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ON SOLVABILITY OF VOLTERRA–HAMMERSTEIN INTEGRAL EQUATIONS IN TWO VARIABLES COORDINATEWISE CONVERGING AT INFINITY

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Abstract. We will investigate the existence of solutions to an infinite system of nonlinear integral equations in two variables of the Volterra-Hammerstein type. The approach we take in our research relates to the construction of an appropriate measure of noncompactness in the space of functions defined, continuous, and bounded on $\mathbb{R}_+ \times \mathbb{R}_+$ with values in the space ℓ_∞ endowed with the standard supremum norm and created by function sequences that are coordinatewise converging to proper limits at infinity. Our research is illustrated with an example.

1. Introduction

This section is for establishing the notation utilized in the paper. We also provide concepts that serve as the foundation for our research, as well as certain information about the theory of measure of noncompactness (MNC) that are pertinent to our concerns.

Integral equations are well-known for their use in the description of a wide range of real-world occurrences, and they form a significant area of nonlinear functional analysis. Obviously, the theory of integral equations and the science of differential equations are intertwined (see [1, 4, 7, 9, 10, 15, 18, 19]). Recently, various effective attempts have been made to apply the idea of measure of noncompactness to the study of the existence and behaviour of nonlinear integral equation solutions (see [5, 6, 12, 17, 15, 18, 19, 20]).

Investigations of infinite systems of integral equations are related with the representation of solutions of those systems in the form of function sequences defined on an interval. The presence of solutions is challenging but not particularly difficult problem when the interval is bounded. However, the situation becomes more challenging when we are looking for function sequences being solutions to an infinite system of integral equations specified on an unbounded interval.

The solutions of infinite systems of integral equations that are defined on $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_+ \times \mathbb{R}_+$ have only recently been the subject of a few works (see [5, 6,

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12, 14]). When examining such solutions, it is necessary to build the right techniques that will allow us to use a fixed point theorem that is applicable to the circumstance under consideration. As it turns out, the technique of appropriate measures of noncompactness constructed in the space of functions defined, continuous, and bounded on the interval $\mathbb{R}_+$ with values in the sequence space, for instance, in the spaces c_0, ℓ_1 and ℓ_∞, can be used as the necessary technique. In view of the expected generality of the results, the sequence space ℓ_∞ appears to be the most appropriate for our needs. Such a direction of investigations was initiated in the papers [5, 6].

With the help of a measure of noncompactness defined in [14], we continue and extend our investigations in the direction of integral equation solutions to infinite systems, and we hope to produce the most fascinating and useful findings. In particular, we demonstrate that, under reasonable assumptions, a solution to a system of integral equations that is thought to be infinite exists. This solution is represented by a function sequence $(x_p(w,s))$ defined on the square $\mathbb{R}_+^2$, where each coordinate $x_p = (x_p(w,s))$ tends to a suitable limit at infinity. Additionally, the sequence created by those suitable limits is a component of the ℓ_∞ sequence space.

2. Prelimaneries and background

We will use the standard notation. Namely, by the symbol $\mathbb{R}$ we will denote the set of real numbers while $\mathbb{N}$ stands for the set of natural numbers.

The Kuratowski measure of noncompactness for a bounded subset D of a metric space X is defined as

$$\alpha(D) = \inf\left\{\delta > 0 : D \subset \cup_{i=1}^{n} D_i, \operatorname{diam}(D_i) \leqslant \delta,\ \text{for } 1 \leqslant i \leqslant m < \infty\right\},$$

where $\operatorname{diam}(D_i)$ denotes diameter of the set D_i.

Another important measure of non-compactness is the Hausdorff measure of noncompactness, which is defined as $\phi(D) = \inf\left\{\varepsilon > 0 : D \text{ has a finite } \varepsilon\text{-net in } E\right\}$.

It can be shown that the Hausdorff measure of noncompactness ϕ is regular and it is equivalent to the Kuratowski measure $\alpha(X)$. More precisely, for an arbitrary set $X \in M_E$, the following inequality holds (see [5]):

$$\phi(X) \leqslant \alpha(X) \leqslant 2\phi(X).$$

Let $(X, ||.||)$ be a Banach space, $\mathbb{R}_+ = [0, \infty)$, the symbols $\bar{X}$ and $\operatorname{Conv}(X)$ denote closure of X and convex closure of X respectively. Let M_E denote the family of non-empty bounded subsets of E and N_E its subfamily consists of relatively compact subsets of E. We now define (MNC) axiomatically given by Banas and Goebel [8].

DEFINITION 2.1. [8] Let X be a Banach space. A function $\phi : M_X \to [0, +\infty)$ is said to be measure of non-compactnes in X if it satisfies the following axioms:

1. The family $\ker \phi = \{E \in M_X : \phi(E) = 0\}$ is a nonempty and $\ker \phi \subset N_X$.

2. $E_1 \subset E_2 \Rightarrow \phi(E_1) \leqslant \phi(E_2)$.

3. $\phi(\overline{E}) = \phi(E)$.

4. $\phi(\mathrm{Conv}(E)) = \phi(E)$.

5. $\phi(\lambda E_1 + (1-\lambda E_2) \leqslant \lambda\phi(E_1) + (1-\lambda)\phi(E_2)$ for all $\lambda \in (0,1)$.

6. If (E_m) is a sequence of closed sets from M_X such that $E_{m+1} \subset E_m$ and $\lim_{m\to\infty} \phi(E_m) = 0$, then the intersection set $E_\infty = \bigcap_{m=1}^{\infty} E_m$ is non-empty.

The family ker ϕ appearing in axiom (i) will be called the kernel of the measure of noncompactness ϕ. Let us notice that the set X_∞ described in axiom (vi) is a member of the family ker ϕ. Indeed, it is a simple consequence of the inclusion $X_\infty \subset X_p$ for $p = 1,2,\ldots$ and axiom (vi) which implies the inequality $\phi(X_\infty) \leqslant \phi(X_p)$ for $p = 1,2,\ldots$. Hence we have $\phi(X_\infty) = 0$. Consequently, $\phi(X_\infty) \in ker\phi$. The above simple observation is quite important in applications.

In what follows let us assume that ϕ is the measure of noncompactness in E. The measure ϕ will be called subadditive if

7. $\phi(X+Y) \leqslant \phi(X) + \phi(Y)$.

8. $\phi(\lambda X) = |\lambda|\phi(X)$, $\lambda \in \mathbb{R}$,

then ϕ is said to be homogenuous. The measure ϕ satisfying both 7 and 8 is called sublinear.

If the measure of noncompactness ϕ satisfies the condition

9. $\phi(X \cup Y) = \max\{\phi(X), \phi(Y)\}$,

then we will say that ϕ has the maximum property. Finally, let us remind [8] that the measure ϕ such that $\ker\phi = N_E$ is called full. If ϕ is sublinear and full measure of noncompactness with maximum property then ϕ is said to be regular.

Further on, we are going to describe a measure of noncompactness used in considerations of this paper. Assume that E is an infinite dimensional Banach space and that ϕ is a measure of noncompactness defined in E.

Consider the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ which consists of functions that are defined, continuous and bounded on $\mathbb{R}_+ \times \mathbb{R}_+$ and have values in the space E. We consider the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ with the supremum norm

$$||x||_\infty = \sup\left\{||x(w,s)||_E : w,s \in \mathbb{R}_+ \times \mathbb{R}_+\right\},$$

where the symbol $||.||_E$ denotes the norm of the space E. $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ is clearly a Banach space with the above mentioned norm.

Simultaneously, we consider the space $C_\zeta = C([0,\zeta]^2, E)$, where $\zeta > 0$ is arbitrarily fixed. Recall, that the C_ζ defines norm as

$$||x||_\zeta = \sup\left\{||x(w,s)||_E : w,s \in [0,\zeta]\right\}.$$

If we take a function $x \in BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$, we can consider the restriction $x|_{[0,\zeta]^2}$ of x to the square $[0,\zeta]^2$ is an element of the space C_ζ.

Let us take an arbitrary and bounded set X, $X \subset BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ for the reminder of this section. Next, let us define the quantity $\Omega^\infty(x,\varepsilon)$ for an arbitrarily fixed function $x \in X$ and for $\varepsilon > 0$ as follows:

$$\Omega^\infty(x,\varepsilon) = \sup\left\{||x(w,s) - x(u,v)||_E : (w,s),(u,v) \in \mathbb{R}_+^2, |w-u| \leqslant \varepsilon, |s-v| \leqslant \varepsilon\right\}. \tag{2.1}$$

Observe that $\lim_{\varepsilon \to 0} \Omega^\infty(x,\varepsilon) = 0$ if and only if the function $x = x(w,s)$ is uniformly continuous on the square $\mathbb{R}_+^2$.

Further, taking into account (2.1), for $X \in M_{BC(\mathbb{R}_+ \times \mathbb{R}_+, E)}$, we define

$$\Omega^\infty(X,\varepsilon) = \sup\left\{\Omega^\infty(x,\varepsilon) : x \in X\right\}, \tag{2.2}$$

$$\Omega_0^\infty(X) = \lim_{\varepsilon \to 0} \Omega^\infty(X,\varepsilon). \tag{2.3}$$

It is self-evident that $\Omega^\infty(X) = 0$ if and only if functions from the set X are equicontinuous on $\mathbb{R}_+ \times \mathbb{R}_+$, or equivalently, functions from X are equiuniformly continuous on $\mathbb{R}_+ \times \mathbb{R}_+$.

Let us have a look at the function $\overline{\phi}_\infty$ defined on the family $X \in M_{BC(\mathbb{R}_+ \times \mathbb{R}_+, E)}$ according to the formula

$$\overline{\phi}_\infty(X) = \lim_{\zeta \to \infty} \overline{\phi}_\zeta(X), \tag{2.4}$$

where

$$\overline{\phi}_\zeta(X) = \sup\left\{\phi(X(w,s)) : w,s \in [0,\zeta]\right\}. \tag{2.5}$$

It is worth noting that the existence of the limit in (2.4) is due to the fact that the function $\zeta \to \overline{\phi}_\zeta(X)$ is nondecreasing and bounded from above on $\mathbb{R}_+ \times \mathbb{R}_+$. Indeed, because the set X is a bounded subset in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$, a constant $c > 0$ exists such that

$$\sup\left\{||x(w,s)|\_E : w,s \in \mathbb{R}_+ \times \mathbb{R}_+\right\} \leqslant c$$

for any $x \in X$. Thus fixing arbitrarily $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ we conclude that $\sup\left\{||x(w,s)||_E : x \in X\right\} \leqslant c$. This implies that the measures of noncompactness $\phi(X(w,s))$ are bounded from above for $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$.

Now, for $\zeta > 0$ let us put

$$\beta_\zeta(X) = \sup_{x \in X}\left\{\sup\{||x(w,s) - x(u,v)||_E : w,s,u,v \geqslant \zeta\right\},$$

$$\beta_\infty(X) = \lim_{\zeta \to \infty} \beta_\zeta(X). \tag{2.6}$$

Finally, by linking (2.3)–(2.6), we can consider the function ϕ_β defined in the following way [14]:

$$\phi_\beta(X) = \Omega_0^\infty(X) + \overline{\phi}_\infty(X) + \beta_\infty(X). \tag{2.7}$$

It can be shown that the function ϕ_β is a measure of noncompactness in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ (cf. [14]). The kernel ker ϕ_β of the measure ϕ_β consists of all nonempty and bounded subsets of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ such that functions from X are uniformly continuous and equicontinuous (equivalently, functions from X are equicontinuous on $\mathbb{R}_+ \times \mathbb{R}_+$) and tend to limits (being elements of E) at infinity with the same rate. Apart from this, all cross sections $X(w,s) = \{x(w,s) : x \in X\}$ of the set X belong to the kernel ker ϕ of the measure of noncompactness ϕ in the Banach space E (cf. [14]). The measure ϕ_β is not full and has the maximum property. If the measure ϕ is sublinear in the space E then the measure ϕ_β defined by (2.7) is also sublinear [14].

Let us mention that in the similar way as above we may define other measures of noncompactness in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, E)$ (see [14]).

Taking into account our further purposes we will consider as the Banach space E the sequence space ℓ_∞ equipped with the standard supremum norm.

Thus, in what follows we consider the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ consisting of functions $x : \mathbb{R}_+ \times \mathbb{R}_+ \to \ell_\infty$ being continuous and bounded on $\mathbb{R}_+ \times \mathbb{R}_+$. If $x \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ then we write this function in the form

$$x(w,s) = (x_p(w,s)) = (x_1(w,s), x_2(w,s), \ldots)$$

for $w,s \in \mathbb{R}_+$, where the sequence $(x_p(w,s))$ is an element of the space ℓ_∞ for any fixed w,s. The norm of the function $x = x(w,s) = (x_p(w,s))$ is defined by the equality

$$||x||_\infty = \sup\left\{||x(w,s)||_E : w,s \in \mathbb{R}_+ \times \mathbb{R}_+\right\} = \sup_{w,s\in\mathbb{R}_+} \{\sup\{|x_p(w,s)| : p = 1,2,\ldots\}\}.$$

Now, we can express the formula for the measure of noncompactness defined by (2.7) in the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$, provided the measure of noncompactness in the space ℓ_∞ is defined in the following way [8]

$$\phi^1(X) = \lim_{p\to\infty}\left\{\sup_{x=(x_i)\in X}\left\{\sup\{|x_l| : l \geqslant p\}\right\}\right\} \tag{2.8}$$

for $X \in M_{\ell_\infty}$. In this case the component $\overline{\phi}_\infty$ defined by (2.4) will be denoted by $\overline{\phi}_\infty^1$.

Thus, our measure of noncompactness ϕ_β defined by (2.7) will be denoted by ϕ_β^1 and is defined as a particular case of (2.7) by the following formula

$$\phi_\beta^1(X) = \Omega_0^\infty(X) + \overline{\phi}_\infty^1(X) + \beta_\infty(X), \tag{2.9}$$

where the components on the right hand side of formula (2.7) are defined in the following way (see [14]):

$$\Omega_0^\infty(X) = \lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left\{\sup_{p\in\mathbb{N}}\{|x_p(w,s)-x_p(u,v)| : p=1,2,\ldots\} :\right.\right.\right.$$
$$\left.\left.\left. w,s,u,v\in\mathbb{R}_+\times\mathbb{R}_+, |w-u|\leqslant\varepsilon, |s-v|\leqslant\varepsilon\right\}\right\}\right\}, \tag{2.10}$$

$$\overline{\phi}_\infty^1(X) = \lim_{\zeta\to\infty}\left\{\sup_{w,s\in[0,\zeta]}\left\{\lim_{p\to\infty}\left\{\sup_{x=x_i\in X}\left\{\sup\{|x_l(w,s)| : l\geqslant p\}\right\}\right\}\right\}\right\}, \tag{2.11}$$

$$\beta_\infty(X) = \lim_{w,s\to\infty}\left\{\sup\left\{\sup\{\sup_{p\in\mathbb{N}}|x_p(w,s)-y_p(w,s)| : x=x(w,s), y=y(w,s)\in X\}\right\}\right\}. \tag{2.12}$$

LEMMA 2.2. [14] *The following equality is satisfied*

$$\overline{\phi}_\infty(X) = \sup\{\phi(X(w,s)) : w,s\in\mathbb{R}_+\times\mathbb{R}_+\},$$

where $\overline{\phi}_\infty$ *is defined by formula* (2.4).

The quantity (2.11) *can be expressed by the formula*

$$\phi_\infty^{-1}(X) = \sup_{w,s\geqslant 0}\left\{\lim_{p\to\infty}\left\{\sup_{x=x_i\in X}\left\{\sup\{|x_l(w,s)| : l\geqslant p\}\right\}\right\}\right\}.$$

REMARK 2.3. Let us keep in mind that the kernel ker ϕ_β^1 of the measure of noncompactness ϕ_β^1 defined by formula (2.9) can be described as the family of all sets $X\in M_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)}$ such that functions $x=x(w,s)=(x_p(w,s))$ from X are equicontinuous on $\mathbb{R}_+\times\mathbb{R}_+$ and tend coordinatewise to proper limits at infinity i.e., for any $p\in\mathbb{N}$ there exists a number $g_p\in\mathbb{R}$ such that $\lim_{w,s\to\infty} x_p(w,s)=g_p$. Obviously, the sequence $g=(g_p)$ is an element of the space ℓ_∞. In addition, let us note that the functions of the sequence $(x_p(w,s))$ tends to limits (g_p) with the same rate. Additionally, let us also note that all cross-sections $X(w,s)$ of the set X belong to the kernel ker ϕ^1 defined by (2.8) being the family of some relatively compact subsets of the space ℓ_∞.

We recall a useful fixed point theorem of Darbo type [8, 13] at the end of this section.

Let us assume that E is a Banach space and ϕ is a measure of noncompactness (as defined in Definition 1) in the space E.

THEOREM 2.4. *Assume that* Q *is a nonempty, bounded, closed and convex subset of a Banach space* E *and* $T:Q\to Q$ *is a continuous operator such that there exists a constant* $k\in[0.1)$ *for which* $\phi(T(X))\leqslant k\phi(X)$ *for an arbitrary nonempty subset* X *of* Q. *Then there exists atleast one fixed point of the operator* T *in the set* Q.

REMARK 2.5. It can be shown that the set Fix T of all fixed points of the operator T belongs to the family ker ϕ.

3. Solvability of infinite system of integral equation in two variables

We will examine the infinite system of Volterra-Hammerstein type nonlinear quadratic integral equations of the form

$$x_p(w,s) = \alpha_p(w,s) + f_p(w,s,x_1(w,s),x_2(w,s),\ldots) \times \int_0^w \int_0^s k_p(w,s,\tau_1,\tau_2) g_p(u,v,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)\, d\tau_1 d\tau_2 \qquad (3.1)$$

for $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ and for $p = 1,2,\ldots$.

Our objective is to demonstrate that infinite system of integral equations (3.1) has a solution $x(w,s) = (x_p(w,s))$ in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ such that there exists a limit $\lim_{w,s\to\infty} x_p(w,s)$. That limit is clearly an element of the space ℓ_∞. As we pointed out in section (2), Remark (1), the functions of the sequence $(x_p(w,s))$ tend coordinatewise to proper limits at infinity (with the same rate). Our considerations are located in the mentioned Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ discussed previously in details. Aside from that, it's important to note that in our study of solutions of infinite system (3.1) we will use the measure of noncompactness $\phi^1_\beta(X)$ expressed by formula (2.9) given in the previous section.

Now we will look at the assumptions that will be used to study the infinite system of integral equations (3.1).

(i) The sequence $(\alpha_p(w,s))$ is an element of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ such that there exists the proper limit $\lim_{w,s\to\infty} \alpha_p(w,s)$ uniformly with respect to $p \in \mathbb{N}$ i.e., the following condition of the Cauchy type is satisfied

$$\forall_{\varepsilon>0}\, \exists_{\zeta>0}\, \forall_{p\in\mathbb{N}}\, \forall_{w,s,u,v\geqslant\zeta} |\alpha_p(w,s) - \alpha_p(u,v)| \leqslant \varepsilon.$$

Moreover, $\lim_{w,s\to\infty} \alpha_p(w,s) = 0$ for any $w,s \in \mathbb{R}_+$.

(ii) The functions $k_p(w,s,\tau_1,\tau_2) = k_p : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are continuous on the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ (p = 1,2,\ldots)$. Apart from this the functions $w,s \to k_p(w,s,\tau_1,\tau_2)$ are equicontinuous on the set $\mathbb{R}_+ \times \mathbb{R}_+$ uniformly with respect to $\tau_1,\tau_2 \in \mathbb{R}_+ \times \mathbb{R}_+$ i.e, the following condition is satisfied

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{p\in\mathbb{N}}\, \forall_{\tau_1,\tau_2\in\mathbb{R}_+\times\mathbb{R}_+}\, \forall_{w_1,w_2,s_1,s_2\in\mathbb{R}_+\times\mathbb{R}_+} \big[|w_2 - w_1| \leqslant \delta, |s_2 - s_1| \leqslant \delta \Longrightarrow |k_p(w_2,s_2,\tau_1,\tau_2) - k_p(w_1,s_1,\tau_1,\tau_2)| \leqslant \varepsilon\big].$$

(iii) There exists a constant $K_1 > 0$ such that

$$\int_o^w \int_o^s |k_p(w,s,\tau_1,\tau_2)|\, d\tau_1, d\tau_2 \leqslant K_1$$

for any $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ and $p = 1,2,\ldots$.

Moreover, $\lim_{w,s\to\infty} \int_o^w \int_o^s |k_p(w,s,\tau_1,\tau_2)|\, d\tau_1, d\tau_2 = 0$ uniformly with respect to $p \in$

$\mathbb{N}$ i.e., the following condition is satisfied

$$\forall_{\varepsilon>0}\ \exists_{\zeta>0}\ \forall_{w,s\geqslant\zeta}\ \forall_{p\in\mathbb{N}} \int_{o}^{w}\int_{o}^{s} |k_p(w,s,\tau_1,\tau_2)|\, d\tau_1, d\tau_2 \leqslant \varepsilon.$$

(iv) The sequence $(k_p(w,s,\tau_1,\tau_2))$ is equibounded on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+$ i.e, there exists a constant $K_2>0$ such that $|k_p(w,s,\tau_1,\tau_2)|\leqslant K_2$ for $w,s,\tau_1,\tau_2\in\mathbb{R}_+\times\mathbb{R}_+$ and $p=1,2,\ldots$.

(v) The functions f_p are defined on the set $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^\infty$ and take real values for $p=1,2,\ldots$. Moreover, the functions $w,s\to f_p(w,s,x_1,x_2,\ldots)$ are equicontinuous on $\mathbb{R}_+\times\mathbb{R}_+$ uniformly with respect to $x=(x_p)\in\ell_\infty$ i.e., the following condition is satisfied

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{p\in\mathbb{N}}\ \forall_{u,v\in\mathbb{R}_+\times\mathbb{R}_+}\ \forall_{w,s,u,v\in\mathbb{R}_+\times\mathbb{R}_+}\Big[||(w,s)-(u,v)||\leqslant\delta \implies |f_p(w,s,x_1,x_2,\ldots)-f_p(u,v,x_1,x_2,\ldots)|\leqslant\varepsilon\Big].$$

(vi) The function sequence $(\overline{f}_p)$ defined by the equality $\overline{f}_p(w,s)=|f_p(w,s,0,0,\ldots)|$ (for $w,s\in\mathbb{R}_+$ and $p=1,2,\ldots$) is bounded on $\mathbb{R}_+\times\mathbb{R}_+$ and $\lim\limits_{p\to\infty}\overline{f}_p(w,s)=0$ for any $w,s\in\mathbb{R}_+$.

(vii) For each $t>0$ there exists a proper limit $\lim\limits_{w,s\to\infty} f_p(w,s,x_1,x_2,\ldots)$ uniformly with respect to $x\in\ell_\infty$ such that $||x||_{\ell_\infty}\leqslant t$ and $p\in\mathbb{N}$ i.e., the following condition is satisfied

$$\forall_{\varepsilon>0}\ \forall_{t>0}\ \exists_{\zeta>0}\ \forall_{w,s\geqslant\zeta}\ \forall_{x\in\ell_\infty,||x||_{\ell_\infty}\leqslant t}\ \ \forall_{p\in\mathbb{N}}\ \ |f_p(w,s,x)-f_p(u,v,x)|\leqslant\varepsilon.$$

(viii) There exists a function $l:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+\times\mathbb{R}_+$ such that l is nondecreasing on $\mathbb{R}_+\times\mathbb{R}_+$, $l(0)=0, l$ is continuous at 0 and the following is satisfied

$$|f_p(w,s,x_1,x_2,\ldots)-f_p(w,s,y_1,y_2,\ldots)|\leqslant l(r)\sup\big\{|x_i-y_i|: i\geqslant p\big\}$$

for any $r>0$, for $x=(x_i), y=(y_i)\in\ell_\infty$ such that $||x||_{\ell_\infty}\leqslant r, ||y||_{\ell_\infty}\leqslant r$ and for all $w,s\in\mathbb{R}_+\times\mathbb{R}_+$ and $p=1,2,\ldots$.

(ix) The functions g_p are defined on the set $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^\infty$ and take real values for $p=1,2,\ldots$. Moreover, the operator g defined on the set $\mathbb{R}_+\times\mathbb{R}_+\times\ell_\infty$ by formula

$$(gx)(w,s)=(g_p(w,s,x))=(g_1(w,s,x),g_2(w,s,x),\ldots)$$

transforms the set $\mathbb{R}_+\times\mathbb{R}_+\times\ell_\infty$ into ℓ_∞ and is such that the family of functions $\{(gx)(w,s)\}_{w,s\in\mathbb{R}_+}$ is equicontinuous on the space ℓ_∞ i.e., for any $\varepsilon>0$ there exists $\delta>0$ such that

$$||(gy)(w,s)-(gx)(w,s)||_{\ell_\infty}\leqslant\varepsilon$$

for any $w,s\in\mathbb{R}_+$ and for all $x,y\in\ell_\infty$ such that $||x-y||_{\ell_\infty}\leqslant\delta$.

(x) The operator g defined on the space $\mathbb{R}_+ \times \mathbb{R}_+ \times \ell_\infty$ by the formula

$$(gx)(w,s) = (g_p(w,s,x)) = (g_1(w,s,x), g_2(w,s,x), \ldots)$$

is bounded i.e., there exists a positive constant $\overline{g}$ such that $||(gx)(w,s)||_{\ell_\infty} \leqslant \overline{g}$ for any $x \in \ell_\infty$ and for each $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$.

(xi) There exists a positive solution r_0 of the inequality

$$A + \overline{F}\overline{g}G_1 + \overline{g}G_1 rl(r) \leqslant r$$

such that

$$\overline{G}K_1 l(r_0) < 1$$

where the constants $\overline{G}, K_1$ were defined above and the constant $A\overline{F}$, was defined in the following way

$$A = \sup\{|\alpha_p(w,s) : w,s \in \mathbb{R}_+ \times \mathbb{R}_+, p = 1,2,\ldots\},$$
$$\overline{F} = \sup\{\overline{f}_p)(w,s) : w,s \in \mathbb{R}_+ \times \mathbb{R}_+, p = 1,2,\ldots\}.$$

Now, we formulate remarks and lemmas concerning some components involved in infinite system (3.1).

REMARK 3.1. Observe that in view of assumptions (i) and (vi) the constants A and F defined above are finite.

REMARK 3.2. The sequence $(\overline{f}_p)$ from assumption (vi) is an element of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$.

In order to prove the above assertion let us first notice that in view of assumption (vi) the sequence $(\overline{f}_p(w,s))$ is an element of the space ℓ_∞ for any $w,s \in \mathbb{R}_+$. Now, we show that the function $\overline{f} : \mathbb{R}_+ \to \ell_\infty$ defined by the equality $\overline{f}(w,s) = (\overline{f}_p(w,s))$, is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$.

To prove this fact let us fix arbitrarily $\varepsilon > 0$. Then for arbitrarily $w,s,u,v \in \mathbb{R}_+$ we have

$$\begin{aligned} |\overline{f}_p(w,s) - \overline{f}_p(u,v)| &= \big||f_p(w,s,0,0,\ldots)| - |f_p(u,v,0,0,\ldots)|\big| \\ &\leqslant |f_p(w,s,0,0,\ldots) - f_p(u,v,0,0,\ldots)|, \end{aligned}$$

for any $p = 1,2,\ldots$. Hence, in view of assumption (v) we can choose a number $\delta > 0$ such that for any $p \in \mathbb{N}$ and for arbitrary $w,s,u,v \in \mathbb{R}_+$ such that $||(w,s) - (u,v)| \leqslant \delta$, we have

$$|\overline{f}_p(w,s) - \overline{f}_(u,v)| \leqslant \varepsilon.$$

This implies that $||\overline{f}_p(w,s) - \overline{f}_p(u,v)||_{\ell_\infty} \leqslant \varepsilon$ for $w,s,u,v \in \mathbb{R}_+, ||(w,s) - (u,v)| \leqslant \delta$. Thus the sequence $\overline{f}(w,s) = (\overline{f}_p(w,s))$ is an element of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$.

LEMMA 3.3. [14] *Let the function* $x(w,s) = (x_p(w,s))$ *be an element of the space* $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$. *Then the sequence* (x_p) *is equibounded and locally equicontinuous on* $\mathbb{R}_+ \times \mathbb{R}_+$.

Since the proof can be accomplished using the same steps as the proof of Lemma 3.1 in [14], it is omitted.

LEMMA 3.4. *Let the function* $x(t,s) = (x_p(t,s))$ *be an element of the space* $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ *such that there exists a proper limit* $\lim_{w,s\to\infty} x_p(w,s)$ *uniformly with respect to* $n \in \mathbb{N}$ *i.e., the following condition is satisfied*

$$\forall_{\varepsilon>0} \exists_{\zeta>0} \forall_{w,s,\tau_1,\tau_2 \geqslant \zeta} \forall_{p\in\mathbb{N}} |x_p(w,s) - x_p(\tau_1,\tau_2)| \leqslant \varepsilon$$

(cf. assumption (i)). Then the sequence (x_p) *is equibounded and equicontinuous on* $\mathbb{R}_+ \times \mathbb{R}_+$.

Proof. The equiboundedness of the sequence (x_p) on $\mathbb{R}_+ \times \mathbb{R}_+$ follows immediately from Lemma 2. In order to prove the equicontinuity of the sequence (x_p) on $\mathbb{R}_+ \times \mathbb{R}_+$ let us fix $\varepsilon > 0$. Then, in view of the assumption imposed in our lemma we can find a number $\zeta > 0$ such that $|x_p(w,s) - x_p(\tau_1,\tau_2)| \leqslant \frac{\varepsilon}{2}$ for $w,s,\tau_1,\tau_2 \geqslant \zeta$ and for $p = 1,2,\ldots$. On the other hand, in virtue of Lemma 2 we infer that the sequence (x_p) is equicontinuous on the interval $[0,\zeta]$. This means that we can find a number $\delta > 0$ such that $|x_p(w_1,s_1) - x_p(w_2,s_2)| \leqslant \frac{\varepsilon}{2}$, for $w_1,w_2,s_1,s_2 \in [0,\zeta]$ such that $|w_2 - w_1| \leqslant \delta$, $|s_2 - s_1| \leqslant \delta$ and for all $p = 1,2,3,\ldots$.

Now, let us take arbitrary numbers $t_1,t_2 \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $|w_2 - w_1| \leqslant \delta, |s_2 - s_1| \leqslant \delta$. Without loss of generality we can assume that $(w_1,s_1) \leqslant (w_2,s_2)$.

If $w_1,s_1,w_2,s_2 \in [0,\zeta]$ then, according to the above established fact we have that

$$|x_p(w_2,s_2) - x_p(w_1,s_1)| \leqslant \frac{\varepsilon}{2}$$

for $p = 1,2,\ldots$.

If $w_1,s_1,w_2,s_2 \geqslant \zeta$ then in view of the made choice of the number ζ we have that

$$|x_p(w_2,s_2) - x_p(w_1,s_1)| \leqslant \frac{\varepsilon}{2}.$$

Further, let us assume that $(w_1,s_1) < \zeta \leqslant (w_2,s_2)$. Then, for an arbitrarily fixed $p \in \mathbb{N}$, taking into account the above established facts we get

$$|x_p(w_2,s_2) - x_p(w_1,s_1)| \leqslant |x_p(w_2,s_2) - x_p(\zeta)| + |x_p(\zeta) - x_p(w_1,s_1)| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that the sequence (x_p) is equicontinuous on the interval $\mathbb{R}_+ \times \mathbb{R}_+$. $\square$

Now we can express our existence result in terms of an infinite system (3.1).

THEOREM 3.5. *Under assumptions (i)–(xi) the infinite system of integral equations* (3.1) *has atleast one solution $x(w,s) = (x_p(w,s))$ in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$. Moreover, the function $x = x(w,s)$ is uniformly continuous on the interval $\mathbb{R}_+ \times \mathbb{R}_+$ and tends at infinity to a limit being an element of the space ℓ_∞.*

Proof. We start with defining three operators F, V, Q on the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ in the following way:

$$(Fx)(w,s) = ((F_p x)(w,s)) = (f_p(w,s,x(w,s))) = (f_p(w,s,x_1(w,s),x_2(w,s),\ldots)),$$

$$(Vx)(w,s) = ((V_p x)(w,s)) = \Big(\int_o^w\int_o^s k_p(w,s,\tau_1,\tau_2)g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(u,v),\ldots)\,d\tau_1 d\tau_2\Big),$$

$$(Qx)(w,s) = ((Q_p x)(w,s)) = (\alpha_p(w,s) + (F_p x)(w,s)(V_p x)(w,s)).$$

At the begining we show that the operator F transforms the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ into itself.

To this end let us choose a function $x = (x_n(w,s)) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$. Then, in view of the imposed asssumptions (vi) and (viii), for any arbitrary fixed $n \in \mathbb{N}$, we obtain

$$\begin{aligned}|(F_p x)(w,s)| &\leqslant |f_p(w,s,x_1(w,s),x_2(w,s),\ldots) - f_p(w,s,0,0,\ldots)| + |f_p(w,s,0,0,\ldots)| \\ &\leqslant l(||x(w,s)||_{\ell_\infty})\sup\{|xI(w,s) : i \geqslant p\} + |\overline{f}_n(w,s)| \\ &\leqslant l(||x||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)})||x||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} + \overline{F}.\end{aligned} \tag{3.2}$$

Hence, we obtain the inequality

$$|(F_p x)(w,s)| \leqslant \overline{F} + l(||x||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)})||x||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)},$$

which implies the following estimate

$$||Fx||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \leqslant l(||x||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)})||x||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \tag{3.3}$$

for any $x \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$. This estimate shows that the function Fx is bounded on $\mathbb{R}_+ \times \mathbb{R}_+$.

Next, we show that the function Fx is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$. In order to show this fact we will utilize the continuity of an arbitrary function

$$x = x(w,s) = (x_p(w,s)) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$$

on the interval $\mathbb{R}_+ \times \mathbb{R}_+$. Let us fix $\varepsilon > 0$. Then, on the basis of assumption (v) we can find a number $\delta = \delta(\varepsilon, ||x||_{\ell_\infty}) > 0$ such that for $w,s,u,v \in \mathbb{R}_+ \times \mathbb{R}_+$ withh $||(w,s)-(u,v)|| \leqslant \delta$ the following inequality holds

$$|f_p(w,s,x_1,x_2,\ldots) - f_p(u,v,x_1,x_2,\ldots)| \leqslant \varepsilon.$$

This implies that

$$||(Fx)(w,s) - (Fx)(u,v)||_{\ell_\infty} \leqslant \varepsilon$$

for all $w,s,u,v \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $||(w,s)-(u,v)|| \leqslant \delta$. This means that the function Fx is continuous (even uniformly continuous) on $\mathbb{R}_+ \times \mathbb{R}_+$. Hence, we infer that that the operator F transforms the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ into itself.

In what follows we show that the operator V acts from the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ into itself.

To this end, similarly as above, let us fix a function $x = x(w,s) = (x_p(w,s))$ belonging to the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$. Next, take arbitrary numbers $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$. Then, for a fixed natural number n, in view of assumptions (iii) and (x), we obtain

$$\begin{aligned} |(u_p x)(w,s)| &\leqslant \int_o^w \int_0^s |k_p(w,s,\tau_1,\tau_2)||g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)|d\tau_1 d\tau_2 \\ &\leqslant \int_o^w \int_0^s |k_p(w,s,\tau_1,\tau_2)|\overline{G} d\tau_1 d\tau_2 \\ &= \overline{G} \int_o^w \int_0^s |k_p(w,s,\tau_1,\tau_2)| d\tau_1 d\tau_2 \leqslant \overline{G} K_1. \end{aligned} \tag{3.4}$$

We now are going to show that the above mentioned operator V transforms the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ into itself. To this end, similarly as above, take a function $x = x(w,s) = (x_n(w,s)) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$. Then, for arbitrarily fixed numbers $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ and $p \in \mathbb{N}$, based on assumptions (iii) and (ix), we get

$$\begin{aligned} |(V_p x)(w,s)| &\leqslant \int_o^w \int_o^s |k_p(w,s,\tau_1,\tau_2)||g_p(u,v,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)| d\tau_1 d\tau_2 \\ &\leqslant \int_o^w \int_o^s |k_p(w,s,\tau_1,\tau_2)|\overline{G} d\tau_1 d\tau_2 \\ &\leqslant \overline{G} \int_o^w \int_o^s |k_p(w,s,\tau_1 d\tau_2)| du dv \leqslant \overline{G} K_1. \end{aligned} \tag{3.5}$$

The derived estimate, in particular, shows that the fuunction Vx is bounded on the interval $\mathbb{R}_+ \times \mathbb{R}_+$. Next, fix $\varepsilon > 0$ and determine a number $\delta > 0$ according to assumption (ii). Then, for arbitrary $w_1, w_2, s_1, s_2 \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $||(w_2,s_2)-(w_1,s_1)|| \leqslant \delta$, on the basis of assumptions (ii) and (ix) (assuming, for example, that $(w_1,s_1) < (w_2,s_2)$, we have

$$\begin{aligned} &|(V_p x)(w_2,s_2) - (V_p x)(w_1,s_1)| \\ &\leqslant \left| \int_o^{w_2} \int_o^{s_2} k_p(w_2,s_2,\tau_1,\tau_2) g_p(u,v,x_1(\tau_1,\tau_2),x_2(u,v),\ldots) d\tau_1 d\tau_2 \right. \\ &\quad \left. - \int_o^{w_2} \int_o^{s_2} k_p(w_1,s_1,\tau_1,\tau_2) g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots) d\tau_1 d\tau_2 \right| \\ &\quad + \left| \int_o^{w_2} \int_o^{s_2} k_p(w_1,s_1,\tau_1,\tau_2) g_p(\tau_1,\tau_2,x_1(u,v),x_2(\tau_1,\tau_2),\ldots) d\tau_1 d\tau_2 \right. \end{aligned}$$

$$
\begin{aligned}
&- \int_{o}^{w_1}\int_{o}^{s_1} k_p(w_1,s_1,\tau_1,\tau_2) g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)\, d\tau_1 d\tau_2 \Bigg| \\
&\leqslant \int_{o}^{w_2}\int_{o}^{s_2} |k_p(w_2,s_2,\tau_1,\tau_2) - k_p(w_1,s_1,\tau_1,\tau_2)||g_p(u,v,x_1(\tau_1,\tau_2),\ldots)|\, d\tau_1 d\tau_2 \\
&\quad + \int_{w_1}^{w_2}\int_{s_1}^{s_2} |k_p(w_1,s_1,\tau_1,\tau_2)||g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),\ldots)|\, d\tau_1 d\tau_2 \\
&\leqslant \int_{w_1}^{w_2}\int_{s_1}^{s_2} \omega_k(\delta)|g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2)\ldots)|\, d\tau_1 d\tau_2 \\
&\quad + \int_{w_1}^{w_2}\int_{s_1}^{s_2} K_2 |g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),\ldots)|\, d\tau_1 d\tau_2,
\end{aligned}
$$

where K_2 is a constant from assumption (iv) and $\omega_k(\delta)$ denotes a common modulus of equicontnuity of the sequence of functions $w,s \to k_p(w,s,u,v)$ (according to the assumption (iii)). Obviously we have $\omega_k(\delta) \to 0$ as $\delta \to 0$.

Let us now notice that, using assumptions (ix) and (x), we can obtain the following estimate from the previous one

$$
|(V_p x)(w_2,s_2) - (V_p x)(w_1,s_1)| \leqslant \overline{G}\omega_k(\delta) + \overline{G}K_2\delta. \tag{3.6}
$$

Hence, we get

$$
||(Vx)(w_2,s_2) - (Vx)(w_1,s_1)||_{\ell_\infty} \leqslant \overline{G}\omega_k(\delta) + \overline{G}K_2\delta.
$$

This shows that the function Vx is continuous on the interval $\mathbb{R}_+ \times \mathbb{R}_+$. We conclude that the operator V transforms the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ into itself by linking the boundedness of the function Vx with its continuity on $\mathbb{R}_+ \times \mathbb{R}_+$.

Taking into account the fact the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ is a Banach algebra in terms of coordinatewise multiplication of function sequences and keeping in mind the definition of the operator Q and assumption (i), we deduce that for an arbitrarily fixed function $x = x(w,s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ the function $(Qx)(w,s) = ((Q_p x)(w,s)) = (\alpha_p(w,s) + (F_p x)(w,s)(V_p x)(w,s))$ transforms the interval $\mathbb{R}_+ \times \mathbb{R}_+$ into the space ℓ_∞.

Indeed, in virtue of the fact that $((F_p x)(w,s)) \in \ell_\infty$ for any $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$ and in the light of estimate (3.5), we get

$$
|(Q_p x)(w,s)| \leqslant |\alpha_p(w,s)| + \overline{G}K_1 |(F_p x)(w,s)|
$$

for any $p \in \mathbb{N}$. In view of (3.2) this yields that $(Qx)(w,s) = ((Q_p x)(w,s)) \in \ell_\infty$ for every $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$.

Next, let us notice that the continuity of the function Qx on $\mathbb{R}_+ \times \mathbb{R}_+$ follows easily from the continuity of the functions Fx and Vx on the interval $\mathbb{R}_+ \times \mathbb{R}_+$. Similarly, if we use assumption (i), we may infer the boundedness of the function Qx on $\mathbb{R}_+ \times \mathbb{R}_+$.

Finally, by combining all the above established properties of the function Qx we infer that the operator Q transforms the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ into itself.

Now, let us observe that in view of estimates (3.2) and (3.5), for an arbitrarily fixed $p \in \mathbb{N}$ and $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$, we have

$$\begin{aligned}|(Q_p x)(w,s)| &\leqslant |\alpha_p(w,s)| + |(F_p x)(w,s)||(V_p x)(w,s)| \\ &\leqslant A + \left[l(||x(w,s||_{\ell_\infty})||x(w,s||_{\ell_\infty}) + \overline{F}\right]\overline{G}K_1.\end{aligned}$$

As a result, we arrive at the following estimate:

$$||Qx||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \leqslant A + \overline{F}\overline{G}K_1 + \overline{G}K_1 l(||x||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)})||x||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)}.$$

Based on the aforementioned estimate and assumption (xi) we conclude that there exists a number $r_0 > 0$ such that the operator Q transforms the ball B_{r_0} (in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$) into itself.

In what follows we show that the operator Q is continuous on the ball B_{r_0}. To achive this, it is sufficient to show the continuity of the operator F and V seperately, taking into account the representation of the operator Q.

So, let us fix an arbitrary $\varepsilon > 0$ and choose $x \in B_{r_0}$. Next, take an arbitrary point $y \in B_{r_0}$ such that $||x-y||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \leqslant \varepsilon$. Then, for a fixed $p \in \mathbb{N}$ and for $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$, in view of assumption (vi), we have

$$\begin{aligned}&|(F_n x)(w,s) - (F_n y)(w,s)| \\ &= |f_n(w,s,x_1(w,s),x_2(w,s),\ldots) - f_n(w,s,y_1(w,s),y_2(w,s),\ldots)| \\ &\leqslant l(r_0)||x-y||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \leqslant l(r_0)\varepsilon.\end{aligned}$$

Hence, we obtain

$$||Fx - Fy||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \leqslant l(r_0)\varepsilon.$$

This sows that the operator F is continuous on the ball B_{r_0}.

To prove the continuity of the operator V on the ball B_{r_0} let us consider the function $\delta = \delta(\varepsilon)$ defined in the following way

$$\delta(\varepsilon) = \sup\{|g_p(w,s,x) - g_p(w,s,y)| : x,y \in \ell_\infty, ||x-y||_{\ell_\infty} \leqslant \varepsilon, w,s \in \mathbb{R}_+, p \in \mathbb{N}\}.$$

Then, in view of assumption (ix) we have $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now, taking $x,y \in B_{r_0}$ such that $||x-y||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \leqslant \varepsilon$ and $w,s \in \mathbb{R}_+$ and fixing $p \in \mathbb{N}$ we obtain

$$\begin{aligned}&|(V_p x)(w,s) - (V_p y)(w,s)| \\ &\leqslant \int_0^w \int_0^s |k_p(w,s,\tau_1,\tau_2)||g_p(u,v,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots) \\ &\quad - g_p(\tau_1,\tau_2,y_1(\tau_1,\tau_2),y_2(\tau_1,\tau_2),\ldots)|d\tau_1 d\tau_2 \\ &\leqslant \int_0^w \int_0^s |k_p(w,s,\tau_1,\tau_2)|d\tau_1 d\tau_2 \delta(\varepsilon) \\ &\leqslant K_1\delta(\varepsilon).\end{aligned}$$

This implies the estimate

$$||Vx - Vy||_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)} \leqslant K_1\delta(\varepsilon).$$

Thus we see that the operator V is continuous on the ball B_{r_0}.

Finally, in the light of the above mentioned statement we conclude that the operator Q is continuous on B_{r_0}.

Further, let us fix an arbitrary number $\varepsilon > 0$ and choose a number $\delta = \delta(\varepsilon, r_0) > 0$ according to assumption (v). Next, take a nonempty subset X of the ball B_{r_0}. Assume that $x \in X$. Then, for arbitrarily fixed $n \in \mathbb{N}$ and $w,s,u,v \in \mathbb{R}_+$ with $||(w,s)-(u,v)| \leqslant \delta$, in view of assumptions (v) and (vii), we obtain

$$\begin{aligned}
&|(F_px)(w,s)-(F_px)(u,v)|\\
&= |f_p(w,s,x_1(w,s),x_2(w,s),\dots) - f_p(u,v,x_1(u,v),x_2(u,v),\dots)|\\
&\leqslant |f_p(w,s,x_1(w,s),x_2(w,s),\dots) - f_p(u.v,x_1(w,s),x_2(w,s),\dots)|\\
&\quad + |f_p(u,v,x_1(w,s),x_2(w,s),\dots) - f_p(u.v,x_1(u,v),x_2(u,v),\dots)|\\
&\leqslant \varepsilon + l(r_0)\sup\{|x_i(w,s)-x_i(u,v) : i \geqslant p\}\\
&\leqslant \varepsilon + l(r_0)\sup\{|x_i(w,s)-x_i(u,v) : i \in \mathbb{N}\}\\
&\leqslant \varepsilon + l(r_0)\omega^\infty(x,\delta).
\end{aligned}$$

The above estimate implies the following one

$$\omega^\infty(Fx,\varepsilon) \leqslant \varepsilon + l(r_0)\omega^\infty(x,\delta). \tag{3.7}$$

Further, similarly as above, let us fix $\varepsilon > 0$ and choose a number $\delta > 0$ according to assumption (ii) (we may choose a number δ with respect to assumptions (ii) and (v)). Next, fix $p \in \mathbb{N}$ and $w,s,u,v \in \mathbb{R}_+$ (say, $(u,v) < (w,s)$) such that $||(w,s)-(u,v)|| = (w,s)-(u,v) \leqslant \delta$. Then, repeating the reasoning conducted in order to obtain estimate (3.6), in view of that estimate, we get

$$|(V_px)(w,s)-(V_px)(u,v)| \leqslant \overline{G}\omega_k(\delta) + K_2\overline{G}\delta,$$

where K_2 is a constant appearing in assumption (iv) and $\omega_k(\delta)$ denotes the above introduced common modulus of continuity of the function sequence $w,s \to k_p(w,s,u,v)$ on the interval $\mathbb{R}_+\times\mathbb{R}_+$ (recall that $\omega_k(\delta) \to 0$ as $\delta \to 0$).

Hence, we derive the following estimate

$$\omega^\infty(Vx,\varepsilon) \leqslant \overline{G}\omega_k(\delta) + \overline{G}K_2\delta. \tag{3.8}$$

Further on, keeping in mind the representation of the operator Q, for an arbitrary function $x \in X$ and for arbitrary $w,s,u,v \in \mathbb{R}_+$, we have

$$\begin{aligned}
||(Qx)(w,s)-(Qx)(u,v)||_{\ell_\infty} \leqslant{}& ||a(w,s)-a(u,v)||_{\ell_\infty}\\
&+ ||(Vx)(w,s)||_{\ell_\infty}||(Fx)(w,s)-(Fx)(u,v)||_{\ell_\infty}\\
&+ ||(Fx)(w,s)||_{\ell_\infty}||(Vx)(w,s)-(Vx)(u,v)||_{\ell_\infty},
\end{aligned}$$

where $a(w,s) = (a_p(w,s))$.

Next, fix $\varepsilon > 0$ and assume that $||(w,s)-(u,v)|| \leqslant \varepsilon$. Then, from the above inequality and estimates (3.7), (3.8), (3.3) and (3.4), we get

$$\begin{aligned}\omega^{\infty}(Qx,\varepsilon) &\leqslant \omega^{\infty}(a,\varepsilon)+\overline{G}K_1\omega^{\infty}(Fx,\varepsilon)+(\overline{F}+r_0m(r_0))(\overline{G}\omega_k(\varepsilon)+\overline{G}K_2\varepsilon)\\ &\leqslant \omega^{\infty}(a,\varepsilon)+\overline{G}K_1m(r_0)\omega^{\infty}(x,\varepsilon)+\overline{G}K_1\varepsilon+(\overline{F}+r_0m(r_0))(\overline{G}\omega_k(\varepsilon)+\overline{G}K_2\varepsilon).\end{aligned}$$

Now, in view of lemma 3 we infer that $\omega^{\infty}(a,\varepsilon) \to 0$ as $\varepsilon \to 0$. Next, taking into account that $\omega_k(\varepsilon) \to 0$ as $\varepsilon \to 0$, from the above obttained estimate we deduce the following inequality

$$\omega_0^{\infty}(QX) \leqslant \overline{G}K_1m(r_0)\omega_0^{\infty}(X). \tag{3.9}$$

In what follows we will investigate the behaviour of the operator Q with respect to second term μ_{∞}^{-1} (cf. formula (2.11)) of the measure of noncompactness μ_b^1 defined by (2.9). To this end take a nonempty subset X of the ball B_{r_0} and choose an element $x = x(w,s) \in X$. Further, fix a natural number p and $\zeta > 0$. Then, for an arbitrarily fixed number $w,s \in [0,\zeta]$, in virtue of the representation of the operator Q and estimates (3.2) and (3.4), we get

$$\begin{aligned}|(Q_px)(w,s)| &\leqslant |a_p(w,s)|+|f_p(w,s,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\dots)|\int_0^w\int_0^s|k_p(w,s,\tau_1,\tau_2)|\\ &\quad\times|g_p(u,v,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\dots)|d\tau_1d\tau_2\\ &\leqslant |a_p(w,s)|+[\overline{f}_p(w,s)+m(||x(w,s)||_{\ell_\infty})\sup\{|x_i(w,s):i\geqslant p\}]\overline{G}K_1.\end{aligned}$$

Now, taking supremum over $x \in X$, from the above estimate we obtain

$$\sup_{x\in X}|(Q_px)(w,s)| \leqslant |a_p(w,s)|+\overline{G}K_1\left[\overline{f}_p(w,s)+m(r_0)\sup_{x\in X}\left\{\sup\{x_i(w,s):i\geqslant p\}\right\}\right].$$

Hence, in view of assumptions (i) and (vi), we derive the following inequality

$$\lim_{p\to\infty}\left\{\sup_{x\in X}|(Q_px)(w,s)|\right\} \leqslant \overline{G}K_1m(r_0)\left\{\lim_{p\to\infty}\left\{\sup_{x\in X}\left\{\sup\{x_i(w,s):i\geqslant p\}\right\}\right\}\right\}.$$

Finally, if we take supremum over $w,s \in [0,\zeta]$ on both sides of the above inequality and if we pass with $\zeta \to \infty$, in view of formula (2.11) we have

$$\mu_{\infty}^{-1}(QX) \leqslant \overline{G}K_1m(r_0)\mu_{\infty}^{-1}(QX). \tag{3.10}$$

Now, we proceed to the study the behaviour of the operator Q with respect to the quantity $b_\infty = b_\infty(X)$ defined by (2.12) which creates the last component of the measure of noncompactness μ_b^{-1} (cf. formula (2.9)).

Thus, take a nonempty subset X of the ball B_{r_0} and an arbitrary number $\zeta > 0$. Next, fix numbers w,s,u,v such that $w,s,u,v \geqslant \zeta$ and $p \in \mathbb{N}$. Then, for an arbitrarily fixed function $x \in X$, we obtain

$$
\begin{aligned}
&|(Q_px)(w,s)-(Q_px)(u,v)| \\
&\leqslant |a_p(w,s)-a_p(u,v)| + |f_p(w,s,x_1(w,s),x_2(w,s),\ldots)\int_0^w\int_0^s k_p(w,s,\tau_1,\tau_2) \\
&\quad \times g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)d\tau_1 d\tau_2 - f_p(u,v,x_1(u,v),x_2(u,v),\ldots) \\
&\quad \times \int_0^u\int_0^v k_p(u,v.\tau_1,\tau_2)g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)d\tau_1 d\tau_2| \\
&\leqslant |a_p(w,s)-a_p(u,v)| + |f_p(w,s,x_1(w,s),x_2(w,s),\ldots)\int_0^w\int_0^s k_p(w,s,\tau_1,\tau_2) \\
&\quad \times g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)d\tau_1 d\tau_2 - f_p(u,v,x_1(u,v),x_2(u,v),\ldots) \\
&\quad \times \int_0^w\int_0^s k_p(u,v.\tau_1,\tau_2)g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)d\tau_1 d\tau_2| \\
&\leqslant |f_p(u,v,x_1(u,v),x_2(u,v),\ldots) \\
&\quad \times \int_0^w\int_0^s k_p(u,v.\tau_1,\tau_2)g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)d\tau_1 d\tau_2 \\
&\quad - f_p(u,v,x_1(u,v),x_2(u,v),\ldots) \\
&\quad \times \int_0^u\int_0^v k_p(u,v.\tau_1,\tau_2)g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)d\tau_1 d\tau_2| \\
&\leqslant |a_p(w,s)-a_p(u,v)| + |f_p(w,s,x_1(w,s),x_2(w,s),\ldots) - f_p(u,v,x_1(u,v),x_2(u,v),\ldots)| \\
&\quad \times \int_0^w\int_0^s |k_p(w,s,\tau_1,\tau_2)||g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)|d\tau_1 d\tau_2 \\
&\quad + |f_p(u,v,x_1(u,v),x_2(u,v),\ldots)|| \\
&\quad \times \int_0^w\int_0^s k_p(w,s,\tau_1,\tau_2)g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)d\tau_1 d\tau_2 \\
&\quad - \int_0^u\int_0^v k_p(u,v.\tau_1,\tau_2)g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)d\tau_1 d\tau_2| \\
&\leqslant |a_p(w,s)-a_p(u,v)| + \big[|f_p(w,s,x_1(w,s),x_2(w,s),\ldots) - f_p(u,v,x_1(u,v),x_2(u,v),\ldots)| \\
&\quad \times |f_p(u,v,x_1(w,s),x_2(w,s),\ldots) - f_p(u,v,x_1(u,v),x_2(u,v),\ldots)|\big] \\
&\quad \times \int_0^w\int_0^s \overline{G}|k_p(w,s,\tau_1,\tau_2)|d\tau_1 d\tau_2 + \left[\overline{f}_p(u,v) + m(||x(u,v)||_{\ell_\infty})\sup\left\{|x_i(u,v) : i \geqslant p\right\}\right] \\
&\times \left\{\int_0^w\int_0^s |k_p(w,s,\tau_1,\tau_2)||g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)|d\tau_1 d\tau_2\right. \\
&\quad \left. + \int_0^u\int_0^v |k_p(u,v,\tau_1,\tau_2)||g_p(\tau_1,\tau_2,x_1(\tau_1,\tau_2),x_2(\tau_1,\tau_2),\ldots)|d\tau_1 d\tau_2\right\} \\
&\leqslant |a_p(w,s)-a_p(u,v)| + \left[\Omega_{r_0}(f,\zeta) + m(r_0)\sup\{|x_i(w,s)-x_i(u,v)| : i \geqslant p\}\right]\overline{G}K_1 \\
&\quad + [\overline{F} + r_0 m(r_0)]\{\int_0^w\int_0^s |k_p(w,s,\tau_1,\tau_2)|\overline{G}d\tau_1 d\tau_2 + \int_0^u\int_0^v |k_p(u,v,\tau_1,\tau_2)|\overline{G}d\tau_1 d\tau_2\},
\end{aligned}
\tag{3.11}
$$

where we denoted

$$\Omega_{r_0}(f,\zeta) = \sup\Big\{ |f_p(w,s,x_1,x_2,\ldots) - f_p(u,v,x_1,x_2,\ldots)| : \\ w,s,u,v \geqslant \zeta, x=(x_i) \in B_{r_0}, p \in \mathbb{N} \Big\}.$$

Observe that $\lim_{\zeta\to\infty} \Omega_{r_0}(f,\zeta) = 0$, in view of assumption (vii).

Further, from estimate (3.11), for $w,s,u,v \geqslant \zeta$ and for $p \in \mathbb{N}$ we obtain

$$\begin{aligned} &|(Q_px)(w,s) - (Q_px)(u,v)| \\ &\leqslant |a_p(w,s) - a_p(u,v)| + \Big[\Omega_{r_0}(f,\zeta) + m(r_0)\sup\{|x_i(w,s) - x_i(u,v)| : i \geqslant p\}\Big]\overline{G}K_1 \\ &\quad + [\overline{F} + r_0 m(r_0)]\{\int_0^w \int_0^s |k_p(w,s,\tau_1,\tau_2)|\overline{G}d\tau_1 d\tau_2 + \int_0^u \int_0^v |k_p(u,v,\tau_1,\tau_2)|\overline{G}d\tau_1 d\tau_2\}. \end{aligned}$$

Now, keeping in mind the above estimate, assumptions (i) and (iii) and the above established facts, in view of formula (2.12), we derive the following inequality

$$b_\infty \leqslant \overline{G}K_1 m(r_0) b_\infty(X). \tag{3.12}$$

Finally, linking estimates (3.9), (3.10), (3.12) and taking account the formula (2.9), we obtain the following inequality for an arbitrary nonempty subset X of the ball B_{r_0}:

$$\mu_b^1(QX) \leqslant \overline{G}K_1 m(r_0)\mu_b^1(X).$$

Hence, combining the fact that the operator Q maps continuously the ball B_{r_0} into itself, assumption (xi) and Theorem (1), we infer that the infinite system of Volterra-Hammerstein integral equation (3.1) has at least one solution $x = x(w,s)$ in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ which belongs to the ball b_{r_0} and is uniformly continuous on the interval $\mathbb{R}_+\mathbb{R}_+$.

Moreover, since the mentioned solution $x = x(w,s)$ of infinite system (3.1) belongs to the kernal ker μ_b^1 we conclude that there exists a limit $\lim_{w,s\to\infty} x(w,s)$ in the space ℓ_∞ i.e., there exists an element $g = (g_p) \in \ell_\infty$ such that $\lim_{w,s\to\infty} x(w,s) = g$. Equivalently this means that if we write $x(w,s) = (x_p(w,s))$ then for any fixed $p \in \mathbb{N}$ there exists a proper limit $\lim_{p\to\infty} x_p(w,s) = g_p$ (cf. Remark 1). Other words this means that the solution $x = x(w,s) = (x_p(w,s))$ is coordinatewise converging at infinity. The proof is complete. □

This outcome is illustrated by the following example:

EXAMPLE 3.6. Let us take a look at the following infinite system of Volterra-

Hammerstein type nonlinear quadratic integral equations.

$$x_p(w,s) = wse^{-2pws} + \left(\frac{x_p^2(w,s)+1}{p+ws} + \frac{x_{p+1}^2(w,s)+1}{p+ws} + \frac{x_{p+2}(w,s)+1}{p+ws}\right)$$
$$\times \int_0^w \int_0^s e^{-\gamma(ws+p)\tau_1\tau_2} \arctan\left(\frac{x_1(\tau_1,\tau_2)+x_p(\tau_1,\tau_2)+x_{p+1}(\tau_1,\tau_2)}{p+ws+\beta}\right) d\tau_1 d\tau_2 \tag{3.13}$$

for $p = 1,2,\ldots$ and $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$. Also, we assume β and γ appearing in the above are positive constants.

Observe that infinite system (3.13) is a particular case of system (3.1) if we put

$$\alpha_p(w,s) = wse^{-2pws}, \tag{3.14}$$

$$f_p(w,s,x_1,x_2,\ldots) = \frac{x_p^2(w,s)+1}{p+ws} + \frac{x_{p+1}^2(w,s)+1}{p+ws} + \frac{x_{p+2}(w,s)+1}{p+ws}, \tag{3.15}$$

$$k_p(w,s,u,v) = e^{-\gamma(ws+p)\tau_1\tau_2}, \tag{3.16}$$

$$g_p(w,s,x_1,x_2,\ldots) = \arctan\left(\frac{x_1(\tau_1,\tau_2)+x_p(\tau_1,\tau_2)+x_{p+1}(\tau_1,\tau_2)}{p+ws+\beta}\right) \tag{3.17}$$

for $p = 1,2,\ldots$ and $w,s \in \mathbb{R}_+ \times \mathbb{R}_+$.

In what follows we are going to show that the infinite system of integral equations (3.13) has a solution $x = x(w,s) = (x_p(w,s))$ in the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ which is coordinatewise converging at infinity in the sense of Remark (1). In order to show that the infinite system of integral equations (3.19) has a solution in the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ it is sufficient to apply Theorem (2). To this end, we have to show that the functions defined by formulas (3.14)–(3.17) satisfy assumptions (i)–(xi) of Theorem (2).

At the begining let us observe that the functions $\alpha_p(w,s)$ defined by (3.20) satisfy the Lipschitz condition with the constant $l = 1 + e^{-1}$ for $p = 1,2,\ldots$. We omit elementary details of the proof.

Hence we infer that the function $\alpha(w,s) = (\alpha_p(w,s))$ is an element of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_\infty)$ and satisfies the Cauchy condition indicated in assumption (i). Moreover, in view of the inequality

$$\alpha_p(w,s) = wse^{-2pws} \leqslant \frac{1}{2p}e^{-1}$$

we infer that $\lim\limits_{p\to\infty} \alpha_p(w,s) = 0$ for any $w,s \in \mathbb{R}_+$. Apart from this we have that

$$|\alpha_p(w,s)| \leqslant \frac{1}{2}e^{-1}$$

for all $w,s \in \mathbb{R}_+$ and $p = 1,2,\ldots$.

Thus, the function sequence $(\alpha_p(w,s))$ is equibounded on $\mathbb{R}_+ \times \mathbb{R}_+$. Moreover, we have

$$A = \sup\{|\alpha_p(w,s)| : p = 1,2,\ldots, w,s \in \mathbb{R}_+ \times \mathbb{R}_+\} = \frac{1}{2}e^{-1}.$$

This shows that the assumption (i) is satisfied.

Further, let us notice that the function $k_p(w,s,\tau_1,\tau_2)$ defined by (3.16) $(p = 1,2,\ldots)$ is continuous on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+$. additionally, uusing standard tools of differential calculus it is easy seen that

$$\frac{\partial k_p}{\partial(w,s)} = -\gamma\tau_1\tau_2 e^{-\gamma(ws+p)\tau_1\tau_2} = -\gamma\tau_1\tau_2 e^{-\gamma(ws)\tau_1\tau_2}e^{-\gamma p\tau_1\tau_2} \tag{3.18}$$

for $p = 1,2,3,\ldots$. It is easy to check that if we consider the function $z_p(\tau_1,\tau_2) = \tau_1\tau_2 e^{-\gamma p\tau_1\tau_2}$ then $z_p(\tau_1,\tau_2) \leqslant \dfrac{1}{\gamma p}$ for any $\tau_1,\tau_2\in\mathbb{R}_+$ and $p = 1,2,\ldots$. Hence in view of (3.18) we deduce that the $\dfrac{\partial k_p}{\partial(w,s)}$ is bounded i.e.,

$$\left|\frac{\partial k_p}{\partial(w,s)}\right| \leqslant 1$$

for $w,s,\tau_1,\tau_2\in\mathbb{R}_+$ and for $p = 1,2,\ldots$. Hence it follows that the function $w,s\to k_p(w,s,\tau_1,\tau_2)$ satisfies the Lipschitz condition on the set $\mathbb{R}_+\times\mathbb{R}_+$ uniformly with respect to $\tau_1,\tau_2\in\mathbb{R}_+\times\mathbb{R}_+$.

Summing up, we see that there is satisfied assumption (ii).

Next, let us observe that for each $p\in\mathbb{N}$ and for arbitrary $w,s,\tau_1,\tau_2\in\mathbb{R}_+\times\mathbb{R}_+$ we have the following estimate

$$\begin{aligned}\int_o^w\int_o^s |k_p(w,s,\tau_1,\tau_2)|\,dudv &= \int_o^w\int_o^s e^{-\gamma(ws+p)}d\tau_1 d\tau_2\\ &= \frac{1}{\gamma(ws+p)}(1-e^{-\gamma(ws+p)ws}) \leqslant \frac{1}{\gamma(ws+p)}.\end{aligned}$$

Hence we see that

$$\lim_{w,s\to\infty}\int_o^w\int_o^s |k_p(w,s,\tau_1,\tau_2)|\,dudv = 0$$

uniformly with respect to $w,s\in\mathbb{R}_+$ and $p = 1,2,\ldots$. Moreover, we have that

$$\int_o^w\int_o^s |k_p(w,s,\tau_1,\tau_2)|\,dud \leqslant \frac{1}{\gamma}$$

for $w,s\in\mathbb{R}_+$ and $p = 1,2,\ldots$. Thus for constant $K_1 = 1$, we infer that assumption (iv) is satisfied.

Further on we show that the function $f_p = f_p(w,s,x_1,x_2,\ldots)$ verifies assumption (v) $(p = 1,2,\ldots)$. To this end fix $\varepsilon > 0, t > 0$ and take an arbitrary element $x = (x_i)\in\ell_\infty$ such that $||x||_{\ell_\infty}\leqslant t$. Then, for arbitrary chosen numbers $w,s,u,v\in\mathbb{R}_+$ and $p\in\mathbb{N}$

we obtain

$$
\begin{aligned}
&|f_p(w,s,x_1,x_2,\ldots)-f_p(u,v,x_1,x_2,\ldots)|\\
&\leqslant \left|\frac{x_p^2+1}{p+ws}-\frac{x_p^2+1}{p+uv}\right|+\left|\frac{x_{p+1}^2+1}{2p+ws}-\frac{x_{p+1}^2+1}{2p+uv}\right|+\left|\frac{x_{p+2}+2}{p^2+ws}-\frac{x_{p+2}+2}{p^2+uv}\right|\\
&\leqslant \frac{|uvx_p^2-wsx_p^2+uv-ws|}{(p+ws)(p+uv)}+x_{p+1}^2\frac{\sqrt{(w-u)^2+(s-v)^2}}{(2p+ws)(2p+uv)}\\
&\quad+\frac{|uvx_{p+2}-wsx_{p+2}+2s-2ws|}{(p^2+ws)(p^3+uv)}\\
&\leqslant \frac{x_p^2\sqrt{(w-u)^2+(s-v)^2}+\sqrt{(w-u)^2+(s-v)^2}}{(p+ws)(p+uv)}\\
&\quad+x_{p+1}^2\frac{\sqrt{(w-u)^2+(s-v)^2}}{(2p+ws)(2p+uv)}\\
&+\frac{|x_{p+2}|\sqrt{(w-u)^2+(s-v)^2}+2\sqrt{(w-u)^2+(s-v)^2}}{(p^2+ws)(p^2+uv)}\\
&\leqslant \frac{t^2\sqrt{(w-u)^2+(s-v)^2}+\sqrt{(w-u)^2+(s-v)^2}}{(1+ws)(1+uv)}+t^2\frac{\sqrt{(w-u)^2+(s-v)^2}}{(2+ws)(2+uv)}\\
&\quad+\frac{t\sqrt{(w-u)^2+(s-v)^2}+2\sqrt{(w-u)^2+(s-v)^2}}{(1+ws)(1+uv)}\\
&\leqslant (t^2+1)\sqrt{(w-u)^2+(s-v)^2}+t^2\sqrt{(w-u)^2+(s-v)^2}\\
&\quad+(t+2)\sqrt{(w-u)^2+(s-v)^2}\\
&=(2t^2+t+3)\sqrt{(w-u)^2+(s-v)^2}.
\end{aligned}
$$

From the above estimate, we conclude that assumption (v) is satisfied.

Now, let us observe that

$$
\overline{f}_p(w,s)=|f_p(w,s,0,0,\ldots)|=\frac{1}{p+ws}+\frac{2}{p^2+ws}.
$$

Hence we infer that $\lim_{p\to\infty}\overline{f}_p(w,s)=0$ for any $w,s\in\mathbb{R}_+$.

Moreover, we have the following estimate

$$
\overline{f}_p(w,s)\leqslant\frac{1}{1+ws}+\frac{2}{1+ws}\leqslant 1+2=3
$$

for any $w,s\in\mathbb{R}_+$ and $p=1,2,\ldots$.

Hence we conclude that the function sequence $(\overline{f}_p)$ satisfies assumption (vi). Moreover, we may accept that $\overline{F}=3$, where the constant $\overline{F}$ is defined in assumption (xi).

In order to verify assumption (vii) let us fix an arbitrary number $t>0$. Take $\varepsilon>0$ and $x\in\ell_\infty$ such that $||x||_{\ell_\infty}\leqslant t$ and choose an arbitrary number $\zeta>0$. Then, for

$w,s,u,v \in \mathbb{R}_+$ such that $w,s,u,v \geqslant \zeta$ and for an arbitrary number $\zeta > 0$. Then, for $w,s,u,v \in \mathbb{R}_+$ such that $w,s,u,v \geqslant \zeta$ and for an arbitrarily fixed natural number n, we obtain

$$
\begin{aligned}
&|f_p(w,s,x) - f_p(u,v,x)| \\
&\leqslant \left|\frac{x_p^2+1}{p+ws} - \frac{x_p^2+1}{p+uv}\right| + \left|\frac{x_{p+1}^2+1}{2p+ws} - \frac{x_{p+1}^2+1}{2p+uv}\right| + \left|\frac{x_{p+2}+2}{p^2+ws} - \frac{x_{p+2}+2}{p^2+uv}\right| \\
&\leqslant \frac{|x_p^2(uv-ws)+(uv-ws)|}{(p+ws)(p+ws)} + x_{p+1}^2\frac{\sqrt{(w-u)^2+(s-v)^2}}{(2p+ws)(2p+uv)} \\
&\quad + \frac{|x_{p+2}(uv-ws)+2(uv-ws)|}{(p^2+ws)(p^2+uv)} \\
&\leqslant (x_p^2+1)\frac{ws+uv}{(p+ws)(p+ws)} + x_{p+1}^2\frac{ws+uv}{(2p+ws)(2p+ws)} + (|x_{p+2}|+2) \\
&\leqslant (t^2+1)\left[\frac{ws}{(p+ws)(p+ws)} + \frac{uv}{(p+ws)(p+ws)}\right] \\
&\quad + t^2\left[\frac{ws}{(2p+ws)(2p+ws)} + \frac{uv}{(2p+ws)(2p+ws)}\right] \\
&\quad + (t+2)\left[\frac{ws}{(p^2+ws)(p^2+ws)} + \frac{uv}{(p^2+ws)(p^2+ws)}\right] \\
&\leqslant (t^2+1)\left(\frac{1}{p+uv} + \frac{1}{p+ws}\right) + t^2\left(\frac{1}{2p+uv} + \frac{1}{2p+ws}\right) \\
&\quad + (t+2)\left(\frac{1}{p^2+uv} + \frac{1}{p^2+ws}\right) \\
&\leqslant (t^2+1)\frac{2}{1+\zeta} + t^2\frac{2}{2+\zeta} + (t+2)\frac{2}{1+\zeta} \\
&\leqslant (t^2+1)\frac{2}{1+\zeta} + t^2\frac{2}{1+\zeta} + (t+2)\frac{2}{1+\zeta} \\
&= (2t^2+t+3)\frac{2}{1+\zeta}.
\end{aligned}
$$

From the above obtained estimate we infer that assumption (vii) is satisfied.

In what follows let us fix $t > 0$ and $p \in \mathbb{N}$. Next, take $x = (x_i)$, $y = y_i \in \ell_\infty$ such that $||x||_{\ell_\infty} \leqslant t$, $||y||_{\ell_\infty} \leqslant t$. Then we get

$$
\begin{aligned}
&|f_p(w,s,x_1,x_2,\dots) - f_p(w,s,y_1,y_2,\dots)| \\
&\leqslant \left|\frac{x_p^2+1}{p+ws} - \frac{y_p^2+1}{p+ws}\right| + \left|\frac{x_{p+1}^2}{2p+ws} - \frac{y_{p+1}^2}{2p+ws}\right| + \left|\frac{x_{p+2}+2}{p+2+ws} - \frac{y_{p+2}+2}{p+2+ws}\right| \\
&\leqslant \frac{1}{p+ws}|x_p^2 - y_p^2| + \frac{1}{2p+ws}|x_{p+1}^2 - y_{p+1}^2| + \frac{1}{p+ws}|x_{p+2} - y_{p+2}|
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{1}{1+ws}|x_p-y_p||x_p+y_p|+\frac{1}{2+ws}|x_{p+1}-y_{p+1}||x_{p+1}+y_{p+1}| \\
&\quad+\frac{1}{1+ws}|x_{p+2}-y_{p+2}| \\
&\leqslant \frac{1}{1+ws}|x_p-y_p|(|x_p+y_p|)+\frac{1}{1+ws}|x_{p+1}-y_{p+1}|(|x_{p+1}+y_{p+1}|) \\
&\quad+\frac{1}{1+ws}|x_{p+2}-y_{p+2}| \\
&\leqslant \frac{1}{1+ws}\left[2t|x_p-y_p|+2t|x_{p+1}-y_{p+1}|+|x_{p+2}-y_{p+2}|\right] \\
&\leqslant 4t+1)\max\{|x_i-y_i| : i=p,p+1,p+2\} \\
&\leqslant (4t+1)\sup\{|x_i-y_i| : i\geqslant p\}.
\end{aligned}
$$

Thus we see that assumption (viii) is satisfied with the function m having the form $m(t)=4t+1$.

Now, keeping in mind formula (3.17) we are going to check assumption (ix). To this end fix $w,s\in\mathbb{R}_+$, $p\in\mathbb{N}$ and take $x=(x_i)$, $y=(y_i)\in\ell_\infty$. Then we get

$$
\begin{aligned}
|g_p(w,s,y)-g_p(w,s,x)| &= |g_p(w,s,y_1,y_2,\ldots)-g_p(w,s,x_1,x_2,\ldots)| \\
&= \left|\arctan\left(\frac{y_1+y_p+y_{p+1}}{p+ws+\beta}\right)-\arctan\left(\frac{x_1+x_p+x_{p+1}}{p+ws+\beta}\right)\right| \\
&\leqslant \left|\frac{y_1+y_p+y_{p+1}}{p+ws+\beta}-\frac{x_1+x_p+x_{p+1}}{p+ws+\beta}\right| \\
&\leqslant (|y_1-x_1|+|y_p-x_p|+|y_{p+1}-x_{p+1}|) \\
&\leqslant \frac{1}{p+ws+\beta}\max\{|y_1-x_1|,|y_p-x_p|,|y_{p+1}-x_{p+1}|\} \\
&\leqslant 3\sup\{|y_p-x_p| : p\in\mathbb{N}\} \\
&= 3||y-x||_{\ell_\infty}.
\end{aligned}
$$

Hence we see that the function $g=(gx)(w,s)$ satisfies assumption (ix).

Next, let us observe that for arbitrarily fixed $w,s\in\mathbb{R}_+, n\in\mathbb{N}$ and $x\in\ell_\infty$ we have

$$
|g_p(w,s,x)|=|g_p(w,s,x_1,x_2,\ldots)|=\left|\arctan\left(\frac{x_1+x_p+x_{p+1}}{p+ws+\beta}\right)\right|\leqslant\frac{\pi}{2}.
$$

Obviously this implies that $||g(x)(w,s)||\leqslant\frac{\pi}{2}$ for any $x\in\ell_\infty$ and $w,s\in\mathbb{R}_+\times\mathbb{R}_+$. Thus the operator g is bounded on the set $\mathbb{R}_+\times\mathbb{R}_+\times\ell_\infty$ and we accept that $\overline{G}=\frac{\pi}{2}$, where $\overline{G}$ is the constant appearing in assumption (x).

Finally, we are going to verify assumption (xi). To this end let us consider the inequality from the assumption. Indeed, taking into account all constants A, $\overline{F},\overline{G},K_1$ established above and keeping in mind that the function $l=l(t)$ has form $l(t)=4r+1$, we obtain that the mentioned inequality has the form

$$\frac{1}{2}e^{-1}+3\frac{\pi}{2}\frac{1}{\gamma}+\frac{\pi}{2\gamma}t(4t+1)\leqslant t.$$

Equivalently, we get te inequality

$$\frac{1}{2}\gamma e^{-1}+\frac{3\pi}{2}+\frac{\pi t}{2}(4t+1)\leqslant \gamma t,$$

which can be written in the form

$$2\pi t^2+\left(\frac{\pi}{2}-\gamma\right)t+\frac{3}{2}\pi+\frac{1}{2}\gamma e^{-1}\leqslant 0. \tag{3.19}$$

It is easy to check that the above inequality has a positive solution for suitable value of parameter γ. For example, for $\gamma=5\pi$ the number $t_0=\frac{9}{8}$ is a solution of inequality (3.19).

Observe that if $t_0>0$ is a solution of inequality (3.19) (equivalently: $t_0>0$ is a solution of the first inequality from assumption (xi)) then we have that

$$\overline{G}K_1 l(t_0)<\frac{1}{t_0}[A+\overline{FG}K_1+\overline{G}K_1t_0l(t_0)]\leqslant\frac{t_0}{t_0}=1.$$

Hence, we infer that the second part of assumption (xi) of Theorem (2) is also satisfied.

Thus, in view of Theorem (2), we conclude that infinite system of integral equations (3.13) has a solution $x(w,s)$ in the Banach space $BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_\infty)$ which is coordinatewise converging at infinity.

Declaration

Conflicts of interests. There is no conflict of interest.

Availibility of data and materials. This paper has no associated data.

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