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CHARACTERIZATIONS OF HOPFIANS SPACES

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Abstract. A Banach space X is called Hopfian, if any bounded linear operator surjective is bijective. The existence of the Banach Hopfians spaces in infinite dimension was established by Gowers and Maury in 1993. In this note we obtain some characterizations of Banach spaces Hopfians by properties of the algebra of bounded linear operators $\mathscr{B}(X)$.

1. Introduction

Throughout this article, let X be a complex infinite dimensional Banach space and denote by $\mathscr{B}(X)$ the Banach algebra of all bounded linear operators on X. Let $T \in \mathscr{B}(X)$, we denote by T^*, $R(T)$, $N(T)$, $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ respectively the adjoint, the range, the kernel, the resolvent set, the spectrum, the point spectrum, the approximate point spectrum and the surjectivity spectrum of T. An operator $T \in \mathscr{B}(X)$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$, abbreviated T has the SVEP at λ_0, if for every neighbourhood $\mathscr{U}$ of λ_0 the only analytic function $f : \mathscr{U} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ is the constant function $f \equiv 0$. For an arbitrary operator $T \in \mathscr{B}(X)$ let $S(T) = \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\}$. Note that $S(T)$ is open and is contained in the interior of the point spectrum $\sigma_p(T)$. An operator T is said to have the SVEP if $S(T)$ is empty (for detail see ([1, 8, 10]).

An operator $T \in \mathscr{B}(X)$ is upper semi-Fredholm (respectively lower semi-Fredholm) if $R(T)$ is closed and $\dim N(T)$ (respectively $\operatorname{codim} R(T)$) is finite. If T is upper or lower semi-Fredholm, then T is called semi-Fredholm. The index of such an operator is given by $\operatorname{ind}(T) = \dim N(T) - \operatorname{codim} R(T)$, and when it is finite we say that T is Fredholm.

An operator $T \in \mathscr{B}(X)$ is upper semi-Weyl (respectively lower semi-Weyl) if T is upper semi-Fredholm (respectively lower semi-Fredholm) and $\operatorname{ind}(T) \leqslant 0$ (respectively $\operatorname{ind}(T) \geqslant 0$). If T is upper or lower semi-Weyl, then T is called Weyl operator.

Recall that the descent and the ascent of T are $d(T) = \min\{q : R(T^q) = R(T^{q+1})\}$ and $a(T) = \min\{q : N(T^q) = N(T^{q+1})\}$ respectively, where if either of the above two sets is empty, its infimum is then defined as ∞, see for example ([1], [9] and [10]). If the ascent and the descent of T are both finite, then $a(T) = d(T) = p$, $X = N(T)^p \oplus$

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$R(T)^p$ and $R(T)^p$ is closed. Similarly we define the essential descent and the essential ascent of T by, $d_e(T) := \inf\{n \in \mathbb{N} : \dim(R(T^n)/R(T^{n+1})) < \infty\}$ and $a_e(T) := \inf\{n \in \mathbb{N} : \dim(N(T^{n+1})/N(T^n)) < \infty\}$ respectively, where the infimum over the empty set is taken to be infinite, see [2, 3].

We say that an operator $T \in \mathscr{B}(X)$ is upper semi-Browder if it is upper semi-Fredholm and has finite ascent. Similarly, T is lower semi-Browder if it is lower semi-Fredholm and has finite descent. An operator T is Browder if it is both lower and upper semi-Browder. Equivalently, this means that T is Fredholm and has finite both ascent and descent, see [10].

An operator T is said to be Drazin invertible, if there exists $S \in B(X)$ and some $m \in \mathbb{N}$ such that

$$T^m = T^m ST, \;\; STS = S \;\text{ and }\; ST = TS.$$

T will be called left Drazin invertible (respectively right Drazin invertible), if $a(T)$ is finite and $R(T^{a(T)+1})$ is closed, (respectively, if $d(T)$ is finite and $R(T^{d(T)})$ is closed), (see [2]).

For every bounded operator $T \in \mathscr{B}(X)$, let us define the upper semi-Weyl spectrum, the lower semi-Weyl spectrum, the Weyl spectrum, the ascent spectrum, the descent spectrum, the essential ascent spectrum, the essential descent spectrum, upper semi-Browder spectrum, lower semi-Browder spectrum, the Browder spectrum, the Drazin spectrum, the right Drazin spectrum, and the left Drazin spectrum of T as follows respectively:

$$\sigma_{lw}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not lower semi-Weyl }\}$$

$$\sigma_{w}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl }\}$$

$$\sigma_{asc}(T) = \{\lambda \in \mathbb{C} : a(T - \lambda) = \infty\}$$

$$\sigma_{desc}(T) = \{\lambda \in \mathbb{C} : d(T - \lambda) = \infty\}$$

$$\sigma^e_{asc}(T) = \{\lambda \in \mathbb{C} : a_e(T - \lambda) = \infty\}$$

$$\sigma^e_{desc}(T) = \{\lambda \in \mathbb{C} : d_e(T - \lambda) = \infty\}$$

$$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Browder }\}$$

$$\sigma_{lb}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not lower semi-Browder }\}$$

$$\sigma_{b}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder }\}$$

$$\sigma_{D}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin invertible }\}$$

$$\sigma_{rd}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not right Drazin }\}$$

$$\sigma_{ld}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left Drazin }\}$$

A Banach space X is said to be Hopfian if every surjective bounded linear operator $T : X \to X$ is bijective. In particular, any Banach space of finite dimension is Hopfian. The first example of Banach space of infinite size Hopfian was built by Gowers and Maury in 1993, (see [5]).

In [6], Haïly et al. have shown that X is a Hopfian space if and only if, for all $T \in \mathscr{B}(X)$, we have $\operatorname{int}(\sigma(T)) \subseteq \sigma_{desc}(T)$. In this work we will characterize the Hopfian spaces by the properties of the algebra of bounded linear operators $\mathscr{B}(X)$.

THEOREM 1. *Let X be a Banach space. Then the following assertions are equivalent:*

1. *X is Hopfian;*
2. *For every $T \in \mathscr{B}(X)$, if D is a closed subset of $\mathbb{C}$ such that $\sigma(T) = \sigma_{su}(T) \cup D$, we have that $int(D) \subseteq \sigma_{desc}(T)$;*
3. *For every $T \in \mathscr{B}(X)$ and for every connected component G of $\rho_{su}(T)$, we have that $G \cap \sigma(T) = \emptyset$;*
4. *For every $T \in \mathscr{B}(X)$ and for every connected component G of $\rho_{desc}(T)$, we have that $G \cap \rho(T) \neq \emptyset$.*

Proof. $1) \Rightarrow 2)$. Let $T \in \mathscr{B}(X)$ and $\lambda \notin \sigma_{des}(T)$, $T - \lambda$ has finite descent. Therefore, we may apply [4, Proposition 1.1] to find $\delta > 0$ such that, for every $\mu \in \mathbb{C}$ with $0 < |\mu - \lambda| < \delta$, $T - \mu$ becomes surjective. Since X is Hopfian, then $T - \mu$ is bijective. Then $\lambda \notin int(D)$.

$2) \Rightarrow 3)$. Let $T \in \mathscr{B}(X)$ and connected component G of $\rho_{su}(T)$. Since $\sigma(T) = \sigma_{su}(T) \cup S(T)$, then $S(T) \subseteq \sigma_{desc}(T)$. Therefore $G \cap S(T) = \emptyset$, this ensures that $G \cap \sigma(T) = \emptyset$.

$3) \Rightarrow 4)$. Let G connected component of $\rho_{desc}(T)$ and $\lambda \in G$. According to [4, Proposition 1.1], there is $\delta > 0$ such that, for every $\mu \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$, the operator $T - \mu$ is surjective. $D^*(\lambda;\delta) = \{\mu \in \mathbb{C} : 0 < |\lambda - \mu| < \delta\}$ is a connected subset of $\rho_{su}(T)$, then there exists a connected component G_0 of $\rho_{su}(T)$ contains $D^*(\lambda;\delta)$. By hypothesis, $G_0 \cap \sigma(T) = \emptyset$. Therefore $G \cap \rho(T) \neq \emptyset$.

$4) \Rightarrow 1)$. Let $T \in \mathscr{B}(X)$ be a surjective operator. Then there is a connected component G_0 of $\rho_{desc}(T)$ containing 0. By hypothesis, we have $G_0 \cap \rho(T) \neq \emptyset$. According to [4, Theorem 1.7], $G_0 \setminus \operatorname{pol}(T) \subseteq \rho(T)$ where $\operatorname{pol}(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is a pole of the resolvent of } T\}$. Consider first the case if $0 \notin \operatorname{pol}(T)$, then $0 \in \rho(T)$, hence T is bijective. On the other hand, if $0 \in \operatorname{pol}(T)$, then T has finite ascent $a(T)$. Since T is surjective, then $a(T) = d(T) = 0$. Therefore T is bijective. □

This theorem has interesting consequences:

COROLLARY 1. *Let X be a Banach space. Then the following assertions are equivalent:*

1. X *is Hopfian;*
2. *For every* $T \in \mathscr{B}(X)$, $S(T) \subseteq \sigma_{desc}(T)$*;*
3. *For every* $T \in \mathscr{B}(X)$, T *has SVEP at every* $\lambda \notin \sigma_{desc}(T)$.

Proof. $1) \Rightarrow 2)$. Since $\sigma(T) = \sigma_{su}(T) \cup S(T)$, Theorem 1 implies that $S(T) \subseteq \sigma_{desc}(T)$.

$2) \Rightarrow 3)$. Obvious.

$3) \Rightarrow 1)$. If $T \in \mathscr{B}(X)$ is surjective, by assumption 3) T has SVEP at 0. Then T is bijective. □

Since $\sigma(T) = \sigma_{su}(T) \cup \sigma_*(T)$ with $\sigma_* \in \{\sigma_{ap}, \sigma_p, \sigma_{ub}, \sigma_{uw}\}$, by Theorem 1, we deduce the followig result:

COROLLARY 2. *Let* X *be a Banach space. Then the following assertions are equivalent:*

1. X *is Hopfian;*
2. *For every* $T \in \mathscr{B}(X)$, $int(\sigma_{ap}(T)) \subseteq \sigma_{desc}(T)$*;*
3. *For every* $T \in \mathscr{B}(X)$, $int(\sigma_p(T)) \subseteq \sigma_{desc}(T)$*;*
4. *For every* $T \in \mathscr{B}(X)$, $int(\sigma_{ub}(T)) \subseteq \sigma_{desc}(T)$*;*
5. *For every* $T \in \mathscr{B}(X)$, $int(\sigma_{uw}(T)) \subseteq \sigma_{desc}(T)$.

PROPOSITION 1. *Let* X *be a Banach space. Then the following assertions are equivalent:*

1. X *is Hopfian;*
2. *For every* $T \in \mathscr{B}(X)$, $\sigma_{ld}(T) \subseteq \sigma_{desc}(T)$*;*
3. *For every* $T \in \mathscr{B}(X)$, $\sigma_{asc}(T) \subseteq \sigma_{desc}(T)$*;*
4. *For every* $T \in \mathscr{B}(X)$, $\sigma^e_{asc}(T) \subseteq \sigma_{desc}(T)$.

Proof. $1) \Rightarrow 2)$. Let $\lambda \notin \sigma_{desc}(T)$, then by [4, Proposition 1.1] there exists $\delta > 0$ such that for every $0 < |\mu - \lambda| < \delta$, $T - \mu$ surjective. Since X is Hopfian, then $T - \mu$ is bijective. Then λ is a pole of the resolvent of T. From [10, Theorem 22.10], the operator $T - \lambda$ is left Drazin invertible.

$2) \Rightarrow 3)$. Since $\sigma_{asc}(T) \subseteq \sigma_{ld}(T)$.

$3) \Rightarrow 4)$. Since $\sigma^e_{asc}(T) \subseteq \sigma_{asc}(T)$.

$4) \Rightarrow 1)$. Let $T \in \mathscr{B}(X)$ be a surjective operator. By hypothesis T has finite essential ascent. According to [10, Lemme 22.11], we have $a_e(T) = 0$. Then T has finite ascent. From [1, Theorem 3.3], we have $a(T) = d(T) = 0$. Therefore T is bijective, and this completes the proof. □

THEOREM 2. *Let X be a Banach space. Then the following assertions are equivalent:*

1. *X^* is Hopfian;*

2. *For every $T \in \mathscr{B}(X)$, if D is a closed subset of $\mathbb{C}$ such that $\sigma(T) = \sigma_{ap}(T) \cup D$, we have that $int(D) \subseteq \sigma_{ld}(T)$;*

3. *For every $T \in \mathscr{B}(X)$ and for every connected component G of $\rho_{ap}(T)$, we have that $G \cap \sigma(T) = \emptyset$;*

4. *For every $T \in \mathscr{B}(X)$ and for every connected component G of $\rho_{ld}(T)$, we have that $G \cap \rho(T) \neq \emptyset$.*

Proof. $1) \Rightarrow 2)$. Let $\lambda \notin \sigma_{ld}(T)$, according to [3, Corollary 2.4], there exists $\delta > 0$ such that for every $0 < |\mu - \lambda| < \delta$, $T - \mu$ is bounded below. Since X^* is Hopfian, then $T - \mu$ is bijective. Then $\lambda \notin int(D)$.

$2) \Rightarrow 3)$. Let $T \in \mathscr{B}(X)$ and connected component G of $\rho_{ap}(T)$. Since $\sigma(T) = \sigma_{ap}(T) \cup S(T^*)$, then $S(T^*) \subseteq \sigma_{ld}(T)$. Therefore $G \cap S(T^*) = \emptyset$. Thus $G \cap \sigma(T) = \emptyset$.

$3) \Rightarrow 4)$. Let G connected component of $\rho_{ld}(T)$ and $\lambda \in G$. According to [3, Corollary 2.4], there is $\delta > 0$ such that, for every $\mu \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$, the operator $T - \mu$ is bounded below. There exists a connected component G_0 of $\rho_{ap}(T)$ contains $D^*(\lambda; \delta)$. By the assumption, $G_0 \cap \sigma(T) = \emptyset$. Hence $G \cap \rho(T) \neq \emptyset$.

$4) \Rightarrow 1)$. Let $T \in \mathscr{B}(X)$ be a bounded below operator. Then there is a connected component G_0 of $\rho_{ld}(T)$ containing 0. By hypothesis, we have $G_0 \cap \rho(T) \neq \emptyset$. Let Ω be a connected component of $\rho^e_{ld}(T) = \{\lambda \in \mathbb{C} : a_e(T - \lambda)$ is finite and $R(T - \lambda)^{a_e(T-\lambda)+1}$is closed $\}$ containing G_0. By [3, Theorem 2.9], $\Omega \setminus$ pol$(T) \subseteq \rho(T)$. We distinguish two cases:

a) Let $0 \notin$ pol(T), then $0 \in \rho(T)$. Thus T is bijective

b) It remains the case $0 \in$ pol(T), then T has finite descent $d(T)$. Since T is bounded below, then $a(T) = d(T) = 0$. Therefore T is bijective. □

Using the previous Theorem, we can get easily the following result.

COROLLARY 3. *Let X be a Banach space. Then the following assertions are equivalent:*

1. *X^* is Hopfian;*

2. *For every $T \in \mathscr{B}(X)$, $S(T^*) \subseteq \sigma_{ld}(T)$;*

3. *For every $T \in \mathscr{B}(X)$, T^* has SVEP at every $\lambda \notin \sigma_{ld}(T)$.*

Using the equality $\sigma(T) = \sigma_{ap}(T) \cup \sigma_*(T)$ with $\sigma_* \in \{\sigma_{su}, \sigma_{lb}, \sigma_{lw}\}$ and by Theorem 2, we can state the following result:

COROLLARY 4. *Let X be a Banach space. Then the following assertions are equivalent:*

1. *X^* is Hopfian;*
2. *For every $T \in \mathscr{B}(X)$, $int(\sigma_{su}(T)) \subseteq \sigma_{ld}(T)$;*
3. *For every $T \in \mathscr{B}(X)$, $int(\sigma_{lb}(T)) \subseteq \sigma_{ld}(T)$;*
4. *For every $T \in \mathscr{B}(X)$, $int(\sigma_{lw}(T)) \subseteq \sigma_{ld}(T)$.*

PROPOSITION 2. *Let X be a Banach space. Then the following assertions are equivalent:*

1. *X^* is Hopfian;*
2. *For every $T \in \mathscr{B}(X)$, $\sigma_{rd}(T) \subseteq \sigma_{ld}(T)$;*
3. *For every $T \in \mathscr{B}(X)$, $\sigma_{desc}(T) \subseteq \sigma_{ld}(T)$;*
4. *For every $T \in \mathscr{B}(X)$, $\sigma^e_{desc}(T) \subseteq \sigma_{ld}(T)$.*

Proof. $1) \Rightarrow 2)$. Let $\lambda \notin \sigma_{ld}(T)$. By [3, Corollary 2.4], there exists $\delta > 0$ such that for every $0 < |\mu - \lambda| < \delta$, $T - \mu$ bounded below. Since X^* is Hopfian, then $T - \mu$ bijective. Then λ is a pole of the resolvent of T. From [10, Theorem 22.10], the operator $T - \lambda$ is left Drazin invertible.

$2) \Rightarrow 3)$ Since $\sigma_{desc}(T) \subseteq \sigma_{rd}(T)$.

$3) \Rightarrow 4)$ Since $\sigma^e_{desc}(T) \subseteq \sigma_{desc}(T)$.

$4) \Rightarrow 1)$ Let $T^* \in \mathscr{B}(X^*)$ be a surjective operator, then T is bounded below. By hypothesis T has finite essential ascent. According to [10, Lemma 22.11], we have $a_e(T) = 0$. Then T has finite ascent. From [1, Theorem 3.3], $a(T) = d(T) = 0$. Therefore T is bijective, and this completes the proof. □

REMARK 1. A Banach space X is indecomposable if there do not exist infinite-dimensional closed subspaces Y and Z of X with $X = Y \oplus Z$, and is hereditarily indecomposable (HI) if every closed subspace is indecomposable. The first hereditary indecomposable Banach spaces were built by Gowers and Maurey in [5], and their complex versions have the property that all their operators are a sum of scalar operators and strictly singular operators. Recall that a bounded operator T on X is strictly singular if the restriction of T to a subspace Y of infinite dimension is never an isomorphism of Y on $T(Y)$. From [5, Theorem 18] the spectrum of any bounded linear operator T from X into X is at most countable. So T and T^* have the SVEP. Then X and X^* are two Hopfian spaces.

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