also,

$$u^2 = (u_1^2, u_2^2, u_3^2);$$
$$w^2 = (w_1^2, w_2^2, w_3^2);$$
$$uw = (u_1 w_1, u_2 w_2, u_3 w_3);$$
$$E(u^2) = \left(\frac{u_1^2 + u_2^2}{2}, \frac{u_1^2 + u_2^2}{2}, u_3^2\right);$$
$$E(w^2) = \left(\frac{w_1^2 + w_2^2}{2}, \frac{w_1^2 + w_2^2}{2}, w_3^2\right);$$
$$E(uw) = \left(\frac{u_1 w_1 + u_2 w_2}{2}, \frac{u_1 w_1 + u_2 w_2}{2}, u_3 w_3\right).$$

Let $u_i \neq 0$, $w_i \neq 0$. put $a = \frac{(u_1^2 + u_2^2)(w_1^2 + w_2^2)}{4}$, $b = \frac{u_1 w_1 + u_2 w_2}{2}$ and $c = u_3 w_3$. Hence

$E(u^2)E(w^2) = (a,a,c^2)$ and $E(uw) = (b,b,c)$. Then we get that

$$T^\dagger = M_{(\frac{1}{a},\frac{1}{a},\frac{1}{c^2})}T^* = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{bmatrix} \begin{bmatrix} \frac{w_1 u_1}{2} & \frac{w_2 u_1}{2} & 0 \\ \frac{w_1 u_2}{2} & \frac{w_2 u_2}{2} & 0 \\ 0 & 0 & w_3 u_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{w_1 u_1}{2a} & \frac{w_2 u_1}{2a} & 0 \\ \frac{w_1 u_2}{2a} & \frac{w_2 u_2}{2a} & 0 \\ 0 & 0 & \frac{w_3 u_3}{c^2} \end{bmatrix}.$$

And

$$\omega(T) = M_{(\frac{1}{a},\frac{1}{a},\frac{1}{c^2})}T = \begin{bmatrix} \frac{w_1 u_1}{2a} & \frac{w_1 u_2}{2a} & 0 \\ \frac{w_2 u_1}{2a} & \frac{w_2 u_2}{2a} & 0 \\ 0 & 0 & \frac{w_3 u_3}{c^2} \end{bmatrix}.$$

Also, since $E(u^2)E(uw) = (d,d,e)$, where $d = \frac{(u_1^2+u_2^2)(u_1 w_1+u_2 w_2)}{4}$ and $e = u_3^2(u_3 w_3)$.
Thus,

$$\omega(\widetilde{T}) = M_{(\frac{1}{d},\frac{1}{d},\frac{1}{e})}M_u E M_u = \begin{bmatrix} \frac{1}{d} & 0 & 0 \\ 0 & \frac{1}{d} & 0 \\ 0 & 0 & \frac{1}{e} \end{bmatrix} \begin{bmatrix} \frac{u_1^2}{2} & \frac{u_1 u_2}{2} & 0 \\ \frac{u_1 u_2}{2} & \frac{u_2^2}{2} & 0 \\ 0 & 0 & u_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{u_1^2}{2d} & \frac{u_1 u_2}{2d} & 0 \\ \frac{u_1 u_2}{2d} & \frac{u_2^2}{2d} & 0 \\ 0 & 0 & \frac{u_3^2}{e} \end{bmatrix}.$$

Now, set $u = (-2,2,-1)$, $w = (1,-1,2)$. Then $E(u^2) = (4,4,1)$, $E(w^2) = (1,1,4)$ and $E(uw) = (-2,-2,-2)$. It is easy to check that $a = 4$, $b = -2$, $c = -2$, $d = -8$, $e = -2$ and

$$T = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad T^\dagger = \begin{bmatrix} \frac{-1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{-1}{4} & 0 \\ 0 & 0 & \frac{-1}{2} \end{bmatrix} = \omega(T) = \omega(\widetilde{T}).$$

Also, we have $E(u^2)E(w^2) = E(uw)^2$. So, $\omega(T)$ is not partial isometry but it is a n-normal operator and also it is a n-power $\dagger$-quasi-normal operator. Note that if in the above we take $u = (1,1,-3)$, $w = (0,1,-1)$. Then $E(u^2) = (1,1,9)$, $E(w^2) = (\frac{1}{2},\frac{1}{2},1)$ and $E(uw) = (\frac{1}{2},\frac{1}{2},3)$. Direct computations show that

$$T = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \qquad T^\dagger = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix},$$

$$\omega(T) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \qquad \omega(\widetilde{T}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

It follows that $\omega(T) \neq \omega(\widetilde{T})$. In this case $E(uw)^2 \neq E(u^2)E(w^2)$. Thus $\omega(T)$ is not n-normal operator and also it is not a n-power $\dagger$-quasi-normal operator.

Now, we introduce the class of $\mathscr{K}(p,n,k)$ as a generalization of the classes of p-hyponormal and p-quasihyponormal operators. In addition, we investigate some characterizations of these classes by $\omega(T)$ on $L^2(\Sigma)$.

DEFINITION 4. Let $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ and let $p > 0$. We denote by $\mathscr{K}(p,n,k)$ the set of all operators such as T on $\mathscr{H}$ that $T^{*k}(T^{*n}T^n)^p T^k \geqslant T^{*k}(T^n T^{*n})^p T^k$.

Note that $T \in \mathscr{K}(p,n,0)$ if and only if T is (p,n)-hyponormal and $T \in \mathscr{K}(p,1,k)$ if and only if T is (p,k)-quasihyponormal.

PROPOSITION 11. *Let $T = M_w E M_u \in B_C(L^2(\Sigma))$. Then the following assertions hold on Q.*
  (a) $\omega(T) \in \mathscr{K}(p,n,0)$ iff $u^2 E(w^2) \geqslant w^2 E(u^2)$;
  (b) $\omega(T) \in \mathscr{K}(p,1,k)$ iff $(E(uw))^2 \geqslant E(u^2)E(w^2)$.

Proof. (a) Let $f \in L^2(\Sigma)$. Thus by Lemma 1, we obtain

$$\{\omega(T)^{*n}\omega(T)^n\}^p(f) = \frac{\chi_K}{(E(u^2)E(w^2))^{2np}} u E(uw)^{(2p-2)n}(E(u^2))^{p-1}(E(w^2))^p E(uf);$$

$$\{\omega(T)^n\omega(T)^{*n}\}^p(f) = \frac{\chi_K}{(E(u^2)E(w^2))^{2np}} w E(uw)^{(2p-2)n}(E(u^2))^p(E(w^2))^{p-1} E(wf).$$

Then $\omega(T) \in \mathscr{K}(p,n,0)$ iff

$$\int_X \left\{ \frac{u E(uw)^{(2p-2)n}E(uf)}{E(u^2)^{(2n-1)p+1}E(w^2)^{(2n-1)p}} - \frac{w E(uw)^{(2p-2)n}E(wf)}{E(u^2)^{(2n-1)p}E(w^2)^{(2n-1)p+1}} \right\} \bar{f} d\mu \geqslant 0. \quad (5)$$

Put $f_n = w\sqrt{E(u^2)}\chi_{A_n}$. After substituting f_n in (5), we obtain

$$\int_{A_n} \left\{ \frac{uw E(uw)^{(2p-2)n}E(uw)\sqrt{E(u^2)}}{E(u^2)^{(2n-1)p+1}E(w^2)^{(2n-1)p}} - \frac{w^2 E(uw)^{(2p-2)n}E(w^2)\sqrt{E(u^2)}}{E(u^2)^{(2n-1)p}E(w^2)^{(2n-1)p+1}} \right\} d\mu \geqslant 0.$$

It follows that $uw E(uw) \geqslant w^2 E(u^2)$. Also for $A_n \in \mathscr{A}$ with $0 < \mu(A_n) < \infty$, put $f_n = u\sqrt{E(w^2)}\chi_{A_n}$. Again by similar argument and after substituting f_n in (5), we obtain $u^2 E(w^2) \geqslant uw E(uw)$. Consequently $u^2 E(w^2) \geqslant w^2 E(u^2)$, on X.
  (b) Suppose $f \in L^2(\Sigma)$. It is easy to check that

$$\omega(T)^{*k}\{\omega(T)^*\omega(T)\}^p\omega(T)^k(f) = \frac{\chi_K u E(uw)^{2p+2k-2}(E(u^2))^{p-1}(E(w^2))^p}{(E(u^2)E(w^2))^{2p+2k}} E(uf);$$

$$\omega(T)^{*k}\{\omega(T)\omega(T)^*\}^p\omega(T)^k(f) = \frac{\chi_K u E(uw)^{2p+2k-4}(E(u^2))^p(E(w^2))^{p+1}}{(E(u^2)E(w^2))^{2p+2k}} E(uf).$$

Then we obtain

$$\langle (\omega(T)^{*k}\{\omega(T)^*\omega(T)\}^p\omega(T)^k - \omega(T)^{*k}\{\omega(T)\omega(T)^*\}^p\omega(T)^k)f, \, f \rangle$$

$$= \int_X \left(\frac{\chi_K uE(uw)^{2p+2k-2}\overline{f}E(uf)}{E(u^2)^{p+2k+1}E(w^2)^{p+2k}} - \frac{\chi_K uE(uw)^{2p+2k-4}\overline{f}E(uf)}{E(u^2)^{p+2k}E(w^2)^{p+2k-1}} \right) d\mu$$

$$= \int_K \left(\frac{\chi_K E(uw)^{2p+2k-2}}{E(u^2)^{p+2k+1}E(w^2)^{p+2k}} - \frac{\chi_K E(uw)^{2p+2k-4}}{E(u^2)^{p+2k}E(w^2)^{p+2k-1}} \right) |E(uf)|^2 d\mu.$$

This implies that if $(E(uw))^2 \geqslant (E(u^2))(E(w^2))$ on K, then $\omega(T) \in \mathscr{K}(p,1,k)$.

Conversely, if $\omega(T) \in \mathscr{K}(p,1,k)$, then

$$\langle (\omega(T)^{*k}\{\omega(T)^*\omega(T)\}^p\omega(T)^k - \omega(T)^{*k}\{\omega(T)\omega(T)^*\}^p\omega(T)^k)f, \, f \rangle \geqslant 0$$

for all $f \in L^2(\Sigma)$. Let $B \in \mathscr{A}$, with $B \subseteq K$ and $0 < \mu(B) < \infty$. By replacing f to χ_B, we have

$$\int_B \left(\frac{E(uw)^{2p+2k-2}}{E(u^2)^{p+2k+1}E(w^2)^{p+2k}} - \frac{E(uw)^{2p+2k-4}}{E(u^2)^{p+2k}E(w^2)^{p+2k-1}} \right) (E(u))^2 d\mu \geqslant 0.$$

Since $B \in \mathscr{A}$ is arbitrary, then $(E(uw))^2 \geqslant E(u^2)E(w^2)$ on K. This completes the proof. $\square$

COROLLARY 3. *Let* $T = M_w E M_u \in B_C(L^2(\Sigma))$. *Then the following assertions hold on Q.*

(a) $\omega(T)$ *is p-hyponormal iff* $w^2 E(u^2) \geqslant u^2 E(w^2)$;

(b) $\omega(T)$ *is p-quasihyponormal iff* $(E(uw))^2 \geqslant E(u^2)E(w^2)$.

EXAMPLE 2. Let $X = [0,1]$, $d\mu = dx$, Σ be the Lebesgue measurable sets and let $\mathscr{A} = \{\emptyset, X\}$. Then $Tf(x) = w(x)E(uf)(x) = w(x)\int_0^1 u(x)f(x)dx$ and $T^*f(x) = u(x)\int_0^1 w(x)f(x)dx$ for all $f \in L^2(\Sigma)$. Put $u(x) = \frac{x}{2\sqrt{2}}$, $w(x) = 5x^2 + 3$. Then $E(uw) = \frac{11}{8\sqrt{2}}$, $E(u^2) = \frac{1}{24}$, $E(w^2) = 24$. Thus, $\omega(T)$ is not n-power †-quasi-normal and also $\omega(T) \notin \mathscr{K}(p,1,k)$. But $\omega(T)$ not only is n-power †-normal but it is also a partial isometry. Moreover, by a direct computation, we get that

$$(\omega(T)f)(x) = \frac{5x^2+3}{2\sqrt{2}} \int_0^1 xf(x)dx = Tf(x);$$

$$(\omega(\widetilde{T})f)(x) = \frac{48x}{11\sqrt{2}} \int_0^1 xf(x)dx;$$

$$(\widetilde{\omega(T)}f)(x) = \frac{33x}{8\sqrt{2}} \int_0^1 xf(x)dx.$$

EXAMPLE 3. Let $X = [-1, 1]$, $d\mu = dx$, Σ be the Lebesgue sets, and let $\mathscr{A} \subseteq \Sigma$ be the σ-algebra generated by the symmetric sets about the origin. Let $0 < a \leqslant 1$ and $f \in L^2(\Sigma)$. Then

$$\int_{-a}^{a} E(f)(x)dx = \int_{-a}^{a} f(x)dx$$

$$= \int_{-a}^{a} \{\frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}\}dx = \int_{-a}^{a} \frac{f(x) + f(-x)}{2}dx.$$

Thus, $E(f)(x) = \frac{f(x)+f(-x)}{2}$. Let $u(x) = 1$, $w(x) = e^x$. So $E(uw)(x) = \cosh(x)$ and $E(w^2)(x) = \cosh(2x)$, $E(u^2) = 1$. Then $\omega(T) \notin \mathscr{K}(p,n,0) \cap \mathscr{K}(p,1,k)$. However if $u(x) = \cos(2x)$ and $w(x) = x^4$ then $E(u^2)(x) = \cos^2(2x)$, $E(w^2)(x) = x^8$ and $E(uw)(x) = x^4\cos(2x)$. Thus $\omega(T) \in \mathscr{K}(p,n,0) \cap \mathscr{K}(p,1,k)$. Finally, if we take $u(x) = x$ and $w(x) = \cos(x)$ then we have $E(u^2)(x) = x^2$, $E(w^2)(x) = \cos^2(x)$ and $E(uw)(x) = 0$. Thus $\omega(T) \in \mathscr{K}(p,n,0) \backslash \mathscr{K}(p,1,k)$.

REFERENCES

[1] A. ANAND, S. CHAVAN, Z. J. JABLONSKI AND J. STOCHEL, *The Cauchy dual subnormality problem for cyclic 2-isometries*, Advances in Operator Theory, 5 (3), 1061–1077.

[2] A. BEN-ISRAEL AND T. N. E. GREVILLE, *Generalized inverses: theory and applications*, sec. ed, Springer, 2003.

[3] S. CHAVAN, *On operators Cauchy dual to 2-hyperexpansive operators*, Proc. Edinburgh Math. Soc., **50** (2007), 637–552.

[4] M. DANA AND R. YOUSEFI, *On the classes of D-normal operators and D-quasi-normal operators*, Operators and Matrices, vol. 12, no. 2 (2018), 465–487.

[5] J. DING AND W. E. HORNOR, *A new approach to Frobenius-Perron operators*, J. Math. Anal. Appl. **187** (1994), 1047–1058.

[6] D. S. DJORDJEVIĆ AND N. C. DINČIĆ, *Reverse order law for the Moore-Penrose inverse*, J. Math. Anal. Appl. **361** (2010) 252–261.

[7] Y. ESTAREMI, M. R. JABBARZADEH, *Weighted lambert type operators on L^p-spaces*, Oper. Matrices **7** (2013), 101–116.

[8] Y. ESTAREMI, M. R. JABBARZADEH, *Weighted composition Lambert-type operators on L^p spaces*, Mediterr. J. Math. **11** (2014), 955–964.

[9] Y. ESTAREMI, *Some classes of weighted conditional type operators and their spectra*, Positivity **19** (2015), 83–93.

[10] Y. ESTAREMI, *Centered weighted conditional type operators*, arxiv.org.

[11] Y. ESTAREMI, *On a class of operators with normal Aluthge transformations*, Filomat **29** (2015), 969–975.

[12] H. EZZAHRAOUI, M. MBEKHTA AND E. H. ZEROUALI, *On the Cauchy dual of closed range operators*, Acta Sci. Math. (Szeged) 85 (2019), 231–248.

[13] T. FURUTA, *Invitation to linear operators*, Taylor & Francis, Ltd., London, 2001.

[14] J. J. GROBLER AND B. DE PAGTER, *Operators representable as multiplication conditional expectation operators*, J. Oper. Theory **48** (2002), 15–40.

[15] J. HERRON, *Weighted conditional expectation operators*, Oper. Matrices **5** (2011), 107–118.

[16] M. R. JABBARZADEH AND M. SOHRABI CHEGENI, *Moore-Penrose inverse of conditional type operators*, Oper. Matrices **11** (2017), 289–299.

[17] M. R. JABBARZADEH, H. EMAMALIPOUR AND M. SOHRABI CHEGENI, *Parallelism between Moore-Penrose inverse and Aluthge transformation of operators*, Applicable Analysis and Discrete Mathematics 12, no. 2 (2018), 318–335.

[18] M. R. JABBARZADEH AND M. SOHRABI, *Moore-Penrose geometric Quasi-Mean for Conditional Operators*, (in press).

[19] M. R. JABBARZADEH, *A conditional expectation type operator on L^p spaces*, Oper. Matrices **4** (2010), 445–453.

[20] I. B. JUNG, E. KO AND C. PEARCY, *Aluthge transforms of operators*, Integral Equations Operator Theory **37** (2000), 437–448.

[21] A. LAMBERT, *L^p-multipliers and nested sigma-algebras*, Oper. Theory Adv. Appl. **104** (1998), 147–153.

[22] A. LAMBERT AND B. WEINSTOCK, *A class of operator algebras induced by probabilistic conditional expectations*, Michigan Math. J. **40** (1993), 359–376.

[23] B. B. MORRELL AND P. S. MUHLY, *Centered operators*, Studia Math. **51** (1974), 251–263.

[24] A. NAEEM AND S. A. MAHMOUD, *On the class of k-quasi-(n,m)-power normal operators*, Hacettepe Journal of Mathematics and Statistics, 1–16.

[25] A. OLOFSSON, *Wandering subspace theorems*, Integral Equ. Oper. Theory, **51** (2005), 395–409.

[26] M. M. RAO, *Conditional measure and applications*, Marcel Dekker, New York, 1993.

[27] S. SHIMORIN, *Wold-type decompositions and wandering subspaces for operators close to isometries*, J. Reine Angew. Math. **531** (2001), 147–189.

[28] T. YAMAZAKI, *On the polar decomposition of the Aluthge transformation and related results*, J. Operator Theory **51** (2004), 303–319.

Morteza Sohrabi
Department of Mathematics
Lorestan University
Khorramabad, Iran
e-mail: sohrabi.mo@lu.ac.ir

Operators and Matrices
www.ele-math.com
oam@ele-math.com