Operators
and
Matrices
Volume 17, Number 2 (2023), 259–277
doi:10.7153/oam-2023-17-19

THE STABILITY OF PROPERTY (gt) UNDER PERTURBATION AND TENSOR PRODUCT

MOHAMMAD H. M. RASHID AND MUNEO CHŌ

(*Communicated by R. Curto*)

Abstract. An operator T acting on a Banach space $\mathscr{X}$ obeys property (gt) if the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues are exactly those points λ of the spectrum for which $T-\lambda$ is an upper semi-B-Fredholm with index less than or equal to 0. In this paper we study the stability of property (gt) under perturbations by finite rank operators, by nilpotent operators and, more generally, by algebraic operators commuting with T. Moreover, we study the transfer of property (gt) from a bounded linear operator T acting on a Banach space $\mathscr{X}$ and a bounded linear operator S acting on a Banach space $\mathscr{Y}$ to their tensor product $T\otimes S$.

1. Introduction

Let $\mathscr{B}(\mathscr{X})$ denote the algebra of all bounded linear operator T acting on a Banach space $\mathscr{X}$. For $T\in\mathscr{B}(\mathscr{X})$, let T^*, $\ker(T)$, $\Re(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote respectively the *adjoint*, the *null space*, the *range*, the *spectrum*, the *point spectrum* and the *approximate point spectrum* of T. Let $\mathbb{C}$ denote the set of *complex numbers*. Let us denote by $\alpha(T)$ the dimension of the kernel and by $\beta(T)$ the codimension of the range. Recall that an operator $T\in\mathscr{B}(\mathscr{X})$ is said to be *upper semi-Fredholm*, $T\in SF_+(\mathscr{X})$, if the range of $T\in\mathscr{B}(\mathscr{X})$ is closed and $\alpha(T)<\infty$, while $T\in\mathscr{B}(\mathscr{X})$ is said to be *lower semi-Fredholm*, $T\in SF_-(\mathscr{X})$, if $\beta(T)<\infty$. An operator $T\in\mathscr{B}(\mathscr{X})$ is said to be *semi-Fredholm* if $T\in SF_+(\mathscr{X})\cup SF_-(\mathscr{X})$ and Fredholm, $T\in\mathfrak{F}$, if $T\in SF_+(\mathscr{X})\cap SF_-(\mathscr{X})$. If T is semi-Fredholm then the *index* of T is defined by $\mathrm{ind}(\mathrm{T})=\alpha(\mathrm{T})-\beta(\mathrm{T})$.

Let $a:=a(T)$ be the *ascent* of an operator T; i.e., the smallest nonnegative integer p such that $\ker(T^p)=\ker(T^{p+1})$. If such integer does not exist we put $a(T)=\infty$. Analogously, let $d:=d(T)$ be the *descent* of an operator T; i.e., the smallest nonnegative integer q such that $\Re(T^q)=\Re(T^{q+1})$, and if such integer does not exist we put $d(T)=\infty$. It is well known that if $a(T)$ and $d(T)$ are both finite then $a(T)=d(T)$ [24, Proposition 38.3]. Moreover, $0<a(T-\lambda I)=d(T-\lambda I)<\infty$ precisely when λ is a pole of the resolvent of T, see Heuser [24, Proposition 50.2].

Mathematics subject classification (2020): 47A10, 47A11, 47A53.
Keywords and phrases: Weyl's theorem, Weyl spectrum, polaroid operators, property (gt).

© ELEMENT, Zagreb
Paper OaM-17-19

A bounded linear operator T acting on a Banach space $\mathscr{X}$ is *Weyl*, $T \in \mathscr{W}$, if it is Fredholm of index zero and Browder, $T \in \mathfrak{B}$, if T is Fredholm of finite ascent and descent. The *Weyl spectrum* $\sigma_w(T)$ and *Browder spectrum* $\sigma_b(T)$ of T are defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}.$$

Let $E^0(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T-\lambda) < \infty\}$ and let $\pi_0(T) := \sigma(T) \setminus \sigma_b(T)$ all *Riesz points* of T. According to Coburn [18], *Weyl's theorem* holds for T if $\Delta(T) = \sigma(T) \setminus \sigma_w(T) = E^0(T)$, and that *Browder's theorem* holds for T if $\sigma_w(T) = \sigma_b(T)$.

Here and elsewhere in this paper, for $A \subset \mathbb{C}$, iso A denotes the set of all isolated points of A and acc A denotes the set of all accumulation points of A.

Let $SF_+^-(\mathscr{X}) = \{T \in SF_+ : \operatorname{ind}(T) \leqslant 0\}$. The *upper semi Weyl spectrum* is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(\mathscr{X})\}$. According to Rakočević [32], an operator $T \in \mathscr{B}(\mathscr{X})$ is said to satisfy *a-Weyl's theorem*, $T \in a\mathscr{W}$, if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$, where

$$E_a^0(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}.$$

It is known [32] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in \mathscr{B}(\mathscr{X})$ and a non negative integer n define $T_{[n]}$ to be the restriction T to $\Re(T^n)$ viewed as a map from $\Re(T^n)$ to $\Re(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $\Re(T^n)$ is closed and $T_{[n]}$ is an upper (resp., lower) semi-Fredholm operator, then T is called *upper* (resp., *lower*) *semi-B-Fredholm* operator. In this case index of T is defined as the index of semi-B-Fredholm operator $T_{[n]}$. A *semi-B-Fredholm operator* is an upper or lower semi-Fredholm operator [11]. Moreover, if $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm* operator [9]. An operator T is called a *B-Weyl* operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not B-Weyl operator$\}$ [12].

An operator $T \in \mathscr{B}(\mathscr{X})$ is called *Drazin invertible* if it has a finite ascent and descent. The *Drazin spectrum* $\sigma_D(T)$ of an operator T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Drazin invertible}\}$. Define also the set $LD(\mathscr{X})$ by $LD(\mathscr{X}) = \{T \in \mathscr{B}(\mathscr{X}) : a(T) < \infty \text{ and } \Re(T^{a(T)+1}) \text{ is closed}\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin LD(\mathscr{X})\}$. Following [14], an operator $T \in \mathscr{B}(\mathscr{X})$ is said to be *left Drazin invertible* if $T \in LD(\mathscr{X})$. We say that $\lambda \in \sigma_a(T)$ is a *left pole* of T if $T - \lambda \in LD(\mathscr{X})$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T-\lambda) < \infty$. Let $\pi_a(T)$ denote the set of all left poles of T and let π_a^0 denote the set of all left poles of T of finite rank. From [14, Theorem 2.8] it follows that if $T \in \mathscr{B}(\mathscr{X})$ is left Drazin invertible, then T is an upper semi-B-Fredholm operator of index less than or equal to 0.

Let $\pi(T)$ be the set of all poles of the resolvent of T and let $\pi^0(T)$ be the set of all poles of the resolvent of T of finite rank, that is $\pi^0(T) = \{\lambda \in \pi(T) : \alpha(T-\lambda) < \infty\}$. According to [24], a complex number λ is a pole of the resolvent of T if and only if $0 < \max\{a(T-\lambda), d(T-\lambda)\} < \infty$. Moreover, if this is true then $a(T-\lambda) = d(T-\lambda)$.

According also to [24], the space $\mathfrak{R}((T-\lambda)^{a(T-\lambda)+1})$ is closed for each $\lambda \in \pi(T)$. Hence we have always $\pi(T) \subset \pi_a(T)$ and $\pi^0(T) \subset \pi_a^0(T)$. We say that *a-Browder's theorem* holds for $T \in \mathscr{B}(\mathscr{X})$, $T \in a\mathfrak{B}$, if $\Delta_a(T) = \pi_a^0(T)$. Following [13], we say that *generalized Weyl's theorem* holds for $T \in \mathscr{B}(\mathscr{X})$, $T \in g\mathscr{W}$ if $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T) = \{\lambda \in \operatorname{iso}\sigma(T) : \alpha(T-\lambda) > 0\}$ is the set of all isolated eigenvalues of T, and that *generalized Browder's theorem* holds for $T \in \mathscr{B}(\mathscr{X})$, $T \in g\mathfrak{B}$, if $\Delta^g(T) = \pi(T)$. It is proved in [6, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem. In [14, Theorem 3.9], it is shown that an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem, but the converse does not hold in general. Nonetheless and under the assumption $E(T) = \pi(T)$, it is proved in [15, Theorem 2.9] that generalized Weyl's theorem is equivalent to Weyl's theorem.

Let $SBF_+(\mathscr{X})$ be the class of all *upper semi-B-Fredholm operators*, $SBF_+^-(\mathscr{X}) = \{T \in SBF_+(\mathscr{X}) : \operatorname{ind}(T) \leqslant 0\}$. The *upper B-Weyl spectrum* of T is defined by $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T-\lambda \notin SBF_+^-(\mathscr{X})\}$. We say that *generalized a-Weyl's theorem* holds for $T \in \mathscr{B}(\mathscr{X})$, $T \in ga\mathscr{W}$, if $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, where $E_a(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : \alpha(T-\lambda) > 0\}$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in \mathscr{B}(\mathscr{X})$ obeys *generalized a-Browder's theorem*, $T \in ga\mathfrak{B}$, if $\Delta_a^g(T) = \pi_a(T)$. It is proved in [6, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [14, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption $E_a(T) = \pi_a(T)$ it is proved in [15, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem.

The organization of the paper is as follows: In section 2, we study the property (gt) in connection with Weyl type theorems. We prove that an operator T possessing property (gt) possesses generalized a-Weyl's theorem, but the converse is not true in general as shown by Example 2.5. And we obtain the equivalence of generalized a-Weyl's theorem and property (gt) if the operator T is a generalized scalar. In section 3, we study the stability of property (gt) under perturbations by finite rank operators, by nilpotent operators and, more generally, by algebraic operators commuting with T. Section 4 is devoted to study the transfer of property (gt) from a bounded linear operator T acting on a Banach space $\mathscr{X}$ and a bounded linear operator S acting on a Banach space $\mathscr{Y}$ to their tensor product $T \otimes S$.

2. Property (gt) for bounded linear operators

Let $\Delta_+^g(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T)$.

DEFINITION 2.1. ([39]) An operator T acting on a Banach space $\mathscr{X}$ obeys property (gt) if the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues are exactly those points λ of the spectrum for which $T-\lambda$ is an upper semi-B-Fredholm with index less than or equal to 0, that is, $T \in \mathscr{B}(\mathscr{X})$ possesses property (gt) if $\Delta_+^g(T) = E(T)$.

THEOREM 2.2. *If $T \in \mathscr{B}(\mathscr{X})$ satisfies property (gt), then $\sigma(T) = \sigma_a(T)$.*

Proof. Since $\sigma_a(T) \subseteq \sigma(T)$ holds for every operator T, we need only to prove $\sigma(T) \subseteq \sigma_a(T)$. Let $\lambda \in \sigma(T)$. Since T satisfies property (gt), we have $\lambda \in E(T)$. If $\lambda \notin \sigma_{SBF_+^-}(T)$, then $\lambda \in \operatorname{iso}\sigma(T) \subseteq \sigma_a(T)$. If $\lambda \in \sigma_{SBF_+^-}(T)$, it is easy to see that $\lambda \in \sigma_a(T)$, that is, $\sigma(T) = \sigma_a(T)$. □

THEOREM 2.3. *If $T \in \mathscr{B}(\mathscr{X})$ obeys property (gt), then $E(T) = E_a(T)$.*

Proof. Suppose that T satisfies property (gt), then $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$, it follows from Theorem 2.2 that $\sigma(T) = \sigma_a(T)$, and so $E(T) = E_a(T)$. □

Combining Theorems 2.2 and 2.3, we have

THEOREM 2.4. *If $T \in \mathscr{B}(\mathscr{X})$ obeys property (gt), then T satisfies generalized a-Weyl's theorem.*

The following example shows that generalized a-Weyl's theorem is weaker than property (gt).

EXAMPLE 2.5. Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the unilateral right shift operator defined by

$$T(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots) \text{ for all } x = (x_1, x_2, \cdots) \in \ell^2(\mathbb{N}).$$

Then $\sigma(T) = \mathbb{D}$, $\sigma_a(T) = \sigma_{SBF_+^-}(T) = \partial\mathbb{D}$ and $E(T) = E_a(T) = \emptyset$, where $\mathbb{D}$ denote the closed unit circle and $\partial\mathbb{D}$ denote the unit circle. It follows that $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, then T satisfies generalized a-Weyl's theorem. Whilst T doesn't obeys property (gt), since $\sigma(T) \setminus \sigma_{SBF_+^-}(T) \neq E(T)$.

THEOREM 2.6. *Let $T \in \mathscr{B}(\mathscr{X})$. Then T obeys property (gt) if and only if the following conditions hold:*

(i) T satisfies generalized a-Browder's theorem;

(ii) $\sigma(T) = \sigma_a(T)$;

(iii) $E(T) = \pi_a(T)$.

Proof. If T obeys property (gt), it follows from [39, Propsition 2.7] and Theorem 2.2 that T satisfies generalized a-Browder's theorem, $E(T) = \pi_a(T)$ and $\sigma(T) = \sigma_a(T)$. Conversely, if T satisfies generalized a-Browder's theorem, $E(T) = \pi_a(T)$ and $\sigma(T) = \sigma_a(T)$, then

$$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi_a(T) = E(T),$$

that is, T obeys property (gt). □

THEOREM 2.7. *Let $T \in \mathscr{B}(\mathscr{X})$. Then T obeys property (gt) if and only if the following conditions hold:*

(i) T satisfies generalized Browder's theorem;

(ii) $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$;

(iii) $E(T) = \pi(T)$.

Proof. If T obeys property (gt), then it follows from [39, Theorem 2.10] that T satisfies generalized Weyl's theorem and $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$ and T satisfies generalized Browder's theorem and $E(T) = \pi(T)$. On the other hand, if T satisfies generalized Browder's theorem, $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$ and $E(T) = \pi(T)$, we have

$$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) = \pi(T) = E(T),$$

that is, T obeys property (gt). $\square$

Following [23] we say that $T \in \mathscr{B}(\mathscr{X})$ has the *single-valued extension property* (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \longrightarrow \mathscr{X}$ which satisfies the equation $(T-\mu)f(\mu) = 0$ is the constant function $f \equiv 0$. An operator $T \in \mathscr{B}(\mathscr{X})$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$. An operator $T \in \mathscr{B}(\mathscr{X})$ has the SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. The identity theorem for analytic functions ensures that for every $T \in \mathscr{B}(\mathscr{X})$, both T and T^* have the SVEP at the points of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, that both T and T^* have the SVEP at every isolated point of $\sigma(T) = \sigma(T^*)$. The SVEP is inherited by the restrictions to closed invariant subspaces, i.e., if $T \in \mathscr{B}(\mathscr{X})$ has the SVEP at λ_0 and M is closed T-invariant subspace then $T|_M$ has SVEP at λ_0. Let $S(T) := \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\}$. Observe that $T \in \mathscr{B}(\mathscr{X})$ has SVEP if and only if $S(T) = \emptyset$.

REMARK 2.8. If $T^* \in \mathscr{B}(\mathscr{X})$ has the SVEP, then it is known from [27, Page 35] that $\sigma(T) = \sigma_a(T)$ and from [38, Corollary 2.9] we have $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. Hence $E_a(T) = E(T)$, $\Delta^g(T) = \Delta_a^g(T)$ and $\Delta_+^g(T) = \Delta^g(T)$. Moreover, it is known that from [3, Theorem 2.6] that if T^* has the SVEP, then $\sigma_{SF_+^-}(T) = \sigma_w(T)$ and hence $E_a^0(T) = E^0(T)$, $\Delta_a(T) = \Delta(T)$ and $\Delta_+(T) = \Delta(T)$.

Let $H_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non-constant on each of the components of its domain. Define, by the classical calculus, $f(T)$ for every $f \in H_{nc}(\sigma(T))$.

A bounded operator $T \in \mathscr{B}(\mathscr{X})$ is said to be *polaroid* (respectively, *a-polaroid*) if $\operatorname{iso}\sigma(T) = \emptyset$ or every isolated point of $\sigma(T)$ is a pole of the resolvent of T (respectively, if $\operatorname{iso}\sigma_a(T) = \emptyset$ or every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T).

THEOREM 2.9. *Let T be a bounded linear operator on $\mathscr{X}$ satisfying the SVEP. If $T-\lambda I$ has finite descent at every $\lambda \in E_a(T)$, then property (gt) holds for $f(T^*)$, for every $f \in H_{nc}(\sigma(T))$.*

Proof. Let $\lambda \in E_a(T)$, then $p = d(T-\lambda I) < \infty$ and since T has the SVEP it follows that $a(T-\lambda I) = d(T-\lambda I) = p$ and hence λ is a pole of the resolvent of T of order p, consequently λ is an isolated point in $\sigma_a(T)$. Then $\mathscr{X} = K(T-\lambda I) \oplus H_0(T-\lambda I)$, with $K(T-\lambda I) = \Re(T-\lambda I)^p$ is closed, Therefore, $\lambda \in \pi_a(T)$. Hence, T is a-polaroid. Now the result follows now from [39, Theorem 3.6]. □

The *quasinilpotent part* $H_0(T-\lambda I)$ and the *analytic core* $K(T-\lambda)$ of $T-\lambda$ are defined by

$$H_0(T-\lambda) := \{x \in \mathscr{X} : \lim_{n \longrightarrow \infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\}.$$

and

$$K(T-\lambda) = \{x \in \mathscr{X} : \text{there exists a sequence } \{x_n\} \subset \mathscr{X} \text{ and } \quad \delta > 0 \text{ for which } \\ x = x_0, (T-\lambda)x_{n+1} = x_n \text{ and} \|x_n\| \leqslant \delta^n \|x\| \text{ for all } \quad n \in \mathbb{N}\}.$$

We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are generally non-closed hyper-invariant subspaces of $T-\lambda$ such that $(T-\lambda)^{-p}(0) \subseteq H_0(T-\lambda)$ for all $p = 0, 1, \cdots$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$.

The class of operators $T \in \mathscr{B}(\mathscr{X})$ for which $K(T) = \{0\}$ was introduced and studied by M. Mbekhta in [28]. It was shown that for such operators, the spectrum is connected and the SVEP holds.

THEOREM 2.10. *Let $T \in \mathscr{B}(\mathscr{X})$. If there exists λ such that $K(T-\lambda) = \{0\}$, then $f(T) \in ga\mathfrak{B}$, for every $f \in H_{nc}(\sigma(T))$. Moreover, if in addition* $\ker(T-\lambda) = 0$, *then property (gt) holds for $f(T)$*

Proof. Since T has the SVEP, then by [6, Theorem 3.2], generalized a-Browder's theorem holds for $f(T)$. Let $\gamma \in \sigma(f(T))$, then

$$f(z) - \gamma I = P(z)g(z),$$

where g is complex-valued analytic function on a neighborhood of $\sigma(T)$ without any zeros in $\sigma(T)$ while P is a complex polynomial of the form $P(z) = \prod_{j=1}^{n}(z-\lambda_j I)^{k_j}$ with distinct roots $\lambda_1, \cdots, \lambda_n \in \sigma(T)$. Since $g(T)$ is invertible, then we deduce that

$$\ker(f(T) - \gamma I) = \ker(P(T)) = \bigoplus_{j=1}^{n} \ker(T-\lambda_j I)^{k_j}.$$

On the other hand, it follows from [28, Proposition 2.1] that $\sigma_p(T) \subseteq \{\lambda\}$. If we assume that $\ker(T-\lambda I) = 0$, then $T-\lambda I$ is an injective and consequently $\sigma_p(T) = \emptyset$. Hence $\ker(f(T) - \lambda I) = 0$. Therefore, $\sigma_p(f(T)) = \emptyset$. Now, we prove that

$$\pi^a(f(T)) = E(f(T)).$$

Obviously, the condition $\sigma_p(f(T)) = \emptyset$ entails that

$$E(f(T)) = E^a(f(T)) = \emptyset.$$

On the other hand, the inclusion $\pi^a(f(T)) \subseteq E^a(f(T))$ holds for every operator $T \in \mathscr{B}(\mathscr{X})$. So also $\pi^a(f(T)) = \emptyset$. Hence property (gw) and generalized a-Weyl's theorem hold for $f(T)$ and so $\sigma_{SBF_+^-}(f(T)) = \sigma_{BW}(f(T)) = \sigma(T) = \sigma_a(T)$. It then follows by [39, Theorem 2.10] that $f(T)$ obeys property (gt). $\square$

In [29] Oudghiri introduced the class $H(p)$ of operators on Banach spaces for which there exists $p := p(\lambda) \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = \ker(T - \lambda I)^p \qquad \text{for all } \lambda \in \mathbb{C}.$$

Let $P(\mathscr{X})$ be the class of operators $T \in \mathscr{B}(\mathscr{X})$ having the property $H(p)$. The class $P(\mathscr{X})$ contains the classes of subscalar, algebraically $wF(p,q,r)$ operators with $p,r > 0$ and $q \geqslant 1$ [37], algebraically w-hyponormal operators [34], algebraically quasi-class (A,k) [33]. It is known that if $H_0(T - \lambda I)$ is closed for every complex number λ, then T has the SVEP (see [1, 25]). So that, the SVEP is shared by all operators of $P(\mathscr{X})$. Moreover, T is polaroid, see [2, Lemma 3.3].

THEOREM 2.11. *Let T be a bounded operator on $\mathscr{X}$. If there exists a function $g \in H_{nc}(\sigma(T))$ such that $g(T^*) \in P(\mathscr{X}^*)$, then property (gt) holds for $f(T)$, for every $f \in H_{nc}(\sigma(T))$.*

Proof. Suppose that $g(T^*) \in P(\mathscr{X}^*)$, then by [29, Theorem 3.4], we have $T^* \in P(\mathscr{X}^*)$. Since T^* has the SVEP, then as it had been already mentioned, we have

$$\sigma_a(T) = \sigma(T),\ \sigma_{SBF_+^-}(T) = \sigma_{BW}(T),\ E_a(T) = E(T) \text{ and } \quad \Delta_+^g(T) = \Delta_+(T),$$

it suffices to show that $\pi_a(T) = E_a(T)$. For this let $\lambda \in E_a(T)$, then λ is an isolated eigenvalue of $\sigma_a(T)$. So $\mathscr{X}^* = H_0(T^* - \bar{\lambda}) \oplus K(T^* - \bar{\lambda})$, where the direct sum is topological. Since $T^* \in P(\mathscr{X}^*)$, then there exists $t = d_\lambda \in \mathbb{N}$ such that $H_0(T^* - \bar{\lambda} I) = \ker(T^* - \bar{\lambda} I)^t$, and hence $\mathscr{X}^* = \ker(T^* - \bar{\lambda})^t \oplus K(T^* - \bar{\lambda})$. Since

$$\Re((T - \bar{\lambda} I)^t) = (T - \bar{\lambda})^t (K(T - \bar{\lambda} I)) = K(T - \bar{\lambda} I),$$

so

$$\mathscr{X} = \ker(T - \bar{\lambda} I)^t \oplus \Re((T - \bar{\lambda} I)^t),$$

which implies, by [1, Theorem 3.6], that $a(T^* - \bar{\lambda} I) = d(T - \bar{\lambda} I) \leqslant t$, hence $\bar{\lambda}$ is a pole of the resolvent of T^*, so that T^* is polaroid. Hence we have $\mathscr{X}^* = \ker(T^* - \bar{\lambda} I)^t \oplus \Re(T^* - \bar{\lambda} I)^t$ and $\Re(T^* - \bar{\lambda} I)^t$ is closed. Therefore, $\Re(T - \lambda I)$ is closed and $\mathscr{X} = \ker(T^* - \bar{\lambda} I)^\perp \oplus \Re(T^* - \bar{\lambda} I)^\perp = \ker(T - \lambda I) \oplus \Re(T - \lambda I)$. So $\lambda \in \pi_a(T)$. As T^* has the SVEP and T is polaroid, then $f(T)$ satisfies property (gt) for every $f \in H_{nc}(\sigma(T))$ by [39, Theorem 3.6]. $\square$

THEOREM 2.12. *Suppose that $T \in \mathscr{B}(\mathscr{X})$ is generalized scalar. Then T satisfies property (gt) if and only if T satisfies generalized Weyl's theorem*

Proof. If T is generalized scalar then both T and T^* has SVEP. Moreover, T is polaroid since every generalized scalar has the property $H(p)$. Then T satisfies property (gt) by [39, Theorem 3.5]. The equivalence then follows from [39, Theorem 2.10]. $\square$

EXAMPLE 2.13. Property (gt), as well as generalized Weyl's theorem, is not transmitted from T to its dual T^*. To see this, consider the weighted right shift $T \in \mathscr{B}(\ell^2(\mathbb{N}))$, defined by

$$T(x_1, x_2, \cdots) := \left(0, \frac{x_1}{2}, \frac{x_2}{3}, \cdots\right) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Then

$$T^*(x_1, x_2, \cdots) := \left(\frac{x_2}{2}, \frac{x_3}{3}, \cdots\right) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Both T and T^* are quasi-nilpotent, and hence are decomposable, T satisfies generalized Weyl's theorem since $\sigma(T) = \sigma_{BW}(T) = \{0\}$ and $E(T) = \pi(T) = \emptyset$ and hence T has property (gt). On the other hand, we have $\sigma(T^*) = \sigma_a(T^*) = \sigma_{SBF_+^-}(T^*) = E_a(T^*) = \sigma_{BW}(T^*) = E(T^*) = \{0\}$ and $\pi_a(T^*) = \emptyset$, so T^* does not satisfy generalized Weyl's theorem (and nor generalized a-Weyl's theorem). Although T^* has SVEP, But T^* does not satisfy property (gt).

3. Property (gt) under perturbations

we shall consider nilpotent perturbations of operators satisfying property (gt). It easy to check that if N is a nilpotent operator commuting with T, then

$$\sigma(T) = \sigma(T+N) \quad \text{and} \quad \sigma_a(T) = \sigma_a(T+N). \tag{3.1}$$

Hence it follows from Equation (3.1)

$$E^0(T) = E^0(T+N), \quad E_a^0(T) = E_a^0(T+N), \quad E(T) = E(T+N) \tag{3.2}$$

and

$$E_a(T) = E_a(T+N). \tag{3.3}$$

By [43, Corollary 3.8] we have

$$\pi_a(T) = \pi_a(T+N) \quad \text{and} \quad \pi(T) = \pi(T+N). \tag{3.4}$$

THEOREM 3.1. *Suppose that $T \in \mathscr{B}(\mathscr{X})$ and $N \in \mathscr{B}(\mathscr{X})$ is a nilpotent operator commuting with T. Then T obeys property (gt) if and only if $T+N$ obeys property (gt).*

Proof. Suppose that T obeys property (gt) we have $\Delta_+^g(T) = E(T)$. It follows from [43, Corollary 3.1] that $\sigma_{SBF_+^-}(T) = \sigma_{SBF_+^-}(T+N)$. Hence

$$E(T+N) = E(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T+N) \setminus \sigma_{SBF_+^-}(T+N).$$

That is, $T+N$ obeys property (gt). The converse follows by symmetry. $\square$

The next example shows that the commutativity hypothesis in Theorem 3.1 is essential.

EXAMPLE 3.2. Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, \cdots) = \left(0, 0, \frac{x_1}{2}, \frac{x_2}{4}, \frac{x_3}{8}, \cdots\right) \quad \text{for all } x = (x_1, x_2, \cdots) \in \ell^2(\mathbb{N})$$

and

$$N(x_1, x_2, \cdots) = \left(0, 0, -\frac{x_1}{2}, 0, 0, \cdots\right) \quad \text{for all } x = (x_1, x_2, \cdots) \in \ell^2(\mathbb{N}).$$

Clearly N is nilpotent, $\sigma(T) = \sigma_{SBF_+^-}(T) = \{0\}$ and $E(T) = \emptyset$. If follows that $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$, i.e., T obeys property (gt). On the other hand, $\sigma(T+N) = \sigma_{SBF_+^-}(T+N) = \{0\}$ and $E(T+N) = \{0\}$, it follows that $\sigma(T+N) \setminus \sigma_{SBF_+^-}(T+N) = \emptyset \neq E(T+N)$, that is, $T+N$ doesn't obeys property (gt).

THEOREM 3.3. *Suppose that $T \in \mathscr{B}(\mathscr{X})$ is polaroid, $N \in \mathscr{B}(\mathscr{X})$ is a nilpotent operator commuting with T. If T^* has SVEP and $f \in H_{nc}(\sigma(T))$ then property (gt) holds for $f(T)+N$.*

Proof. By [39, Theorem 3.5], T satisfies property (gt). The SVEP for T^* implies that $\sigma(T) = \sigma_a(T)$, so every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T. It follows from [39, Theorem 3.7] that property (gt) holds for $f(T)$. Since $\sigma(f(T)) = f(\sigma(T)) = f(\sigma_a(T)) = \sigma_a(f(T))$ we have by Theorem 3.1 $f(T)+N$ satisfies property (gt). $\square$

REMARK 3.4. It is somewhat meaningful to ask what we can say about the operators $f(T+N)$, always under the assumptions of Theorem 3.3. Now, if T is polaroid then $T+N$ is polaroid, by [3, Theorem 2.10]. Moreover, by $T^* + N^* = (T+N)^*$ has SVEP by [1, Corollary 2.12]. Hence by [39, Theorem 3.7] $f(T+N)$ satisfies property (gt) for every $f \in H_{nc}(\sigma(T))$.

Note that Theorem 3.1 does not extend to commuting finite rank operators as shown by the following example.

EXAMPLE 3.5. Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be an injective quasinilpotent operator and let $U : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by $U(x_1, x_2, \cdots) = (-\frac{x_1}{2}, 0, 0, \cdots)$ for all $x = (x_1, x_2, \cdots) \in \ell^2(\mathbb{N})$. Define

$$T = \begin{pmatrix} \frac{1}{2}I & 0 \\ 0 & S \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\sigma(T)=\sigma_{SBF_+^-}(T)=\{0,\frac{1}{2}\}$ and $E(T)=\emptyset$. It follows that $\sigma(T)\setminus\sigma_{SBF_+^-}(T)=E(T)$, i.e., T obeys property (gt). On the other hand, since $\sigma(T+F)=\sigma_{SBF_+^-}(T+F)=\{0,\frac{1}{2}\}$ and $E(T+F)=\{0\}$, then $\sigma(T+F)\setminus\sigma_{SBF_+^-}(T+F)=\emptyset\neq E(T+F)$, i.e., $T+F$ does not obeys property (gt). Note that F is finite rank operator commuting with T.

LEMMA 3.6. *Suppose that $T\in\mathscr{B}(\mathscr{X})$ obeys property (gt) and F is a finite operator commuting with T such that $\sigma_a(T+F)=\sigma_a(T)$. Then $\pi_a(T+F)\subseteq E(T+F)$.*

Proof. As T obeys property (gt) then it follows from [39, Theorem 2.4] that T obeys property (gw) and hence the result then follows by [36, Lemma 2.13]. $\square$

Recall that a bounded operator $T\in\mathscr{B}(\mathscr{X})$ is said to be *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T.

THEOREM 3.7. *Suppose that $T\in\mathscr{B}(\mathscr{X})$ is isoloid, F is an operator that commutes with T and for which there exists a positive integer n such that F^n is finite rank. If T satisfies property (gt), then $T+F$ satisfies property (gt).*

Proof. Suppose that T obeys property (gt). It follows from [39, Proposition 2.8] that T satisfies generalized Browder's theorem and $\sigma_{BW}(T)=\sigma_{SBF_+^-}(T)$, and hence $T+F$ satisfies generalized Browder's theorem and $\sigma_{BW}(T+F)=\sigma_{SBF_+^-}(T+F)$. By Theorem 2.7, in order to show that $T+F$ satisfies property (gt), we need only to show $\pi(T+F)=E(T+F)$. Since $\pi(T+F)\subseteq E(T+F)$ holds for every operator, it is sufficient to prove $E(T+F)\subseteq\pi(T+F)$. Let $\lambda\in E(T+F)$. If $T-\lambda$ is invertible, then $T+F-\lambda$ is B-Fredholm, and hence $\lambda\in E(T+F)$. If $\lambda\in\sigma(T)$, it follows from [30, Lemma 2.3] that $\lambda\in\text{iso}\,\sigma(T)$. Since T is isoloid, we have $0<\alpha(T+F)$ as F^n is a finite rank operator commuting with T, $(T+F-\lambda)^n|_{\ker(T-\lambda)}=F^n|_{\ker(T-\lambda)}$ has finite-dimension range and kernel, it is easy to obtain that $\alpha(\lambda-T)<\infty$, i.e., $\lambda\in E^0(T)$. We have $\lambda\in\pi^0(T)$ by Theorem 2.7, then $\lambda-T$ is Browder. It follows that $\lambda-(T+F)$ is also Browder, hence $\lambda\in\sigma(T+F)\setminus\sigma_b(T+F)=\pi^0(T+F)$, i.e., $T+F$ satisfies property (gt). $\square$

The following example shows that Theorem 3.7 fails if we do not assume that T is isoloid.

EXAMPLE 3.8. Let T be defined as in Example 3.5. Then $\sigma(T)=\sigma_{SBF_+^-}(T)=\{0,\frac{1}{2}\}$ and $E(T)=\emptyset$, it follows that $\sigma(T)\setminus\sigma_{SBF_+^-}(T)=E(T)$, i.e., T obeys property (gt). On the other hand, since $\sigma(T+F)=\sigma_{SBF_+^-}(T+F)=\{0,\frac{1}{2}\}$ and $E(T+F)=\{0\}$, then $\sigma(T+F)\setminus\sigma_{SBF_+^-}(T+F)=\emptyset\neq E(T+F)$, i.e., $T+F$ does not obey property (gt). It is easy to verify that F^n is a finite rank operator commuting with T and T is not isoloid.

COROLLARY 3.9. *Suppose that $T \in \mathscr{B}(\mathscr{X})$ is isoloid, F is a finite rank operator that commutes with T. If T satisfies property (gt), then $T+F$ satisfies property (gt).*

THEOREM 3.10. *Suppose $T \in \mathscr{B}(\mathscr{X})$ and $\operatorname{iso}\sigma_a(T) = \emptyset$. If T obeys property (gt) and F is a finite rank operator commuting with T, then $T+F$ obeys property (gt).*

Proof. Suppose that T obeys property (gt). It follows from Theorem 2.7 that T satisfies generalized Browder's theorem, $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$ and $\pi(T) = E(T)$ and hence $T+F$ satisfies generalized Browder's theorem and so

$$\begin{aligned}\sigma_{BW}(T+F) &= \sigma_D(T+F) = \sigma_D(T) = \sigma_{BW}(T) = \sigma_{SBF_+^-}(T)\\ &= \sigma_{LD}(T) = \sigma_{LD}(T+F) = \sigma_{SBF_+^-}(T+F),\end{aligned}$$

i.e., $\sigma_{BW}(T+F) = \sigma_{SBF_+^-}(T+F)$. In order to show that $T+F$ obeys property (gt), we need only to show $\pi(T+F) = E(T+F)$. Since $\pi(T+F) \subseteq E(T+F)$ holds for every operator, it is sufficient to prove $E(T+F) \subseteq \pi(T+F)$. Since $\operatorname{iso}\sigma_a(T) = \emptyset$ and F is a finite rank operator commuting with T, by [5, Theorem 2.8] that $\sigma_a(T) = \sigma_a(T+F)$, then $\operatorname{iso}\sigma_a(T+F) = \emptyset$. Since $\operatorname{iso}\sigma(T+F) \subseteq \operatorname{iso}\sigma_a(T+F)$, $\operatorname{iso}\sigma(T+F) = \emptyset$. It follows that $E(T+F) = \emptyset$ and so $E(T+F) \subseteq \pi(T+F)$, i.e., $T+F$ obeys property (gt). $\square$

Recall that $T \in \mathscr{B}(\mathscr{X})$ is said to be a *Riesz operator* if $T - \lambda \in \mathfrak{F}$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Evidently, quasi-nilpotent operators and compact operators are Riesz operators.

EXAMPLE 3.11. In general property (gt) is not transmitted from an operator to a commuting quasinilpotent perturbation as the following example shows.

If we consider on the Hilbert space $\ell^2(\mathbb{N})$ the operators $T = 0$ and Q defined by

$$Q(x_1, x_2, \cdots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \cdots\right) \quad \text{for all } x_n \in \ell^2(\mathbb{N}).$$

Then Q is quasinilpotent operator commuting with T. Moreover, we have $\sigma(T) = \{0\}, \sigma_{SBF_+^-}(T) = \emptyset$, $E(T) = \{0\}$. Hence T obeys property (gt). But property (gt) fails for $T+Q = Q$. Indeed, $\sigma_{SBF_+^-}(T+Q) = \{0\}, E(T+Q) = E(T) = \{0\}$ and $\sigma(T+Q) = \{0\}$.

REMARK 3.12. It is well-known that if Q is a quasi-nilpotent operator commuting with T then

$$\sigma(T+Q) = \sigma(T) \quad \text{and} \quad \sigma_a(T+Q) = \sigma_a(T).$$

THEOREM 3.13. *Let $T \in \mathscr{B}(\mathscr{X})$ and Q is a quasinilpotent which is commutes with T. If T obeys property (gt) and has the SVEP, then $T+Q$ obeys property (gt).*

Proof. It follows from [10, Lemma 2.5, Lemma 2.7] that $\sigma_{SBF_+^-}(T) = \sigma_{SBF_+^-}(T+Q)$ and $E(T) = E(T+Q)$. As T obeys property (gt) we have $\Delta_+^g(T) = E(T)$. Hence $E(T+Q) = E(T) = \Delta_+^g(T) = \Delta_+^g(T+Q)$. That is, $T+Q$ obeys property (gt). □

THEOREM 3.14. *Let $T \in \mathscr{B}(\mathscr{X})$ and $F \in \mathscr{B}(\mathscr{X})$ be a finite rank operator commuting with T. If T obeys property (gt), then the following assertions are equivalent.*

(a) $T+F$ obeys property (gt);

(b) $E(T+F) = \pi_a(T+F)$;

(c) $E(T+F) \cap \sigma(T) \subset E(T)$.

Proof. (a)$\Leftrightarrow$ (b) If $T+F$ obeys property (gt), then from [39, Proposition 2.8], $E(T+F) = \pi_a(T+F)$. Conversely, assume that $E(T+F) = \pi_a(T+F)$, since T obeys property (gt), then by [39, Proposition 2.8], T satisfies generalized a-Browder's theorem and hence $\sigma_{LD}(T) = \sigma_{SBF_+^-}(T)$. Since F is a finite rank, from [16, Lemma 2.3] we have $\sigma_{LD}(T+F) = \sigma_{BF}(T+F)$. As T commutes with F, from [17, Theorem 2.1] we have $\sigma_{LD}(T) = \sigma_{LD}(T+F)$. So $\sigma_{LD}(T+F) = \sigma_{SBF_+^-}(T+F)$. As $E(T+F) = \pi_a(T+F)$, then from [7, Theorem 2.6], $T+F$ satisfies property (gw). Since $\sigma(T) = \sigma(T+F)$, it then follows by [39, Theorem 2.4] that $T+F$ obeys property (gt).

(c)$\Rightarrow$(b) Assume that $E(T+F) \cap \sigma(T) \subset E(T)$. Let $\lambda \in E(T+F)$. If $\lambda \notin \sigma(T)$, then $\lambda \notin \sigma_{LD}(T)$. Since F commutes with T, from [14, Theorem 4.2] we have $\sigma_{LD}(T) = \sigma_{LD}(T+F)$. As $\lambda \in \sigma(T+F)$, then $\lambda \in \pi_a(T+F)$. If $\lambda \in \sigma(T)$, then $\lambda \in E(T+F) \cap \sigma(T)$ and by hypothesis we have $\lambda \in E(T)$. As T obeys property (gt), it follows that $\lambda \in \pi_a(T)$. As $\sigma_{LD}(T) = \sigma_{LD}(T+F)$ and $\lambda \in \sigma(T+F)$ then $\lambda \in \pi_a(T+F)$. Finally we have $E(T+F) \subseteq \pi_a(T+F)$. Conversely, assume that $\lambda \in \pi_a(T+F)$, then $\lambda \notin \sigma_{LD}(T+F) = \sigma_{LD}(T)$. As T obeys property (gt) then $\lambda \notin \sigma_{SBF_+^-}(T) = \sigma_{SBF_+^-}(T+F)$. and $E(T) = \pi_a(T)$ Hence λ is an isolated point of $\sigma(T) = \sigma(T^*)$ and therefore, both T and T^* have SVEP at λ. Since $T - \lambda I \in ga\mathfrak{B}$ it then follows that $0 < m = a(T-\lambda I) = d(T-\lambda I) < \infty$. Furthermore, since $\lambda \in E(T)$ we also have $\alpha(T-\lambda I) > 0$, thus $T - \lambda I \in ga\mathfrak{B}$ and hence also $T + Q - \lambda I \in ga\mathfrak{B}$, by [26, Theorem 2.1]. Hence λ is an isolated point of $\sigma(T+Q)$ and $\alpha(T+Q-\lambda I) > 0$. On the other hand, $(T+Q-\lambda I)^{m+1}$ has closed range and since $\lambda \in \sigma_a(T+Q)$ it then follows that $\alpha(T+Q-\lambda I) > 0$. Thus $\lambda \in E(T+Q)$.

(b)$\Rightarrow$ (c) Assume that $E(T+F) = \pi_a(T+F)$ and let $\lambda \in \pi_a(T+F) \cap \sigma(T)$, then $\lambda \in E(T+F) \cap \sigma(T)$. Therefore $\lambda \notin \sigma_{LD}(T+F)$. As $\sigma_{LD}(T+F) = \sigma_{LD}(T)$ and $\lambda \in \sigma(T)$, then $\lambda \in \pi_a(T)$. As T obeys property (gt) we have $\lambda \in E(T)$. □

In the next theorem, we consider an operator $T \in \mathscr{B}(\mathscr{X})$ obeying property (gt), a nilpotent operator commuting with T, and we give a necessary and sufficient condition for $T+N$ to obey property (gt).

THEOREM 3.15. *Let $T \in \mathscr{B}(\mathscr{X})$ and $N \in \mathscr{B}(\mathscr{X})$ be a nilpotent operator commuting with T. If T obeys property (gt), then the following assertions are equivalent.*

(a) $T+N$ *obeys property* (gt)*;*

(b) $\sigma_{SBF_+^-}(T+N)=\sigma_{SBF_+^-}(T)$*;*

(c) $E(T)=\pi_a(T+N)$ *and* $\sigma_a(T+N)=\sigma(T+N)$*.*

Proof. (a)$\Leftrightarrow$(b) Assume that $T+N$ obeys property (gt), then

$$\sigma(T+N)\setminus\sigma_{SBF_+^-}(T+N)=E(T+N).$$

As $\sigma(T+N)=\sigma(T)$ and $E(T)=E(T+N)$, then $\sigma(T)\setminus\sigma_{SBF_+^-}(T+N)=E(T)$. Since T obeys property (gt), then $\sigma(T)\setminus\sigma_{SBF_+^-}(T)=E(T)$. So $\sigma_{SBF_+^-}(T+N)=\sigma_{SBF_+^-}(T)$. Conversely, assume that $\sigma_{SBF_+^-}(T+N)=\sigma_{SBF_+^-}(T)$, then as T obeys property (gt) it follows that $T+N$ obeys property (gt).

(a)$\Leftrightarrow$(c) Assume that $T+N$ obeys property (gt), then from [39, Proposition 2.8] that $E(T+N)=\pi_a(T+N)$. Hence $E(T)=\pi_a(T+N)$. By [39, Theorem 2.6], we give $\sigma(T+N)=\sigma_a(T+N)$. Conversely, assume that $E(T)=\pi_a(T+N)$. Since T obeys property (gt), then by [39, Theorem 2.4] that T obeys property (gw). By [43, Theorem 3.1] we have $\sigma_{SBF_+^-}(T+N)=\sigma_{SBF_+^-}(T)$. Hence

$$\begin{aligned}E(T+N)=E(T)&=\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)=\sigma_a(T+N)\setminus\sigma_{SBF_+^-}(T+N)\\&=\sigma(T+N)\setminus\sigma_{SBF_+^-}(T+N).\end{aligned}$$

That is $T+N$ obeys property (gt). □

DEFINITION 3.16. A bounded linear operator T is said to be *algebraic* if there exists a non-trivial polynomial h such that $h(T)=0$.

From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set. A nilpotent operator is a trivial example of an algebraic operator. Also finite rank operators K are algebraic; more generally, if K^n is a finite rank operator for some $n\in\mathbb{N}$ then K is algebraic. Clearly, if T is algebraic then its dual T^* is algebraic, as well as T' in the case of Hilbert space operators.

THEOREM 3.17. *Suppose that* $T\in\mathscr{B}(\mathscr{X})$ *and* $K\in\mathscr{B}(\mathscr{X})$ *is an algebraic operator which commutes with* T*.*

(i) If T^* *is hereditarily polaroid and has SVEP, then* $T+K$ *obeys property* (gt)*.*

(ii) If T *is hereditarily polaroid and has SVEP, then* T^*+K^* *obeys property* (gt)*.*

Proof. (i) Obviously, K^* is algebraic and commutes with T^*. Moreover, by [5, Theorem 2.15], we have T^*+K^* is polaroid, or equivalently, $T+K$ is polaroid. Since T^* has SVEP then by [3, Theorem 2.14], we have T^*+K^* has SVEP. Therefore, $T+K$ obeys property (gt) by [39, Theorem 3.5 (i)].

(ii) It follows from the proof of Theorem 2.15 of [5] that $T+K$ is polaroid and hence by duality T^*+K^* is polaroid. Since T has SVEP then it follows from [3, Theorem 2.14] that $T+K$ has SVEP. Therefore, T^*+K^* obeys property (gt) by [39, Theorem 3.5 (ii)]. $\square$

THEOREM 3.18. *Suppose that $T \in \mathscr{B}(\mathscr{X})$ and $K \in \mathscr{B}(\mathscr{X})$ is an algebraic operator which commutes with T.*

(i) If T^* is hereditarily polaroid and has SVEP, then $f(T+K)$ obeys property (gt) for all $f \in H_{nc}(\sigma(T))$.

(ii) If T is hereditarily polaroid and has SVEP, then $f(T^*+K^*)$ obeys property (gt) for all $f \in H_{nc}(\sigma(T))$.

Proof. (i) We conclude from [5, Theorem 2.15] that $T+K$ is polaroid and hence by [4, Lemma 3.11], we have $f(T+K)$ is polaroid and from [3, Theorem 2.14] that T^*+K^* has SVEP. The SVEP of T^*+K^* entails the SVEP for $f(T^*+K^*)$ by [1, Theorem 2.40]. So, $f(T+K)$ obeys property (gt) by [39, Theorem 3.7 (i)].

(ii) The proof of part (ii) is analogous. $\square$

4. Property (gt) and tensor products

Let $\sigma_s(S)=\{\lambda \in \sigma(S) : S-\lambda \text{ is not surjective}\}$ denote, the surjectivity spectrum. Let $\Psi_-(\mathscr{X})$ be the class of all lower semi B-Fredholm operators, $\Psi_-^+(\mathscr{X})=\{S \in \Psi_-(\mathscr{X}) : ind(S-\lambda) \geqslant 0\}$. The lower semi B-Weyl spectrum of S is defined by $\sigma_{SBF_-^+}(S)=\left\{\lambda \in \mathbb{C} : S-\lambda \notin \Psi_-^+(\mathscr{X})\right\}$. Define $RD(\mathscr{X})=\{S \in \mathscr{B}(\mathscr{X}) : dsc(S)=d<\infty \text{ and } \Re(S^{d+1}) \text{ is closed}\}$. The right Drazin spectrum is defined by $\sigma_{RD}(S)=\{\lambda \in \mathbb{C} : S-\lambda \notin RD(\mathscr{X})\}$. It is not difficult to see that $\sigma_D(S)=\sigma_{LD}(S)\cup\sigma_{RD}(S)$. Moreover, $\sigma_{LD}(S)=\sigma_{RD}(S^*)$ [8]. Then S satisfies generalized s-Browder's theorem if $\sigma_{SBF_-^+}(S)=\sigma_{RD}(S)$. Apparently, S satisfies generalized s-Browder's theorem if and only if S^* satisfies generalized a-Browder's theorem. A necessary and sufficient condition for S to satisfy generalized a-Browder's theorem is that S has SVEP at every $\lambda \in \Delta_a^g(S)$ [20, Theorem 3.1]; by duality, S satisfies generalized s-Browder's theorem if and only if S^* has SVEP at every $\lambda \in \sigma_s(S)\setminus\sigma_{SBF_-^+}(S)$. More generally, if either of S and S^* has SVEP, then S and S^* satisfy both generalized a-Browder's theorem and generalized s-Browder's theorem. Either of generalized a-Browder's theorem and generalized s-Browder's theorem implies generalized Browder's theorem, but the converse is false. generalized a-Browder's theorem fails to transfer from A and B to $A\otimes B$ [21, Example 1].

The problem of transferring property (Bb), property (Sw), generalized Weyl's theorem and Property (gw) from operators T and S to their tensor product $T\otimes S$ was considered in [41], [40], [42]. The main objective of this section is to study the transfer of property (gt) from a bounded linear operator T acting on a Banach space $\mathscr{X}$ and a bounded linear operator S acting on a Banach space $\mathscr{Y}$ to their tensor product $T\otimes S$.

THEOREM 4.1. *Let $T \in \mathscr{B}(\mathscr{X})$ and $S \in \mathscr{B}(\mathscr{Y})$ such that T and S are isoloid and $0 \notin iso\,\sigma(T \otimes S)$. If property (gt) holds for T and S, then the following statements are equivalent.*

(a) $T \otimes S$ *satisfies property* (gt).

(b) $\sigma_{SBF_+^-}(T \otimes S) = \sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma(T)\sigma_{SBF_+^-}(S)$.

Proof. (a)$\Longrightarrow$ (b): Assume that $T \otimes S$ satisfies property (gt). Let

$$\lambda \in E(T \otimes S) = \sigma(T)\sigma_{SBF_+^-}(S) \cup \sigma_{SBF_+^-}(T)\sigma(S).$$

Since $0 \notin \operatorname{iso}\sigma(T \otimes S)$, then $\lambda \neq 0$. Hence $\lambda \in \operatorname{iso}\sigma(T \otimes S) = \operatorname{iso}\sigma(T)\operatorname{iso}\sigma(S)$. That is, $\lambda = \mu\nu$ with $\mu \in \operatorname{iso}\sigma(T)$ and $\nu \in \operatorname{iso}\sigma(S)$. Since T and S are isoloid, then $\mu \in E(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T)$ and $\nu \in E(S) = \sigma(S) \setminus \sigma_{SBF_+^-}(S)$, and hence $\lambda = \mu\nu \notin \sigma_{SBF_+^-}(T)\,\sigma(S) \cup \sigma(T)\,\sigma_{SBF_+^-}(S)$. Thus

$$\sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma(T)\sigma_{SBF_+^-}(S) \subseteq \sigma_{SBF_+^-}(T \otimes S).$$

Conversely, let $\lambda \in \sigma(T \otimes S) \setminus (\sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma(T)\sigma_{SBF_+^-}(S))$, then for $\lambda = \mu\nu$ we have that $\mu \in \sigma(T)$ and $\nu \in \sigma(S)$, hence $\mu \in E(T)$ and $\nu \in E(S)$. Thus $\lambda = \mu\nu \in E(T \otimes S) = \sigma(T \otimes S) \setminus \sigma_{SBF_+^-}(T \otimes S)$. Therefore,

$$\sigma_{SBF_+^-}(T \otimes S) = \sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma(T)\sigma_{SBF_+^-}(S).$$

(b)$\Longrightarrow$ (a): Since T and S obey property (gt), then

$$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T) \quad \text{and} \quad \sigma(S) \setminus \sigma_{SBF_+^-}(S) = E(S).$$

Assume that

$$\sigma_{SBF_+^-}(T \otimes S) = \sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma(T)\sigma_{SBF_+^-}(S).$$

Let $\lambda \in E(T \otimes S)$. Then there exists $\mu \in \operatorname{iso}\sigma(T)$ and $\nu \in \operatorname{iso}\sigma(S)$ such that $\lambda = \mu\nu$. Since T and S are isoloid, then $\mu \in E(T)$ and $\nu \in E(S)$. Hence $\mu \notin \sigma_{SBF_+^-}(T)$ and $\nu \notin \sigma_{SBF_+^-}(S)$. Then $\lambda \notin \sigma_{SBF_+^-}(T \otimes S)$. Thus

$$E(T \otimes S) \subseteq \sigma(T \otimes S) \setminus \sigma_{SBF_+^-}(T \otimes S).$$

Conversely, assume that $\lambda \notin \sigma(T \otimes S) \setminus \sigma_{SBF_+^-}(T \otimes S)$, then there exists $\mu \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$ and $\nu \in \sigma(S) \setminus \sigma_{SBF_+^-}(S)$ such that $\lambda = \mu\nu$. Since

$$T \otimes S = (T - \mu) \otimes S + \mu I \otimes (S - \nu),$$

then we can see that $\lambda \in E(T \otimes S)$. Hence $T \otimes S$ obeys property (gt). $\square$

EXAMPLE 4.2. Let T be a non-zero nilpotent operator and let S be a quasinilpotent which is not nilpotent. Then it is easy to see that

$$\sigma(T)=\{0\}=E(T),\ \sigma_{SBF_+^-}(T)=\emptyset \qquad \text{and} \qquad \sigma(S)=\sigma_{SBF_+^-}(S)=\{0\}, E(S)=\emptyset.$$

Hence T and S satisfy property (gt). Since $T\otimes S$ is nilpotent then 0 is a pole and then $\sigma_{SBF_+^-}(T\otimes S)=\emptyset$. Hence $T\otimes S$ satisfies property (gt). However

$$\sigma(T)\sigma_{SBF_+^-}(S)\cup\sigma(S)\sigma_{SBF_+^-}(T)=\{0\}\neq\sigma_{SBF_+^-}(T\otimes S).$$

Here $0\in\operatorname{iso}\sigma(T\otimes S)$.

The following example show that there exist two operators $T,S\in\mathscr{B}(\mathscr{X})$ such that $T\otimes S$ obeys property (gt) but T and S do not obey the property (gt).

EXAMPLE 4.3. Let $S=U+U^*$, where U is the unilateral shift on ℓ^2. Since S is self-adjoint, then

$$\sigma(S)=\{\lambda\in\mathbb{C}:|\lambda|\leqslant 1\}$$

and hence from [1] that

$$\sigma_{BW}(S)=\sigma_{SBF_+^-}(S)=\{\lambda\in\mathbb{C}:|\lambda|=1\}.$$

Hence

$$\sigma(S)\setminus\sigma_{SBF_+^-}(S)=\{\lambda\in\mathbb{C}:|\lambda|\leqslant 1\}\setminus\{\lambda\in\mathbb{C}:|\lambda|=1\}.$$

Since $E(S)=\emptyset$, then property (gt) fails for S. In the other hand, if I is the identity acting on ℓ^2, then $I\otimes S$ is self-adjoint, hence

$$\sigma(I\otimes S)=\sigma_{SBF_+^-}(I\otimes S)=\{\lambda\in\mathbb{C}:|\lambda|\leqslant 1\}.$$

Therefore,

$$\sigma(I\otimes S)\setminus\sigma_{SBF_+^-}(I\otimes S)=\emptyset=E(I\otimes S).$$

Thus $I\otimes S$ obeys property (gt).

$T\in\mathscr{B}(\mathscr{X})$ is polaroid implies T^* polaroid. It is well known that if T or T^* has SVEP and T is polaroid, then T and T^* satisfy generalized Weyl's theorem. Note the well known fact, [39, Theorem 3.5], that if T is polaroid and T^* (resp., T) has SVEP, then T (resp., T^*) satisfies property (gt). The following theorem is the tensor product analogue of this result.

THEOREM 4.4. *Suppose that operators $T\in\mathscr{B}(\mathscr{X})$ and $S\in\mathscr{B}(\mathscr{Y})$ are polaroid.*

(i) If T^* and S^* have SVEP, then $T\otimes S$ satisfies property (gt).

(ii) If T and S have SVEP, then $T^*\otimes S^*$ satisfies property (gt).

Proof. (i) The hypotheses T^* and S^* have SVEP implies

$$\sigma(T)=\sigma_a(T), \qquad \sigma(S)=\sigma_a(S), \qquad \sigma_{SBF_+^-}(T)=\sigma_{BW}(T), \qquad \sigma_{SBF_+^-}(S)=\sigma_{BW}(S)$$

and

$$T^*,S^* \qquad \text{and } T^*\otimes S^* \text{ satisfy generalized } s\text{-Browder's theorem.}$$

Thus generalized s-Browder's theorem and generalized Browder's theorem (generalized s-Browder's theorem $\Longrightarrow$ generalized Browder's theorem) transfer from T^* and S^* to $T^*\otimes S^*$ [40]. Hence

$$\begin{aligned}\sigma_{SBF_+^-}(T\otimes S)&=\sigma_{SBF_-^+}(T^*\otimes S^*)=\sigma_s(T^*)\sigma_{SBF_-^+}(S^*)\cup\sigma_{SBF_-^+}(T^*)\sigma_s(S^*)\\&=\sigma_a(T)\sigma_{SBF_+^-}(S)\cup\sigma_{SBF_+^-}(T)\sigma_a(S)=\sigma(T)\sigma_{BW}(S)\cup\sigma_{BW}(T)\sigma(S),\end{aligned}$$

and

$$\begin{aligned}\sigma_{BW}(T\otimes S)&=\sigma_{BW}(T^*\otimes S^*)=\sigma_{BW}(T^*)\sigma(S^*)\cup\sigma_{BW}(S^*)\sigma(T^*)\\&=\sigma(T)\sigma_{BW}(S)\cup\sigma(S)\sigma_{BW}(T).\end{aligned}$$

Consequently,

$$\sigma_{SBF_+^-}(T\otimes S)=\sigma_{BW}(T\otimes S).$$

Evidently, $T\otimes S$ is polaroid [22, Lemma 2]; combining this with $T\otimes S$ satisfies generalized Browder's theorem, it follows that $T\otimes S$ satisfies generalized Weyl's theorem, i.e., $\sigma(T\otimes S)\setminus\sigma_{BW}(T\otimes S)=E(T\otimes S)$. But then

$$\sigma(T\otimes S)\setminus\sigma_{SBF_+^-}(T\otimes S)=\sigma(T\otimes S)\setminus\sigma_{BW}(T\otimes S)=E(T\otimes S),$$

i.e., $T\otimes S$ satisfies property (gt).

(ii) In this case $\sigma(T)=\sigma_a(T^*)$, $\sigma(S)=\sigma_a(S^*)$, $\sigma_{BW}(T^*)=\sigma_{SBF_+^-}(T^*)$, $\sigma_{BW}(S^*)=\sigma_{SBF_+^-}(S^*)$, $\sigma(T^*\otimes S^*)=\sigma_a(T^*\otimes S^*)$, polaroid property transfer from T and S to $T^*\otimes S^*$, and both generalized s-Browder's theorem and generalized Browder's theorem transfer from T and S to $T\otimes S$. Hence

$$\begin{aligned}\sigma_{SBF_+^-}(T^*\otimes S^*)&=\sigma_{SBF_-^+}(T\otimes S)=\sigma_s(T)\sigma_{SBF_-^+}(S)\cup\sigma_{SBF_-^+}(T)\sigma_s(S)\\&=\sigma_a(T^*)\sigma_{SBF_+^-}(S^*)\cup\sigma_{SBF_+^-}(T^*)\sigma_a(S^*)\\&=\sigma(T)\sigma_{BW}(S)\cup\sigma_{BW}(T)\sigma(S)\\&=\sigma_{BW}(T\otimes S)=\sigma_{BW}(T^*\otimes S^*).\end{aligned}$$

Thus, since $T^*\otimes S^*$ polaroid and $T\otimes S$ satisfies generalized Browder's theorem imply $T^*\otimes S^*$ satisfies generalized Weyl's theorem,

$$\sigma(T^*\otimes S^*)\setminus\sigma_{SBF_+^-}(T^*\otimes S^*)=\sigma(T^*\otimes S^*)\setminus\sigma_{BW}(T^*\otimes S^*)=E(T^*\otimes S^*),$$

i.e., $T^*\otimes S^*$ satisfies property (gt). $\square$

REFERENCES

[1] P. AIENA, *Fredholm and local spectral theory with applications to multipliers*, Kluwer Acad. Publishers, Dordrecht, 2004.

[2] P. AIENA, C. CARPINTERO, *Weyl's theorem, a-Weyl's theorem and single-valued extension property*, Extracta Math. **20** (2005) 25–41.

[3] P. AIENA, J. R. GUILLEN AND P. PEÑA, *Property (w) for perturbations of polaroid operators*, Linear Alg. Appl. **428** (2008), 1791–1802.

[4] P. AIENA, E. APONTE, *Polaroid type operators under perturbations*, Studia Math. **214**, (2), (2013), 121–136.

[5] P. AIENA, E. APONTE, J. R. GUILLEN AND P. PEÑA, *Property (R) under perturbations*, Mediterr. J. Math. **10** (1) (2013), 367–382.

[6] M. AMOUCH, H. ZGUITTI, *On the equivalence of Browder's and generalized Browder's theorem*, Glasg. Math. J. **48** (2006), 179–?85.

[7] M. AMOUCH, M. BERKANI, *on the property (gw)*, Mediterr. J. Math. **5** (2008), 371–378.

[8] M. AMOUCH, H. ZGUITTI, *B-Fredholm and Drazin invertible operators through localized SVEP*, Math. Bohemica 136 (2011) 39–49.

[9] M. BERKANI, *On a class of quasi-Fredholm operators*, Integral Equations and Operator Theory. **34** (1999), no. 2, 244–249.

[10] M. AMOUCH, *Polaroid operators with SVEP and perturbations of Property (gw)*, Mediterr. J. Math. **6** (2009), 461–?70.

[11] M. BERKANI, M. SARIH, *On semi B-Fredholm operators*, Glasg. Math. J. **43** (2001), 457–465.

[12] M. BERKANI, *Index of B-Fredholm operators and gereralization of a Weyl Theorem*, Proc. Amer. Math. Soc. **130** (2001), 1717–1723.

[13] M. BERKANI, *B-Weyl spectrum and poles of the resolvent*, J. Math. Anal. Appl. **272** (2002), 596–603.

[14] M. BERKANI AND J. KOLIHA, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. **69** (2003), 359–376.

[15] M. BERKANI, *On the equivalence of Weyl theorem and generalized Weyl theorem*, Acta Math. Sinica **272** (2007), 103–110.

[16] M. BERKANI, M. AMOUCH, *Preservation of property (gw) under perturbations*, Acta Sci. Math. (Szeged) **74** (2008), 767–779.

[17] M. BERKANI AND H. ZARIOUH, *Generalized a-Weyl's theorem and perturbations*, Functional Analysis, Approximation and computation **2** (1) (2010), 7–18.

[18] L. A. COBURN, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288.

[19] H. R. DOWSON, *Spectral theory of linear operator*, Academic press, London, 1978.

[20] B. P. DUGGAL, *SVEP and generalized Weyl's theorem*, Mediterr. J. Math. 4 (2007) 309–320.

[21] B. P. DUGGAL, S. V. DJORDJEVIÈ, C. S. KUBRUSLY, *On the a-Browder and a-Weyl spectra of tensor products*, Rend. Circ. Mat. Palermo 59 (2010), 473–481.

[22] B. P. DUGGAL,, *Tensor product and property (w)*, Rend. Circ. Mat. Palermo, doi:10.1007/s12215-011-0023-9.

[23] J. K. FINCH, *The single valued extension property on a Banach space*, Pacific J. Math. **58** (1975), 61–69.

[24] H. HEUSER, *Functional analysis*, Marcel Dekker, New York, 1982.

[25] K. B. LAURSEN, *Operators with finite ascent*, Pacific J. Math. **152** (1992), 323–336.

[26] M. LAHROUZ, M. ZOHRY, *Weyl type theorems and the approximate point spectrum*, Irish Math. Soc. Bulletin 55 (2005) 41–51.

[27] K. B. LAURSEN, M. M. NEUMANN, *An introduction to local spectral theory*, Oxford. Clarendon, 2000.

[28] M. MBEKHTA, *Sur la th'eorie spectrale locale et limite de nilpotents*, Proc. Amer. Math. Soc. **3** (1990), 621–631.

[29] M. OUDGHIRI, *Weyl's and Browder's theorem for operators satysfying the SVEP*, Studia Math. **163** (2004), 85–101.

[30] M. OUDGHIRI, *Weyl's theorem and perturbations*, Integr. Equ. Oper. Theory **70** (2011), 561–568.

[31] V. RAKOČEVIĆ, *On a class of operators*, Math. Vesnik **37** (1985), 423–426.

[32] V. RAKOČEVIĆ, *Operators obeying a-Weyl's theorem*, Rev. Roumaine Math. Pures Appl. **10** (1986), 915–919.

[33] M. H. M. RASHID, *Property* (w) *and quasi-class* (A,k) *operators*, Revista De Le Unión Math. Argentina **52** (2011), 133–142.
[34] M. H. M. RASHID AND M. S. M. NOORANI, *Weyl's type theorems for algebraically* w*-hyponormal operators*, Arab. J. Sci. Eng. **35** (2010), 103–116.
[35] M. H. M RASHID, *Property* (gb) *and perturbations*, J. Math. Anal. Appl. **383** (2011), 82–94.
[36] M. H. M. RASHID, *Property* (gw) *and perturbations*, Bull. Belg. Math. Soc. Simon Stevin **18** (2011), 635–654.
[37] M. H. M. RASHID, *Weyl's theorem for algebraically* $wF(p,r,q)$ *operators with* $p,r > 0$ *and* $q \geqslant 1$, Ukrainian Math. J. **63** (8) (2011), 1256–1267.
[38] M. H. M. RASHID, *Weyl's type theorems and hypercyclic operators*, Acta Math. Sci. **32** (2) (2012), 539–551.
[39] M. H. M. RASHID, *Properties* (t) *and* (gt) *For Bounded Linear Operators*, Mediterr. J. Math. (2014) 11: 729, doi:10.1007/s00009-013-0313-x.
[40] M. H. M. RASHID, *Generalized Weyl's theorem and tensor product*, Ukrainian Math. J. **64** (9) (2013), 1289–1296.
[41] M. H. M. RASHID AND T. PRASAD, *Property* (Bb) *and Tensor product*, Filomat **27** (7) (2013), 1297–1303.
[42] M. H. M. RASHID AND T. PRASAD, *The stability of variants of weyl type theorems under tensor product*, Ukrainian Math. J. **68** (4) (2016), 612–624.
[43] Q. ZENG, Q. JIANG, AND H. ZHONG, *Spectra originated from semi-*B*-Fredholm theory and commuting perturbations*, arXiv:1203.2442vl[math. FA] 12 Mar 2012.

(Received March 5, 2021)

Mohammad H. M. Rashid
Department of Mathematics& Statistics
Faculty of Science P. O Box (7), Mu'tah University
Al-Karak-Jordan
e-mail: malik_okasha@yahoo.com

Muneo Chō
Department of Mathematics
Faculty of Science, Kanagawa University
Hiratsuka 259-1293, Japan
e-mail: chiyom01@kanagawa-u.ac.jp