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DUALITY OF GENERALIZED HARDY AND BMO SPACES ASSOCIATED WITH SINGULAR PARTIAL DIFFERENTIAL OPERATOR

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Abstract. First we define and study the generalized bounded mean ossilation space $\mathfrak{B}mo_\alpha$ associated with the Riemann-Liouville operator $\mathscr{R}_\alpha$. Next we prove the duality between $\mathfrak{B}mo_\alpha$ and the genralized Hardy space $\mathcal{H}^1_\alpha$ associated with $\mathscr{R}_\alpha$.

1. Introduction

The theory of Hardy spaces $H^p(\mathbb{R}^n)$ plays a very important role in harmonic analysis and operator theory and it's shown that it has many interesting applications in different fields. These spaces were studied and developed by of C. Fefferman, E. M. Stein [11], R. Coifman and G. Weiss [9]. There are many equivalent definitions of these spaces $H^p(\mathbb{R}^n)$ either by using the Poisson maximal function or by using the atomic decomposition or also by mean of Littlewood-Paley g-function [20]. In [1], the authors introduced the Riemann-Liouville operator and defined the generalized Hardy spaces $\mathcal{H}^p_\alpha$ connected with. They showed mainely the equivalence between the maximal function and the atomic decomposition definitions. The bounded mean oscillation spaces in the Euclidean setting emerged in the sixties and played a central role in both of harmonic analysis and partial differential equations. It's shown by Fefferman and Stein [11] that in the Euclidean case the BMO space is the dual of the Hardy space $H^1(\mathbb{R}^n)$.

In [5], Baccar, Ben Hamadi and Rachdi have considered the singular partial differential operators defined by

$$\begin{cases} \Delta_1 = \dfrac{\partial}{\partial x}, \\ \Delta_2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}; \quad (r,x) \in]0,+\infty[\times\mathbb{R}, \quad \alpha \geqslant 0 \end{cases}$$

and they associated to Δ_1 and Δ_2 the following integral transform $\mathscr{R}_\alpha$, called the Riemann-Liouville operator defined on $\mathscr{C}_e(\mathbb{R}^2)$ (The space of continuous functions on

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$\mathbb{R}^2$, even with respect to the first variable), by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}\,dt\,ds, & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)\dfrac{dt}{\sqrt{(1-t^2)}}; & \text{if } \alpha = 0. \end{cases}$$

The Riemann-Liouville operator $\mathscr{R}_\alpha$ generalizes the spherical mean operator given by

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\sin\theta, x+r\cos\theta)\,d\theta,$$

which plays an important role in image processing of the so-called synthetic aperture radar (SAR) data and in the linearized inverse scattering problem in acoustics as well as in the interpretation of many physical phenomena in quantum mechanics. [10, 12, 13].

According to [5], the Fourier transform $\mathscr{F}_\alpha$ associated with the Riemann-Liouville operator is defined by

$$\forall (s,y) \in \Upsilon,\ \mathscr{F}_\alpha(f)(s,y) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\mathscr{R}_\alpha\left(\cos(s.)e^{-iy.}\right)(r,x)\frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}drdx,$$

for a suitable integrable function, where Υ is a set that will be defined later.

Many harmonic analysis results have been already proved by Baccar, Ben Hamadi, Rachdi, Rouz and Omri for the Riemann-Liouville operator and its Fourier transform [3, 4, 5, 6, 7, 17]. Hleili, Mejjaoli, Omri and Rachdi have also established several uncertainty principles for the same Fourier transform $\mathscr{F}_\alpha$ [14, 16, 18, 19].

Our purpose in this paper is to define and to study the bounded mean oscillation space $\mathfrak{B}mo_\alpha$ associated with the Riemann-Liouville operator $\mathscr{R}_\alpha$ and to prove that $\mathfrak{B}mo_\alpha$ coincides with the dual of the genralized Hardy space $\mathcal{H}^1_\alpha$.

The paper is organized as follows. In the second section we give some classical harmonic analysis results related to the Riemann-Liouville operator, the third section is devoted to the characterization of the Hardy and BMO spaces related to $\mathscr{R}_\alpha$ by using its Poisson maximal function. In the last section we introduce the atomic decomposition which will allow us to characterize the space $\mathcal{H}^1_\alpha$ and to prove the main result of this work that is the duality between Hardy and BMO spaces.

2. Riemann-Liouville operator

In this section we give and develop some harmonic analysis results related to the Riemann-Liouville operator that we will use later. All theses results are well known and for more details about their proofs and globaly about the harmonic analysis related to the operator $\mathscr{R}_\alpha$, we refere the reader to [5, 7, 17].

In [5], Baccar, Ben Hamadi and Rachdi considered the following system

$$\begin{cases} \triangle_1 u = -i\lambda u(r,x) \\ \triangle_2 u = -\mu^2 u(r;x) \\ u(0,0) = 1,\ \frac{\partial u}{\partial x}(0,x), \forall x \in \mathbb{R} \end{cases}$$

and showed that for all (μ,λ), this system admits a unique infinitely differentiable solution given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha(r\sqrt{\mu^2+\lambda^2})e^{-i\lambda x},$$

where j_α is the modified Bessel function of first kind and index α, see [15, 22].

The function $\varphi_{\mu,\lambda}$ is bounded on $[0,+\infty[\times\mathbb{R}$ if and only if (μ,λ) belongs to the set

$$\Upsilon = \mathbb{R}^2 \cup \big\{(ir,x),\ (r,x)\in\mathbb{R}^2,\ |r|\leqslant|x|\big\}.$$

In this case we have

$$\sup_{(r,x)\in\mathbb{R}^2} |\varphi_{\mu,\lambda}(r,x)| = 1.$$

We introduce the following notations

- ν_α the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}drdx.$$

- $L^p(d\nu_\alpha)$, $p\in[1,+\infty]$, is the Lebesgue space of all measurable functions f on $[0,+\infty[\times\mathbb{R}$ such that $\|f\|_{p,\nu_\alpha}<+\infty$, where

$$\|f\|_{p,\nu_\alpha} = \begin{cases} \left(\displaystyle\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|^p d\nu_\alpha(r,x)\right)^{\frac{1}{p}}, & \text{if } p\in[1,+\infty[\\ \displaystyle\operatorname*{ess\,sup}_{(r,x)\in[0,+\infty[\times\mathbb{R}} |f(r,x)|, & \text{if } p=+\infty. \end{cases}$$

- $L^1_{loc}(d\nu_\alpha)$ the space of measurable functions on $[0,+\infty[\times\mathbb{R}$ that are locally integrable on $[0,+\infty[\times\mathbb{R}$ with respect to the measure ν_α.

According to [2], the eigenfunction $\varphi_{\mu,\lambda}$ satisfies the following product formula

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \Big(\varphi_{\mu,\lambda}(\sqrt{r^2+s^2+2rs\cos\theta},x+y)\Big)\sin^{2\alpha}\theta d\theta.$$

This allows us to define the translation operators as follows.

DEFINITION 2.1. For every $(r,x)\in[0,+\infty[\times\mathbb{R}$, the translation operator $\mathcal{T}_{(r,x)}$ associated with the operator $\mathscr{R}_\alpha$ is defined on $L^1(d\nu_\alpha)$ by, for all $(s,y)\in[0,+\infty[\times\mathbb{R}$,

$$\mathcal{T}_{(r,x)}(f)(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f(\sqrt{r^2+s^2+2rs\cos\theta},x+y)\sin^{2\alpha}\theta d\theta,$$

whenever the integral an the right hand side is well defined.

PROPOSITION 2.2. *Let f be in $L^1(d\nu_\alpha)$, then for all $(r,x)\in[0,+\infty[\times\mathbb{R}$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}}\mathcal{T}_{(r,x)}(f)(s,y)d\nu_\alpha(s,y)=\int_0^{+\infty}\int_{\mathbb{R}}f(s,y)d\nu_\alpha(s,y).$$

PROPOSITION 2.3. *For every $f\in L^p(d\nu_\alpha)$, $1\leqslant p\leqslant+\infty$, and for every $(r,x)\in[0,+\infty[\times\mathbb{R}$, the function $\mathcal{T}_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$ and we have*

$$\|\mathcal{T}_{(r,x)}(f)\|_{p,\nu_\alpha}\leqslant\|f\|_{p,\nu_\alpha}. \tag{2.1}$$

DEFINITION 2.4. The convolution product of two measurable functions f and g on $[0,+\infty[\times\mathbb{R}$ is defined by

$$f*g(r,x)=\int_0^{+\infty}\int_{\mathbb{R}}\mathcal{T}_{(r,-x)}(\check{f})(s,y)g(s,y)d\nu_\alpha(s,y),\ \ \forall(r,x)\in[0,+\infty[\times\mathbb{R},$$

where $\check{f}(s,y)=f(s,-y)$, whenever the integral an the right hand side is well defined.

THEOREM 2.5. *If $p,q,r\in[1,+\infty]$ are such that $\frac{1}{r}+1=\frac{1}{p}+\frac{1}{q}$ then, for every function $f\in L^p(d\nu_\alpha)$ and $g\in L^q(d\nu_\alpha)$, $f*g$ belongs to $L^r(d\nu_\alpha)$ and we have the Young's inequality*

$$\|f*g\|_{r,d\nu_\alpha}\leqslant\|f\|_{p,\nu_\alpha}\|g\|_{q,\nu_\alpha}.$$

DEFINITION 2.6. The Fourier transform $\mathscr{F}_\alpha$ associated with the operator $\mathscr{R}_\alpha$ is defined for every integrable function f on $[0,+\infty[\times\mathbb{R}$ with respect to the measure ν_α, by

$$\forall(\mu,\lambda)\in\Upsilon,\ \ \mathscr{F}_\alpha(f)(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x).$$

PROPOSITION 2.7.

(i) Let $f\in L^1(d\nu_\alpha)$ and $(r,x)\in[0,+\infty[\times\mathbb{R}$ we have

$$\forall(\mu,\lambda)\in\Upsilon,\ \ \mathscr{F}_\alpha(\mathcal{T}_{(r,-x)}(f))(\mu,\lambda)=\varphi_{\mu,\lambda}(r,x)\mathscr{F}_\alpha(f)(\mu,\lambda).$$

(ii) Let $f,g\in L^1(d\nu_\alpha)$, then we have

$$\forall(\mu,\lambda)\in\Upsilon,\ \ \mathscr{F}_\alpha(f*g)(\mu,\lambda)=\mathscr{F}_\alpha(f)(\mu,\lambda)\mathscr{F}_\alpha(g)(\mu,\lambda).$$

In the following we denote by

- Υ_+ the subspace of Υ given by

$$\Upsilon_+=[0,+\infty[\times\mathbb{R}\cup\big\{(ir,x),\ (r,x)\in[0,+\infty[\times\mathbb{R}\,|0\leqslant r\leqslant|x|\big\}.$$

- B_{Υ_+} the $\sigma-$algebra defined on Υ_+ by

$$B_{\Upsilon_+} = \{\theta^{-1}(B), B \in Bor([0,+\infty[\times\mathbb{R})\},$$

where $Bor([0,+\infty[\times\mathbb{R})\}$ is the usual Borel $\sigma-$ algebra on $[0,+\infty[\times\mathbb{R}$ and θ is the bijective function defined by

$$\begin{aligned}\theta : \Upsilon_+ &\longrightarrow [0,+\infty[\times\mathbb{R}\\ (\mu,\lambda) &\longmapsto (\sqrt{\mu^2+\lambda^2},\lambda).\end{aligned}$$

- γ_α the mesure defined on B_{Υ_+} by

$$\gamma_\alpha(A) = \nu_\alpha(\theta(A)),\ \forall A \in B_{\Upsilon_+}.$$

- $L^p(d\gamma_\alpha), p \in [1,+\infty]$ is the Lebesgue space on Υ_+ with respect to the measure γ_α equipped with the L^p-norm denoted by $\|.\|_{p,\gamma_\alpha}$, where

$$\|f\|_{p,\gamma_\alpha} = \begin{cases}\left(\displaystyle\int_{\Upsilon_+} |f(\mu,\lambda)|^p d\gamma_\alpha(\mu,\lambda)\right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty[\\ \operatorname*{ess\ sup}\limits_{(\mu,\lambda)\in\Upsilon_+} |f(\mu,\lambda)|, & \text{if } p = +\infty.\end{cases}$$

- $\mathscr{S}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreassing together with all their derivatives, even with respect the first variable. The space $\mathscr{S}_e(\mathbb{R}^2)$ is equipped with the topology associated to the following countable family of normes

$$\forall m \in \mathbb{N},\ \rho_m(\varphi) = \sup_{\substack{(r,x)\in[0,+\infty[\times\mathbb{R}\\ k+|\beta|\leqslant m}} (1+r^2+x^2)^k |D^\beta(\varphi)(r,x)|.$$

- $\mathscr{D}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$ with compact support, even with respect the first variable.

THEOREM 2.8. (Inversion formula for $\mathscr{F}_\alpha$) *Let $f \in L^1(d\nu_\alpha)$ such that $\mathscr{F}_\alpha(f)$ belongs to $L^1(d\gamma_\alpha)$. Then for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}$, we have*

$$f(r,x) = \int\int_{\Upsilon_+} \mathscr{F}_\alpha(f)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda).$$

THEOREM 2.9. (Plancherel's theorem) *The Fourier transform $\mathscr{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$.*

3. Generalized $\mathfrak{B}mo$ and Hardy spaces associated with the Riemann-Liouville operator

We introduce the following notations

- A cube of $[0,+\infty[\times\mathbb{R}$ denotes a subset of $[0,+\infty[\times\mathbb{R}$ of the form
$$Q=[a_0,b_0]\times[a_1,b_1],$$
where $b_0-a_0=b_1-a_1=L>0$.
- $\mathcal{C}$ the set of cubes of $[0,+\infty[\times\mathbb{R}$.

DEFINITION 3.1. Let $f\in L^1_{loc}(d\nu_\alpha)$ and $Q\in\mathcal{C}$. The average of the function f on Q is defined by
$$\underset{Q}{\mathrm{Avg}}(f)=\frac{1}{\nu_\alpha(Q)}\int_Q f(r,x)d\nu_\alpha(r,x).$$

DEFINITION 3.2. ($\mathfrak{B}mo_\alpha$ space) Let $f\in L^1_{loc}(d\nu_\alpha)$. The function f is said to be of bounded mean oscillation if f satisfies
$$\|f\|_{\mathfrak{B}mo_\alpha}=\sup_{Q\in\mathcal{C}}\frac{1}{\nu_\alpha(Q)}\int_Q|f(r,x)-\underset{Q}{\mathrm{Avg}}(f)|d\nu_\alpha(r,x)<+\infty.$$
We designate by $\mathfrak{B}mo_\alpha$ the set of all bounded mean oscillation functions $f\in L^1_{loc}(d\nu_\alpha)$.

REMARK 3.3. We can easily see that, the map $f\longmapsto\|f\|_{\mathfrak{B}mo_\alpha}$ is a semi-norm on $\mathfrak{B}mo_\alpha$.

PROPOSITION 3.4. *$L^\infty(d\nu_\alpha)\subset\mathfrak{B}mo_\alpha$ and for every function $f\in L^\infty(d\nu_\alpha)$, we have*
$$\|f\|_{\mathfrak{B}mo_\alpha}\leqslant 2\|f\|_{\infty,\nu_\alpha}. \tag{3.1}$$

Proof. It is obvious that $L^\infty(d\nu_\alpha)\subset L^1_{loc}(d\nu_\alpha)$. Let $f\in L^\infty(d\nu_\alpha)$, then for every $Q\in\mathcal{C}$, we have
$$\begin{aligned}\frac{1}{\nu_\alpha(Q)}\int_Q|f(r,x)-\underset{Q}{\mathrm{Avg}}\,f|d\nu_\alpha(r,x)&\leqslant\frac{1}{\nu_\alpha(Q)}\int_Q|f(r,x)|d\nu_\alpha(r,x)+|\underset{Q}{\mathrm{Avg}}\,f|\\&\leqslant 2\frac{1}{\nu_\alpha(Q)}\int_Q|f(r,x)|d\nu_\alpha(r,x)\\&\leqslant 2\|f\|_{\infty,\alpha}.\end{aligned}$$
Consequently
$$\|f\|_{\mathfrak{B}mo_\alpha}\leqslant 2\|f\|_{\infty,\alpha}.\quad\square$$

LEMMA 3.5. *Let $f \in L^1_{loc}(d\nu_\alpha)$. If there exists $A > 0$ such that for every cube $Q \in \mathcal{C}$ there exists c_Q such that*

$$\frac{1}{\nu_\alpha(Q)} \int_Q |f(r,x) - c_Q| d\nu_\alpha(r,x) \leqslant A.$$

Then $f \in \mathfrak{B}mo_\alpha$ and $\|f\|_{\mathfrak{B}mo_\alpha} \leqslant 2A$.

Proof. Let $K \subset [0,+\infty[\times\mathbb{R}$ be a compact and $Q_K \in \mathcal{C}$ such that $K \subset Q_K$ then

$$\begin{aligned}
\int_K |f(r,x)| d\nu_\alpha(r,x) &\leqslant \int_K |f(r,x) - c_{Q_K}| d\nu_\alpha(r,x) + \int_K |c_{Q_K}| d\nu_\alpha(r,x) \\
&\leqslant \int_{Q_K} |f(r,x) - c_{Q_K}| d\nu_\alpha(r,x) + c_{Q_K} \nu_\alpha(K) \\
&\leqslant A\nu_\alpha(Q_K) + c_{Q_K} \nu_\alpha(K) \\
&< +\infty.
\end{aligned}$$

Then $f \in L^1_{loc}(d\nu_\alpha)$.

$$\begin{aligned}
|f(r,x) - \operatorname*{Avg}_Q f| &\leqslant |f(r,x) - c_Q| + |\operatorname*{Avg}_Q f - c_Q| \\
&\leqslant |f(r,x) - c_Q| + \left| \frac{1}{\nu_\alpha(Q)} \int_Q f(r,x) - c_Q d\nu_\alpha(r,x) \right| \\
&\leqslant |f(r,x) - c_Q| + \frac{1}{\nu_\alpha(Q)} \int_Q |f(r,x) - c_Q| d\nu_\alpha(r,x) \\
&\leqslant |f(r,x) - c_Q| + A.
\end{aligned}$$

Then

$$\frac{1}{\nu_\alpha(Q)} \int_Q |f(r,x) - \operatorname*{Avg}_Q f| d\nu_\alpha(Q) \leqslant \frac{1}{\nu_\alpha(Q)} \int_Q |f(r,x) - c_Q| d\nu_\alpha(r,x) + A.$$

Consequently

$$\|f\|_{\mathfrak{B}mo_\alpha} \leqslant A + A = 2A. \quad \square$$

PROPOSITION 3.6. *Let $f \in \mathfrak{B}mo_\alpha$. Then $\|f\|_{\mathfrak{B}mo_\alpha} = 0$ if and only if f is constant.*

Proof. Let $f \in L^1_{loc}(d\nu_\alpha)$. It is clear that if $f = c$ we have $\|f\|_{\mathfrak{B}mo_\alpha} = 0$. Conversely if $\|f\|_{\mathfrak{B}mo_\alpha} = 0$, then

$$\forall Q \in \mathcal{C},\ |f - \operatorname*{Avg}_Q f| = 0.$$

In particular, for every $N \in \mathbb{N}^*$, we have

$$|f - \operatorname*{Avg}_{[0,N]\times[-N,N]} f| = 0.$$

This implies that for all $N \in \mathbb{N}^*$ and $(r,x) \in [0,N] \times [-N,N]$, we have

$$f(r,x) = c_N,$$

where $c_N = \underset{[0,N]\times[-N,N]}{\mathrm{Avg}} f$. Also for every $N \in \mathbb{N}$,

$$[0,N] \times [-N,N] \subset [0,N+1] \times [-N-1,N+1].$$

Then $c_N = c_{N+1}$. This implies that f is constant. $\square$

THEOREM 3.7. *For every $f \in \mathfrak{B}mo_\alpha$ the function $|f|$ belongs to $\mathfrak{B}mo_\alpha$ and we have*

$$\||f|\|_{\mathfrak{B}mo_\alpha} \leqslant 2\|f\|_{\mathfrak{B}mo_\alpha}.$$

Proof. Let $f \in \mathfrak{B}mo_\alpha$ and $Q \in \mathcal{C}$.

$$\forall (r,x) \in Q,$$

$$\begin{aligned}
||f(r,x)| - \underset{Q}{\mathrm{Avg}}|f|| &\leqslant \left||f(r,x)| - |\underset{Q}{\mathrm{Avg}} f|\right| + \left||\underset{Q}{\mathrm{Avg}} f| - \underset{Q}{\mathrm{Avg}}|f|\right| \\
&\leqslant \left||f(r,x)| - |\underset{Q}{\mathrm{Avg}} f|\right| + \left||\underset{Q}{\mathrm{Avg}} f| - \frac{1}{\nu_\alpha(Q)}\int_Q |f(y)| d\nu_\alpha(s,y)\right| \\
&\leqslant \left||f(r,x)| - |\underset{Q}{\mathrm{Avg}} f|\right| + \left|\frac{1}{\nu_\alpha(Q)}\int_Q |\underset{Q}{\mathrm{Avg}} f| - |f(s,y)| d\nu_\alpha(r,x)\right| \\
&\leqslant \left||f(r,x)| - |\underset{Q}{\mathrm{Avg}} f|\right| + \frac{1}{\nu_\alpha(Q)}\int_Q \left||\underset{Q}{\mathrm{Avg}} f| - |f(s,y)|\right| d\nu_\alpha(s,y) \\
&\leqslant |f(r,x) - \underset{Q}{\mathrm{Avg}} f| + \frac{1}{\nu_\alpha(Q)}\int_Q |\underset{Q}{\mathrm{Avg}} f - f(s,y)| d\nu_\alpha(s,y).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{\nu_\alpha Q)}\int_Q \left||f(r,x)| - \underset{Q}{\mathrm{Avg}}|f|\right| d\nu_\alpha(r,x) \leqslant{} & \frac{1}{\nu_\alpha(Q)}\int_Q |f(r,x) - \underset{Q}{\mathrm{Avg}} f| d\nu_\alpha(r,x) \\
& + \frac{1}{\nu_\alpha(Q)}\int_Q |f(r,x) - \underset{Q}{\mathrm{Avg}} f| d\nu_\alpha(r,x).
\end{aligned}$$

This implies that

$$\||f|\|_{\mathfrak{B}mo_\alpha} \leqslant 2\,\|f\|_{\mathfrak{B}mo_\alpha}. \quad \square$$

PROPOSITION 3.8. *Let f and $g \in \mathfrak{B}mo_\alpha$. Then we have*

(i) If f and g are two real-valued functions then $\max(f,g) \in \mathfrak{B}mo_\alpha$ *and we have*

$$\|\max(f,g)\|_{\mathfrak{B}mo_\alpha} \leqslant \frac{3}{2}(\|f\|_{\mathfrak{B}mo_\alpha} + \|g\|_{\mathfrak{B}mo_\alpha}). \tag{3.2}$$

(ii) If f and g are two real-valued functions then $\min(f,g) \in \mathfrak{B}mo_\alpha$ *and we have*

$$\|\min(f,g)\|_{\mathfrak{B}mo_\alpha} \leqslant \frac{3}{2}(\|f\|_{\mathfrak{B}mo_\alpha} + \|g\|_{\mathfrak{B}mo_\alpha}). \tag{3.3}$$

Proof. Let f and g be two real functions belonging to the space $\mathfrak{B}mo_\alpha$.

(i) We know that $\max(f,g) = \frac{f+g+|f-g|}{2}$. Using Remark 3.3 and Theorem 3.7, we obtain

$$\begin{aligned}
\|\max(f,g)\|_{\mathfrak{B}mo_\alpha} &= \left\|\frac{f+g+|f-g|}{2}\right\|_{\mathfrak{B}mo_\alpha} \\
&\leqslant \frac{1}{2}(\|f+g\|_{\mathfrak{B}mo_\alpha} + \||f-g|\|_{\mathfrak{B}mo_\alpha}) \\
&\leqslant \frac{1}{2}(\|f\|_{\mathfrak{B}mo_\alpha} + \|g\|_{\mathfrak{B}mo_\alpha} + 2\|f-g\|_{\mathfrak{B}mo_\alpha}) \\
&\leqslant \frac{3}{2}(\|f\|_{\mathfrak{B}mo_\alpha} + \|g\|_{\mathfrak{B}mo_\alpha}).
\end{aligned}$$

(ii) We know that $\min(f,g) = \frac{f+g-|f-g|}{2}$. Using Remark 3.3 and Theorem 3.7, we obtain

$$\begin{aligned}
\|\min(f,g)\|_{\mathfrak{B}mo_\alpha} &= \left\|\frac{f+g-|f-g|}{2}\right\|_{\mathfrak{B}mo_\alpha} \\
&\leqslant \frac{1}{2}(\|f+g\|_{\mathfrak{B}mo_\alpha} + \||f-g|\|_{\mathfrak{B}mo_\alpha}) \\
&\leqslant \frac{1}{2}(\|f\|_{\mathfrak{B}mo_\alpha} + \|g\|_{\mathfrak{B}mo_\alpha} + 2\|f-g\|_{\mathfrak{B}mo_\alpha}) \\
&\leqslant \frac{3}{2}(\|f\|_{\mathfrak{B}mo_\alpha} + \|g\|_{\mathfrak{B}mo_\alpha}). \qquad \square
\end{aligned}$$

DEFINITION 3.9. For every $t > 0$, the Poisson kernel p_t associated with the Riemann-Liouville operator $\mathscr{R}_\alpha$ is defined on $\mathbb{R}^2$ by

$$\begin{aligned}
p_t(r,x) &= \int\int_{\Upsilon_+} e^{-t\sqrt{s^2+2y^2}}\,\overline{\varphi_{s,y}(r,x)}\,d\gamma_\alpha(s,y) \\
&= \mathscr{F}_\alpha^{-1}(e^{-t\sqrt{.^2+2.^2}})(r,x).
\end{aligned}$$

DEFINITION 3.10. (Bounded distribution) Let $v \in \mathscr{S}'_e(\mathbb{R}^2)$. We say that v is a bounded tempered distribution if

$$\forall \varphi \in \mathscr{S}_e(\mathbb{R}^2), \varphi * v \in L^\infty(d\nu_\alpha)$$

and if the operator

$$\begin{aligned}
\phi_v : \mathscr{S}_e(\mathbb{R}^2) &\longrightarrow L^\infty(d\nu_\alpha) \\
\varphi &\longmapsto \varphi * v
\end{aligned}$$

is bounded.

DEFINITION 3.11. Let $f \in \mathscr{S}'_e(\mathbb{R}^2)$ be a bounded tempered distribution. The Poisson maximal function $\mathcal{P}^{\alpha}_f$ associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$ is defined on $\mathbb{R}^2$ by

$$\mathcal{P}^{\alpha}_f(r,x) = \sup_{t>0} |p_t * f(r,x)|.$$

DEFINITION 3.12. (Hardy space) For every $p \in [1,+\infty[$, the Hardy space $\mathcal{H}^p_\alpha$ associated with the Riemann- Liouville operator is the space of all the bounded tempered distributions f on $\mathbb{R}^2$ satisfying

$$\mathcal{P}^{\alpha}_f \in L^p(d\nu_\alpha).$$

We set

$$\|f\|_{\mathcal{H}^p_\alpha} = \|\mathcal{P}^{\alpha}_f\|_{p,\nu_\alpha}. \tag{3.4}$$

DEFINITION 3.13. (Atomic Decomposition) A function f is called an L^∞-atom of $\mathcal{H}^1_\alpha$, if there exists a cube $Q \in \mathcal{C}$ satisfying

(i) $Supp(f) \subset Q$.

(ii) $\|f\|_{\infty,\nu_\alpha} \leqslant \frac{1}{\nu_\alpha(Q)}$.

(iii) $\displaystyle\int_Q f(x,r)d\nu_\alpha(x,r) = 0$.

DEFINITION 3.14. A function $f \in L^1(d\nu_\alpha)$ belongs to the set $\mathcal{H}^{atomic}_\alpha$ if there exists a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \in l^1(\mathbb{N})$ and a sequence $\{f_i\}_{i\in\mathbb{N}}$ of L^∞-atom of $\mathcal{H}^1_\alpha$ such that

$$f = \sum_{i=1}^{+\infty} \lambda_i f_i \in L^1(d\nu_\alpha).$$

We set

$$\|f\|_{\mathcal{H}^{atomic}_\alpha} = \inf\Big\{ \sum_{i=1}^{+\infty} |\lambda_i| : f = \sum_{i=1}^{+\infty} \lambda_i f_i \quad \text{where } \{\lambda_i\}_{i\in\mathbb{N}} \in l^1(\mathbb{N}) \text{ and } \{f_i\}_{i\in\mathbb{N}}\ L^\infty\text{-atom for } \mathcal{H}^1_\alpha \Big\}.$$

We have recently established that

PROPOSITION 3.15. *There exists a constant $C > 0$ such that for every $f \in L^1(d\nu_\alpha)$ we have*

$$\frac{1}{C}\|f\|_{\mathcal{H}^1_\alpha} \leqslant \|f\|_{\mathcal{H}^{atomic}_\alpha} \leqslant C\|f\|_{\mathcal{H}^1_\alpha} \tag{3.5}$$

Proof. [1] □

4. Duality between $\mathcal{H}_\alpha^{atomic}$ and $\mathfrak{B}mo_\alpha$

- For any normed $\mathbb{C}$-vector space E, we denote by E^* its topological dual. E^* be provided with the dual topology associated to the norm defined by
$$\forall f \in E^*,\ \|f\|_{E\to\mathbb{C}} = \sup_{\substack{x\in E\\ \|x\|=1}} |f(x)|.$$
- For every $Q\in\mathcal{C}$, we denote by
  - $L^2_Q(d\nu_\alpha)$ the space of functions $f\in L^2(d\nu_\alpha)$ with support on Q.
  - $L^2_{0,Q}(d\nu_\alpha)$ the space of functions $f\in L^2_Q(d\nu_\alpha)$ such that $\underset{Q}{\mathrm{Avg}}\, f = 0$.

PROPOSITION 4.1. *$\mathcal{H}_\alpha^{atomic}\subset L^1(d\nu_\alpha)$ and we have*
$$\|f\|_{1,\nu_\alpha} \leqslant 2\|f\|_{\mathcal{H}_\alpha^{atomic}}.$$

Proof. Let $f\in\mathcal{H}_\alpha^{atomic}$, we can find a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ and a sequence $\{f_i\}_{i\in\mathbb{N}}$ of L^∞-atom of $\mathcal{H}_\alpha^1$ such that
$$\sum_{j=1}^{+\infty}|\lambda_i| \leqslant 2\|f\|_{\mathcal{H}_\alpha^{atomic}}.$$
We have
$$\begin{aligned}|f(r,x)| &= |\sum_{j=1}^{+\infty}\lambda_i f_i(r,x)|\\ &\leqslant \sum_{j=1}^{+\infty}|\lambda_i||f_i(r,x)|\\ &\leqslant \sum_{j=1}^{+\infty}\frac{|\lambda_i|}{\nu_\alpha(Q_i)}\chi_{Q_i}(r,x).\end{aligned}$$
Then
$$\|f\|_{1,\nu_\alpha} \leqslant \sum_{j=1}^{+\infty}|\lambda_i| \leqslant 2\|f\|_{\mathcal{H}_\alpha^{atomic}}. \quad\square$$

In the following we denote by $\mathcal{H}^1_{0,\alpha}$ the vector space of all finite linear combinations of L^∞-atom of $\mathcal{H}^1_\alpha$.

PROPOSITION 4.2. *Let $b\in\mathfrak{B}mo_\alpha$ and Φ_b the mapping defined by*
$$\begin{aligned}\Phi_b :&\mathcal{H}^1_{0,\alpha}\to\mathbb{C}\\ &f\mapsto\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)b(r,x)d\nu_\alpha(r,x).\end{aligned}$$
Then $\Phi_b\in(\mathcal{H}^1_{0,\alpha})^*$.

Proof. For evrey $g \in \mathcal{H}^1_{0,\alpha}$, there exists $m \in \mathbb{N}$ such that

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R},\ \ g(r,x) = \sum_{i=1}^{m} \lambda_i g_i(r,x),$$

where for every $1 \leqslant i \leqslant m$, $\lambda_i \in \mathbb{C}$ and g_i is a L^∞-atom for $\mathcal{H}^1_\alpha$. For every $c \in \mathbb{C}$ and $b \in \mathfrak{B}mo_\alpha$, we have

$$\begin{aligned}
\Phi_{b+c}(g) &= \int_0^{+\infty}\int_{\mathbb{R}} g(r,x)(b(r,x)+c)d\nu_\alpha(r,x)\\
&= \int_0^{+\infty}\int_{\mathbb{R}} \sum_{i=1}^{m} \lambda_i g_i(r,x)(b(r,x)+c)d\nu_\alpha(r,x)\\
&= \sum_{i=1}^{m} \lambda_i \left(\int_0^{+\infty}\int_{\mathbb{R}} g_i(r,x)b(r,x)d\nu_\alpha(r,x) + c\int_{\mathbb{R}^d} g_i(r,x)d\nu_\alpha(r,x)\right)\\
&= \sum_{i=1}^{m} \lambda_i \int_0^{+\infty}\int_{\mathbb{R}} g_i(r,x)b(r,x)d\nu_\alpha(r,x)\\
&= \int_0^{+\infty}\int_{\mathbb{R}} g(r,x)b(r,x)d\nu_\alpha(r,x)\\
&= \Phi_b(g).
\end{aligned}$$

Therefore

$$\forall g \in \mathcal{H}^1_{0,\alpha},\ \Phi_{b+c}(g) = \Phi_b(g). \tag{4.1}$$

Let $f \in \mathcal{H}^1_{0,\alpha}$ and $Q \in \mathcal{C}$ such that $Supp(f) \subset Q$. Using the Relation (4.1), we obtain

$$\begin{aligned}
\forall b \in \mathfrak{B}mo_\alpha,\ \ |\Phi_b(f)| &= |\Phi_{b-\mathrm{Avg}_Q b}(f)|\\
&\leqslant \int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)||b(r,x) - \operatorname*{Avg}_Q b|d\nu_\alpha(r,x)\\
&= \int_Q |f(r,x)||b(r,x) - \operatorname*{Avg}_Q b|d\nu_\alpha(r,x)\\
&\leqslant \|f\|_{\infty,\nu_\alpha}\int_Q |b(r,x) - \operatorname*{Avg}_Q b|d\nu_\alpha(r,x)\\
&\leqslant \|f\|_{\infty,\nu_\alpha}\nu_\alpha(Q)\|b\|_{\mathfrak{B}mo_\alpha}\\
&= C\|b\|_{\mathfrak{B}mo_\alpha}\\
&< +\infty.
\end{aligned}$$

Then Φ_b is well defined. Moreover, it is clear that Φ_b is linear, since $|\Phi_b(f)| \leqslant C\|b\|_{\mathfrak{B}mo_\alpha}$, we deduce that Φ_b is bounded. □

PROPOSITION 4.3. *Let $b \in L^\infty(d\nu_\alpha)$ and Φ_b the mapping defined by*

$$\begin{aligned}
\Phi_b :\mathcal{H}^{atomic}_\alpha &\longrightarrow \mathbb{C}\\
f &\mapsto \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)b(r,x)d\nu_\alpha(r,x).
\end{aligned}$$

Then $\Phi_b \in (\mathcal{H}_\alpha^{atomic})^*$.

Proof. The linearity of Φ_b is obvious. In addition, we use the Remark 3.4, we deduce that for any $f \in \mathcal{H}_\alpha^{atomic}$, we have

$$\begin{aligned}|\Phi_b(f)| &\leqslant \int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)||b(r,x)|d\nu_\alpha(r,x)\\ &\leqslant \|b\|_{\infty,\nu_\alpha}\|f\|_{1,\alpha}\\ &\leqslant 2\|b\|_{\infty,\nu_\alpha}\|f\|_{\mathcal{H}_\alpha^{atomic}}.\end{aligned}$$

Therefore

$$\|\Phi_b\|_{\mathcal{H}_\alpha^{atomic}\to\mathbb{C}} \leqslant \|b\|_{\infty,\nu_\alpha}. \quad \square$$

LEMMA 4.4. *Let* $f \in \mathfrak{B}mo_\alpha$ *be a real-valued function and* $K, L \in \mathbb{R}$ *such that* $K \leqslant L$. *We define the function* f_{KL} *on* $[0,+\infty[\times\mathbb{R}$ *by*

$$f_{KL} = \max(K, \min(f, L)).$$

Then

$$\|f_{KL}\|_{\mathfrak{B}mo_\alpha} \leqslant \frac{9}{4}\|f\|_{\mathfrak{B}mo_\alpha}. \tag{4.2}$$

Proof. Let $f \in \mathfrak{B}mo_\alpha$ be the real-valued function and $K, L \in \mathbb{R}$ such that $k \leqslant L$. It is clear that

$$f_{KL} = \max(K, \min(f, L)).$$

Using Relations (3.2) and (3.3) and the Proposition 3.6, we obtain

$$\begin{aligned}\|f_{KL}\|_{\mathfrak{B}mo_\alpha} &= \|\max(K,\min(f,L))\|_{\mathfrak{B}mo_\alpha}\\ &\leqslant \frac{3}{2}\left(\|K\|_{\mathfrak{B}mo_\alpha} + \|\min(f,L)\|_{\mathfrak{B}mo_\alpha}\right)\\ &= \frac{3}{2}\|\min(f,L)\|_{\mathfrak{B}mo_\alpha}\\ &\leqslant \frac{3}{2}\left(\frac{3}{2}\left(\|L\|_{\mathfrak{B}mo_\alpha} + \|f\|_{\mathfrak{B}mo_\alpha}\right)\right)\\ &= \frac{9}{4}\|f\|_{\mathfrak{B}mo_\alpha}. \quad \square\end{aligned}$$

LEMMA 4.5. *There exists a constant* C *such that for every* $b \in L^\infty(d\nu_\alpha)$, *we have*

$$\|\Phi_b\|_{\mathcal{H}_\alpha^{atomic}\to\mathbb{C}} \leqslant C\|b\|_{\mathfrak{B}mo_\alpha}, \tag{4.3}$$

where Φ_b *is the function defined in Proposition* 4.3.

Proof. Let $f \in \mathcal{H}_\alpha^{atomic}$, then there exist a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of $\mathbb{C}$ and a sequence $(f_{Q_k})_{k\in\mathbb{N}}$ of L^∞-atom for $\mathcal{H}_\alpha^1$ supported respectively in Q_k such that

$$f = \sum_{k=1}^{+\infty} \lambda_k f_{Q_k}.$$

Using the the Proposition 4.1, we have

$$\begin{aligned}
\left|\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)b(r,x)d\nu_\alpha(r,x)\right| &\leqslant \|b\|_{\infty,\nu_\alpha}\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|d\nu_\alpha(r,x)\\
&\leqslant \|b\|_{\infty,\nu_\alpha}\|f\|_{1,\nu_\alpha}\\
&\leqslant 2\|b\|_{\infty,\nu_\alpha}\|f\|_{\mathcal{H}_\alpha^{atomic}}\\
&< +\infty.
\end{aligned}$$

Thus we use the term by term Integration Theorem, we obtain

$$\begin{aligned}
|\Phi_b(f)| &= \left|\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)b(r,x)d\nu_\alpha(r,x)\right|\\
&= \left|\sum_{k=1}^{+\infty}\lambda_k\int_{Q_k} f_{Q_k}(r,x)b(r,x)d\nu_\alpha(r,x)\right|\\
&= \left|\sum_{k=1}^{+\infty}\lambda_k\int_{Q_k} f_{Q_k}(r,x)(b(r,x)-\operatorname*{Avg}_{Q_k} b)d\nu_\alpha(r,x)\right|\\
&= \left|\sum_{k=1}^{+\infty}\lambda_k\|f_{Q_k}\|_{\infty,\nu_\alpha}\int_{Q_k} (b(r,x)-\operatorname*{Avg}_{Q_k} b)d\nu_\alpha(r,x)\right|\\
&\leqslant \sum_{k=1}^{+\infty}|\lambda_k|\|b\|_{\mathfrak{B}mo_\alpha}\\
&\leqslant 2\|f\|_{\mathcal{H}_\alpha^{atomic}}\|b\|_{\mathfrak{B}mo_\alpha}.
\end{aligned}$$

Thus

$$\|\Phi_b\|_{\mathcal{H}_\alpha^{atomic}\to\mathbb{C}} \leqslant 2\|b\|_{\mathfrak{B}mo_\alpha}. \qquad \square$$

THEOREM 4.6. *For every $b \in \mathfrak{B}mo_\alpha$, the linear form Φ_b extends to a continuous linear form $\widetilde{\Phi}_b$ on $\mathcal{H}_\alpha^{atomic}$. Moreover, there exists $C > 0$ such that*

$$\|\widetilde{\Phi}_b\|_{\mathcal{H}_\alpha^{atomic}\to\mathbb{C}} \leqslant C\|b\|_{\mathfrak{B}mo_\alpha}.$$

Proof. Let $b \in \mathfrak{B}mo_\alpha$, $M \in \mathbb{N}$ and $b_M = \max(-M, \min(|b|, M))$. Then, $b_M \in L^\infty(d\nu_\alpha)$ and $\|b_M\|_{\infty,\nu_\alpha} \leqslant M$. Moreover, we use the Theorem 3.7 ainsi and the Relation (4.2), we have

$$\|b_M\|_{\mathfrak{B}mo_\alpha} \leqslant \frac{9}{4}\||b|\|_{\mathfrak{B}mo_\alpha} \leqslant \frac{9}{2}\|b\|_{\mathfrak{B}mo_\alpha}. \tag{4.4}$$

Moreover, by using Relations (4.3) and (4.4) we deduce that

$$\forall M \in \mathbb{N},\ \|\Phi_{b_M}\|_{\mathcal{H}_\alpha^{atomic}\to\mathbb{C}} \leqslant C\|b_M\|_{\mathfrak{B}mo_\alpha} \leqslant \frac{9C}{2}\|b\|_{\mathfrak{B}mo_\alpha}.$$

This implies that the sequence $(\Phi_{b_M})_{M\in\mathbb{N}}$ belongs to the ball $\overline{B(0,\frac{9C}{2}\|b\|_{\mathfrak{B}mo_\alpha})}$ of $(\mathcal{H}_\alpha^{atomic})^*$ which is compact from the Banch-Alaoglu-Bourbaki theorem in the weak* topology. Then, there exists a subsequence $(\Phi_{b_{\rho(M)}})_{M\in\mathbb{N}}$ such that

$$\forall f \in \mathcal{H}_\alpha^{atomic},\ \lim_{M\longrightarrow+\infty}\Phi_{b_{\rho(M)}}(f) = \widetilde{\Phi}_b(f). \tag{4.5}$$

Let f_Q be a L^∞-atom for $\mathcal{H}_\alpha^1$ and $Q\in\mathcal{C}$ such that $Supp(f_Q)\subset Q$. Then we have

$$\begin{aligned}|\Phi_{b_{\rho(M)}}(f_Q)-\Phi_b(f_Q)| &= \left|\int_Q f_Q(r,x)b_{\rho(M)}(r,x)d\nu_\alpha(r,x) - \int_Q f_Q(r,x)b(r,x)d\nu_\alpha(r,x)\right|\\ &= \left|\int_Q f_Q(r,x)\left(b_{\rho(M)}(r,x)-b(r,x)\right)d\nu_\alpha(r,x)\right|\\ &\leqslant \|f_Q\|_{\infty,\nu_\alpha}\int_Q\left|b_{\rho(M)}(r,x)-b(r,x)\right|d\nu_\alpha(r,x).\end{aligned}$$

However

$$\lim_{M\longrightarrow+\infty}\int_Q\left|b_{\rho(M)}(r,x)-b(r,x)\right|d\nu_\alpha(r,x)=0.$$

Then

$$\lim_{M\to+\infty}|\Phi_{b_{\rho(M)}}(f_Q)-\Phi_b(f_Q)|=0.$$

As a result,

$$\forall g\in\mathcal{H}_{0,\alpha}^1,\ \lim_{M\to+\infty}\Phi_{b_{\rho(M)}}(g)=\Phi_b(g). \tag{4.6}$$

Thus by combining the Relations (4.5) and (4.6), we deduce that

$$\forall g\in\mathcal{H}_{0,\alpha}^1,\ \widetilde{\Phi}_b(g)=\Phi_b(g).$$

In the other hand, we know that $\overline{\mathcal{H}_{0,\alpha}^1}=\mathcal{H}_\alpha^{atomic}$, therefore $\widetilde{\Phi}_b$ is the only extension of Φ_b on $\mathcal{H}_\alpha^{atomic}$. Using the Relation (4.4) we obtain

$$\lim_{M\to+\infty}\|\Phi_{b_{\rho(M)}}\|_{\mathcal{H}_\alpha^{atomic}\to\mathbb{C}}\leqslant\frac{9}{2}\|b\|_{\mathfrak{B}mo_\alpha}.$$

Consequently

$$\|\widetilde{\Phi}_b\|_{\mathcal{H}_\alpha^{atomic}\to\mathbb{C}}\leqslant C_d\|b\|_{\mathfrak{B}mo_\alpha}.\quad\square$$

PROPOSITION 4.7. *Let $Q\in\mathcal{C}$ and $u\in L^1_{loc}(d\nu_\alpha)$ such that $Supp(u)\subset Q$. If for every $v\in L^2_Q(d\nu_\alpha)$ we have*

$$\int_Q u(r,x)v(r,x)d\nu_\alpha(r,x)=0. \tag{4.7}$$

Then $u=0$ a.e.

Proof. Let $u \in L^1_{loc}(d\nu_\alpha)$ and $Q \in \mathcal{C}$ such $Supp(u) \subset Q$. It is clear that $u \in L^1(d\nu_\alpha)$.

Let $(\mu,\lambda) \in [0,+\infty[\times\mathbb{R}$ and $v_{\mu,\lambda}$ the function defined on $[0,+\infty[\times\mathbb{R}$ by

$$v_{\mu,\lambda}(r,x) = \varphi_{\mu,\lambda}(r,x)\chi_Q(r,x).$$

We have

$$\begin{aligned}
\|v_{\mu,\lambda}\|_{2,\alpha} &= \left(\int_Q (\varphi_{\mu,\lambda}(r,x))^2 d\nu_\alpha(r,x)\right)^{\frac{1}{2}} \\
&\leqslant \sup_{(r,x)\in[0,+\infty[\times\mathbb{R}} |\varphi_{\mu,\lambda}(r,x)|\sqrt{\nu_\alpha(Q)} \\
&= \sqrt{\nu_\alpha(Q)}.
\end{aligned}$$

Then $v_{\mu,\lambda} \in L^2_Q(d\nu_\alpha)$. Which implies that

$$\begin{aligned}
\mathscr{F}_\alpha(u)(\mu,\lambda) &= \int_0^{+\infty}\int_{\mathbb{R}} u(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x) \\
&= \int_Q u(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x) \\
&= \int_Q u(r,x)v_{\mu,\lambda}(r,x)d\nu_\alpha(r,x) \\
&= 0.
\end{aligned}$$

Thus by using the Inversion formula we deduce that $u = 0$ a.e. $\square$

COROLLARY 4.8. *Let $Q \in \mathcal{C}$ and $u \in L^1_{loc}(d\nu_\alpha)$ such that $Supp(u) \subset Q$. If for every $v \in L^2_{0,Q}(d\nu_\alpha)$, we have*

$$\int_Q u(r,x)v(r,x)d\nu_\alpha(r,x) = 0. \tag{4.8}$$

Then u is constant a.e.

Proof. Let $v \in L^2_Q(d\nu_\alpha)$ and ω_v the function defined on $[0,+\infty[\times\mathbb{R}$ by

$$\omega_v(r,x) = v(r,x) - \operatorname*{Avg}_Q v,$$

it is clear that $\omega_v \in L^2_Q(d\nu_\alpha)$. Moreover, we have

$$\begin{aligned}
\int_Q \omega_v(r,x)d\nu_\alpha(r,x) &= \int_Q v(r,x) - \operatorname*{Avg}_Q v\, d\nu_\alpha(r,x) \\
&= \int_Q v(r,x)d\nu_\alpha(r,x) - \nu_\alpha(Q)\operatorname*{Avg}_Q v \\
&= 0.
\end{aligned}$$

Then

$$\omega_v \in L^2_{0,Q}(d\nu_\alpha).$$

Using the Relation (4.8) we deduce that

$$\int_Q \omega_v(r,x)u(r,x)d\nu_\alpha(r,x) = 0,$$

since

$$\begin{aligned}\int_Q \omega_v(r,x)u(r,x)d\nu_\alpha(r,x) &= \int_Q v(r,x)u(r,x)d\nu_\alpha(r,x) - \operatorname*{Avg}_Q v \int_Q u(r,x)d\nu_\alpha(r,x)\\ &= \int_Q v(r,x)u(r,x)d\nu_\alpha(r,x) - \operatorname*{Avg}_Q u \int_Q v(r,x)d\nu_\alpha(r,x)\\ &= \int_Q v(r,x)\left(u(r,x) - \operatorname*{Avg}_Q u\right)d\nu_\alpha(r,x).\end{aligned}$$

We get

$$\int_Q v(r,x)\left(u(r,x) - \operatorname*{Avg}_Q u\right)d\nu_\alpha(r,x) = 0.$$

Then by using Proposition 4.7 we deduce that

$$u - \operatorname*{Avg}_Q u = 0 \ \text{ a.e.}$$

Consequently

$$u = \operatorname*{Avg}_Q u \ \text{ a.e.} \quad \square$$

PROPOSITION 4.9. *For every* $Q \in \mathcal{C}$, $L^2_Q(d\nu_\alpha) \subset \mathcal{H}^{atomic}_\alpha$ *and there exists* $c > 0$ *such that for every* $f \in L^2_Q(d\nu_\alpha)$

$$\|f\|_{\mathcal{H}^{atomic}_\alpha} \leqslant c\sqrt{\nu_\alpha(Q)}\|f\|_{2,\alpha}. \tag{4.9}$$

THEOREM 4.10. *Let* Φ *be a continuous linear form on* $\mathcal{H}^{atomic}_\alpha$. *Then there exists* $b \in \mathfrak{B}mo_\alpha$ *and a constant* $C > 0$ *such that*

$$\forall f \in \mathcal{H}^1_{0,\alpha},\ \Phi(f) = \Phi_b(f),$$

and

$$\|b\|_{\mathfrak{B}mo_\alpha} \leqslant C\|\Phi_b\|_{\mathcal{H}^{atomic}_\alpha \to \mathbb{C}}.$$

Proof. Let Φ be a continuous linear form on $\mathcal{H}^{atomic}_\alpha$ and $Q \in \mathcal{C}$. Then, we use the Relation (4.9), we deduce that for every $f \in L^2_{0,Q}(d\nu_\alpha)$ not identically null, we have

$$\begin{aligned}|\Phi(f)| &\leqslant \|\Phi\|_{\mathcal{H}^{atomic}_\alpha \to \mathbb{C}}\|f\|_{\mathcal{H}^{atomic}_\alpha}\\ &\leqslant c\sqrt{\nu_\alpha(Q)}\|\Phi\|_{\mathcal{H}^{atomic}_\alpha \to \mathbb{C}}\|f\|_{2,\alpha}.\end{aligned}$$

Consequently

$$\|\Phi\|_{L^2_{0,Q}(d\nu_\alpha)\to\mathbb{C}} \leqslant c\sqrt{\nu_\alpha(Q)}\|\Phi\|_{\mathcal{H}^{atomic}_\alpha\to\mathbb{C}}. \tag{4.10}$$

Thus, Φ est a bounded linear function on $L^2_{0,Q}(d\nu_\alpha)$ which is a Hilbert space, therefore using the Riesz representation theorem, we deduce that there exists a unique function $F^Q \in L^2_{0,Q}(d\nu_\alpha)$ such that

$$\forall f \in L^2_{0,Q}(d\nu_\alpha),\ \Phi(f) = \int_Q F^Q(r,x)f(r,x)d\nu_\alpha(r,x).$$

and

$$\|F^Q\|_{2,\alpha} \leqslant \|\Phi\|_{L^2_{0,Q}(d\nu_\alpha)\to\mathbb{C}}. \tag{4.11}$$

Moreover, such as $\operatorname*{Avg}_Q f = 0$, we deduce that for every $c \in \mathbb{C}$, we have

$$\int_Q (F^Q(r,x)+c)f(r,x)d\nu_\alpha(r,x) = \int_Q F^Q(r,x)f(r,x)d\nu_\alpha(r,x).$$

Let $Q' \in \mathcal{C}$ such that $Q \subset Q'$. Then, in view of the above, we know that there exists $F^{Q'} \in L^2_{0,Q'}(d\nu_\alpha)$ such that

$$\forall f \in L^2_{0,Q'}(d\nu_\alpha),\ \Phi(f) = \int_{Q'} F^{Q'}(r,x)f(r,x)d\nu_\alpha(r,x).$$

In particular

$$\forall f \in L^2_{0,Q}(d\nu_\alpha),\ \Phi(f) = \int_Q F^{Q'}(r,x)f(r,x)d\nu_\alpha(r,x).$$

Then

$$\forall f \in L^2_{0,Q}(d\nu_\alpha),\ \int_Q \left(F^{Q'}(r,x) - F^Q(r,x)\right) f(r,x)d\nu_\alpha(r,x) = 0.$$

Using the Corollary 4.8, we obtain that $F^{Q'} - F^Q$ is a constant on Q. For every $m \in \mathbb{N}^*$, Let $Q_m = [0,m]\times[-m,m]$. Let b the function defined on $[0,+\infty[\times\mathbb{R}$ by

$$\forall (r,x) \in Q_m,\ b(r,x) = F^{Q_m}(r,x) - \operatorname*{Avg}_{Q_1} F^{Q_m}.$$

We will show now in the following that the value b does not depend of m and therefore the function b is well defined. Indeed, let $n,m \in \mathbb{N}^*$ such that $n < m$, it is clear that $Q_n \subset Q_m$ and $L^2_{0,Q_n}(d\nu_\alpha) \subset L^2_{0,Q_m}(d\nu_\alpha)$ and then we have $F^{Q_n} - F^{Q_m}$ is constant on Q_n. Thus,

$$\forall (r,x) \in Q_n,\ F^{Q_n}(r,x) - \operatorname*{Avg}_{Q_1} F^{Q_n} = F^{Q_m}(r,x) - \operatorname*{Avg}_{Q_1} F^{Q_m}.$$

Let $Q \in \mathcal{C}$ and Q_s, $s \in \mathbb{N}^*$ the smallest cube that contains Q. Then,

$$C_Q = -\left(F^Q(r,x) - F^{Q_s}(r,x) + \operatorname*{Avg}_{Q_1} F^{Q_s}\right)$$

is a constant on Q and we have

$$\forall (r,x) \in Q,\ b(r,x) = F^{Q}(r,x) + C_{Q}.$$

Now, we will show that $b \in L^1_{loc}(d\nu_\alpha)$. Let $K \subset [0,+\infty[\times\mathbb{R}$ be a compact and $Q \in \mathcal{C}$ such that $K \subset Q$, then

$$\begin{aligned}
\int_K |b(r,x)|d\nu_\alpha(r,x) &\leqslant \int_Q |b(r,x)|d\nu_\alpha(r,x)\\
&\leqslant \int_Q |F^{Q}(r,x)|d\nu_\alpha(r,x) + C_Q\nu_\alpha(Q)\\
&\leqslant \sqrt{\nu_\alpha(Q)}\|F^{Q}\|_{2,\alpha} + C_Q\nu_\alpha(Q)\\
&< +\infty
\end{aligned}$$

Using Relations (4.10) and (4.11) and Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
\frac{1}{\nu_\alpha(Q)}\int_Q |b(r,x) - C_Q|d\nu_\alpha(r,x) &= \frac{1}{\nu_\alpha(Q)}\int_Q |F^{Q}(r,x)|d\nu_\alpha(r,x)\\
&\leqslant \frac{1}{\sqrt{\nu_\alpha(Q)}}\|F^{Q}\|_{2,\alpha}\\
&\leqslant \|\Phi\|_{L^2_{0,Q}(d\nu_\alpha)\to\mathbb{C}}\\
&\leqslant C\|\Phi\|_{\mathcal{H}_\alpha^{atomic}\to\mathbb{C}}\\
&< +\infty.
\end{aligned}$$

Then using the Lemma 3.5 we obtain

$$b \in \mathfrak{B}mo_\alpha$$

and

$$\|b\|_{\mathfrak{B}mo_\alpha} \leqslant c\|\Phi\|_{\mathcal{H}_\alpha^{atomic}\to\mathbb{C}}.$$

It is clear that for every $Q \in \mathcal{C}$ and $g \in L^2_{0,Q}$, we have

$$\begin{aligned}
\Phi_b(g) &= \int_Q b(r,x)g(r,x)d\nu_\alpha(r,x)\\
&= \int_Q (F^{Q}(r,x) + C^{Q})g(r,x)d\nu_\alpha(r,x)\\
&= \int_Q F^{Q}(r,x)g(r,x)d\nu_\alpha(r,x)\\
&= \Phi(g).
\end{aligned}$$

Let $h \in \mathcal{H}^1_{0,\alpha}$ such that

$$h(r,x) = \sum_{i=1}^{k} \lambda_i a_i(r,x),$$

where

$$\lambda_i \in \mathbb{C} \text{ and } a_i \text{ is an } L^\infty - atom\,,\ 1 \leqslant i \leqslant k.$$

Then

$$\Phi(h) = \sum_{i=1}^{k} \lambda_i \Phi(a_i) = \sum_{i=1}^{k} \lambda_i \Phi_b(a_i) = \Phi_b(h).$$

Since $\mathcal{H}^1_{0,\alpha}$ is dense in $\mathcal{H}^{atomic}_\alpha$ we have

$$\Phi = \Phi_b. \quad \square$$

THEOREM 4.11. (Duality of $\mathcal{H}^{atomic}_\alpha$) *$(\mathcal{H}^{atomic}_\alpha)^*$ is isomorphic to $\mathfrak{B}mo_\alpha$.*

Proof. Let

$$\Psi : \mathfrak{B}mo_\alpha \to (\mathcal{H}^{atomic}_\alpha)^*$$
$$b \mapsto \Phi_b.$$

Ψ is a linear map. In the one hand, using the Theorem 4.10, the wap Ψ is surjective. In the one hand, let $b_1, b_2 \in \mathfrak{B}mo_\alpha$ such that

$$\Psi(b_1) = \Psi(b_2),$$

then for every $f \in \mathcal{H}^{atomic}_\alpha$, we have

$$\Phi_{b_1}(f) = \Phi_{b_2}(f),$$

then

$$\int_0^{+\infty}\int_{\mathbb{R}} (b_1(r,x) - b_2(r,x)) f(r,x) d\nu_\alpha(r,x) = 0.$$

Using Proposition 4.7, we get that $b_1 - b_2$ is constant. $\square$

COROLLARY 4.12. *$(\mathcal{H}^1_\alpha)^*$ is isomorphic to $\mathfrak{B}mo_\alpha$.*

Proof. Using the fact that there exists C such that

$$\frac{1}{C}\|f\|_{\mathcal{H}^1_\alpha} \leqslant \|f\|_{\mathcal{H}^{atomic}_\alpha} \leqslant C\|f\|_{\mathcal{H}^1_\alpha}. \quad \square$$

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