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WEAK–STAR DENTABILITY, QUASI–WEAK–STAR NEAR DENTABILITY AND CONTINUITY OF METRIC PROJECTOR IN BANACH SPACES

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Abstract. The relations between the w^* dentability and Chebyshev set or the continuity of metric projection operator are given. Let X^* be the conjugate space of Banach space X, the conditions of a point (x^*,y^*) on the unit sphere of product space $X^*\times X^*$ to be w^* denting point of closed unit ball of product space $X^*\times X^*$ are given. Also, a notion of quasi-w^* near dentability in conjugate space X^* is introduced and the relations between the quasi-w^* nearly denting point of closed unit ball of X^* and a certain slice of closed unit ball of X^* are given.

1. Introduction and preliminaries

Some concepts of dentability in Banach spaces are known. Among them dentability, weak-star (denote by w^*) dentability, near dentability and weak-star near dentability are some major notions. One of the reasons is that these properties are strongly related to Radon-Nikodym property, convexity, smoothness, approximative compactness, continuity of metric projection operator and the geometric properties of sets in Banach spaces. Moreover, the metric projection operator plays an important role in optimization, computational mathematics, and approximation theory (see [1], [9]–[11], [21], [32]). The aim of this paper is to study further w^* denting point and give some important results concerning w^* dentability in conjugate space X^*. In addition, we introduce a notion of quasi-near w^* dentability in conjugate space X^* and discuss the relations between the quasi-nearly w^* denting point of closed unit ball of X^* and the slice of closed unit ball of X^*. The topic of this paper is related to the topic of [1–33].

Let X be an infinite dimensional real Banach space. X^* and X^{**} denote the conjugate and quadratic conjugate space of X, respectively. $B(X)$, $B^\circ(X)$ and $S(X)$ denote the closed unit ball of X, the interior of $B(X)$ and the unit sphere of X, respectively. $J_X : X \to X^{**}$ denote the the natural embedding of X into X^{**}. The symbol (X^*,w^*) denotes the weak* topology of X^*. The open set, closed set, compact set, neighborhood and accumulation point with respect to weak* topology is said to be w^* open set, w^* closed set, w^* compact set, w^* neighborhood and w^* accumulation point, respectively. $x_n^* \xrightarrow{w^*} x^*$ (resp. $x_n^* \longrightarrow x^*$) denotes that the sequence $\{x_n^*\}_{n=1}^\infty \subset S(X^*)$

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weakly* (resp. strongly) convergent to an element $x^* \in X^*$. The symbol $\mathrm{co}M$ and $\overline{Q}^{w^*}$ denote the convex hull of set $M \subset X$ and the w^* closure of set $Q \subset X^*$, respectively. The symbol $S(x^{**}, \lambda, B(X^*))$ denotes the slice of $B(X^*)$ generated by $x^{**} \in X^{**}$ and scalar $\lambda > 0$, where

$$S(x^{**}, \lambda, B(X^*)) = \{y^* : y^* \in B(X^*), x^{**}(y^*) \geqslant \sup_{x^* \in B(X^*)} x^{**}(x^*) - \lambda\}.$$

In what follows, we will need some known notions and a new geometric property $(w^*\mathrm{M})$.

A subset $D^* \subset X^*$ is said to be w^* hyperplane (see [24]), if there exist $x \in S(X)$ and real number $\lambda > 0$ such that $D^* = \{x^* : x^* \in X^*, x^*(x) = \lambda\}$.

A point $x_0^* \in B(X^*)$ is said to be w^* exposed point of $B(X^*)$ (see [20], [33]), if there exists $x_0 \in S(X)$ such that $x_0^*(x_0) > x^*(x_0)$ for all $x^* \in B(X^*) \backslash \{x_0^*\}$.

A functional x^* is said to be support functional of unit sphere $S(X)$ at point x (see [2]), if for $x \in S(X)$, there exists a continuous linear functional $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*(x) = 1$.

The closed unit ball $B(X^*) \subset X^*$ is said to have the property $(w^*\mathrm{M})$: If any $x^* \in X^*$, $\{x_n^*\} \subset B(X^*)$ satisfying the condition that $\lim_{n\to\infty} \|x_n^* + x^*\|$ exists, then there exist $x_0 \in S(X)$ and subsequence $\{x_{n_k}^*\} \subset \{x_n^*\}$ such that $\lim_{k\to\infty} x_{n_k}^*(x_0) = \sup_{z^* \in B(X^*)} |z^*(x_0)|$.

DEFINITION 1.1. (see [31]) Let X be a Banach space, $M \subset X$ be a nonempty subset of X. Then the set-valued mapping $P_M : X \to 2^M$

$$P_M(x) = \{y : y \in M : \|x - y\| = \mathrm{dist}(x, M) = \inf_{y \in M} \|x - y\|\}$$

is called the metric projection operator X onto M.

DEFINITION 1.2. (see [15]) A subset $M \subset X$ is said to be proximinal, if $P_M(x) \neq \emptyset$ for all $x \in X$. M is said to be semi-Chebyshev, if $P_M(x)$ is at most a singleton for all $x \in X$. M is said to be Chebyshev, if it is proximinal and semi-Chebyshev.

The concept of dentabe set was first introduced by Rieffel [22] in 1966 and the following result related to dentability has been given therein. i.e. X has the Radon-Nikodym property whenever every bounded subset of X is dentable. This result, later improved by Maynard [14] in 1973, is as follows: *X has Radon-Nikodym property if and only if X is dentable.*

DEFINITION 1.3. (see [22]) A subset $M \subset X$ is said to be dentable set, if for any $\varepsilon > 0$ there exists a $x_\varepsilon \in M$ such that $x_\varepsilon \notin \overline{\mathrm{co}}(M \setminus B(x_\varepsilon, \varepsilon))$, where $B(x_\varepsilon, \varepsilon) = \{x : x \in X : \|x - x_\varepsilon\| < \varepsilon\}$.

The property (G) was given by Fan and Glicksberg [4] in 1958. Banach space X has property (G) if and only if every point $x \in S(X)$ is the denting point of $B(X)$. i.e. for all $x \in S(X)$ and $\varepsilon > 0$, we have $x \notin \overline{\mathrm{co}} D(x, \varepsilon)$, where $D(x, \varepsilon) = \{y : y \in X, \| y - x \| \geqslant \varepsilon\}$.

In 1993, Wu and Li [29] introduced the concept of strong convexity in Banach spaces and gave some results concerning the relations between property (G) and strong convexity.

A Banach space X is called strongly convex space (see [29]), if for any $x \in S(X)$ and sequence $\{x_n\}_{n=1}^{\infty} \subset S(X)$, there exists $x^* \in A(x)$ satisfying $x^*(x_n) \to 1$, then $x_n \to x$, where $A(x) = \{x^* : x^* \in S(X^*),\ x^*(x) = 1\}$.

In view of the connection with dentable set and property (G), if we replace the property (G) by its equivalent statement, then the results obtained by Wu and Li in [29] can be written as follows:

(i) *If X is strongly convex space, then every point of $S(X)$ is denting point of $B(X)$;*

(ii) *If X is reflexive and every point of $S(X)$ is denting point of $B(X)$, then X is strongly convex space.*

The concept of w^* denting point of $B(X^*)$ was given by K. Fan and I. Glicksburg [4].

DEFINITION 1.4. (see [4], [13], [33]) A point $x^* \in S(X^*)$ is said to be w^* denting point of $B(X^*)$, if $x^* \notin \overline{\text{co}}^{w^*}(B(X^*) \setminus B(x^*, \varepsilon))$ holds for each $\varepsilon > 0$, where $B(x^*, \varepsilon) = \{y^* : y^* \in X^*, \| y^* - x^* \| < \varepsilon\}$.

The concept of strong smoothness in Banach spaces was defined in [8], which is strongly related to w^* dentability of Banach spaces.

A Banach space X is called strongly smooth space (see [8]), if any $x \in S(X)$, $\{x_n^*\}_{n=1}^{\infty} \subset S(X^*)$ satisfying $x_n^*(x) \longrightarrow 1$, then there exists some $x^* \in S(X^*)$, such that $x_n^* \longrightarrow x^*$.

The concept of approximative compactness was first given by Jefimow and Stechkin in [7] as a property of Banach spaces, which is also strongly related to w^* dentability and guarantees the existence of the best approximation element in a nonempty closed convex set C for any $x \in X$. In 1972, Oshman [18] proved that the metric projection operator P_C is upper semi-continuous if set C is approximatively compact subset of a Banach space X. In 1987, Montesinos [16] proved that Banach space X is approximately compact if and only if X has the drop property. In 2007, Chen et al. [3] proved that a nonempty closed convex set C of a midpoint locally uniformly rotund space is approximately compact if and only if C is a Chebyshev set and the metric projection operator P_C is continuous.

A subset $C \subset X$ is called approximatively compact (see [7]), if for any sequence $\{x_n\}_{n=1}^{\infty} \subset C$ and $y \in X$ satisfying $\|x_n - y\| \longrightarrow \text{dist}(y, C) = \inf\{\|x - y\| : x \in C\}$, then $\{x_n\}_{n=1}^{\infty}$ has a subsequence converging to an element in C. X is called approximatively compact, if every nonempty closed convex subset of X is approximatively compact.

Some results relating to w^* dentability, approximative compactness and strong smoothness have been given by Shang et al. [25] i.e. The following statement are equivalent:

(i) *X is strongly smooth space;*

(ii) *Every w^* closed convex set of X^* is approximative compact Chebyshev set;*

(iii) *If $x^* \in S(X^*)$ is norm attainable on $S(X)$, then x^* is w^* denting point of $B(X^*)$.*

In 2011, Shang et al. [26] introduced the notion of nearly dentability in Banach space X.

DEFINITION 1.5. (see [26]) A Banach space X is said to be nearly dentable, if for arbitrary $x^* \in S(X^*)$, $A_{x^*} \neq \emptyset$ and any open set $U_{A_{x^*}} \supset A_{x^*}$, we have $A_{x^*} \cap \overline{\mathrm{co}}(B(X) \backslash U_{A_{x^*}}) = \emptyset$, where $A_{x^*} = \{x : x \in B(X),\ x^*(x) = \|x^*\|\}$.

The some results concerning nearly dentability, approximative compactness and metric projection operator P_M have been given in [26] as follows:

(i) *X is approximatively compact if and only if X is nearly dentable and every closed convex subset of $S(X)$ is compact;*

(ii) *If X is nearly dentable, then for any closed convex set $M \subset X$, metric projection operator P_M is upper semi-continuous.*

In 2015, Shang and Cui [24] introduced the notion of w^* near dentability in conjugate space X^*.

DEFINITION 1.6. (see [24]) The conjugate space X^* is said to be w^* nearly dentable, if for any $x \in S(X)$ and open set $U_{A(x)} \supset A(x)$, we have $A(x) \cap \overline{\mathrm{co}}^{w^*}(B(X^*) \backslash U_{A(x)}) = \emptyset$, where $A(x) = \{x^* : \ x^* \in S(X^*),\ x^*(x) = 1\}$.

Some interesting results concerning Radon-Nikodym property, approximative compactness and continuity metric projection operator in w^* nearly dentable Banach space have been given in [24] as follows:

(i) *Let X^* is w^* nearly dentable. Then, X^* has the Radon-Nikodym property if and only if $A_E(x) = \{x^* : x^* \in S(E^*), x^*(x) = 1 = \|x\|\}$ is a separable subset of E^*, where E is a separable closed subspace of X;*

(ii) *Let X^* is w^* nearly dentable. Then, for any w^* open convex set C, metric projection operator $P_{\overline{C}^{w^*}}$ is upper semi-continuous;*

(iii) *X^* is w^* nearly dentable. Then, for any w^* hyperplane D^*, metric projection operator P_{D^*} is upper semi-continuous.*

2. Main results

Considering the idea of introducing w^* near dentability in Banach spaces, we refer to the ideas of Shang and Cui [24] and give a new notion of quasi-w^* near dentability.

DEFINITION 2.1. A point $x_0^* \in S(X^*)$ is called quasi-w^* nearly denting point of $B(X^*)$, if for any $\varepsilon > 0$ and support functional x^{**} of unit sphere $S(X^*)$ at point x_0^*, we have

$$H_{x^{**}} \cap \overline{\mathrm{co}}^{w^*}(B(X^*) \backslash B(H_{x^{**}}, \varepsilon)) = \emptyset.$$

The conjugate space X^* is called quasi-w^* nearly dentabe, if every point $x^* \in S(X^*)$ is quasi-w^* nearly denting point of $B(X^*)$. Where $H_{x^{**}} = \{x^* : \ x^* \in B(X^*),\ x^{**}(x^*) = x^{**}(x_0^*)\}$, $B(H_{x^{**}}, \varepsilon) = \{x^* : x^* \in B(X^*),\ \mathrm{dist}(x^*, H_{x^{**}}) < \varepsilon\}$, respectively.

THEOREM 2.2. *Let X^* be conjugate space of X. If $B(X^*)$ has $(w^*\mathrm{M})$ property and every $x^* \in S(X^*)$ is w^* denting point of $B(X^*)$, then*

(1) $B(X^*)$ is Chebyshev set,

(2) Metric projection operator $P_{B(X^*)}: X^* \to 2^{B(X^*)}$ is norm-norm continuous, where $\|x^* - P_{B(X^*)}(x^*)\| = \operatorname{dist}(x^*, B(X^*))$.

Proof. (1) For each $x^* \in X^* \backslash B(X^*)$, it is easy to see that there exists a sequence $\{x_n^*\}_{n=1}^\infty$ such that $\lim_{n\to\infty} \|x^* - x_n^*\| = \operatorname{dist}(x^*, B(X^*))$. We may even assume that $\{x_n^*\}_{n=1}^\infty \subset S(X^*)$, $x_n^* \neq x_m^*$, $n \neq m$.

Because $B(X^*)$ is a w^* compact set and $\{x_n^*\}$ is an infinite set, there exists $x_0^* \in B(X^*)$ such that x_0^* is w^* accumulation point of $\{x_n^*\}_{n=1}^\infty$. Let $\Delta = \{U_{x_0^*} : U_{x_0^*}$ is w^* neighborhood of point $x_0^*\}$ and define a partial order by inclusive relation. i.e. $U_{x_0^*} \subset V_{x_0^*}$ if and only if $U_{x_0^*} \succ V_{x_0^*}$, then Δ is a direct set. Furthermore, we construct a family of sets $\{U_{x_0^*} \cap \{x_n^*\}_{n=1}^\infty : U_{x_0^*}$ is w^* neighborhood of point $x_0^*\}$.

By Zermelo principle, there is a mapping f such that

$$f(U_{x_0^*} \cap \{x_n^*\}_{n=1}^\infty) \in U_{x_0^*} \cap \{x_n^*\}_{n=1}^\infty \subset \{x_n^*\}_{n=1}^\infty.$$

Put $x_\alpha^* = f(U_{x_0^*} \cap \{x_n^*\}_{n=1}^\infty)$, then $\{x_\alpha^*\}_{\alpha\in\Delta} \subset \{x_n^*\}_{n=1}^\infty$ is a net. From the structure of this net, we know that $x_\alpha^* \xrightarrow{w^*} x_0^*$.

Now we are going to prove that $x_0^* \in S(X^*)$. Suppose that $x_0^* \in B^\circ(X^*)$. Since $B(X^*)$ has $(w^*\mathrm{M})$ property and $\lim_{n\to\infty} \|x_n^* - x^*\| = \operatorname{dist}(x^*, B(X^*))$, there exists $x_0 \in S(X)$ and subsequence $\{x_{n_k}^*\}_{k=1}^\infty \subset \{x_n^*\}_{n=1}^\infty$ such that $\lim_{k\to\infty} x_{n_k}^*(x_0) = \sup_{z^*\in B(X^*)} |z^*(x_0)|$.

Without loss of generality, we may assume that $\lim_{n\to\infty} x_n^*(x_0) = \sup_{z^*\in B(X^*)} |z^*(x_0)| = R$. Since $x_0^* \in B^\circ(X^*)$, there exists $t > 1$ such that $tx_0^* \in B^\circ(X^*)$. Clearly,

$$r = |x_0^*(x_0)| < |tx_0^*(x_0)| \leqslant \sup_{z^*\in B(X^*)} |z^*(x_0)| = R.$$

This allows us to construct a w^* neighborhood of point x_0^* such as

$$\Big\{y^* : \ |y^*(x_0) - x_0^*(x_0)| < \frac{1}{2}(R-r)\Big\}.$$

Since $x_\alpha^* \xrightarrow{w^*} x_0^*$, there exists α_1 such that for $\alpha > \alpha_1$ we have

$$x_\alpha^* \in \Big\{y^* : \ |y^*(x_0) - x_0^*(x_0)| < \frac{1}{2}(R-r)\Big\}.$$

From $\lim_{n\to\infty} x_n^*(x_0) = R$ we know that $\{x_n^* : \ |x_n^*(x_0)| < \frac{1}{2}(R-r) + R\}$ is a finite set, so there exists a w^* neighborhood V of point x_0^* such that

$$\{x_n^* : \ |x_n^*(x_0)| < \frac{1}{2}(R-r) + R\} \cap V = \emptyset.$$

Since $x_\alpha^* \xrightarrow{w^*} x_0^*$, there exists α_2 such that for $\alpha > \alpha_2$ we have $x_\alpha^* \in V$. Take α_0 such that $\alpha_0 > \alpha_1$, $\alpha_0 > \alpha_2$, then we know that, for $\alpha > \alpha_0$, there holds

$$x_\alpha^* \in \Big\{y^* : \ |y^*(x_0) - x_0^*(x_0)| < \frac{1}{2}(R-r)\Big\} \cap V.$$

From $x_\alpha^* \in \left\{y^* : |y^*(x_0) - x_0^*(x_0)| < \frac{1}{2}(R-r)\right\}$ we know that $|x_0^*(x_0)| < \frac{1}{2}(R-r)+R$. Also, from $x_\alpha^* \in V$ we know that $|x_\alpha^*(x_0)| \geqslant \frac{1}{2}(R-r)+R$, it follows that $|x_0^*(x_0)| \geqslant \frac{1}{2}(R-r)+R$, a contradiction. Hence $x_0^* \in S(X^*)$.

Firstly, we will prove that $x_\alpha^* \to x_0^*$ and $\|x^* - x_0^*\| = \operatorname{dist}(x^*, B(X^*))$. If $x_\alpha^* \not\to x_0^*$, then there exists $\varepsilon_0 > 0$ such that, for any $\alpha \in \triangle$, there is $\beta > \alpha$ so that $\|x_\beta^* - x_0^*\| > \varepsilon_0$. Thus, we construct a subnet $\{x_\beta^*\}_{\beta \in \triangle_1 \subset \triangle}$ of net $\{x_\alpha^*\}_{\alpha \in \triangle}$ satisfying $\{x_\beta^*\}_{\beta \in \triangle_1} \cap B(x_0^*, \varepsilon_0) = \emptyset$. It follows that $\{x_\beta^*\} \subset B(X^*) \backslash B(x_0^*, \varepsilon_0)$. By the condition given here, we know that $x_0^* \in S(X^*)$ is w^* denting point of $B(X^*)$, this means that $x_0^* \notin \overline{\mathrm{co}}^{w^*}(B(X^*) \backslash B(x_0^*, \varepsilon_0))$. By separating theorem, we know that there exists $y_0 \in X$ such that

$$x_0^*(y_0) > \sup_{z^* \in \overline{\mathrm{co}}^{w^*}(B(X^*) \backslash B(x_0^*, \varepsilon_0))} z^*(y_0).$$

Moreover, we choose a scalar $b > 0$ such that

$$x_0^*(y_0) - \sup_{z^* \in \overline{\mathrm{co}}^{w^*}(B(X^*) \backslash B(x_0^*, \varepsilon_0))} z^*(y_0) > b.$$

Hence we have $x_0^*(y_0) - x_\beta^*(y_0) > b$. This leads to $x_\beta^* \overset{w^*}{\not\longrightarrow} x_0^*$. By $x_\alpha^* \xrightarrow{w^*} x_0^*$, we know that, for any w^* neighborhood V', there exists α' such that $x_\alpha^* \in V'$ when $\alpha > \alpha'$. From the structure of this subnet $\{x_\beta^*\}_{\beta \in \triangle_1}$, we know that there exists $\beta' > \alpha'$ such that for $\beta > \beta'$, there holds $x_\beta^* \in V'$. This means that $x_\beta^* \xrightarrow{w^*} x_0^*$, a contradiction. So we prove the fact that $x_\alpha^* \to x_0^*$. Consequently, by $x_\alpha^* \to x_0^*$ we know that x_0^* is accumulation point of $\{x_n^*\}_{n=1}^\infty$. Hence there exists $\{x_{n_k}^*\} \subset \{x_n^*\}$ such that $x_{n_k}^* \to x_0^*$. Thus we have

$$\operatorname{dist}(x^*, B(X^*)) \leqslant \|x^* - x_0^*\| \leqslant \|x^* - x_{n_k}^*\| + \|x_{n_k}^* - x_0^*\| \to \operatorname{dist}(x^*, B(X^*)).$$

It follows that $\|x^* - x_0^*\| = \operatorname{dist}(x^*, B(X^*))$.

Secondly, we will prove that each $x^* \in X^* \backslash B(X^*)$ has a unique proximinal point in $B(X^*)$. Suppose that there exists an another point $y_0^* \in B(X^*)$ such that

$$\|x^* - x_0^*\| = \|x^* - y_0^*\| = \operatorname{dist}(x^*, B(X^*)).$$

Repeating the same procedure as proving $x_0^* \in S(X^*)$, we get $y_0^* \in S(X^*)$. Since $B(X^*)$ is closed subset of X^*, this leads to $\frac{1}{2}(x_0^* + y_0^*) \in B(X^*)$ and

$$\operatorname{dist}(x^*, B(X^*)) \leqslant \|x^* - \frac{1}{2}(x_0^* + y_0^*)\| \leqslant \frac{1}{2}\|x^* - x_0^*\| + \frac{1}{2}\|x^* - y_0^*\| = \operatorname{dist}(x^*, B(X^*)).$$

Hence $\frac{1}{2}(x_0^* + y_0^*) \in S(X^*)$. By the condition given here, we know that $\frac{1}{2}(x_0^* + y_0^*)$ is w^* denting point of $B(X^*)$.

Let

$$\varepsilon_1 = \left\|x_0^* - \frac{1}{2}(x_0^* + y_0^*)\right\| = \left\|\frac{1}{2}(x_0^* - y_0^*)\right\| = \left\|y_0^* - \frac{1}{2}(x_0^* + y_0^*)\right\|.$$

Clearly, $x_0^*, y_0^* \notin B(\frac{1}{2}(x_0^*+y_0^*), \frac{\varepsilon_1}{2})$, it follows that $x_0^*, y_0^* \in B(X^*)\backslash B(\frac{1}{2}(x_0^*+y_0^*), \frac{\varepsilon_1}{2})$. Hence

$$\frac{1}{2}(x_0^*+y_0^*) \in \operatorname{co}\left[B(X^*)\backslash B\left(\frac{1}{2}(x_0^*+y_0^*), \frac{\varepsilon_1}{2}\right)\right] \subset \overline{\operatorname{co}}^{w^*}\left[B(X^*)\backslash B\left(\frac{1}{2}(x_0^*+y_0^*), \frac{\varepsilon_1}{2}\right)\right].$$

This shows that $\frac{1}{2}(x_0^*+y_0^*)$ is not a w^* denting point of $B(X^*)$, a contradiction. Up to now, we have completed the proof that $B(X^*)$ is Chebyshev set.

(2) Let $x^* \in X^*$ and sequences $\{x_n^*\}, \{y_n^*\} \subset X^*$ satisfying $\lim\limits_{n\to\infty} \|x_n^* - x^*\| = 0$, $P_{B(X^*)}(x_n^*) = y_n^*$, $n = 1, 2, \ldots$.

Now we will prove that $\{y_n^*\}$ is a convergent sequence.

Since $\varphi(x^*) = \operatorname{dist}(x^*, B(X^*))$ is continuous functional on X^*, considering the following inequality

$$\|x^* - y_n^*\| \leqslant \|x^* - x_n^*\| + \|x_n^* - y_n^*\| = \|x^* - x_n^*\| + \operatorname{dist}(x_n^*, B(X^*)),$$

we have

$$\overline{\lim_{n\to\infty}} \|x^* - y_n^*\| \leqslant \operatorname{dist}(x^*, B(X^*)). \tag{2.1}$$

On the other hand, from the inequality $\|x^* - y_n^*\| \geqslant \operatorname{dist}\,(x^*, B(X^*))$, we also have

$$\underline{\lim_{n\to\infty}} \|x^* - y_n^*\| \geqslant \operatorname{dist}(x^*, B(X^*)). \tag{2.2}$$

Combining inequalities (2.1) and (2.2), we have

$$\lim_{n\to\infty} \|x^* - y_n^*\| = \operatorname{dist}(x^*, B(X^*)).$$

If dist $(x^*, B(X^*)) = 0$, then $\{y_n^*\}$ is a convergent sequence. If dist $(x^*, B(X^*)) \neq 0$, from the proof of (1), we know that $\{y_n^*\}$ has accumulation point y^*. Suppose that $y_n^* \not\to y^*$, then there exist $\eta > 0$ and subsequence $\{y_{n_k}^*\} \subset \{y_n^*\}$ such that $\|y_{n_k}^* - y^*\| \geqslant \eta$, but $\lim\limits_{k\to\infty} \|x^* - y_{n_k}^*\| = $ dist $(x^*, B(X^*))$. From the proof of (1), we know that $\{y_{n_k}^*\}$ has accumulation point y_1^*. Obviously, $y^* \neq y_1^*$. It is easy to prove that $\|x^* - y^*\| = \|x^* - y_1^*\| = $ dist $(x^*, B(X^*))$. This contradicts that $B(X^*)$ is Chebyshev set. So $y_n^* \to y^*$ and $P_{B(X^*)}(x^*) = y^*$.

If metric projection operator $P_{B(X^*)}: X^* \to 2^{B(X^*)}$ is not norm-norm continuous at some point x_0^*, then there exists $\zeta > 0$ and a sequence $\{x_n^*\}$ such that $\lim\limits_{n\to\infty} \|x_n^* - x_0^*\| = 0$, but $\|y_0^* - y_n^*\| \geqslant \zeta$, where $P_{B(X^*)}(x_n^*) = y_n^*$, $n = 1, 2, \ldots$ $P_{B(X^*)}(x_0^*) = y_0^*$. From the proof as above, we know that $y_n^* \to y_0^*$, a contradiction. Hence, metric projection operator $P_{B(X^*)}: X^* \to 2^{B(X^*)}$ is norm-norm continuous. $\square$

THEOREM 2.3. *Let X be a separable Banach space and X^* be its conjugate space. If x_0^*, y_0^* are both w^* exposed point and w^* denting point of $B(X^*)$, then (x_0^*, y_0^*) is w^* denting point of $B(X^* \times X^*)$.*

Proof. Because x_0^*, y_0^* are w^* denting point of $B(X^*)$, so $x_0^*, y_0^* \in S(X^*)$. It follows that $\|(x_0^*, y_0^*)\| = (\|x_0^*\|^2 + \|y_0^*\|^2)^{\frac{1}{2}} = 1$, this means that $(x_0^*, y_0^*) \in S(X^* \times X^*)$. Since

x_0^* is w^* exposed point of $B(X^*)$, there exists $x_0 \in S(X)$ such that $x_0^*(x_0) > x^*(x_0)$ for all functional $x^* \in B(X^*)\backslash\{x_0^*\}$. Let $x_n^* \subset B(X^*)$ such that $x_n^*(x_0) \to x_0^*(x_0)$. Since X is separable Banach space, X has a countable dense subset. Let us denote a countable dense subset in X by $\{x_i\}_{i=1}^{\infty}$. Because $B(X^*)$ is norm-bounded subset, $\{x_n^*(x_i)\}_{n=1}^{\infty}$ is a bounded sequence of numbers for any i. By diagonal rule, there exists subsequence $\{x_{n_k}^*\} \subset \{x_n^*\}$, such that, for any i, $\{x_{n_k}^*(x_i)\}$ to be a Cauchy sequence of numbers.

For any $x \in X$ and $\varepsilon > 0$, there exists $x_i \in \{x_i\}_{i=1}^{\infty}$ such that $\|x_i - x\| \leqslant \frac{\varepsilon}{3}$ and there exists $K > 0$ such that $|(x_{n_{k_1}}^* - x_{n_{k_2}}^*)(x_i)| < \frac{\varepsilon}{3}$ when $k_1, k_2 > K$. Therefore,

$$\begin{aligned}
|(x_{n_{k_1}}^* - x_{n_{k_2}}^*)(x_i)|| &\leqslant |x_{n_{k_1}}^*(x - x_i)| + |(x_{n_{k_1}}^* - x_{n_{k_2}}^*)(x_i)| + |x_{n_{k_2}}^*(x - x_i)| \\
&\leqslant \|x_{n_{k_1}}^*\|\|x - x_i\| + |(x_{n_{k_1}}^* - x_{n_{k_2}}^*)(x_i)| + \|x_{n_{k_2}}^*\|\|x - x_i\| \\
&\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

This shows that, for any $x \in X$, $\{x_{n_k}^*(x)\}_{n=1}^{\infty}$ is a Cauchy sequence of numbers.

For $x \in X$, define $z^*(x) = \lim\limits_{k\to\infty} x_{n_k}^*(x)$. Then, it is obvious that z^* is a linear functional on X and $\|z^*\| \leqslant \lim\limits_{k\to\infty} \|x_{n_k}^*\| \leqslant 1$. Hence, $z^* \in X^*$ and $x_{n_k}^* \xrightarrow{w^*} z^*$. Since $x_n^*(x_0) \to x_0^*(x_0)$, we have $z^*(x_0) = x_0^*(x_0)$. Because $B(X^*)$ is w^* closed subset, so $z^* \in B(X^*)$. Since $x_0^*(x_0) > x^*(x_0)$ holds for all $x^* \in B(X^*)\backslash\{x_0^*\}$, it is easy to see that $z^* = x_0^*$. This means that $x_{n_k}^* \xrightarrow{w^*} x_0^*$.

Now we are going to prove that $x_n^* \to x_0^*$. Suppose that $x_n^* \not\longrightarrow x_0^*$, then there exist $\varepsilon_0 > 0$ and subsequence $\{x_{n_k}^*\}_{k=1}^{\infty} \subset \{x_n^*\}_{n=1}^{\infty}$ such that $\|x_{n_k}^* - x_0^*\| \geqslant \varepsilon_0$. Since x_0^* is w^* denting point of $B(X^*)$, by separating theorem, there exist $x_1 \in X$ and scalar r such that

$$x_0^*(x_1) > r > \sup_{x^* \in \overline{\mathrm{co}}^{w^*}(B(X^*)\backslash B(x_0^*,\varepsilon_0))} x^*(x_1).$$

In view of $\|x_{n_k}^* - x_0^*\| \geqslant \varepsilon_0$, we have $x_{n_k}^* \in B(X^*)\backslash B(x_0^*, \varepsilon_0)$. It follows that there exists a scalar $t > 0$ such that $x_0^*(x_1) - x_{n_k}^*(x_1) > t$. This shows that any subsequence of $\{x_{n_k}^*\}$ does not w^* convergent to x_0^*. On the other hand, repeating the proof procedure as above, we know that there exists a subsequence $\{x_{n_{k_l}}^*\} \subset \{x_{n_k}^*\}$ such that $x_{n_{k_l}}^* \xrightarrow{w^*} x_0^*$. A contradiction. Hence, when $x_n^*(x_0) \to x_0^*(x_0)$, there must be $x_n^* \to x_0^*$. Therefore, for any $\varepsilon > 0$, there exists a scalar $b > 0$ such that for $x^* \in B(X^*)\backslash B(x_0^*, \varepsilon_0)$, there holds $x_0^*(x_0) > x^*(x_0) + b$. Otherwise, there exist $x_n^* \in B(X^*)\backslash B(x_0^*, \varepsilon_0)$ such that $\lim\limits_{n\to\infty} x_n^*(x_0) = x_0^*(x_0)$. It follows that $x_n^* \to x_0^*$. This contradicts that $x_n^* \in B(X^*)\backslash B(x_0^*, \varepsilon_0)$. So we have proved that, for any $\varepsilon > 0$, there exists a scalar $b > 0$ such that

$$x_0^*(x_0) - b > \sup_{x^* \in \overline{\mathrm{co}}^{w^*}(B(X^*)\backslash B(x_0^*,\varepsilon_0))} x^*(x_0).$$

Similarly, for any $\varepsilon > 0$, there exists a scalar $b' > 0$ such that

$$y_0^*(y_0) - b' > \sup_{x^* \in \overline{\mathrm{co}}^{w^*}(B(X^*)\backslash B(x_0^*,\varepsilon_0))} x^*(y_0).$$

For any $(x^*,y^*)\in X^*\times X^*$, define $(x_0,y_0)((x^*,y^*))=x^*(x_0)+y^*(y_0)$. Clearly, (x_0,y_0) is linear functional on $X^*\times X^*$ and

$$\begin{aligned}|(x_0,y_0)((x^*,y^*))| &= |x^*(x_0)+y^*(y_0)|\\ &\leqslant |x^*(x_0)|+|y^*(y_0)|\\ &\leqslant \|x^*\|\|x_0\|+\|y^*\|\|y_0\|\\ &\leqslant (\|x_0\|^2+\|y_0\|^2)^{\frac{1}{2}}\cdot(\|x^*\|^2+\|y^*\|^2)^{\frac{1}{2}}\\ &= (\|x_0\|^2+\|y_0\|^2)^{\frac{1}{2}}\|(x^*,y^*)\|.\end{aligned}$$

Hence (x_0,y_0) is linear continuous functional on $X^*\times X^*$. On the other hand, since $X^*\times X^*$ and $(X\times X)^*$ are linearly homeomorphic, this leads to that (x_0,y_0) is linear continuous functional on $(X\times X)^*$. By $(x_0,y_0)\in X\times X$, we know that (x_0,y_0) is continuous functional under the topology $((X\times X)^*,w^*)$.

For any $\varepsilon>0$ and $B((x_0^*,y_0^*),\varepsilon)$, it is easy to see that there exists $\varepsilon_1>0$ such that $B(x_0^*,\varepsilon_1)\times B(y_0^*,\varepsilon_1)\subset B((x_0^*,y_0^*),\varepsilon)$. Hence, if $(x^*,y^*)\in B(X^*)\times B(X^*)\backslash B((x_0^*,y_0^*),\varepsilon)$, then we have

$$(x^*,y^*)\in B(X^*)\times B(X^*)\backslash B(x_0^*,\varepsilon_1)\times B(y_0^*,\varepsilon_1).$$

Without loss of generality, we may assume that $x^*\in B(X^*)\backslash B(x_0^*,\varepsilon_1)$, then

$$(x_0,y_0)((x_0^*,y_0^*))-(x_0,y_0)((x^*,y^*))=x_0^*(x_0)-x^*(x_0)+y_0^*(y_0)-y^*(y_0)>b.$$

Therefore

$$\begin{aligned}(x_0,y_0)((x_0^*,y_0^*)) &> \sup_{(x^*,y^*)\in B(X^*)\times B(X^*)\backslash B(x_0^*,\varepsilon_1)\times B(y_0^*,\varepsilon_1)}(x_0,y_0)((x^*,y^*))\\ &\geqslant \sup_{(x^*,y^*)\in B(X^*)\times B(X^*)\backslash B((x_0^*,y_0^*),\varepsilon)}(x_0,y_0)((x^*,y^*))\\ &= \sup_{(x^*,y^*)\in \mathrm{co}^{w^*}(B(X^*)\times B(X^*)\backslash B((x_0^*,y_0^*),\varepsilon))}(x_0,y_0)((x^*,y^*))\\ &= \sup_{(x^*,y^*)\in \overline{\mathrm{co}}^{w^*}(B(X^*)\times B(X^*)\backslash B((x_0^*,y_0^*),\varepsilon))}(x_0,y_0)((x^*,y^*))\\ &= \sup_{(x^*,y^*)\in \overline{\mathrm{co}}^{w^*}(B(X^*)\times B(X^*)\backslash B((x_0^*,y_0^*),\varepsilon))}(x_0,y_0)((x^*,y^*)).\end{aligned}$$

It follows that $(x_0^*,y_0^*)\notin\overline{\mathrm{co}}^{w^*}(B(X^*\times X^*)\backslash B((x_0^*,y_0^*),\varepsilon))$; i.e. (x_0^*,y_0^*) is w^* denting point of $B(X^*\times X^*)$. $\square$

THEOREM 2.4. *Let X be a Banach space and X^* be its conjugate space. Then the following hold:*

(1) If $x_0^*\in S(X^*)$ is quasi-w^* nearly denting point of $B(X^*)$ and there exists support functional x^{**} of unit sphere $S(X^*)$ at point x_0^* such that $x^{**}\in J_X(X)$, then for any $\varepsilon>0$, there exists a slice $S(x^{**},\lambda,B(X^*))\subset B(H_{x^{**}},\varepsilon)$;

(2) If $x_0^*\in S(X^*)$, $x^{**}\in J_X(X)$ is support functional of unit sphere $S(X^*)$ at point x_0^* and for any $\varepsilon>0$ there exists a slice $S(x^{**},\lambda,B(X^*))\subset B(H_{x^{**}},\varepsilon)$, then

$H_{x^{**}} \cap \overline{\text{co}}^{w^*}(B(X^*)\backslash B(H_{x^{**}},\varepsilon)) = \emptyset$. Where $H_{x^{**}}$ and $B(H_{x^{**}},\varepsilon)$ are the same as symbols defined in Definition 2.1.

Proof. If x_0^* is quasi-w^* nearly denting point of $B(X^*)$ and there is support functional x^{**} of unit sphere $S(X^*)$ at point x_0^* such that $x^{**} \in J_X(X)$. Then by Definition 2.1 we know that, for any $\varepsilon > 0$, there holds

$$H_{x^{**}} \cap \overline{\text{co}}^{w^*}(B(X^*)\backslash B(H_{x^{**}},\varepsilon)) = \emptyset.$$

Since $H_{x^{**}}$ is w^* closed subset of X^* and $B(X^*)$ is norm-bounded subset of X^*, so $H_{x^{**}}$ and $\overline{\text{co}}^{w^*}(B(X^*)\backslash B(H_{x^{**}},\varepsilon))$ are both w^* compact set. Hence, there exist scalar $\lambda_0 > 0$ such that, for all $u^* \in H_{x^{**}}$, there holds

$$x^{**}(u^*) > x^{**}(u^*) - \lambda_0 > \sup_{x^* \in \overline{\text{co}}^{w^*}(B(X^*)\backslash B(H_{x^{**}},\varepsilon))} x^{**}(x^*). \tag{2.3}$$

Since $x_0^* \in H_{x^{**}}$, it follows from inequality (2.3) that

$$x^{**}(x_0^*) - \lambda_0 > \sup_{x^* \in \overline{\text{co}}^{w^*}(B(X^*)\backslash B(H_{x^{**}},\varepsilon))} x^{**}(x^*). \tag{2.4}$$

Let $\lambda = \sup_{x^* \in B(X^*)} x^{**}(x^*) - x^{**}(x_0^*) + \lambda_0$, then $x^{**}(x_0^*) > \sup_{x^* \in B(X^*)} x^{**}(x^*) - \lambda$. It follows that $x_0^* \in S(x^{**},\lambda,B(X^*))$.

Now we will prove that $S(x^{**},\lambda,B(X^*)) \subset B(H_{x^{**}},\varepsilon)$. For any $z^* \in S(x^{**},\lambda,B(X^*))$, we have

$$\begin{aligned} x^{**}(z^*) &\geqslant \sup_{x^* \in B(X^*)} x^{**}(x^*) - \lambda \\ &= \sup_{x^* \in B(X^*)} x^{**}(x^*) - \sup_{x^* \in B(X^*)} x^{**}(x^*) + x^{**}(x_0^*) - \lambda_0. \\ &= x^{**}(x_0^*) - \lambda_0. \end{aligned}$$

From the inequality (2.4) we know that $z^* \notin \overline{\text{co}}^{w^*}(B(X^*)\backslash B(H_{x^{**}},\varepsilon))$. It follows that $z^* \in B(H_{x^{**}},\varepsilon)$. By the arbitrary of $z^* \in S(x^{**},\lambda,B(X^*))$, we obtained the desired result that $S(x^{**},\lambda,B(X^*)) \subset B(H_{x^{**}},\varepsilon)$.

(2) Let $x_0^* \in S(X^*)$, $x^{**} \in J_X(X)$ be a support functional of unit sphere $S(X^*)$ at point x_0^*. Suppose that for any $\varepsilon > 0$, there exists a slice $S(x^{**},\lambda,B(X^*)) \subset B(H_{x^{**}},\varepsilon)$. Clearly,

$$x^{**}(x_0^*) \geqslant \sup_{x^* \in B(X^*)} x^{**}(x^*) - \lambda,$$

this means that $x_0^* \in S(x^{**},\lambda,B(X^*))$. For arbitrary element $x^* \in B(X^*)\backslash B(H_{x^{**}},\varepsilon)$, from the given condition that $S(x^{**},\lambda,B(X^*)) \subset B(H_{x^{**}},\varepsilon)$, we have

$$\begin{aligned} B(X^*)\backslash B(H_{x^{**}},\varepsilon) &\subset B(X^*)\backslash S(x^{**},\lambda,B(X^*)) \\ &\subset \{y^* : y^* \in B(X^*), x^{**}(y^*) < \sup_{x^* \in B(X^*)} x^{**}(x^*) - \lambda\}. \end{aligned}$$

Moreover, it is easy to see that

$$\mathrm{co}\,(B(X^*)\backslash B(H_{x^{**}},\varepsilon)) \subset \{y^* : y^* \in B(X^*), x^{**}(y^*) < \sup_{x^* \in B(X^*)} x^{**}(x^*) - \lambda\}.$$

Since x^{**} is a continuous functional under the topology (X^*,w^*), we have

$$\overline{\mathrm{co}}^{w^*}(B(X^*)\backslash B(H_{x^{**}},\varepsilon)) \subset \{y^* : y^* \in B(X^*), x^{**}(y^*) < \sup_{x^* \in B(X^*)} x^{**}(x^*) - \lambda\}. \quad (2.5)$$

Combining $x_0^* \in S(x^{**},\lambda,B(X^*))$ and inequality (2.5), we have

$$\{x_0^*\} \cap \overline{\mathrm{co}}^{w^*}(B(X^*)\backslash B(H_{x^{**}},\varepsilon)) = \emptyset.$$

Similarly, for arbitrary element $u^* \in H_{x^{**}}$, we can prove that

$$\{u^*\} \cap \overline{\mathrm{co}}^{w^*}(B(X^*)\backslash B(H_{x^{**}},\varepsilon)) = \emptyset.$$

So we prove the desired result that

$$H_{x^{**}} \cap \overline{\mathrm{co}}^{w^*}(B(X^*)\backslash B(H_{x^{**}},\varepsilon)) = \emptyset. \quad \square$$

COROLLARY 2.5. *Let X be a Banach space and X^* be its conjugate space. Then the following hold: If $x_0^* \in S(X^*)$ is quasi-w^* nearly denting point of $B(X^*)$, then for any support functional x^{**} of unit sphere $S(X^*)$ at point x_0^*, there exists a slice $S(x^{**},\lambda,B(X^*)) \subset B(H_{x^{**}},\varepsilon)$; Conversely, If $x_0^* \in S(X^*)$ and for any support functional x^{**} of unit sphere $S(X^*)$ at point x_0^*, there exists a slice $S(x^{**},\lambda,B(X^*)) \subset B(H_{x^{**}},\varepsilon)$, then x_0^* is quasi-w^* nearly denting point of $B(X^*)$. Where $H_{x^{**}}$ and $B(H_{x^{**}},\varepsilon)$ are the same as symbols defined in Definition 2.1.*

REMARK 2.6. Let X be a reflexive Banach space, then quasi-w^* nearly dentale spaces coincide with w^* nearly dentale spaces.

In fact, if X is a reflexive Banach space, then $X^{**} = X$. Therefore, it can considered that the points of X^{**} and X are the same. We denote $x^{**} = x$. Suppose that $x_0^* \in S(X^*)$ and x^{**} is support functional of unit sphere $S(X^*)$ at point x_0^*, then we have

$$\begin{aligned} H_{x^{**}} &= \{x^* :\, x^* \in B(X^*),\, x^{**}(x^*) = x^{**}(x_0^*) = 1\} \\ &= \{x^* :\, x^* \in B(X^*),\, x^*(x) = x_0^*(x) = 1\} \\ &= \{x^* :\, x^* \in S(X^*),\, x^*(x) = x_0^*(x) = 1\} \\ &= A(x). \end{aligned}$$

Hence, for any $\varepsilon > 0$, set $B(H_{x^{**}},\varepsilon)$ is open set containing $A(x)$. These mean that quasi-w^* nearly dentale spaces coincide with w^* nearly dentale spaces.

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