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THE ROOTS OF ELEMENTS OF $\mathsf{Aut}(\mathsf{SH}_2)$

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Abstract. We study the roots of automorphisms on the Siegel upper half plane of complex dimension three. We use the normal form of any element of $\mathsf{Sp}(2,\mathbb{R})$ under the conjugation in $\mathsf{Sp}(2,\mathbb{R})$ to show that some of automorphisms have roots and that some of them do not have. As an application, we generalize the Siegel unit disk of the same dimension.

1. Introduction and preliminaries

The square matrix A is said to be (square) root if there exists a matrix B such that $B^2 = A$. The study of this issue of solving matrix equations has been of interest to many mathematicians [6, 1, 3]. In this article, we investigate the rootabliyt of automorphisms on the Siegel unit disk. We denote by $\mathsf{GL}(n,\mathbb{R})$ the set of all $n\times n$ invertible matrices on the field $\mathbb{R}$. Let $\mathsf{Sym}(n,\mathbb{R})$ be the space of $n\times n$ symetric matrices. Let $\mathsf{SD}_n = \{Z\in \mathsf{Sym}(n,\mathbb{R}) : \|Z\|_2 < 1\}$ be the Siegel n-disk. Also we consider $\overline{\mathsf{SD}}_n = \{Z\in \mathsf{Sym}(n,\mathbb{R}) : \|Z\|_2 \leqslant 1\}$ and $\partial\mathsf{SD}_n = \{Z\in \mathsf{Sym}(n,\mathbb{R}) : \|Z\|_2 = 1\}$, the Shilov boundary of SD_n. Moreover, we set $\mathsf{USym}(n) = \mathsf{U}_n \cap \mathsf{Sym}(n,\mathbb{R})$, the set of $n\times n$ unitary symmetric matrices. Let $\mathsf{SH}_n = \{Z\in \mathsf{Sym}(n,\mathbb{R}) : \mathrm{Im} Z > 0\}$ be the Siegel upper half plane and let $\mathrm{Cl}(\mathsf{SH}_n)$ denote the compactification of SH_n, which is diffeomorphic to $\overline{\mathsf{SD}}_n$; for details, see [7].

The symplectic group $\mathsf{Sp}(n,\mathbb{R})$ is defined as

$$\mathsf{Sp}(n,\mathbb{R}) = \{M\in \mathsf{GL}(2n,\mathbb{R}) : M^T J_n M = J_n\},$$

in which

$$J_n = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} \in \mathsf{SL}(2n,\mathbb{R}),$$

where $\mathsf{SL}(2n,\mathbb{R})$ is the set of all matrices such as $A\in\mathbb{R}^{2n\times 2n}$ with $\det A = 1$. It is seen that if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

$$M\in \mathsf{Sp}(n,\mathbb{R}) \Longleftrightarrow M^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix},$$

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which is equivalent that A^TC and B^TD are symmetric and $A^TD - C^TB = I_n$. Recall that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathsf{Sp}(n,\mathbb{R})$ acts on SH_n as follows:

$$M(Z) := (AZ+B)(CZ+D)^{-1} \quad (Z \in \mathsf{SH}_n).$$

We call these maps generalized Möbius transformations on SH_n, where the action M and $-M$ coincide. It is easy to see that $\mathsf{Sp}(n,\mathbb{R})$ is a group and that $\{\pm I_{2n}\}$ is a normal subgroup. Furthermore, $\mathsf{PSp}(n,\mathbb{R}) = \mathsf{Sp}(n,\mathbb{R})/\{\pm I_{2n}\}$ is equal to the group of biholomorphisms of SH_n. The action of $M \in \mathsf{Sp}(n,\mathbb{R})$ can be extended to $\mathsf{Cl}(\mathsf{SH}_n)$. The following theorem explains the normal forms of conjugacy in $\mathsf{SL}(2,\mathbb{R}) = \mathsf{Sp}(1,\mathbb{R})$; see [6, 7].

THEOREM 1.1. [7, Theorem 3.2] *Let $X \in SL(2,\mathbb{R})$ with $X \neq \pm I_2$. Then X is conjugate to one and only one of the following normal forms in $SL(2,\mathbb{R})$:*

(1) $\begin{pmatrix} 1/\alpha & 0 \\ 0 & \alpha \end{pmatrix}$, where $|\alpha| > 1$,

(2) $\pm\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$,

(3) $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where $a^2 + b^2 = 1$.

Also $X \in \mathsf{SL}(2,\mathbb{R})$ is called hyperbolic, parabolic, and elliptic if X is conjugate to one of the forms in $(1), (2)$, and (3), respectively. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and let $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathbb{R}^{2\times 2}$. Then define $A \odot B$ by

$$A \odot B := \begin{pmatrix} a & 0 & b & 0 \\ 0 & e & 0 & f \\ c & 0 & d & 0 \\ 0 & g & 0 & h \end{pmatrix}.$$

In [6], it is shown that

$$\theta : \mathsf{SL}(2,\mathbb{R})^n \to \mathsf{Sp}(n,\mathbb{R}), \quad \theta(M_1 \times \cdots \times M_n) = M_1 \odot \cdots \odot M_n$$

is an isomorphism from $\mathsf{SL}(2,\mathbb{R})^n$ onto a subgroup of $\mathsf{Sp}(n,\mathbb{R})$.

REMARK 1.2. It follows from [7, Theorem 3.3] that each $M \in \mathsf{Sp}(2,\mathbb{R})$ is conjugated to one of the following matrices:

type 1. $\left(\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right)$, where $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and $B = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix}$,

type 2. $\left(\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right)$, where $A = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}$ and $B = \begin{pmatrix} \alpha^{-1} & -\alpha^{-2} \\ 0 & \alpha^{-1} \end{pmatrix}$, type 3. $\begin{pmatrix} A & O \\ C & B \end{pmatrix}$, where $A = \begin{pmatrix} 0 & 1 \\ -1 & 2a \end{pmatrix}$, $B = \begin{pmatrix} 2a & 1 \\ -1 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & \delta \\ 0 & 0 \end{pmatrix}$ with $|a| \leqslant 1$ and $\delta = 0, \pm 1$,

type 4. $\begin{pmatrix} A & O \\ C & B \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, and $C = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}$ with $\alpha \neq \pm 1$ and $\delta = 0, \pm 1$,

type 5. $\begin{pmatrix} A & O \\ C & A \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ and $C = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$ with $\alpha = \pm 1$, $\delta_1, \delta_2 = 0, \pm 1$,

type 6. $A \odot B$, where $A = \begin{pmatrix} a_1 & -c_1 \\ c_1 & a_1 \end{pmatrix}$ and $B = \begin{pmatrix} a_2 & 0 \\ \delta & a_2 \end{pmatrix}$ with $a_1^2 + c_1^2 = 1$, $\delta = 0, \pm 1$, and if $\delta \neq 0$, then $a_2 = \pm 1$,

type 7. $A \odot B$, where $A = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}$ and $B = \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}$ with $a_1^2 + b_1^2 = a_2^2 + b_2^2 = 1$.

2. Main results

Let H be the upper half plane in $\mathbb{R}^2$. Then we set

$$\mathsf{PSL}(2,\mathbb{R}) := \mathsf{SL}(2,\mathbb{R})/\{\pm I_2\} = \mathsf{PSp}(1,\mathbb{R}) = \mathsf{Aut}(H).$$

In this section, we examine the roots of some elements of $\mathsf{PSp}(2,\mathbb{R})$. First, we give the concept of root-approximable in a topological group.

DEFINITION 2.1. Let G be a topological group with unit e. An element x in G is called root-approximable if there exists a sequence (x_n) in G such that

(i) $x_n^{2^n} = x, \quad n = 0, 1, 2, \ldots,$

(ii) $\lim_{n\to\infty} x_n = e.$

The topological group G is root-approximable if each $x \in G$ is root-approximable.

In the next theorem, we show that $\mathsf{Aut}(H)$ is root-approximable. Let $y = a^{-1}xa$ and let $y_n = a^{-1}x_n a$. Then $\lim y_n = \lim(a^{-1}x_n a) = a^{-1}\lim x_n a = a^{-1}ea = e$ and $y_n^{2^n} = (a^{-1}x_n a)^{2^n} = a^{-1}x_n^{2^n}a = a^{-1}xa = y$, so we have the following lemma.

LEMMA 2.2. *If $x \in G$ is root-approximable and $a \in G$, then so is $a^{-1}xa$.*

THEOREM 2.3. *The group $\mathsf{Aut}(H)$ is root-approximable.*

Proof. Let $M \in Aut(H)$. Making use of Theorem 1.1 and Lemma 2.2, it suffices to consider the following cases:

(1) $[A] = \left\{ \pm \begin{pmatrix} 1/\alpha & 0 \\ 0 & \alpha \end{pmatrix} \right\}$, where $\alpha > 1$,

(2) $[B]=\left\{\pm\begin{pmatrix}1 & 0\\ \pm 1 & 1\end{pmatrix}\right\}$,

(3) $[C]=\left\{\pm\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right\}$.

If M is equal to $[A]$, $[B]$ or $[C]$, then

$$[A]^{1/2^n}=\left\{\pm\begin{pmatrix}1/\sqrt[2^n]{\alpha} & 0\\ 0 & \sqrt[2^n]{\alpha}\end{pmatrix}\right\},$$
$$[B]^{1/2^n}=\left\{\pm\begin{pmatrix}1 & 0\\ \pm 1/2^n & 1\end{pmatrix}\right\},$$
$$[C]^{1/2^n}=\left\{\pm\begin{pmatrix}\cos\frac{\theta}{2^n} & \sin\frac{\theta}{2^n}\\ -\sin\frac{\theta}{2^n} & \cos\frac{\theta}{2^n}\end{pmatrix}\right\}.$$

By setting M_n as $[A]^{1/2^n}$, $[B]^{1/2^n}$, or $[C]^{1/2^n}$, we obtain $M_n \longrightarrow [I_2]=\{\pm I_2\}$ and $M_n^{2^n}=M$, $n=0,1,2,\ldots$. □

COROLLARY 2.4. *Aut(D) is root-approximable, where D is the unit ball in $\mathbb{C}$.*

Note that
$$\mathsf{Aut}(\mathsf{SH}_2)=\mathsf{PSp}(2,\mathbb{R})=\mathsf{Sp}(2,\mathbb{R})/\{\pm I_4\}.$$

For root-approximability and roots, according to Lemma 2.2, it is sufficient to examine only the canonical form, which is stated in the first section. To get the main result, we need the following lemmas.

LEMMA 2.5. *If A and B in $\mathbb{R}^{2\times 2}$ have roots, then $M=A\oplus B\in R^{4\times 4}$ has a root.*

Proof. Let A and B have roots. That is, there exist A_1 and B_1 are in $\mathbb{R}^{2\times 2}$ such that $A=A_1^2$ and $B=B_1^2$. Hence $M=A\oplus B=A_1^2\oplus B_1^2=(A_1\oplus B_1)^2$. □

LEMMA 2.6. *Let $M=X\odot Y$ be in $Sp(2,\mathbb{R})$ where $X=\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}$ and $Y=\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}$. Then X and Y belong to $SL(2,\mathbb{R})$.*

Proof. Since $M=I_{2,3}(X\oplus Y)I_{2,3}$ where $I_{2,3}$ is a root of I_4 ($I_{2,3}^2=I_4$),

$$I_{2,3}=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

It is seen that $M^T=I_{2,3}(X^T\oplus Y^T)I_{2,3}$, therefore $(X^T\odot Y^T)J_2(X\odot Y)=J_2$ and $X^T\odot Y^T=I_{2,3}(X^T\oplus Y^T)I_{2,3}$. Then we have

$$I_{2,3}(X^T\oplus Y^T)I_{2,3}J_2I_{2,3}(X\oplus Y)I_{2,3}=J_2,$$

Thus $(X^T \oplus Y^T)I_{2,3}J_2I_{2,3}(X \oplus Y) = I_{2,3}J_2I_{2,3}$. In adition, $I_{2,3}J_2I_{2,3} = J_1 \oplus J_1$, where $J_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. So $(X^T \oplus Y^T)(J_1 \oplus J_1)(X \oplus Y) = J_1 \oplus J_1$. Hence

$$X^T J_1 X = Y^T J_1 Y = J_1.$$

As a result, $X, Y \in \mathsf{SL}(2,\mathbb{R})$. $\square$

LEMMA 2.7. *Let M be a symplectic matrix in $Sp(2,\mathbb{R})$ and there exist two matrices A and B belong to $SL(2,\mathbb{R})$ such that $I_{2,3}MI_{2,3} = A \oplus B$. Then M has a root in $Sp(2,\mathbb{R})$ if and only if A and B have roots in $SL(2,\mathbb{R})$.*

Proof. Assume that A and B have root. If $N = I_{2,3}MI_{2,3} = A \oplus B$, then $N^{1/2} = A^{1/2} \oplus B^{1/2}$, therefore $M^{1/2} = (I_{2,3}NI_{2,3})^{1/2} = I_{2,3}N^{1/2}I_{2,3}$. Therefore we have

$$\begin{aligned}
(M^{1/2})^T J_2 M^{1/2} &= I_{2,3}(N^{1/2})^T I_{2,3}J_2I_{2,3}N^{1/2}I_{2,3} \\
&= I_{2,3}((A^{1/2})^T \oplus (B^{1/2})^T)(J_1 \oplus J_1)(A^{1/2} \oplus B^{1/2})I_{2,3} \\
&= I_{2,3}((A^{1/2})^T J_1 A^{1/2} \oplus (B^{1/2})^T J_1 B^{1/2})I_{2,3} \\
&= I_{2,3}(J_1 \oplus J_1)I_{2,3} = J_2.
\end{aligned}$$

Therefore $M^{1/2} \in \mathsf{Sp}(2,\mathbb{R})$. Now we assume that M has a root in $\mathsf{Sp}(2,\mathbb{R})$, then

$$M^{\frac{1}{2}} = (I_{2,3}A \oplus BI_{2,3})^{\frac{1}{2}} = I_{2,3}(A \oplus B)^{\frac{1}{2}}I_{2,3}) = I_{2,3}(A^{\frac{1}{2}} \oplus B^{\frac{1}{2}})I_{2,3}.$$

Hence, by Lemma 2.6, $A^{1/2}$ and $B^{1/2}$ belong to $\mathsf{SL}(2,\mathbb{R})$. $\square$

Employing the Remark 1.2, we get the following theorems.

THEOREM 2.8. *Let M be conjugated with type 1. Then M has a root unless*

$$\alpha\beta < 0, |\alpha| \neq 1 \quad \textit{and} \quad |\beta| \neq 1.$$

Proof. Let

$$M = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & \beta^{-1} \end{pmatrix}.$$

In this case, $M = A \oplus B$, where $A = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$ and $B = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix}$. If $\alpha\beta > 0$, then either A and B or $-A$ and $-B$ have roots. Utilizing Lemma 2.5, we deduce that $[M]$ has a root. For $\alpha\beta < 0$, we first assume $|\alpha| = 1$ or $|\beta| = 1$. Then $I_{2,3}MI_{2,3} = C \oplus D$, where C is diagonal with positive entries and D is $-I_2$ (otherwise, we consider $-C$ and $-D$). It is seen that $-I_2$ has a root and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a root of $-I_2$. It follows from

Lemma 2.5 that $[M]$ has a root. Second, if $|\alpha| \neq 1$ and $|\beta| \neq 1$, then we show that $[M]$ does not have any root. In fact, if

$$\begin{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} & O \\ O & \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \end{pmatrix} = I_{2,3}MI_{2,3} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^2$$

$$= \begin{pmatrix} X^2+YZ & XY+YT \\ ZX+TZ & ZY+T^2 \end{pmatrix}. \tag{2.1}$$

It can be assumed that $\alpha > 0$ and that $\beta < 0$. Hence

$$X^2+YZ = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad ZY+T^2 = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \quad XY = -YT, \quad ZX = -TZ.$$

Hence we get

$$XYZ = -YTZ = YZX.$$

This yields that $X^3+XYZ = X^3+YZX$, and thus $X\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}X$. Similarly $T\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}T$, therefore $X = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$ and $T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$. If $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$ and $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$, then

$$X^2+YZ = \begin{pmatrix} x_1^2+y_1z_1+y_2z_3 & y_1z_2+y_2z_4 \\ y_3z_1+y_4z_3 & x_2^2+y_3z_2+y_4z_4 \end{pmatrix}, \tag{2.2}$$

$$XY+YT = \begin{pmatrix} y_1(x_1+t_1) & y_2(x_1+t_2) \\ y_3(x_2+t_1) & y_4(x_2+t_2) \end{pmatrix}, \tag{2.3}$$

$$ZX+TZ = \begin{pmatrix} z_1(x_1+t_1) & z_2(x_1+t_2) \\ z_3(x_2+t_1) & z_4(x_2+t_2) \end{pmatrix}, \tag{2.4}$$

$$ZY+T^2 = \begin{pmatrix} t_1^2+z_1y_1+z_2y_3 & z_1y_2+z_2y_4 \\ z_3y_1+z_4y_3 & t_2^2+z_3y_2+z_4y_4 \end{pmatrix}. \tag{2.5}$$

Assuming different modes on y_1 and y_2, such as whether they are zero or not, we get four cases.

Case 1. If $y_1 = y_2 = 0$ then by (2.5) and (2.1), we have $z_2y_3 \neq 0$ and $z_4y_4 \neq 0$. Hence $z_2y_4 \neq 0$. On the other hand, by employing (2.5), we have $z_2y_4 = 0$, which is a contradiction.

Case 2. If $y_1 = 0$ and $y_2 \neq 0$, then by (2.1), (2.2) and (2.3), $t_2 = -x_1$, $y_2z_4 = 0$. Hence $z_4 = 0$. Again, by making use of (2.3) and (2.5), we arrive at $x_1^2+y_2z_3 = \alpha$, and $x_1^2+z_3y_2 = \beta^{-1}$. This is a contradiction.

Case 3. If $y_1 \neq 0$ and $y_2 = 0$ then by (2.1), (2.2) and (2.3), $t_1 = -x_1$, $y_1z_2 = 0$. Thus $z_2 = 0$, hence $x_1^2+y_1z_1 = \alpha$ and $x_1^2+y_1z_1 = \beta$ is contradict.

Case 4. If $y_1 \neq 0$, $y_2 \neq 0$, then (2.1) and (2.3) ensure that $t_1 = t_2 = -x_1$. If $x_1 = x_2$, by employing (2.1), (2.2) and (2.5) we have $y_2z_3 - y_3z_2 = \alpha - \beta > 0$ and $y_2z_3 - y_3z_2 = \beta^{-1} - \alpha^{-1} < 0$, which is a contradiction. Now $x_1 \neq x_2$. Then by (2.4), $z_3 = z_4 = 0$, therefore by (2.1) and (2.5), give $t_2^2 = \beta^{-1} < 0$ which is a contradiction. □

THEOREM 2.9. *Let M be conjugated with one of the type 2, type 6 or type 7. Then M has a root in $\mathsf{Aut}(SH_2)$.*

Proof. type 2. Let

$$[M]=\left\{\pm\begin{pmatrix}\alpha & 0 & 0 & 0\\ 1 & \alpha & 0 & 0\\ 0 & 0 & \alpha^{-1} & -\alpha^{-2}\\ 0 & 0 & 0 & \alpha^{-1}\end{pmatrix}\right\}.$$

Then, without loss of generality, assume $\alpha>0$. Hence $[M]^{1/2}=\{\pm A\oplus B\}$, where $A=\begin{pmatrix}\sqrt{\alpha} & 0\\ \frac{1}{2\sqrt{\alpha}} & \sqrt{\alpha}\end{pmatrix}$ and $B=\begin{pmatrix}\frac{1}{\sqrt{\alpha}} & \frac{-1}{2\alpha\sqrt{\alpha}}\\ 0 & \frac{1}{\sqrt{\alpha}}\end{pmatrix}$. Moreover $[M]^{1/2}\in\mathsf{PSp}(2,\mathbb{R})$.

type 6. Let

$$M=\begin{pmatrix}a_1 & 0 & -c_1 & 0\\ 0 & a_2 & 0 & 0\\ c_1 & 0 & a_1 & 0\\ 0 & \delta & 0 & a_2^{-1}\end{pmatrix},\quad a_1^2+c_1^2=1,\ \ \delta=0,\pm1,$$

and let

$$N=I_{2,3}MI_{2,3}=\begin{pmatrix}A & O\\ O & B\end{pmatrix}=A\oplus B,$$

where $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ and $B=\begin{pmatrix}a_2 & 0\\ \delta & \frac{1}{a_2}\end{pmatrix}$. An application of Theorem 2.3 gives us that A always has roots. If $a_2>0$, then $B^{1/2}=\begin{pmatrix}\sqrt{a_2} & 0\\ \frac{\delta}{\sqrt{a_2}+\sqrt{a_2^{-1}}} & \frac{1}{\sqrt{a_2}}\end{pmatrix}$. If $a_2<0$, then $-A$ and $-B$ have roots. From Lemma 2.7 we conclude that for each $a_2\neq 0$, $[M]$ has a root in $\mathsf{PSp}(2,\mathbb{R})$.

type 7. Let $a_1^2+b_1^2=a_2^2+b_2^2=1$, and let

$$\begin{aligned}M=\begin{pmatrix}a_1 & 0 & b_1 & 0\\ 0 & a_2 & 0 & b_2\\ -b_1 & 0 & a_1 & 0\\ 0 & -b_2 & 0 & a_2\end{pmatrix}&=I_{2,3}\begin{pmatrix}a_1 & b_1\\ -b_1 & a_1\end{pmatrix}\oplus\begin{pmatrix}a_2 & b_2\\ -b_2 & a_2\end{pmatrix}I_{2,3}\\
&=I_{2,3}\begin{pmatrix}a_1' & b_1'\\ -b_1' & a_1'\end{pmatrix}^2\oplus\begin{pmatrix}a_2' & b_2'\\ -b_2' & a_2'\end{pmatrix}^2 I_{2,3}\qquad\text{(By Theorem 2.3)}\\
&=\left[I_{2,3}\begin{pmatrix}a_1' & b_1'\\ -b_1' & a_1'\end{pmatrix}\oplus\begin{pmatrix}a_2' & b_2'\\ -b_2' & a_2'\end{pmatrix}I_{2,3}\right]^2\\
&=\begin{pmatrix}a_1' & 0 & b_1' & 0\\ 0 & a_2' & 0 & b_2'\\ -b_1' & 0 & a_1' & 0\\ 0 & -b_2' & 0 & a_2'\end{pmatrix}^2,\end{aligned}$$

where ${a_1'}^2+{b_1'}^2={a_2'}^2+{b_2'}^2=1$. Hence M has a root belonging to $\mathsf{Sp}(2,\mathbb{R})$. □

THEOREM 2.10. *Let M be conjugated with type 3. Then*

(i) If $-1 < a \leqslant 1$ and $\delta = 0, \pm 1$, then M has a root.

(ii) If $a = -1$ and $\delta = \pm 1$, then M has no root.

Proof. Let $M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2a & 0 & 0 \\ 0 & \delta & 2a & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

(i). Let $-1 < a \leqslant 1$ and let $\delta = 0, \pm 1$. If we set

$$A = \frac{1}{\sqrt{2(a+1)}} \begin{pmatrix} 1 & 1 \\ -1 & 2a+1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2(a+1)}} \begin{pmatrix} 2a+1 & 1 \\ -1 & 1 \end{pmatrix} \text{ and}$$
$$C = \frac{\sqrt{a+1}\delta}{\sqrt{2}[1+(2a+1)(2a+3)]} \begin{pmatrix} 1 & 2a+3 \\ 1 & 1 \end{pmatrix},$$

then $A^2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 2a \end{smallmatrix}\right)$ and $B^2 = \left(\begin{smallmatrix} 2a & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $CA + BC = \left(\begin{smallmatrix} 0 & \delta \\ 0 & 0 \end{smallmatrix}\right)$. Hence $M = \left(\begin{smallmatrix} A & O \\ C & B \end{smallmatrix}\right)^2$ with $\left(\begin{smallmatrix} A & O \\ C & B \end{smallmatrix}\right)$ is in $\mathsf{Sp}(2,\mathbb{R})$.

(ii). Let $a = -1$ and let $\delta = \pm 1$. Then we show that M does not have any root. If

$$M = \begin{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ -1 & -2 \end{smallmatrix}\right) & O \\ \left(\begin{smallmatrix} 0 & \delta \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} -2 & 1 \\ -1 & 0 \end{smallmatrix}\right) \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^2 = \begin{pmatrix} X^2 + YZ & XY + YT \\ ZX + TZ & ZY + T^2 \end{pmatrix}. \tag{2.6}$$

It follows from (2.6) that $XYZ = YTZ$ and

$$X \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} = X(X^2 + YZ) = X^3 + XYZ = X^3 - YTZ, \tag{2.7}$$

again by (2.6) we have

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} X &= (X^2 + YZ)X = X^3 + YZX \\ &= X^3 + Y\left(\begin{pmatrix} 0 & \delta \\ 0 & 0 \end{pmatrix} - TZ\right) \\ &= X^3 + Y \begin{pmatrix} 0 & \delta \\ 0 & 0 \end{pmatrix} - YTZ. \end{aligned} \tag{2.8}$$

By employing (2.7) and (2.8), we get

$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} X - X \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} = Y \begin{pmatrix} 0 & \delta \\ 0 & 0 \end{pmatrix}, \tag{2.9}$$

similarly

$$T \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} X - \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} T = \begin{pmatrix} 0 & \delta \\ 0 & 0 \end{pmatrix} Y. \tag{2.10}$$

Let

$$X=\begin{pmatrix} x_4 & x_2 \\ x_3 & x_1 \end{pmatrix},\ Y=\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix},\ T=\begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}.$$

In light of (2.9) and (2.10), we get

$$X=\begin{pmatrix} x_1+2x_2 & x_2 \\ -x_2 & x_1 \end{pmatrix},\ \ Y=\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix},\ \ T=\begin{pmatrix} t_1 & t_2 \\ -t_2 & t_1+2t_2 \end{pmatrix}. \tag{2.11}$$

In this case we have

$$\begin{pmatrix} 0 & y(x_1+2x_2) \\ 0 & -x_2y \end{pmatrix}+\begin{pmatrix} -t_2y & y(t_1+2t_2) \\ 0 & 0 \end{pmatrix}=XY+YT=O. \tag{2.12}$$

Then (2.12) gives $t_2y=x_2y=y(t_1+x_1)=0$. If $y\neq 0$ then $x_2=t_2=0$, $t_1=-x_1$, thus $X=\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$, $T=\left(\begin{smallmatrix} -x & 0 \\ 0 & -x \end{smallmatrix}\right)$. If $Z=\left(\begin{smallmatrix} z_1 & z_2 \\ z_3 & z_4 \end{smallmatrix}\right)$, then by (2.6) we have

$$\left(\begin{smallmatrix} 0 & 1 \\ -1 & -2 \end{smallmatrix}\right)=X^2+YZ=\left(\begin{smallmatrix} x^2 & 0 \\ 0 & x^2 \end{smallmatrix}\right)+\left(\begin{smallmatrix} yz_3 & yz_4 \\ 0 & 0 \end{smallmatrix}\right)\Rightarrow x^2=-2$$

which is impossible. Therefore $y=0$ and then $Y=O$,

$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}=X^2=\begin{pmatrix} (x_1+2x_2)^2-x_2^2 & 2x_2(x_1+x_2) \\ -2x_2(x_1+x_2) & x_1^2-x_2^2 \end{pmatrix},$$

then $0=(x_1+2x_2)^2-x_2^2=(x_1+x_2)(x_1+3x_2)$, and $x_1+x_2\neq 0$, therefore $x_1=-3x_2$ and hence $-2=-x_2^2+x_1^2=8x_2^2$, which is a contradiction. □

THEOREM 2.11. *Let M be conjugated with type 4. Then*

(i). If $\alpha>0$, then M has a root.

(ii). If $\alpha<0$, then M has no root.

Proof. Let

$$M=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ \delta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\alpha \end{pmatrix}\ \alpha\neq\pm 1,\ \ \delta=\pm 1.$$

(i). If $\alpha>0$, then $M=\left(\begin{smallmatrix} A & O \\ C & B \end{smallmatrix}\right)^2$, where $A=\left(\begin{smallmatrix} 1 & 0 \\ 0 & \sqrt{\alpha} \end{smallmatrix}\right)$, $B=\left(\begin{smallmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\alpha}} \end{smallmatrix}\right)$ and $C=\left(\begin{smallmatrix} \frac{\delta}{2} & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Hence

$$M^{1/2}=\begin{pmatrix} A & O \\ C & B \end{pmatrix}\in \mathsf{SP}(2,\mathbb{R}).$$

(ii). Let $-1\neq\alpha<0$, let $\delta=\pm 1$, and let

$$N=I_{2,3}MI_{2,3}=\begin{pmatrix} C & O \\ O & D \end{pmatrix}=C\oplus D,$$

where $C = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$ and $D = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}$. If $N = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^2$, then by a same computation in type 3 we get $X = \begin{pmatrix} x_1 & 0 \\ x_2 & x_1 \end{pmatrix}$ and $T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$. If $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$ and $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$, then we get

$$X^2 + YZ = \begin{pmatrix} x_1^2 + y_1 z_1 + y_2 z_3 & y_1 z_2 + y_2 z_4 \\ 2x_1 x_2 + y_3 z_1 + y_4 z_3 & x_1^2 + y_3 z_2 + y_4 z_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}, \tag{2.13}$$

$$XY + YT = \begin{pmatrix} y_1(x_1 + t_1) & y_2(x_1 + t_2) \\ x_2 y_1 + y_3(x_1 + t_1) & x_1 y_2 + y_4(x_1 + t_2) \end{pmatrix} = O, \tag{2.14}$$

$$ZX + TZ = \begin{pmatrix} z_1(x_1 + t_1) + x_2 z_2 & z_2(x_1 + t_1) \\ z_3(x_1 + t_2) + x_2 z_4 & z_4(x_1 + t_2) \end{pmatrix} = O, \tag{2.15}$$

$$ZY + T^2 = \begin{pmatrix} t_1^2 + z_1 y_1 + z_2 y_3 & z_1 y_2 + z_2 y_4 \\ z_3 y_1 + z_4 y_3 & t_2^2 + z_3 y_2 + z_4 y_4 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}. \tag{2.16}$$

If z_1 and z_2 are zero or nonzero, we calculated like type 1, we get four cases.

Case 1. If $z_1 = z_2 = 0$, then by (2.16), $t_1^2 = \alpha < 0$, which it is a contradiction.

Case 2. If $z_1 = 0$ and $z_2 \neq 0$, then by (2.16), $x_2 = 0$. Hence by (2.13), $y_4 z_3 = \delta \neq 0$ thus $y_4 \neq 0$ and $z_2 y_4 = 0$, so note that $y_4 = 0$, this is a contradiction.

Case 3. If $z_1 \neq 0$ and $z_2 = 0$, then by (2.15) and (2.16), $t_1 = -x_1$, $y_2 z_1 = 0$, $x_1^2 + y_1 z_1 + y_2 z_3 = 1$ and $x_1^2 + y_1 z_1 = \alpha$. Therefore $y_2 z_3 = 1 - \alpha > 0$, hence $y_2 \neq 0$ thus $z_1 = 0$. It is a contradiction.

Case 4. If $z_1 \neq 0$ and $z_2 \neq 0$, then by (2.15), $t_1 = -x_1$ and $x_2 = 0$. If $z_3 \neq 0$, then $t_2 = x_1$. Therefore, (2.13) and (2.16) entail that $y_2 z_3 - y_3 z_2 = 1 - \alpha > 0$ and $y_2 z_3 - y_3 z_2 = \alpha^{-1} - 1 < 0$. In this case, we get a contradiction. If $z_3 = 0$, then (2.13) and (2.16) imply $z_4 y_3 = 0$ and $y_3 z_1 = \delta \neq 0$. Thus $y_3 \neq 0$ and $z_4 = 0$ therefore $t_2^2 = \alpha^{-1} < 0$, this is a contradiction. □

THEOREM 2.12. *Let M be conjugated with type 5. Then M has a root unless*

$$\alpha = -1 \quad \textit{and} \quad \delta_1 \delta_2 \neq 0.$$

Proof. Let

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ \delta_1 & 0 & 1 & 0 \\ 0 & \delta_2 & 0 & \alpha \end{pmatrix} \quad \alpha = \pm 1,\ \delta_1, \delta_2 = 0, \pm 1.$$

Put

$$N = I_{2,3} M I_{2,3} = \begin{pmatrix} A & O \\ O & B \end{pmatrix} = A \oplus B,$$

where $A = \begin{pmatrix} 1 & 0 \\ \delta_1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} \alpha & 0 \\ \delta_2 & \alpha \end{pmatrix}$. If $\alpha = \pm 1$ and $\delta_1 \delta_2 = 0$, then make use of the proof of Theorem 2.3 (by setting $n = 1$) to get either A and B or $-A$ and $-B$ have roots. Note that in the case $\alpha = -1$ and $\delta_1 = \delta_2 = 0$, we have

$$B = -I_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2.$$

An application of Lemma 2.7 yields that $[M]$ has a root in $\mathsf{PSp}(2,\mathbb{R})$. For $\alpha = 1$ and $\delta_1\delta_2 \neq 0$, we get

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \delta_1 & 0 & 1 & 0 \\ 0 & \delta_2 & 0 & 1 \end{pmatrix}$$

then

$$N = I_{2,3}MI_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \delta_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \delta_1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ \delta_2 & 1 \end{pmatrix}$$
$$= \left[\begin{pmatrix} 1 & 0 \\ \delta_1/2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ \delta_2/2 & 1 \end{pmatrix} \right]^2 .$$

It follows from Lemma 2.7 that M has a root. Moreover, for $\alpha = -1$ and $\delta_1\delta_2 \neq 0$, we show that M is not has a root. In this case, if $N = I_{2,3}MI_{2,3}$, then

$$N = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta_1 & 1 \end{pmatrix} & O \\ O & \begin{pmatrix} -1 & 0 \\ \delta_2 & -1 \end{pmatrix} \end{pmatrix}.$$

If $N = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^2$, then

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta_1 & 1 \end{pmatrix} & O \\ O & \begin{pmatrix} -1 & 0 \\ \delta_2 & -1 \end{pmatrix} \end{pmatrix} = N = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^2 = \begin{pmatrix} X^2 + YZ & XY + YT \\ ZX + TZ & ZY + T^2 \end{pmatrix}. \tag{2.17}$$

By the same computation in type 3, we have $X\begin{pmatrix} 1 & 0 \\ \delta_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \delta_1 & 1 \end{pmatrix}X$ and $T\begin{pmatrix} -1 & 0 \\ \delta_2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ \delta_2 & -1 \end{pmatrix}T$ so $X = \begin{pmatrix} x_1 & 0 \\ x_2 & x_1 \end{pmatrix}$ and $T = \begin{pmatrix} t_1 & 0 \\ t_2 & t_1 \end{pmatrix}$. If $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$ and $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$ then by (2.17), we get

$$\begin{cases} x_1^2 + y_1z_1 + y_2z_3 &= 1 \\ y_1z_2 + y_2z_4 &= 0 \\ 2x_1x_2 + y_3z_1 + y_4z_3 &= \delta_1 \\ x_1^2 + y_3z_2 + y_4z_4 &= 1 \end{cases}, \quad \begin{cases} t_1^2 + y_1z_1 + y_3z_2 &= -1 \\ y_2z_1 + y_4z_2 &= 0 \\ 2t_1t_2 + y_1z_3 + y_3z_4 &= \delta_2 \\ t_1^2 + y_2z_3 + y_4z_4 &= -1, \end{cases} \tag{2.18}$$

also

$$\begin{cases} x_1y_2 &= -y_2t_1 \\ x_1y_1 &= -y_1t_1 - y_2t_2 \\ x_2y_1 + x_1y_3 &= -y_3t_1 - y_4t_2 \\ x_2y_2 + x_1y_4 &= -y_4t_1 \end{cases}, \quad \begin{cases} x_1z_2 &= -t_1z_2 \\ x_1z_1 + x_2z_2 &= -t_1z_1 \\ x_1z_3 + x_2z_4 &= -t_2z_1 - t_1z_3 \\ x_1z_4 &= t_2z_2 - t_1z_4. \end{cases} \tag{2.19}$$

If $y_2 \neq 0$ then by (2.19) $x_1 = -t_1$ thus (2.18) gives $y_2z_3 - y_3z_2 = 2$ and $y_2z_3 - y_3z_2 = -2$ which is a contradiction. If $y_2 = 0$, by (2.19) we get $x_1y_4 = -y_4t_1$. If $y_4 \neq 0$, by (2.18)

$z_2 = 0$, thus (2.18) gives $x_1^2 + y_1 z_1 = t_1^2 + y_1 z_1 = 1$ and $x_1^2 + y_1 z_1 = t_1^2 + y_1 z_1 = -1$ hence we get a contradiction. If $y_4 = 0$, then by (2.18) we have $t_1^2 = -1$ which is imposible. Therefore N and hence M has no root. □

The above theorems give the following result:

THEOREM 2.13. *Some elements of $Aut(SH_2)$ have a root in $Aut(SH_2)$. Thus it has root-approximable subsets.*

Making use of Lemmas 2.5, 2.7, and the proof of Theorem 2.8, we get the following corollaries.

COROLLARY 2.14. *Let M be a conjugation of $\begin{pmatrix} A & O \\ O & A^{-T} \end{pmatrix}$, where $A = \operatorname{diag}(\alpha, \beta)$ such that either $\alpha, \beta \in \mathbb{R}$, $\alpha\beta > 0$, or $\alpha\beta < 0$ with $\alpha = \pm 1$ or $\beta = \pm 1$. Then M is a root-approximable subset in $PSp(2, \mathbb{R})$.*

COROLLARY 2.15. *Suppose that $G = \{X \odot Y : X, Y \in SL(2, \mathbb{R}) \text{ and } X, Y \text{ have roots}\}$. Then G is a root-approximable subset of $Sp(2, \mathbb{R})$.*

In the following remark, we are generalizing a previous result, about getting roots of matrices in $\mathsf{Sp}(n, \mathbb{R})$ from roots in $\mathsf{SL}(2, \mathbb{R})$.

REMARK 2.16. Let $A_1, A_2, \ldots, A_n$ be in $\mathsf{SL}(2, \mathbb{R})$. Since $U(A_1 \oplus A_2 \oplus \cdots \oplus A_n)U^T = A_1 \odot A_2 \odot \cdots \odot A_n \in \mathsf{Sp}(n, \mathbb{R})$, where

$$U = (e_1, e_3, \ldots, e_{2n-1}, e_2, e_4, \ldots, e_{2n})^T$$

is an orthogonal matrix. If $A_1, A_2, \ldots, A_n$ have roots in $\mathsf{SL}(2, \mathbb{R})$, then $A_1 \odot A_2 \odot \cdots \odot A_n$ is a root-approximable in $\mathsf{Sp}(n, \mathbb{R})$. Because if, by use of the following representation

$$\bigodot_{i=1}^{n} A_i = A_1 \odot A_2 \odot \ldots \odot A_n, \text{ and} \bigoplus_{i=1}^{n} A_i = A_1 \oplus A_2 \oplus \ldots \oplus A_n,$$

we have

$$\bigodot_{i=1}^{n} A_i = U(\bigoplus_{i=1}^{n} A_i)U^T,$$

then

$$(\bigodot_{i=1}^{n} A_i)^{1/2} = U(\bigoplus_{i=1}^{n} A_i)^{1/2} U^T = U(\bigoplus_{i=1}^{n} A_i^{1/2})U^T,$$

hence

$$\begin{aligned}
[(\bigodot_{i=1}^{n} A_i)^{1/2}]^T J_n (\bigodot_{i=1}^{n} A_i)^{1/2} &= U(\bigoplus_{i=1}^{n} (A_i^{1/2})^T)U^T J_n U \bigoplus_{i=1}^{n} (A_i)^{1/2} U^T \\
&= U(\bigoplus_{i=1}^{n} (A_i^{1/2})^T)(\bigoplus_{i=1}^{n} J_1)(\bigoplus_{i=1}^{n} A_i^{1/2})U^T \\
&= U(\bigoplus_{i=1}^{n} [(A_i^{1/2})^T J_1 A_i^{1/2}])U^T = U(\bigoplus_{i=1}^{n} J_1)U^T = J_n.
\end{aligned}$$

Hence, it is a root-approximable element in $\mathsf{Sp}(n,\mathbb{R})$. It follows from Theorem 2.3 that A or $-A$ has a root in $\mathsf{SL}(2,\mathbb{R})$ for all $A \in \mathsf{SL}(2,\mathbb{R})$. If either $A_1, A_2, \dots, A_n$ or $-A_1, -A_2, \dots, -A_n$ have roots in $\mathsf{SL}(2,\mathbb{R})$, then $A_1 \odot A_2 \odot \cdots \odot A_n$ is root-approximable in $\mathsf{PSp}(n,\mathbb{R})$.

REMARK 2.17. We examine the automorphisms on SD_n. Define a map $\Phi : SH_n \to SD_n$ by $Z \mapsto (Z-iI)(Z+iI)^{-1}$. Hence, it follows from [6] that Φ is a biholomorphic from SH_n onto SD_n with

$$\Phi^{-1} : \mathsf{SD}_n \longrightarrow \mathsf{SH}_n \text{ by } Z \mapsto i(I+Z)(I-Z)^{-1}.$$

Then all biholomorphisms on SD_n are given by the generalized Möbius transformation. That is,

$$\mathsf{Aut}(\mathsf{SD_n}) = \{\Phi \circ \Psi \circ \Phi^{-1};\ \Psi \in \mathsf{Aut}(\mathsf{SH_n})\}.$$

In the case where $n=2$, $[\Phi]M[\Phi^{-1}]$ has the following form: $[\Phi] = \left(\begin{smallmatrix} I & -iI \\ I & iI \end{smallmatrix}\right)$, $[\Phi] = i\left(\begin{smallmatrix} I & I \\ -I & I \end{smallmatrix}\right)$, where $I = I_2$. If $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathsf{Sp}(2,\mathbb{R})$, then

$$\begin{aligned}\widetilde{M} = [\Phi]M[\Phi^{-1}] &= i\begin{pmatrix} I & -iI \\ I & iI \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & I \\ -I & I \end{pmatrix} \\ &= \begin{pmatrix} C-D+i(A-B) & C+D+i(A+B) \\ -C+D+i(A-B) & -C-D+i(A+B) \end{pmatrix}.\end{aligned}$$

Theorem 2.4 ensures that all automorphisms on SD_2 have roots except those of the following forms:

1)

$$\begin{aligned}\widetilde{M}_1 &= \begin{pmatrix} -\alpha^{-1}+i\alpha & 0 & \alpha^{-1}+i\alpha & 0 \\ 0 & -\beta^{-1}+i\beta & 0 & \beta^{-1}+i\beta \\ \alpha^{-1}+i\alpha & 0 & -\alpha^{-1}+i\alpha & 0 \\ 0 & \beta^{-1}+i\beta & 0 & -\beta^{-1}+i\beta \end{pmatrix} \\ &= \begin{pmatrix} -\alpha^{-1}+i\alpha & \alpha^{-1}+i\alpha \\ \alpha^{-1}+i\alpha & -\alpha^{-1}+i\alpha \end{pmatrix} \odot \begin{pmatrix} -\beta^{-1}+i\beta & \beta^{-1}+i\beta \\ \beta^{-1}+i\beta & -\beta^{-1}+i\beta \end{pmatrix},\end{aligned}$$

where $\alpha\beta < 0$, $\alpha \neq \pm 1$, and $\beta \neq \pm 1$.

2)

$$\widetilde{M}_2 = \begin{pmatrix} 2 & \delta-1+i & -2 & \delta+1+i \\ 1-i & -2i & -1-i & -2i \\ -2 & -\delta+1+i & 2 & -\delta-1+i \\ -1-i & -2i & 1-i & -2i \end{pmatrix}, \quad \text{where } \delta = \pm 1,$$

3)

$$\widetilde{M}_3 = \begin{pmatrix} \delta-1+i & 0 & \delta+1+i & 0 \\ 0 & -\alpha^{-1}+i\alpha & 0 & \alpha^{-1}+i\alpha \\ -\delta+1+i & 0 & -\delta-1+i & 0 \\ 0 & \alpha^{-1}+i\alpha & 0 & -\alpha^{-1}+i\alpha \end{pmatrix},$$

where $-1 \neq \alpha < 0$, and $\delta = \pm 1$.

4)

$$\widetilde{M}_4 = \begin{pmatrix} \delta_1 - 1 + i & 0 & \delta_1 + 1 + i & 0 \\ 0 & \delta_2 + 1 - i & 0 & \delta_2 - 1 - i \\ -\delta_1 + 1 + i & 0 & -\delta_1 - 1 + i & 0 \\ 0 & -\delta_2 - 1 - i & 0 & -\delta_2 + 1 - i \end{pmatrix},$$

where $\delta_1, \delta_2 = \pm 1$.

REFERENCES

[1] D. A. BINI, N. J. HIGHAM, B. MEINI, *Algorithms for the matrix pth root*, Numer. Algorithms, **39** (2005) 349–378.

[2] A. CHADEMAN, F. MIRZAPOUR, *Midconvex functions in locally compact groups*, Proc. Amer. Math. Soc. **127** (1999), no. 10, 2961–2968.

[3] S. H. CHENG, N. J. HIGHAM, C. S. KENNEY, A. J. LAUB, *Approximating the logarithm of a matrix to specified accuracy*, Siam J. Matrix Anal. Appl. **22** (2001), no. 4, 1112–1125.

[4] S. MAGHSOUDI, F. MIRZAPOUR, *Root-Approximability of the group of automorphisms of the unit ball in* $\mathbb{R}^n$, Bull. Malays. Math. Sci. Soc. **39** (2016) 1477–1485.

[5] F. MIRZAPOUR, *A note on the group of automorphisms of unit disc*, Inter. J. of App. Math. **14**, (2003), no. 1 51–55.

[6] C. SHMUEL FRIEDLAND, PEDRO J. FREITAS, *Revisiting the Siegel upper half plane I*, Linear Algebra Appl. **376** (2004) 19–44.

[7] C. PEDRO J. FREITAS, SHMUEL FRIEDLAND, *Revisiting the Siegel upper half plane II*, Linear Algebra Appl. **376** (2004) 45–67.

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