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ERRATUM TO *SPECTRAL ANALYSIS OF CERTAIN SPHERICALLY HOMOGENEOUS GRAPHS*, OPERATORS AND MATRICES, 7 (2013), 825—847

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Abstract. We correct an error in Theorem 2.6 in [1], pointed out to us by Ram Band and Gregory Berkolaiko. We note that this error does not affect any of the examples treated in the paper.

In this erratum we correct a mistake in Theorem 2.6 in [1] which does not hold under the assumption as stated. We note that while the notion of family preserving graphs includes examples for which the decomposition algorithm described in that theorem fails to hold, all examples treated in the paper are correct. We start by elaborating on the error and secondly present a modified definition under which the statement of Theorem 2.6 is correct.

Using the notation in that paper, the error lies in the fact that the proof relies on the existence of a basis of W_1 (and similarly W_ℓ for all $\ell \geqslant 1$) whose members are all mutual eigenvectors of the operators $\Lambda_{n,+j}$, V_n, and $E_n^T E_{n+1}^T \cdots E_{n+k}^T V_{n+k} E_{n+k} \cdots E_{n+1} E_n$ ($j,k = 1,2,\ldots$). While [1, Lemma 3.2] guarantees the commutativity of $\Lambda_{n,+j}$ and $E_n^T E_{n+1}^T \cdots E_{n+j}^T V_{n+j} E_{n+j} \cdots E_{n+1} E_n$ (with the same j), the rest of the operators in this set do not necessarily commute with each other and so there does not necessarily exist such a basis.

This error was pointed out to us by Ram Band and Gregory Berkolaiko, who also provided an example for a path commuting and family preserving graph for which there does not exist such a basis (in the case of the sphere of radius 1 around the root). With their permission it is presented in Figure 1.

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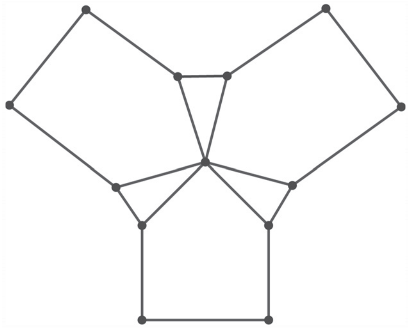

Figure 1: *A rooted family preserving graph for which the decomposition algorithm described in* [1, Theorem 2.6] *fails. We thank Ram Band and Gregory Berkolaiko for this example.*

As mentioned above this error is not relevant for any of the examples treated in the paper, as we shall explain below. But first, in order to fix the error, we modify the definitions of 'path commuting graph' and of 'family preserving graph'. In order to avoid confusion with the old definitions, we will use the prefix 'R' (for 'reinforced') in our new definitions. In everything that follows, we use the notation and terminology of [1].

DEFINITION 1. Let $G=(V,E)$ be a rooted graph with root o, and let $n\in\mathbb{Z}_+=\{0,1,\ldots\}$.

(i) For $k\in\mathbb{Z}$ a path of type $\mathbf{I}_k$, based at $S_n=\{v\in V \mid d(v,o)=n\}=$(the sphere of radius n about o), is a path $(v_0,\ldots,v_{2|k|})$ such that $v_0,v_{2|k|}\in S_n$ and $v_{|k|}\in S_{n+k}$. Thus, a path of type $\mathbf{I}_k$ starts at S_n and takes $|k|$ steps moving away from/towards the root and then takes k steps towards/away from the root to end up back in S_n.

(ii) For $k\in\mathbb{Z}_+$ a path of type $\mathbf{II}_k$, based at S_n, is a path $(v_0,\ldots,v_{2k+1})$ such that $v_0,v_{2k+1}\in S_n$ and $v_k,v_{k+1}\in S_{n+k}$. Thus, a path of type $\mathbf{II}_k$ is a path that moves away from the root for k steps, then takes a step within S_{n+k} and finally takes k steps to return to S_n. Note that $k=0$ means the path consists of one step within S_n.

Given two paths $(v_0,\ldots,v_k)$ and $(w_0,\ldots,w_\ell)$ such that $v_k=w_0$, their *concatenation* is the path $(v_0,\ldots,v_k,w_1,\ldots,w_\ell)$.

DEFINITION 2. Let $G=(V,E)$ be a rooted graph $n\in\mathbb{Z}_+$ and $x,y\in S_n$. For A,B being $\mathbf{I}$ or $\mathbf{II}$ and $k,\ell\in\mathbb{Z}$ let $P_{A_k,B_\ell}(x,y)$ be the set of all paths obtained as concatenations of a path of type A_k beginning at x and a path of type B_ℓ ending at y.

We say that G is *R-path commuting* if for any n, any $x,y\in S_n$ and any values of A,B,k,ℓ,

$$\#P_{A_k,B_\ell}(x,y)=\#P_{A_k,B_\ell}(y,x)$$

(i.e. the numbers of paths in these sets are equal).

The definition of ‘strongly path commuting’ is modified in the obvious way: a rooted graph is *R-strongly path commuting* if it is R-path commuting and the degree of a vertex is a function of its distance from the root.

REMARK. To put these notions into perspective with the notation in the paper, observe the following correspondence:

- A path of type $\mathbf{I}_k$ is a k-forward path for $k>0$ and a k-backward path for $k<0$.
- A concatenation of two paths of type $\mathbf{I}_k$ and $\mathbf{I}_\ell$ is a k,ℓ-forward-backward path if $k>0$ and $\ell<0$ and a k,ℓ-backward-forward path if $k<0$ and $\ell>0$.
- A concatenation of two paths of type $\mathbf{I}_k$ and $\mathbf{II}_0$ is a tailed-k-forward path if $k>0$ and a tailed-k-backward path if $k<0$.
- A concatenation of two paths of type $\mathbf{II}_0$ and $\mathbf{I}_k$ is a headed-k-forward path if $k>0$ and a headed-k-backward path if $k<0$.

The difference between path commuting and R-path commuting is that we now also consider concatenation of paths in $\mathbf{I}_k$ and $\mathbf{II}_\ell$ with $\ell>0$.

PROPOSITION 3. *Let G be a rooted graph. If G is R-path commuting then G is path commuting.*

Proof. Fix $x,y\in S_n$ and $k,\ell\in\mathbb{N}$. The set $P_{\mathbf{I}_k,\mathbf{I}_{-\ell}}(x,y)$ is precisely the set of k,ℓ-forward-backward paths from x to y. Since G is R-path commuting

$$\#P_{\mathbf{I}_k,\mathbf{I}_{-\ell}}(x,y)=\#P_{\mathbf{I}_k,\mathbf{I}_{-\ell}}(y,x).$$

But $P_{\mathbf{I}_k,\mathbf{I}_{-\ell}}(y,x)$ is the set of ℓ,k-backward-forward paths from x to y (by the bijection between paths from x to y and paths from y to x in the reverse direction). The other equalities in the definition of path commuting graphs follow in a similar way. □

The reason the definition given above fixes the problem is summarized in the following lemma.

LEMMA 4. *Let G be an R-path commuting rooted graph. Then, with the notation of Section 3 of the paper, for any n, the operators $\Lambda_{n,\pm j}$, V_n, and $E_n^TE_{n+1}^T\cdots E_{n+k}^TV_{n+k}E_{n+k}\cdots E_{n+1}E_n$ $(j,k=1,2,\dots)$ all commute with each other.*

Proof. For any $x,y\in S_n$ and j,k,

$$\begin{aligned}
&E_n^TE_{n+1}^T\cdots E_{n+k}^TV_{n+k}E_{n+k}\cdots E_{n+1}E_n\Lambda_{n,+j}(x,y)\\
&\qquad=\#P_{\mathbf{I}_j,\mathbf{II}_k}(x,y)=\#P_{\mathbf{I}_j,\mathbf{II}_k}(y,x)=\#P_{\mathbf{II}_k,\mathbf{I}_j}(x,y)\\
&\qquad\qquad=\Lambda_{n,+j}E_n^TE_{n+1}^T\cdots E_{n+k}^TV_{n+k}E_{n+k}\cdots E_{n+1}E_n(x,y),
\end{aligned}$$

where in the third equality we used ‘time’ reversal symmetry. The other commutation relations can be shown similarly. □

Now the proof of [1, Theorem 2.6] for R-path commuting graphs follows as in the paper and so the modification to the statement of that theorem is that 'path commuting' should be replaced with 'R-path commuting'.

Note that if G is a graph such that for every n there are no edges within S_n, then the definitions of R-path commuting and path commuting are the same (this follows, e.g., by Lemma 3.1 in [1]). Thus, we define an *R-family preserving graph* to be a family preserving graph with no edges within spheres. Antitrees are thus R-family preserving graphs. Also, all the examples treated in [2] are R-family preserving.

Finally, we comment on the second main example of the paper: spherically symmetric trees modified so that certain spheres, S_n, are complete graphs. Clearly, for such graphs for any $x,y \in S_n$ there is a rooted graph isometry, τ such that $\tau(x) = y$ and $\tau(y) = x$. It follows immediately that such graphs are R-path commuting.

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