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INTERPOLATING SEQUENCES FOR THE BANACH ALGEBRAS GENERATED BY A CLASS OF TEST FUNCTIONS

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Abstract. The problem of characterizing interpolating sequences in a bounded domain $\Omega \subset \mathbb{C}^n$ for the Banach algebra $H^\infty(\Omega)$ of bounded holomorphic functions is well-studied in the literature. For the unit disc $\mathbb{D}$, the bidisc $\mathbb{D}^2$ and the symmetrized bidisc $\mathbb{G}^2$, there is a way to such a characterization via *the realization formula* that the function algebras $H^\infty(\Omega)$ possess in these cases. Our aim in this article is to present such a characterization of interpolating sequences in a more general setting for a class of Banach algebras that possess such a realization formula. The closed unit ball of these Banach algebras are known as the Schur–Agler-class associated to a class of test functions Ψ on Ω. We shall also note that the case of $\mathbb{D}$, $\mathbb{D}^2$ and $\mathbb{G}^2$ are special cases of our main result. A few other examples of function algebras is also mentioned where our main result applies leading to a characterization of interpolating sequences.

1. Introduction

1.1. Interpolating sequences: an overview

Let $\mathscr{A}$ be a Banach algebra of bounded functions on a domain Ω in $\mathbb{C}^n$, for some positive integer n, with the norm $||\bullet||_{\mathscr{A}}$ which has the property that $||f||_{\mathscr{A}} \geqslant ||f||_\infty := \sup_{z\in\Omega}|f(z)|$ for every $f \in \mathscr{A}$. Let $l^\infty(\mathbb{N})$ denote the Banach algebra of all bounded sequences with sup-norm. Given a sequence $\{z_i\}_{i\in\mathbb{N}} \subset \Omega$, we consider the following linear map

$$L : \left(\mathscr{A}, ||\bullet||_{\mathscr{A}}\right) \longrightarrow l^\infty(\mathbb{N}), \text{ defined by } L(\phi) := \{\phi(z_i)\}_{i\in\mathbb{N}} \;\forall\; \phi \in \mathscr{A}. \tag{1}$$

Observe that $\sup_{i\in\mathbb{N}}|\phi(z_i)| \leqslant ||\phi||_\infty \leqslant ||\phi||_{\mathscr{A}}$, hence L is a bounded linear operator on $\mathscr{A}$. Consider the following abstract interpolation problem:

(IS) Given a Banach algebra $(\mathscr{A}, ||\bullet||_{\mathscr{A}})$ of bounded functions on Ω with the property that $||\phi||_{\mathscr{A}} \geqslant ||\phi||_\infty$ for every $\phi \in \mathscr{A}$. Characterize those sequences $\{z_i\}_{i\in\mathbb{N}} \subset \Omega$ for which the bounded linear map L, as defined above, is a surjective map.

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A sequence $\{z_i\}_{i\in\mathbb{N}} \subset \Omega$ for which the map L is surjective will be called an *interpolating sequence for the algebra $\mathscr{A}$*.

Note, if $\{z_i\}_{i\in\mathbb{N}}$ is an interpolating sequence for $\mathscr{A}$ it follows from the open mapping theorem that there exists a $\delta > 0$ such that $l_1^\infty(\mathbb{N}) :=$ the closed unit ball of $l^\infty(\mathbb{N})$, is contained in $L(\delta\mathscr{A}_1)$, where $\mathscr{A}_1$ denotes the closed unit ball of $\mathscr{A}$ in its norm. The smallest of such a δ is called *the constant of interpolation* for the interpolating sequence $\{z_i\}_{i\in\mathbb{N}} \subset \Omega$. This, in particular, implies that for each $i \in \mathbb{N}$, there exists $\phi_i \in \mathscr{A}$ with $||\phi_i||_{\mathscr{A}} \leqslant M$, such that $\phi_i(z_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol and M is the constant of interpolation associated to $\{z_i\}_{i\in\mathbb{N}}$. A sequence $\{z_i\}_{i\in\mathbb{N}}$ for which there exists a sequence $\{\phi_i\}_{i\in\mathbb{N}} \subset \mathscr{A}$ that is uniformly bounded and has the property that $\phi_i(z_j) = \delta_{i,j}$ is called a *strongly separated sequences*. So every interpolating sequence $\{z_i\}_{i\in\mathbb{N}}$ for $\mathscr{A}$ is strongly separated. Given a sequence $\{z_i\}_{i\in\mathbb{N}}$, it is called *weakly separated by $\mathscr{A}$* if there exists $R > 0$ such that for each pair $i \neq j$ there exists $\phi_{i,j} \in \mathscr{A}$ with $||\phi_{i,j}||_{\mathscr{A}} \leqslant R$ and $\phi_{i,j}(z_i) = 1,\ \phi_{i,j}(z_j) = 0$.

The problem (IS) originated in the case $\Omega = \mathbb{D}$, where $\mathbb{D}$ denotes the open unit disc in the complex plane $\mathbb{C}$ centered at 0, and with $\mathscr{A} = H^\infty(\mathbb{D}) :=$ the set of bounded holomorphic functions in the unit disc with the sup-norm. Carleson in 1958 proved the following theorem.

RESULT 1. (Carleson, [13]) *Let $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{D}$ be a sequence in $\mathbb{D}$. Then the following are equivalent.*

1. *$\{\lambda_i\}_{i\in\mathbb{N}}$ is an interpolating sequence for $H^\infty(\mathbb{D})$.*
2. *$\{\lambda_i\}_{i\in\mathbb{N}}$ is weakly separated and the atomic measure $\sum_{i\in\mathbb{N}}\left(1-|\lambda_i|^2\right)\delta_i$ is a Carleson measure for the Hardy space $H^2(\mathbb{D})$.*
3. *$\{\lambda_i\}_{i\in\mathbb{N}}$ is strongly separated.*

The reader is referred to [2, Chapter 9] for the definition of Carleson measure. What is essential here is that the Carleson measure condition can be equivalently stated in terms of the boundedness of the Grammian operator on l^2 corresponding to the Szegő kernel on $\mathbb{D}$. For this purpose let us introduce the Grammian associated with a positive kernel k on Ω. (See Section 2 for the definition of a positive kernel k and the reproducing kernel Hilbert space $\mathscr{H}_k$ that is associated to it.) Given a positive kernel k on Ω and a sequence $\{z_i\}_{i\in\mathbb{N}} \subset \Omega$, let us denote by k_i the kernel function at z_i, i.e., $k(\bullet, z_i)$ and write $k_{i,j} := \langle k_j, k_i\rangle$. Let $g_i := k_i/||k_i||$ be the normalized kernel functions. The *Grammian associated to the sequence* $\{z_i\}_{i\in\mathbb{N}}$ is the infinite matrix G given by

$$G_{i,j} := \langle g_j, g_i\rangle = \frac{k_{i,j}}{||k_i||\,||k_j||}.$$

It is a fact that the Grammian associated to a sequence $\{z_i\}_{i\in\mathbb{N}}$ is bounded on l^2 if and only if the measure $\sum_{i=1}^n ||k_i||^{-2}\delta_i$ is a Carleson measure for $\mathscr{H}_k$; see [2, Proposition 9.5]. The Hardy space on the unit disk, $H^2(\mathbb{D})$, is a reproducing kernel Hilbert space with the kernel k being the Szegő kernel. So the Carleson measure condition in the above result is equivalent to boundedness of the Grammian matrix associated to the Szegő kernel and the sequence $\{\lambda_i\}_{i\in\mathbb{N}}$.

Shapiro–Shield in [18] gave an alternative proof of Carleson's result by replacing the notions of Carleson measure condition and strong separation by conditions on the the Grammian matrix associated to a sequence. They also considered interpolating sequences for many other holomorphic function spaces on the unit disc.

A very important case of the problem (IS) is when $\mathscr{A}$ is the multiplier algebra, denoted by $Mult(\mathscr{H}_k)$, of a reproducing kernel Hilbert space $\mathscr{H}_k$ associated to a kernel k, together with the multiplier norm. This was initiated by Marshall–Sundberg [17] and by C. Bishop [12]. They also introduced a notion of interpolating sequences for $\mathscr{H}_k$ and observed that the set of interpolating sequences for multiplier algebras is contained in the set of interpolating sequences for the Hilbert space $\mathscr{H}_k$. Moreover, if the kernel satisfies the *scalar* Pick property, then the two notions of interpolating sequences coincide; see [2, Theorem 9.19]. This is important since interpolating sequences for separable reproducing kernel Hilbert spaces are exactly those for which the Grammian is both bounded from above and below.

A well studied class of positive kernels is the family of complete Nevanlinna–Pick kernels (see [2] for the definition) that satisfy a stronger form of the Pick property. In this case, $\mathscr{H}_k$ is called a *complete Pick space*. A characterization of the interpolating sequences for the multiplier algebra for this class of kernels is now completely known.

RESULT 2. *Let $\{\lambda_i\}_{i\in\mathbb{N}} \subset \Omega$ be a sequence and let k be an irreducible complete Nevanlinna–Pick kernel. Then the following are equivalent:*

(IM) *the sequence is interpolating for $Mult(\mathscr{H}_k)$,*

(IH) *the sequence is interpolating for $\mathscr{H}_k$,*

$(S+C)$ *the sequence is weakly separated and the Grammian associated to the sequences is bounded.*

As mentioned before the equivalence of (IM) and (IH) above was established by Marshall-Sundberg and they proved the equivalence under the weaker condition that k has the scalar Pick property. The implication (IH) $\Longrightarrow$ (S+C) holds, in general, for any reproducing kernel Hilbert space; see e.g. [2] or [19]. The implication (S+C) $\Longrightarrow$ (IM) for irreducible complete Nevanlinna-Pick kernels is established in a recent article by Aleman-Hartz–McCarthy–Richter [5], where they applied Marcus–Spielman–Srivastava theorem, a path-breaking result that established the Kadison-Singer conjecture.

The condition (S+C) implies strong separation with respect to $Mult(\mathscr{H}_k)$ for kernels having the Pick property; see [2, Theorem 9.43]. On the other hand, both Marshall-Sundberg [17] and Bishop [12] have shown that strong separation does not imply (IM) in the case of the Dirichlet space of the unit disc which is a complete Pick space.

1.2. Test functions

In this article, we shall address the problem (IS) with $\mathscr{A}$ being those Banach algebras of bounded functions that are obtained by taking intersections of multiplier algebras of certain reproducing kernel Hilbert spaces associated with a class of test functions. These are algebras which are not necessarily multiplier algebras of a reproducing kernel Hilbert spaces and were first introduced by Jim Agler. We begin with the definition of test functions.

Let Ω be a bounded domain in $\mathbb{C}^n$ and let Ψ be a family of functions on Ω. We say Ψ is a collection of *test functions* on Ω if the following conditions hold:

1. $\sup\,\{|\psi(x)| : \psi \in \Psi\} < 1$ for each $x \in \Omega$.

2. For each finite subset F of Ω, the collection $\{\psi|_F : \psi \in \Psi\}$ together with unity generates the algebra of all $\mathbb{C}$-valued functions on F.

The second condition is not essential part of the definition, but it makes some situations simpler (see [8] and [15]). The collection Ψ is a natural topological subspace of $\overline{\mathbb{D}}^{\Omega}$ equipped with the product topology. For every $x \in \Omega$, there is an element $E(x)$ in $\mathscr{C}_b(\Psi)$, *the* C^**-algebra of all bounded functions on* Ψ, such that $E(x)(\psi) = \psi(x)$. Clearly, $\|E(x)\| = \sup_{\psi\in\Psi}|\psi(x)| < 1$ for each $x \in \Omega$. The functions $E(x)$ will be used at several places in this paper.

Given Ω and a collection of test functions Ψ, let us denote by $\mathscr{K}_\Psi(\mathbb{C})$ the set of all $\mathbb{C}$-valued positive kernels k on Ω for which the operator $M_\psi : \mathscr{H}_k \longrightarrow \mathscr{H}_k$ defined by $M_\psi(f) := f\psi$, for all $f \in \mathscr{H}_k$, is a contraction for each $\psi \in \Psi$. Recall, a contraction on a Hilbert space is a bounded linear operator whose operator norm is atmost 1. We now introduce the Banach algebra associated to a class of test functions as alluded to in the first paragraph of this section.

DEFINITION 1.1. Let us denote by $H^\infty_\Psi(\mathbb{C})$ the collection of such $\mathbb{C}$-valued functions $\phi : \Omega \longrightarrow \mathbb{C}$ for which there exists a constant $C > 0$ having the following property:

$(*)$ for each $k \in \mathscr{K}_\Psi(\mathbb{C})$, the bounded linear operator $M_\phi : \mathscr{H}_k \longrightarrow \mathscr{H}_k$ defined by $M_\phi(f) := f\phi$ for all $f \in \mathscr{H}_k$, is a bounded linear operator with $||M_\phi||_{\mathscr{H}_k} \leqslant C$.

Given $\phi \in H^\infty_\Psi(\mathbb{C})$, define:

$$||\phi||_\Psi := \inf\big\{C : C \text{ satisfying the property } (*) \text{ above}\big\}. \tag{2}$$

It turns out that $H^\infty_\Psi(\mathbb{C})$ is a Banach algebra with norm $||\bullet||_\Psi$. The *scalar-valued* Ψ*-Schur–Agler class*, denoted by $\mathscr{SA}_\Psi(\mathbb{C})$, is defined by $\mathscr{SA}_\Psi(\mathbb{C}) := \big\{\phi \in H^\infty_\Psi(\mathbb{C}) : ||\phi||_\Psi \leqslant 1\big\}$. It is easy to see that if $\phi \in H^\infty_\Psi(\mathbb{C})$ then $||\phi||_\infty := \sup\{|\phi(z)| : z \in \Omega\} \leqslant ||M_\phi||_{\mathscr{H}_k}$ for all $k \in \mathscr{K}_\Psi$. It follows from this that $||\phi||_\infty \leqslant ||\phi||_\Psi$. We now present the main result of this article concerning the problem (IS) in the case when $\mathscr{A} = H^\infty_\Psi(\mathbb{C})$.

THEOREM 1.2. *Let Ω be a bounded domain and let Ψ be a family of test functions. Consider the Banach algebra $H^\infty_\Psi(\mathbb{C})$ consisting of bounded functions with the norm $||\bullet||_\Psi$ as above. Let $\{w_j\}_{j\in\mathbb{N}} \subset \Omega$ be a sequence in Ω. Then the following are equivalent.*

1. *The sequence $\{w_j\}_{j\in\mathbb{N}}$ is an interpolating sequence for $H^\infty_\Psi(\mathbb{C})$.*

2. *For all admissible kernels* $k \in \mathscr{K}_\Psi$*, the normalized Grammians* G_k *associated to the sequence* $\{w_j\}_{j\in\mathbb{N}}$ *are uniformly bounded from below, i.e., for all* $k \in \mathscr{K}_\Psi$ *there exists* $N > 0$*, independent of* k*, such that* $G_k \geqslant (1/N)\mathbb{I}$.

3. *The sequence* $\{w_j\}_{j\in\mathbb{N}}$ *is strongly separated and for all kernels* $k \in \mathscr{K}_\Psi$*, the normalized Grammians* G_k *associated to the sequence* $\{w_j\}_{j\in\mathbb{N}}$ *are uniformly bounded, i.e., for all* $k \in \mathscr{K}_\Psi$ *there exists* $M > 0$*, independent of* k*, such that* $G_k \leqslant M\mathbb{I}$.

4. (2) *and* (3) *above holds together.*

Here, and elsewhere in the article $\mathbb{I}$ shall denote the identity operator.

As noted after Result 2 for reproducing kernel Hilbert spaces with irreducible complete Nevanlinna–Pick kernel a sequence is interpolating for the multiplier algebra if and only if it is strongly separated and the associated Grammian is bounded. The equivalence of (1) and (3) above establishes an analogous result for $H^\infty_\Psi(\mathbb{C})$.

REMARK. Let $k, l \in \mathscr{K}_\Psi$ be such that $l = gk$ for some positive kernel g. Then if $c\mathbb{I} - G_k \geqslant 0$ for some $c > 0$ then $c\mathbb{I} - G_l \geqslant 0$. To see that, note that $G_g \geqslant 0$ and by the Schur-product theorem we get $(c\mathbb{I} - G_k) \odot G_g \geqslant 0$. Notice now the Schur product of $\mathbb{I}$ and G_g is $\mathbb{I}$ and $G_l = G_k \odot G_g$ whence the conclusion. Proceeding similarly if $G_k - d\mathbb{I} \geqslant 0$ for some $d > 0$, then $G_l - d\mathbb{I} \geqslant 0$, where $l, k \in \mathscr{K}_\Psi$ are as before.

When $\Omega = \mathbb{D}$ and $\Psi = \{z\}$ then every $k \in \mathscr{K}_\Psi$ is of the form $k = sg$, where s denotes the Szegő kernel and g is some positive kernel. It follows from the remark above that conditions (2) and (3) in Theorem 1.2 have to be satisfied only for the Szegő kernel. It is a fact that $H^\infty_\Psi(\mathbb{C})$ is $H^\infty(\mathbb{D})$ in this case. This leads to a characterization of interpolating sequences for $H^\infty(\mathbb{D})$ which is equivalent to Result 1 ([2, Section 9.5]) by Carleson.

We shall present the proof of Theorem 1.2 in Section 4. At the heart of our proof is a result, namely Proposition 3.4 in Section 3, that relates the boundedness of the Grammian with the existence of vector-valued interpolants. Our proof of this proposition and our main theorem is inspired from the ideas as in [3]. Later, in Section 5, we shall provide several other examples where the above theorem is applied in characterizing interpolating sequences for those algebras that are realized as $H^\infty_\Psi(\mathbb{C})$ for some appropriately chosen Ψ.

2. Preliminaries

In this section, we prove a few lemmas and gather certain basic tools that will be needed in upcoming sections. We begin with the definition of Ψ-Schur–Agler class functions in operator-valued setting. To do that, we need to recall various notions of positive kernel on a domain Ω.

A positive kernel k on a set Ω is a function $k : \Omega \times \Omega \to \mathbb{C}$ such that for any $n \geqslant 1$, any n points $x_1, \ldots, x_n$ in Ω and any n complex numbers $c_1, \ldots, c_n$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} c_j k(x_i, x_j) \geqslant 0.$$

Let $\mathscr{E}$ is a Hilbert space and $k : \Omega \times \Omega \to B(\mathscr{E})$ is a function, then k is called a positive kernel if for any $n \geqslant 1$, any n points $x_1, \dots, x_n$ in Ω and any n vectors $\mathbf{e_1}, \dots, \mathbf{e_n}$ in $\mathscr{E}$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \langle k(x_i, x_j)\mathbf{e_j}, \mathbf{e_i} \rangle \geqslant 0. \tag{3}$$

We also recall the notion of completely positive kernels here. Let $\mathscr{A}$ and $\mathscr{B}$ be two C^*-algebras and let Γ be a function on $\Omega \times \Omega$ taking values in $B(\mathscr{A}, \mathscr{B})$ (space of all bounded linear operators from $\mathscr{A}$ to $\mathscr{B}$). Γ is called a *completely positive kernel* if

$$\sum_{i,j=1}^{n} b_i^* \, \Gamma(x_i, x_j)(a_i^* a_j) b_j \geqslant 0 \tag{4}$$

for all $n \geqslant 1$, $a_1, a_2, \dots, a_n \in \mathscr{A}$, $b_1, b_2, \dots, b_n \in \mathscr{B}$ and $x_1, x_2, \dots, x_n \in \Omega$.

2.1. The Ψ-Schur–Agler class: general case

Given a Hilbert space $\mathscr{E}$ and a $B(\mathscr{E})$-valued kernel K *(satisfying (3))* on Ω, there is a Hilbert space $\mathscr{H}(K)$ of $\mathscr{E}$-valued functions on Ω such that span of the set

$$\{K(\cdot, \omega)\mathbf{e} : \mathbf{e} \in \mathscr{E}, \, \omega \in \Omega\}$$

is dense in $\mathscr{H}(K)$ and for any $\mathbf{e} \in \mathscr{E}$, $\omega \in \Omega$ and $h \in \mathscr{H}(K)$, we have

$$\langle h, K(\cdot, \omega)\mathbf{e} \rangle_{\mathscr{H}(K)} = \langle h(\omega), \mathbf{e} \rangle_{\mathscr{E}}.$$

Given a set of test functions Ψ on Ω, a kernel $K : \Omega \times \Omega \to B(\mathscr{E})$ is said to be Ψ-*admissible* if the map M_ψ, sending each element $h \in \mathscr{H}(K)$ to $\psi \cdot h$, is a contraction on $\mathscr{H}(K)$. We denote the set of all $B(\mathscr{E})$-valued Ψ-admissible kernels by $\mathscr{K}_\Psi(\mathscr{E})$. For two Hilbert spaces $\mathscr{U}$ and $\mathscr{Y}$, we say that $S : \Omega \to B(\mathscr{U}, \mathscr{Y})$ *is in* $H_\Psi^\infty(\mathscr{U}, \mathscr{Y})$ if there is a constant C such that the $B(\mathscr{Y} \otimes \mathscr{Y})$-valued function

$$(C^2 I_{\mathscr{Y}} - S(x)S(y)^*) \otimes k(x, y) \tag{5}$$

is a positive $B(\mathscr{Y} \otimes \mathscr{Y})$-valued kernel for every k in $\mathscr{K}_\Psi(\mathscr{Y})$. If S is in $H_\Psi^\infty(\mathscr{U}, \mathscr{Y})$, then we denote by $\|S\|_\Psi$ the smallest C which satisfies (5). The Ψ-Schur–Agler class, denoted by $\mathscr{SA}_\Psi(\mathscr{U}, \mathscr{Y})$, is the set of those $S \in H_\Psi^\infty(\mathscr{U}, \mathscr{Y})$ for which $\|S\|_\Psi \leqslant 1$. Also observe when $\mathscr{U} = \mathbb{C} = \mathscr{Y}$, then $H_\Psi^\infty(\mathscr{U}, \mathscr{Y}) = H_\Psi^\infty(\mathbb{C})$. We begin with the following lemma.

LEMMA 2.1. *Let $\mathscr{X}, \mathscr{Y}, \mathscr{H}$ be Hilbert spaces such that either $\mathscr{H} = \mathbb{C}$ or $\mathscr{H} = \mathscr{Y}$. Let $g \in \mathscr{SA}_\Psi(\mathscr{X}, \mathscr{Y})$ and $f \in \mathscr{SA}_\Psi(\mathscr{Y}, \mathscr{H})$. Then if we set $fg(z) := f(z)g(z)$ for all $z \in \Omega$ then $fg \in \mathscr{SA}_\Psi(\mathscr{X}, \mathscr{H})$.*

Proof. Notice that for each $z \in \Omega$, $fg(z) \in B(\mathscr{X}, \mathscr{H})$. We need to show that for each $k \in \mathscr{K}_\Psi(\mathscr{H})$, $\big(I_{\mathscr{H}} - f(z)g(z)g(w)^*f(w)^*\big) \otimes k(z,w)$ is a positive $B(\mathscr{H} \otimes \mathscr{H})$-valued kernel. Note that

$$\begin{aligned}\big(I_{\mathscr{H}} - (fg)(z)(fg)(w)^*\big) \otimes k(z,w) &= \big(I_{\mathscr{H}} - f(z)g(z)g(w)^*f(w)^*\big) \otimes k(z,w) \\ &= \big(I_{\mathscr{H}} - f(z)f(w)^*\big) \otimes k(z,w) \\ &\quad + f(z)\big(I_{\mathscr{Y}} - g(z)g(w)^*\big)f(w)^* \otimes k(z,w). \qquad (6)\end{aligned}$$

The expression $\big(I_{\mathscr{H}} - f(z)f(w)^*\big) \otimes k(z,w)$ is positive as $f \in \mathscr{S}\mathscr{A}_\Psi(\mathscr{Y}, \mathscr{H})$. We now consider $f(z)\big(I_{\mathscr{Y}} - g(z)g(w)^*\big)f(w)^* \otimes k(z,w)$ in two cases:

Case 1. $\mathscr{H} = \mathbb{C}$.

In this case, the above expression becomes $\big(f(z)(I_{\mathscr{Y}} - g(z)g(w)^*)f(w)^*\big)k(z,w)$. Now we make the following claim.

Claim. For a $\mathbb{C}$-valued Ψ-admissible kernel k, if we let $K := kI_{\mathscr{Y}}$ then K is a $B(\mathscr{Y})$-valued kernel that is Ψ-admissible.

To see that $K := kI_{\mathscr{Y}}$ is a kernel, choose $y_1, \ldots, y_n \in \mathscr{Y}$ and $z_1, \ldots, z_n \in \Omega$ and compute

$$\sum_{i,j=1}^{n} \langle K(z_i,z_j)y_j, y_i\rangle = \sum_{i,j=1}^{n} \langle k(z_i,z_j)y_j, y_i\rangle = \sum_{i,j=1}^{n} k(z_i,z_j)\langle y_j, y_i\rangle.$$

Now fix a basis $\{e_\alpha\}$ of $\mathscr{Y}$ and write $y_i = \sum_\alpha y_{\alpha,i}e_\alpha$. If we define $\bar{y}_i = \sum \bar{y}_{\alpha,i}e_\alpha$, then the last expression above is equal to

$$\sum_{i,j=1}^{n} k(z_i,z_j)\langle \bar{y}_j, \bar{y}_i\rangle$$

which is positive. Consequently K above is positive too. We now show that K is Ψ-admissible. Choose $\psi \in \Psi$ and $z_1, \ldots, z_n \in \Omega$ and $y_1, \ldots, y_n \in \mathscr{Y}$ and compute

$$\begin{aligned}&||\sum_{j=1}^{n} K(\bullet, z_j)y_j||^2 - ||M_\psi^*(\sum_{j=1}^{n} K(\bullet, z_j)h_j)||^2 \\ &= \sum_{i,j=1}^{n} \big(\langle k(z_i,z_j)y_j, y_i\rangle - \psi(z_i)\psi(z_j)\langle k(z_i,z_j)y_j, y_i\rangle\big) \\ &= \sum_{i,j=1}^{n} \big(1 - \psi(z_i)\psi(z_j)\big)k(z_i,z_j)\langle h_j, h_i\rangle.\end{aligned}$$

The last expression above is nonnegative and hence $||M_\psi^*|| = ||M_\psi|| \leqslant 1$. Since the above holds for any $\psi \in \Psi$, the claim is established.

Coming back to the proof of Case 1 above, using the claim we see that $(I_{\mathscr{Y}} - g(z)g(w)^*) \otimes (kI_{\mathscr{Y}})$ is a $B(\mathscr{Y} \otimes \mathscr{Y})$-valued kernel. Choose a basis element e of $\mathscr{Y}$

and consider $u_i = f(z_i)^* \otimes (c_i e)$. Then

$$
\begin{aligned}
0 &\leqslant \sum_{i,j=1}^{n} \Big\langle \Big(\big(I_{\mathscr{Y}} - g(z_i)g(z_j)^*\big) \otimes \big(k(z_i,z_j)I_{\mathscr{Y}}\big) \Big) u_j, u_i \Big\rangle \\
&= \sum_{i,j=1}^{n} \big\langle \big(I_{\mathscr{Y}} - g(z_i)g(z_j)\big) f(z_j)^*(1), f(z_i)^*(1) \big\rangle k(z_i,z_j) \langle c_j e, c_i e\rangle \\
&= \sum_{i,j=1}^{n} \bar{c}_i c_j \big\langle f(z_i)\big(I_{\mathscr{Y}} - g(z_i)g(z_j)\big) f(z_j)^*(1), 1 \big\rangle k(z_i,z_j) \\
&= \sum_{i,j=1}^{n} \bar{c}_i c_j \Big(f(z_i)\big(I_{\mathscr{Y}} - g(z_i)g(z_j)^*\big) f(z_j)^* \Big) k(z_i,z_j)
\end{aligned}
$$

Thus $f(z)\big(I_{\mathscr{Y}} - g(z)g(w)^*\big)f(w)^*\, k(z,w)$ is a positive kernel and we are done in this case.

Case 2. $\mathscr{H} = \mathscr{Y}$.

So $f \in \mathscr{S}\mathscr{A}_{\Psi}(\mathscr{Y},\mathscr{Y})$ and we need to show that $fg \in \mathscr{S}\mathscr{A}_{\Psi}(\mathscr{X},\mathscr{Y})$. Consider any $B(\mathscr{Y})$-valued Ψ-admissible kernel K and consider $f(z)\big(I_{\mathscr{Y}} - g(z)g(w)^*\big)f(w)^* \otimes K(z,w)$ which has to be shown a positive $B(\mathscr{Y}\otimes\mathscr{Y})$-valued kernel. For this purpose take $y_{i1}\otimes y_{i2} \in \mathscr{Y}\otimes\mathscr{Y}$, $1 \leqslant i \leqslant n$, and compute:

$$
\begin{aligned}
&\sum_{i,j=1}^{n} \Big\langle \Big[\Big(f(z_i)\big(I_{\mathscr{Y}} - g(z_i)g(z_j)^*\big)\Big) f(z_j) \otimes K(z_i,z_j)\Big](y_{j1}\otimes y_{j2}), y_{i1}\otimes y_{i2} \Big\rangle \\
&= \sum_{i,j=1}^{n} \Big\langle f(z_i)\big(I_{\mathscr{Y}} - g(z_i)g(z_j)^*\big) f(z_j)^* y_{j1}, y_{i1} \Big\rangle \langle K(z_i,z_j)y_{j2}, y_{i2}\rangle \\
&= \sum_{i,j=1}^{n} \Big\langle \big(I_{\mathscr{Y}} - g(z_i)g(z_j)^*\big) V_j, V_i \Big\rangle \langle K(z_i,z_j)y_{j2}, y_{i2}\rangle,
\end{aligned}
$$

where in the last expression $V_i = f(z_i)^* y_{i1}$ for all i, $1 \leqslant i \leqslant n$. The last expression can be written as

$$
\sum_{i,j=1}^{n} \Big\langle \big[\big(I_{\mathscr{Y}} - g(z_i)g(z_j)^*\big) \otimes K(z_i,z_j)\big] (V_j \otimes y_{j2}), (V_i \otimes y_{i2} \Big\rangle,
$$

which is non-negative owing to the fact that $g \in \mathscr{S}\mathscr{A}_{\Psi}(\mathscr{X},\mathscr{Y})$ and K is a $B(\mathscr{Y})$-valued Ψ-admissible positive kernel. Thus $f(z)\big(I_{\mathscr{Y}} - g(z)g(w)^*\big)f(w)^* \otimes K(z,w)$ is a positive kernel for any K that is Ψ-admissible and we are done in this case too. $\square$

2.2. Ψ-unitary colligations and realization of Ψ-Schur–Agler class functions

When $\Omega = \mathbb{D}^n$ and Ψ being the collection of co-ordinate functions on $\mathbb{D}^n$, Agler [1] showed that the Ψ-Schur-Agler class functions are precisely those that admit a decomposition (now called Agler decoposition). He also showed that this latter class of functions coincides with those functions that admit a transfer function realization.

When Ψ has infinitely many test functions, an analogous result holds true. Before we state such a result, we first introduce, following Ambrozie [4], a Ψ-unitary colligation and the transfer function associated to it.

Given a collection of test functions Ψ and Hilbert spaces $\mathscr{X}$, $\mathscr{U}$, $\mathscr{Y}$, a Ψ*-unitary colligation* is a pair (U,ρ) where U is a unitary operator from $\mathscr{X}\oplus\mathscr{U}$ to $\mathscr{X}\oplus\mathscr{Y}$ and $\rho:\mathscr{C}_b(\Psi)\to B(\mathscr{X})$ is a $*$-representation. If we write U as

$$\begin{array}{cc} & \begin{array}{cc}\mathscr{X} & \mathscr{U}\end{array} \\ U = \begin{array}{c}\mathscr{X}\\ \mathscr{Y}\end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \end{array}$$

then we can define a bounded $B(\mathscr{U},\mathscr{Y})$ valued function on Ω, given by

$$f(x) = D + C\rho(E(x))(\mathbb{I}_{\mathscr{X}} - A\rho(E(x)))^{-1}B \ \ \forall\, x\in\Omega, \tag{7}$$

equivalently,

$$f(x) = D + C(\mathbb{I}_{\mathscr{X}} - \rho(E(x))A)^{-1}\rho(E(x))B \ \ \forall\, x\in\Omega. \tag{8}$$

This f is called the *transfer function* associated with (U,ρ). Since U^* is also a unitary, we have that

$$g(x) = D^* + B^*(\mathbb{I}_{\mathscr{X}} - \rho(E(x))A^*)^{-1}\rho(E(x))C^*$$

is the transfer function of the colligation (U^*,ρ). We now state the result alluded to as above that was established in [11]. We must mention that variants of this result exist in literature; see e.g. [8] and [15].

RESULT 3. *Consider a function S_0 on some subset Ω_0 of Ω with values in $B(\mathscr{U},\mathscr{Y})$. Then the following conditions are equivalent.*

1. *There exists an S in $H^\infty_\Psi(\mathscr{U},\mathscr{Y})$ with $\|S\|_\Psi \leqslant 1$ such that $S|_{\Omega_0} = S_0$.*
2. *S_0 has an* Agler decomposition *on Ω_0, that is, there exists a completely positive kernel $\Gamma:\Omega_0\times\Omega_0\to B(\mathscr{C}_b(\Psi),B(\mathscr{Y}))$ so that*
$$I_{\mathscr{Y}} - S_0(z)S_0(w)^* = \Gamma(z,w)(1-E(z)E(w)^*) \ \textit{for all } z,w\in\Omega_0.$$
3. *There exists a Hilbert space $\mathscr{X}$, a $*$-representation $\rho:\mathscr{C}_b(\Psi)\to B(\mathscr{X})$ and a Ψ-unitary colligation (V,ρ) such that writing V as*
$$\begin{array}{cc} & \begin{array}{cc}\mathscr{X} & \mathscr{U}\end{array} \\ V = \begin{array}{c}\mathscr{X}\\ \mathscr{Y}\end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \end{array}$$
one has
$$S_0(z) = D + C(\mathbb{I}_{\mathscr{X}} - \rho(E(z))A)^{-1}\rho(E(z))B \ \textit{for all } z\in\Omega_0. \tag{9}$$

4. *There exists a Hilbert space $\mathscr{X}$, a $*$-representation $\rho : \mathscr{C}_b(\Psi) \to B(\mathscr{X})$ and a Ψ-unitary colligation (W, ρ) such that writing W as*

$$\begin{array}{cc} & \begin{array}{cc} \mathscr{X} & \mathscr{Y} \end{array} \\ W = \begin{array}{c} \mathscr{X} \\ \mathscr{U} \end{array} & \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \end{array}$$

one has

$$S_0(z)^* = D_1 + C_1(\mathbb{I}_{\mathscr{X}} - \rho(E(z))^* A_1)^{-1} \rho(E(z))^* B_1 \ \textit{for all } z \in \Omega_0. \tag{10}$$

The following lemma is proved in [11] which was one of tools in establishing the Result 3 above. The proof involves a cone separation argument; many of such arguments are present in the literature, see e.g. [2, Theorem 11.26]. Since it is needed in the proof of our main theorem, we state it here without its proof.

LEMMA 2.2. *Let $J : \Omega \times \Omega \to B(\mathscr{Y})$ be a self-adjoint function. Suppose*

$$J \oslash K : (z, w) \mapsto J(z, w) \otimes K(z, w) \tag{11}$$

is a positive kernel for every $B(\mathscr{Y})$-valued admissible kernel K, then there is a completely positive kernel $\Gamma : \Omega \times \Omega \to B(\mathscr{C}_b(\Psi), B(\mathscr{Y}))$ such that

$$J(z, w) = \Gamma(z, w)(1 - E(z)E(w)^*) \ \textit{for all } z, w \in \Omega.$$

We now end this section with another lemma that shows an application of Result 3 to the interpolation problem of Pick–Nevanlinna type.

LEMMA 2.3. *Let $\underline{w} = \{w_j : j \in \mathbb{N}\} \subset \Omega$ be a sequence in Ω and let $\underline{x} = \{x_j : \in \mathbb{N}\}$ be a sequence of complex numbers. Then there exists $f \in H^\infty_\Psi(\mathbb{C})$ with $||f||_\Psi \leqslant C_{\underline{w}}$ and $f(w_j) = x_j$ if and only if for every $n \in \mathbb{N}$ the matrix*

$$\Big((C_{\underline{w}}^2 - x_i \overline{x_j}) k(w_i, w_j) \Big)_{i,j=1}^n$$

is positive semi-definite for every $k \in \mathscr{K}_\Psi$.

Proof. First assume that there exists $f \in H^\infty_\Psi(\mathbb{C})$ with $||f||_\Psi \leqslant C_{\underline{w}}$ and such that $f(w_j) = x_j$. Then it follows – from the implication (1) $\implies$ (2) of Result 3 – that $\big((C_{\underline{w}}^2 - x_i \overline{x_j}) k(w_i, w_j) \big) \geqslant 0$ for all $k \in \mathscr{K}_\Psi$. Conversely, assume that the functional $(i, j) \mapsto \big(C_{\underline{w}}^2 - x_i \overline{x_j} \big) k(w_i, w_j)$ is positive semi-definite for all $k \in \mathscr{K}_\Psi$. Let us denote by $\Omega_0 := \{w_i : i \in \mathbb{N}\}$ and $y_i = x_i / C_{\underline{w}}$. Then the function $\mathscr{J} : \Omega_0 \times \Omega_0 \longrightarrow \mathbb{C}$ defined by $\mathscr{J}(w_i, w_j) = (1 - y_i \bar{y}_j)$ satisfies the following:

$$\mathscr{J}(w_i, w_j)^* = \mathscr{J}(w_j, w_i) \quad \text{and} \quad \mathscr{J}(w_i, w_j) k(w_i, w_j) \geqslant 0 \quad \forall \ k \in \mathscr{K}_\Psi.$$

Now by Lemma 2.2, by taking $\mathscr{Y} = \mathbb{C}$, there exists a completely positive kernel $\Gamma : \Omega_0 \times \Omega_0 \longrightarrow \mathscr{C}_b(\Psi)^*$ such that

$$\mathscr{J}(w_i, w_j) = (1 - y_i\overline{y_j}) = \Gamma(w_i, w_j)\big(1 - E(w_i)E(w_j)^*\big), \quad \forall\, i, j.$$

Let $S_0 : \Omega_0 \longrightarrow \mathbb{C}$ be defined by $S_0(w_j) := y_j$. Then from the equivalence of part (1) and (2) in Result 3, we get that there exists $S \in H^\infty_\Psi(\mathbb{C})$ with $||S||_\Psi \leqslant 1$ and such that $S|_{\Omega_0} = S_0$. Observe that $\phi = C_{\underline{w}}S$ that has the desired properties. $\square$

3. Boundedness of Grammian and existence of vector-valued interpolants

In this section we shall prove a crucial proposition: namely Proposition 3.4 below. The proposition is an analogue, in our setting, of Lemma 4.1 and Lemma 4.2 in [3] in the case when $\Omega = \mathbb{D}^2$ due to Agler-McCarthy; also see [2, Theorem 9.46]. This proposition is at the heart of our proof of Theorem 1.2. Before we present this proposition, we need a few lemmas.

LEMMA 3.1. *Let $\mathscr{U}$, $\mathscr{Y}$, $\mathscr{H}$ be Hilbert spaces with bases $\{u_\alpha\}$, $\{y_\beta\}$, $\{h_\delta\}$, respectively. For $A \in B(\mathscr{U}, \mathscr{Y})$, $B \in B(\mathscr{Y}, \mathscr{H})$, $C \in B(\mathscr{U}, \mathscr{H})$, define $A^t \in B(\mathscr{Y}, \mathscr{U})$, $B^t \in B(\mathscr{H}, \mathscr{Y})$, $C^t \in B(\mathscr{H}, \mathscr{U})$ by first setting*

$$\begin{aligned}
\langle A^t y_\beta, u_\alpha\rangle &:= \langle A u_\alpha, y_\beta\rangle\\
\langle B^t h_\delta, y_\beta\rangle &:= \langle B y_\beta, h_\delta\rangle\\
\langle C^t h_\delta, u_\alpha\rangle &:= \langle C u_\alpha, h_\delta\rangle
\end{aligned}$$

and then extending linearly on linear combinations of basis elements. Then $(BA)^t = A^t B^t$ with respect to these given bases.

Proof. Note that

$$\Big\langle A^t \sum d_\alpha y_\alpha, \sum a_\beta u_\beta \Big\rangle := \sum_{\alpha,\beta} d_\alpha \bar{a}_\beta \langle A u_\beta, y_\alpha\rangle = \Big\langle A\Big(\sum_\beta \bar{a}_\beta u_\beta\Big), \sum_\alpha d_\alpha y_\alpha \Big\rangle.$$

From this it follows that A^t is bounded linear operator, and so are B^t, $(BA)^t$. Now

$$\langle (BA)^t h_\delta, u_\alpha\rangle = \langle (BA)u_\alpha, h_\delta\rangle = \Big\langle B\sum_\beta \langle A u_\alpha, y_\beta\rangle y_\beta, h_\delta \Big\rangle = \sum_\beta \langle A u_\alpha, y_\beta\rangle \langle B y_\beta, h_\delta\rangle.$$

Also

$$\begin{aligned}
\langle A^t B^t h_\delta, u_\alpha\rangle &= \Big\langle A^t \sum_\beta \langle B^t h_\delta, y_\beta\rangle y_\beta, u_\alpha \Big\rangle\\
&= \sum_\beta \langle B^t h_\delta, y_\beta\rangle \langle A^t y_\beta, u_\alpha\rangle = \sum_\beta \langle B y_\beta, h_\delta\rangle \langle A u_\alpha, y_\beta\rangle.
\end{aligned}$$

Thus $(BA)^t = A^t B^t$. $\square$

LEMMA 3.2. *Let $\mathscr{U}$, $\mathscr{Y}$ be Hilbert spaces with bases $\{u_\alpha\}$ and $\{y_\beta\}$ respectively. Then for any $\Phi \in \mathscr{S}\mathscr{A}_\Psi(\mathscr{U}, \mathscr{Y})$, there exists $\Phi^t \in \mathscr{S}\mathscr{A}_\Psi(\mathscr{Y}, \mathscr{U})$ such that*

$$\langle \Phi^t(z) y_\beta, u_\alpha \rangle = \langle \Phi(z) u_\alpha, y_\beta \rangle \tag{12}$$

for all $z \in \Omega$ and α, β.

Proof. Since $\Phi \in \mathscr{S}\mathscr{A}_\Psi(\mathscr{U}, \mathscr{Y})$, there exist a Hilbert space $\mathscr{H}$ (with basis $\{h_\delta\}$), a unital $*$-representation $\rho : \mathscr{C}_b(\Psi) \longrightarrow B(\mathscr{H})$ and a unitary V, where if we write

$$V = \begin{array}{c c} & \begin{array}{cc} \mathscr{H} & \mathscr{U} \end{array} \\ \begin{array}{c} \mathscr{H} \\ \mathscr{Y} \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array},$$

then

$$\Phi(z) = D + C\rho(E(z))\big(\mathbb{I}_{\mathscr{H}} - A\rho(E(z))\big)^{-1}B.$$

Consider now A^t, B^t, C^t, D^t and V^t, $\rho(E(z))^t$ with respect to the bases above. Observe V^t is a unitary operator such that

$$V^t = \begin{array}{c c} & \begin{array}{cc} \mathscr{H} & \mathscr{Y} \end{array} \\ \begin{array}{c} \mathscr{H} \\ \mathscr{U} \end{array} & \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \end{array},$$

It is easy to see that $\rho^t : \mathscr{C}_b(\Psi) \longrightarrow B(\mathscr{H})$ is a unital $*$-representation. Now consider

$$\Phi^t(z) := D^t + B^t\rho^t(E(z))\big(\mathbb{I}_{\mathscr{H}} - A^t\rho^t(E(z))\big)^{-1}C^t.$$

Since V^t is unitary, we know that $\Phi^t \in \mathscr{S}\mathscr{A}_\Psi(\mathscr{Y}, \mathscr{U})$. The identity (12) follows from this latter observation. □

LEMMA 3.3. *Let $\{\phi_n\} \in \mathscr{S}\mathscr{A}_\Psi(\mathbb{C})$ be a sequence of $\mathbb{C}$-valued functions. Then there exists a $\Phi \in \mathscr{S}\mathscr{A}_\Psi(l^2, l^2)$ such that*

$$\langle \Phi(z) e_i, e_j \rangle = \phi_i(z)\,\delta_{i,j} \quad \forall i, j \quad \text{and } z \in \Omega.$$

Here, $\{e_i : i \in \mathbb{N}\}$ is the standard orthonormal basis of l^2.

The proof of the above lemma is a routine exercise. Therefore, we omit it here. We only mention that the function Φ above is the direct sum of the functions ϕ_n constructed via the transfer function realization of each ϕ_n of an associated Ψ-unitary colligation (U_n, ρ_n).

We are now ready to state the principal result of this section.

PROPOSITION 3.4. *Let Ω be a bounded domain in $\mathbb{C}^n$ and let Ψ be a family of test functions and consider the set $\mathscr{K}_\Psi$ of Ψ-admissible $\mathbb{C}$-valued kernels on Ω. Let $\{w_i\}_{i\in\mathbb{N}} \subset \Omega$ be a sequence. Then*

1. *There exists $N > 0$ such that the Grammian corresponding to $\{w_i\}_{i\in\mathbb{N}}$, $G_k \geqslant (1/N)\mathbb{I}$ for all $k \in \mathscr{K}_\Psi$ if and only if there exists $\Phi \in H^\infty_\Psi(\mathbb{C}, l^2)$ such that $\Phi(w_i)(1) = e_i$ with $||\Phi||_\Psi \leqslant \sqrt{N}$.*

2. *There exists an $M > 0$ such that the Grammian corresponding to $\{w_i\}_{i\in\mathbb{N}}$, $G_k \leqslant M\mathbb{I}$ for all $k \in \mathscr{K}_\Psi$ if and only there exists $\varphi \in H^\infty_\Psi(l^2, \mathbb{C})$ such that $\phi(w_i)(e_i) = 1$ with $||\varphi||_\Psi \leqslant \sqrt{M}$.*

Proof. Let us start with establishing part (1) above.

1. Suppose there is an $N > 0$ such that $G_k \geqslant \frac{1}{N} \cdot \mathbb{I}$ for all $k \in \mathscr{K}_\Psi$. This is equivalent to

$$(N - \delta_{ij})k(w_i, w_j) \geqslant 0$$

for all $k \in \mathscr{K}_\Psi$. By Lemma 2.2 there is a completely positive kernel $\Gamma : \Omega_0 \times \Omega_0 \to \mathscr{C}_b(\Psi)^*$, $(\Omega_0 = \{w_j : j \geqslant 1\})$ such that

$$N - \delta_{ij} = \Gamma(w_i, w_j)(1 - E(w_i)E(w_j)^*) \tag{13}$$

Now we follow the standard lurking isometry argument, see e.g. [15]. First, by [15, Proposition 3.3], there is a Hilbert space $\mathscr{E}$, a function $L : \Omega_0 \longrightarrow B(\mathscr{C}_b(\Psi), \mathscr{E})$ and a unital $*$-representation $\rho : \mathscr{C}_b(\Psi) \longrightarrow B(\mathscr{E})$ such that

$$\Gamma(x,y)(fg^*) = \langle L(x)f, L(y)g\rangle \text{ and } L(x)(fg) = \rho(f)L(x)(g)$$

for all $f, g \in \mathscr{C}_b(\Psi), x, y \in \Omega_0$. So (13) can be rewritten as

$$\left\langle \begin{pmatrix} \rho(E(w_i))L(w_i)(1) \\ \sqrt{N} \end{pmatrix}, \begin{pmatrix} \rho(E(w_j))L(w_j)(1) \\ \sqrt{N} \end{pmatrix} \right\rangle_{\mathscr{E}\oplus\mathbb{C}}$$
$$= \left\langle \begin{pmatrix} L(w_i)(1) \\ e_i \end{pmatrix}, \begin{pmatrix} L(w_j)(1) \\ e_j \end{pmatrix} \right\rangle_{\mathscr{E}\oplus l^2}$$

for all i, j where $\{e_j : j \geqslant 1\}$ is the standard basis for l^2.

It is easy to see that there is a unitary operator $V : \mathscr{E} \oplus \mathbb{C} \to \mathscr{E} \oplus l^2$ (adding an infinite dimensional Hilbert space to $\mathscr{E}$ if necessary) that sends $\begin{pmatrix} \rho(E(w_i))L(w_i)(1) \\ \sqrt{N} \end{pmatrix}$ to $\begin{pmatrix} L(w_i)(1) \\ e_i \end{pmatrix}$ for all $i \geqslant 1$. Let us write

$$V = \begin{array}{cc} & \begin{array}{cc} \mathscr{E} & \mathbb{C} \end{array} \\ \begin{array}{c} \mathscr{E} \\ l^2 \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array},$$

and take $F(z) = D + C(\mathbb{I}_{\mathscr{E}} - \rho(E(z))A)^{-1}\rho(E(z))B$. Then $F \in \mathscr{S}\mathscr{A}_{\Psi}(\mathbb{C}, l^2)$. If we set $\Phi = \sqrt{N}F$, then Φ satisfies

$$||\Phi||_{\Psi} \leqslant \sqrt{N} \text{ and } \Phi(w_i)(1) = e_i$$

for all $i \geqslant 1$.

Conversely, let there be an element $\phi' \in H^{\infty}_{\Psi}(\mathbb{C}, l^2)$ such that $||\phi'||_{\Psi} \leqslant \sqrt{N}$ and $\phi'(w_i)(1) = e_i$ for all $i \geqslant 1$. So $F' = \phi'/\sqrt{N} \in \mathscr{S}\mathscr{A}_{\Psi}(\mathbb{C}, l^2)$ and consequently there is a Hilbert space $\mathscr{H}$, a unital $*$-representation $\mu : \mathscr{C}_b(\Psi) \to B(\mathscr{H})$ and a unitary

$$U = \begin{array}{cc} & \begin{array}{cc} \mathscr{H} & \mathbb{C} \end{array} \\ \begin{array}{c} \mathscr{H} \\ l^2 \end{array} & \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \end{array}$$

such that $F'(z) = D' + C'(\mathbb{I}_{\mathscr{H}} - \mu(E(z))A')^{-1}\mu(E(z))B'$ for all $z \in \Omega$. Set $h_i = (\mathbb{I}_{\mathscr{H}} - \mu(E(w_i))A')^{-1}B'\sqrt{N}$ for all i. Then we have that the unitary U sends $\begin{pmatrix} \mu(E(w_i))h_i \\ \sqrt{N} \end{pmatrix}$ to $\begin{pmatrix} h_i \\ e_i \end{pmatrix}$. Using these facts we obtain

$$N - \delta_{ij} = \langle \mu(1 - E(w_i)E(w_j)^*)h_i, h_j \rangle. \tag{14}$$

It is easy to see that the map $\Gamma' : \Omega \times \Omega \to \mathscr{C}_b(\Psi)^*$ defined by

$$\Gamma'(z,w)(f) = \langle \mu(f)h(z), h(w) \rangle \tag{15}$$

where $h(z) = (\mathbb{I}_{\mathscr{H}} - \mu(E(z))A')^{-1}B'\sqrt{N}$, is a completely positive kernel. Now define $\widetilde{\phi} : \Omega_0 \to B(l^2, \mathbb{C})$ by $\widetilde{\phi}(w_i)^* = B_i^*$ where $B_i(\sum l_i e_i) = l_i$, $l_i \in \mathbb{C}$. Then (14) takes the form

$$1 - \frac{\widetilde{\phi}(w_i)}{\sqrt{N}} \frac{\widetilde{\phi}(w_j)^*}{\sqrt{N}} = \frac{1}{N}\Gamma'(1 - E(w_i)E(w_j)^*)$$

for all i, j. By Result 3, (the equivalence of part (1) and (2)), $\widetilde{\phi}/\sqrt{N}$ can be extended to an element of $\mathscr{S}\mathscr{A}_{\Psi}(l^2, \mathbb{C})$ and the definition of $\mathscr{S}\mathscr{A}_{\Psi}(l^2, \mathbb{C})$ yields that

$$(N - \delta_{ij})k(w_i, w_j) \geqslant 0$$

for all admissible kernel k. This completes the proof of the first part.

2. For $M > 0$ and an admissible kernel $k \in \mathscr{K}_{\Psi}$, the condition $G_k \leqslant M \cdot \mathbb{I}$ is equivalent to $(M\delta_{ij} - 1)k(w_i, w_j) \geqslant 0$. So when $G_k \leqslant M \cdot \mathbb{I}$ for every admissible kernel k, following the same procedure as in part 1, we can show that there is a $\phi' \in H^{\infty}_{\Psi}(l^2, \mathbb{C})$ such that

$$||\phi'||_{\Psi} \leqslant \sqrt{M} \text{ and } \phi'(w_j)(e_j) = 1.$$

Conversely, let there be a $\phi \in H^\infty_\Psi(l^2,\mathbb{C})$ such that

$$||\phi||_\Psi \leqslant \sqrt{M} \text{ and } \phi(w_j)(e_j) = 1.$$

So $F = \phi/\sqrt{M} \in \mathscr{S}\mathscr{A}_\Psi(l^2,\mathbb{C})$ and consequently, we can find a Hilbert space $\mathscr{H}$, a unital $*$-representation $\rho : \mathscr{C}_b(\Psi) \to B(\mathscr{H})$ and a unitary

$$V = \begin{array}{cc} & \begin{array}{cc} \mathscr{H} & l^2 \end{array} \\ \begin{array}{c} \mathscr{H} \\ \mathbb{C} \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}$$

such that $F(z) = D + C\rho(E(z))(\mathbb{I}_\mathscr{H} - A\rho(E(z)))^{-1}B$ for all $z \in \Omega$. Let $h_j = (I_\mathscr{H} - A\rho(E(w_i)))^{-1}B\sqrt{M}(e_i)$. Then the unitary operator V sends $\begin{pmatrix} \rho(E(w_i))h_i \\ \sqrt{M}e_i \end{pmatrix}$ to $\begin{pmatrix} h_i \\ 1 \end{pmatrix}$. From this we deduce

$$M\delta_{ij} - 1 = \langle \rho(1 - E(w_i)E(w_j)^*)h_i, h_j \rangle. \tag{16}$$

Define $\Gamma : \Omega_0 \times \Omega_0 \to \mathscr{C}_b(\Psi)^*$ by

$$\Gamma(w_i,w_j)(\delta) = \langle \rho(\delta)h_i, h_j \rangle, \delta \in \mathscr{C}_b(\Psi). \tag{17}$$

From (15), we know that it is completely positive.

Claim. $(M\delta_{ij} - 1)k(w_i,w_j) = \Gamma(w_i,w_j)(1 - E(w_i)E(w_j)^*)k(w_i,w_j)$ is positive for every admissible kernel k.

Fix a $k \in \mathscr{K}_\Psi$ and define $\Gamma_k : \Omega_0 \times \Omega_0 \to B(\mathscr{C}_b(\Psi))$ by

$$\Gamma_k(w_i,w_j)(\delta) = (1 - E(w_i)E(w_j)^*)k(w_i,w_j)\delta, \quad \delta \in \mathscr{C}_b(\Psi).$$

We claim that Γ_k is completely positive. To see that, take $a_1, a_2, \dots a_n, b_1, b_2, \dots b_n \in \mathscr{C}_b(\Psi)$ and consider the expression

$$\sum_{i,j=1}^n b_i^* \Gamma_k(w_i,w_j)(a_i^* a_j) b_j = \sum_{i,j=1}^n (a_i b_i)^*(a_j b_j)(1 - E(w_i)E(w_j)^*)k(w_i,w_j).$$

It is an element of $\mathscr{C}_b(\Psi)$. Evaluating it at any $\psi \in \Psi$ and using the fact that k is admissible, we find that $\sum_{i,j=1}^n b_i^* \Gamma_k(w_i,w_j)(a_i^* a_j) b_j$ is a positive element of $\mathscr{C}_b(\Psi)$. This proves the claim that Γ_k is completely positive.

Now for k as above and any i,j, define $F_k(i,j) = (1 - E(w_i)E(w_j)^*)k(w_i,w_j)$. Similar argument as above yields that the matrix $F_k = (F_k(i,j))_{1 \leqslant i,j \leqslant n}$ is a positive matrix with entries in $\mathscr{C}_b(\Psi)$ (see Lemma IV.3.2 in [20]). Again by Lemma IV.3.1. in [20], F_k can be written as a sum of finitely many matrices of the form

$C = (a_i^* a_j)_{1 \leqslant i,j \leqslant n}$, $a_i \in \mathscr{C}_b(\Psi)$. That is, there is a positive integer l such that for any i and j

$$F_k(i,j) = \sum_{m=1}^{n} a_{m_i}^* a_{m_j}$$

with $a_{m_i} \in \mathscr{C}_b(\Psi)$. Now if $\Gamma : \Omega_0 \times \Omega_0 \to \mathscr{C}_b(\Psi)^*$ is completely positive, then for $w_1, w_2, \dots w_n \in \Omega_0$ and $\alpha_1, \alpha_2, \dots \alpha_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{n} \overline{\alpha_i} \alpha_j \Gamma(w_i, w_j)(F_k(i,j)) = \sum_{m=1}^{l} \Big\{ \sum_{i,j=1}^{n} \overline{\alpha_i} \alpha_j \Gamma(w_i, w_j)(a_{m_i}^* a_{m_j}) \Big\}$$

Since Γ is completely positive, we get that the last expression is non-negative. Hence $(M\delta_{ij} - 1)k(w_i, w_j) = \Gamma(w_i, w_j)(1 - E(w_i)E(w_j)^*)k(w_i, w_j)$ is positive and this completes the proof. $\square$

4. Proof of Theorem 1.2

Proof. To see how one (1) implies (4), start with an interpolating sequence for $H_\Psi^\infty(\mathbb{C})$ and then proceeding exactly as in the first half of the proof of [2, Theorem 9.19], one gets both the conditions (2) and (3). Now suppose (2) and (3) hold together. Then for every kernel $k \in \mathscr{K}_\Psi$ and $a = (a_i) \in l^2$ we have

$$\frac{1}{N} \sum_{i \in \mathbb{N}} |a_i|^2 \leqslant || \sum_{i \in \mathbb{N}} a_i g_i ||^2 \leqslant M \sum_{i \in \mathbb{N}} |a_i|^2,$$

where $g_i = k_i / ||k_i||$. Given $\{c_j : j \in \mathbb{N}\} \in l_1^\infty(\mathbb{N})$, we have:

$$|| \sum_{i=1}^{N} a_i \bar{c}_i g_i ||^2 \leqslant M \sum_{i=1}^{N} |a_i \bar{c}_i|^2 \leqslant M \sum_{i=1}^{N} |a_i|^2 \leqslant MN || \sum_{i=1}^{N} a_i g_i ||^2$$

whence $|| \sum_{i=1}^{N} a_i \bar{c}_i g_i || \leqslant \sqrt{MN} || \sum_{i=1}^{N} a_i g_i ||$. It follows from this that the map $R : \operatorname{span}\{g_i : i \in \mathbb{N}\} \longrightarrow \operatorname{span}\{g_i : i \in \mathbb{N}\}$ that maps g_i to $\bar{c}_i g_i$ is a bounded linear operator with norm $\leqslant \sqrt{MN}$ and hence it extends to $\overline{\operatorname{span}\{g_i : i \in \mathbb{N}\}}$ with norm $\leqslant \sqrt{MN}$. This implies that the expression

$$\sum_{i,j \in \mathbb{N}} \big(MN - c_j \bar{c}_i \big) k(w_i, w_j)$$

is positive semi-definite for every $k \in \mathscr{K}_\Psi$. By Lemma 2.3, there exists a $\phi \in H_\Psi^\infty(\mathbb{C})$ with $||\phi||_\Psi \leqslant \sqrt{MN}$ such that $\phi(w_j) = c_j$ for all $j \in \mathbb{N}$. This proves (4) implies (1).

We shall now show that (2) is equivalent to (3). By Proposition 3.4, we have to show the following statements are equivalent.

(i) There is a $\phi \in H_\Psi^\infty(\mathbb{C}, l^2)$ with $||\phi||_\Psi \leqslant \sqrt{N}$ and $\phi(w_j) = e_j$ for all $j \geqslant 1$, where $\{e_j : j \geqslant 1\}$ is the standard basis for l^2.

(ii) $\{w_j : j \geqslant 1\}$ is strongly separated and there is a $\widetilde{\phi} \in H^\infty_\Psi(l^2,\mathbb{C})$ such that $||\widetilde{\phi}||_\Psi \leqslant \sqrt{M}$ and $\widetilde{\phi}(w_j)(e_j) = 1$ for all $j \geqslant 1$.

Suppose (i) holds. So with respect to the standard basis of l^2 we can write

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \end{pmatrix}$$

such that $\phi_i(w_j) = \delta_{ij}$.

Claim. Suppose

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \end{pmatrix} \in H^\infty_\Psi(\mathbb{C}, l^2)$$

with $||f||_\Psi \leqslant D$, then each $f_j \in H^\infty_\Psi(\mathbb{C})$ with $||f_j||_\Psi \leqslant D$.

To see this, note that

$$f(z)f(w)^* = \left(f_i(z)\overline{f_j(w)}\right)$$

is an infinite matrix and for every $B(l^2)$-valued admissible kernel K,

$$(D \cdot I_{l^2} - f(z)f(w)^*) \otimes K(z,w)$$

is a positive $B(l^2 \otimes l^2)$-valued kernel. Let k be a $\mathbb{C}$-valued admissible kernel. Then, by the claim in Lemma 2.1, $k \cdot I_{l^2}$ is a $B(l^2)$-valued admissible kernel. Take $K = k \cdot I_{l^2}$, $c_1, c_2 \ldots, c_n \in \mathbb{C}$, $z_1, z_2, \ldots, z_n \in \Omega$, $u_i = e_m$ for some $m \geqslant 1$ and $v_i = c_i e_1$. Then

$$\begin{aligned} &\sum_{i,j=1}^n \left\langle (D^2 \cdot I_{l^2} - f(z_i)f(z_j)^*) \otimes K(z_i,z_j) u_j \otimes v_j, u_i \otimes v_i \right\rangle \\ &= \sum_{i,j=1}^n c_j \overline{c_i} (D^2 - f_m(z_i)\overline{f_m(z_j)}) k(z_j, z_i). \end{aligned}$$

Since $f \in H^\infty_\Psi(\mathbb{C}, l^2)$ with $||f||_\Psi \leqslant D$, the expression at the right-hand side above is non-negative. Hence each $f_m \in H^\infty_\Psi(\mathbb{C})$ with $||f_m||_\Psi \leqslant D$.

So $\phi_i \in H^\infty_\Psi(\mathbb{C})$ and $||\phi_i|| \leqslant \sqrt{N}$ and consequently, $\{w_j : j \geqslant 1\}$ is strongly separated. Also $\phi(w_j)^t(e_j) = \phi_j(w_j) = 1$, where $\phi(w_j)^t$ is as in Lemma 3.2. Thus there there is an map $\widetilde{\phi} = \phi^t \in H^\infty_\Psi(l^2, \mathbb{C})$ such that $||\widetilde{\phi}||_\Psi \leqslant \sqrt{N}$ and $\widetilde{\phi}(w_j)(e_j) = 1$ for all $j \geqslant 1$. So (ii) holds.

Now let (ii) hold. So there is a $\widetilde{\phi} \in H^\infty_\Psi(l^2, \mathbb{C})$ such that $||\widetilde{\phi}||_\Psi \leqslant \sqrt{M}$ and $\widetilde{\phi}(w_j)(e_j) = 1$ for all $j \geqslant 1$. Again by Lemma 3.2,

$$\widetilde{\phi}^t = \begin{pmatrix} \widetilde{\phi}_1 \\ \widetilde{\phi}_2 \\ \cdot \\ \cdot \end{pmatrix} \in H^\infty_\Psi(\mathbb{C}, l^2)$$

and $||\widetilde{\phi}^t|| \leqslant \sqrt{M}$. Since $\{w_j : j \geqslant 1\}$ is strongly separated, there is an $L > 0$ and a sequence $\{\phi_j : j \geqslant 1\}$ in H^∞_Ψ such that $\phi_j(w_j) = \delta_{ij}$ and $||\phi_j||_\Psi \leqslant L$ for all i, j. Consider

$$\Phi_1 = \begin{pmatrix} \phi_1 \widetilde{\phi}_1 \\ \phi_2 \widetilde{\phi}_2 \\ \cdot \\ \cdot \end{pmatrix} = diag(\phi_1, \phi_2 \cdots) \cdot \begin{pmatrix} \widetilde{\phi}_1 \\ \widetilde{\phi}_2 \\ \cdot \\ \cdot \end{pmatrix}.$$

By Lemma 3.3, $diag(\phi_1, \phi_2 \cdots) \in H^\infty_\Psi(l^2, l^2)$ with Ψ-norm atmost L. Also $||\widetilde{\phi}||_\Psi \leqslant \sqrt{M}$. Hence by Lemma 2.1, $\Phi_1 \in H^\infty_\Psi(\mathbb{C}, l^2)$ with $||\Phi_1||_\Psi \leqslant L\sqrt{M}$. Clearly, $\Phi_1^t \in H^\infty_\Psi(l^2, \mathbb{C})$ with Ψ-norm atmost $L\sqrt{M}$ and $\Phi_1^t(w_j) = e_j$. This is *(i)* and our proof is complete. □

We also present the following sufficient condition for a sequence to be interpolating.

PROPOSITION 4.1. *Let $\{w_j : j \in \mathbb{N}\}$ be a sequence of points in Ω. Given $\psi \in \Psi$, define $z_{\psi,j} = \psi(w_j)$. If there is an $\varepsilon > 0$ (that depends on ψ) such that*

$$\prod_{j \neq m} \frac{|z_{\psi,j} - z_m|}{|1 - z_{\psi,j}\overline{z_m}|} \geqslant \varepsilon, \text{ for all } m \geqslant 1, \tag{18}$$

then $\{w_j : j \in \mathbb{N}\}$ is an interpolating sequence for $H^\infty_\Psi(\mathbb{C})$.

Proof. It is not very difficult to see that the condition (18) above is equivalent to the condition that the sequence $\{z_{\psi,j}\}$ in $\mathbb{D}$ is strongly separated. Hence by Result 1 of Carleson, we know that the sequence $\{z_{\psi,j}\}$ is interpolating for $H^\infty(\mathbb{D})$. Now given an arbitrary sequence $\{c_j : j \in \mathbb{N}\} \in l^\infty(\mathbb{N})$, choose $f \in H^\infty(\mathbb{D})$ such that $f(z_{\psi,j}) = c_j$ for all j.

Claim. $f \circ \psi \in H^\infty_\Psi(\mathbb{C})$.

To see the claim above, assume without loss of generality that $||f||_\infty \leqslant 1$. Then there exists a positive kernel Γ on $\mathbb{D}$ such that

$$1 - f(z)\overline{f(w)} = (1 - z\overline{w})\Gamma(z, w).$$

Now let $k \in \mathscr{K}_\Psi$ be given then

$$\begin{aligned} \big(1 - f(\psi(z))\overline{f(\psi(w))}\big) k(z,w) &= (1 - \psi(z)\overline{\psi(w)})\Gamma(\psi(z), \psi(w))k(z, w) \\ &= (1 - \psi(z)\overline{\psi(w)})k(z, w)\Gamma(\psi(z), \psi(w)). \end{aligned}$$

Note that $(1 - \psi(z)\overline{\psi(w)})k(z, w)$ is positive from the definition that k is a Ψ admissible kernel. Of course $\Gamma(\psi(z), \psi(w))$ is positive and hence it follows from this that $\big(1 - f(\psi(z))\overline{f(\psi(w))}\big) k(z,w)$ is positive for every Ψ-admissible kernel k from which the claim follows.

Now notice that $f \circ \psi(w_j) = c_j$ for all j and since $\{c_j : j \in \mathbb{N}\}$ is arbitrary, we are done. □

5. Examples

In this section, we present several cases where Theorem 1.2 could be applied to give a characterization of interpolating sequences. A few of the cases that appear below have already been addressed in the literature.

1. **The case of Polydisc.** This is the case when $\Omega = \mathbb{D}^n$, $n \geqslant 1$. The class of test functions that we consider in this case is $\Psi = \{z_1, \dots, z_n\}$. The case $n = 1$ has already been discussed after the statement of Theorem 1.2.

   In the case $n = 2$, due to Ando's inequality, we know that $H^\infty_\Psi(\mathbb{C}) = H^\infty(\mathbb{D}^2) :=$ the set of all bounded holomorphic functions on the bidisc, with the norm $||\bullet||_\Psi$ being sup-norm. Therefore, Theorem 1.2 provides a characterization of interpolating sequences in the bidisc for the Banach algebra of bounded holomorphic functions with the sup-norm. This case was already considered by Agler–McCarthy in [3]. In general, when $n \geqslant 3$, the Banach algebra $H^\infty_\Psi(\mathbb{C})$ does not coincide with the algebra of bounded holomorphic functions on $\mathbb{D}^n$; see e.g. [21]. One can still apply Theorem 1.2 to give a characterization of interpolating sequences in the polydisc for the algebra $H^\infty_\Psi(\mathbb{C})$ with $\Psi = \{z_1, \dots, z_n\}$.

2. **The case of multiply connected planar domains.** Let Ω be a bounded domain in the complex plane with boundary consisting of $m+1$ disjoint smooth Jordan curves $\partial_0, \partial_1, \dots, \partial_m$ where ∂_0 denotes the boundary of the unbounded component of the complement of Ω. Then there exists a collection of test functions $\Psi = \{\psi_{\mathbf{x}} : \mathbf{x} \in \mathbb{T}_\Omega\}$, indexed by the so-called Ω-torus $\mathbb{T}_\Omega := \partial_0 \times \partial_1 \times \dots \times \partial_m$ (see [8, Section 4.1] and [14]), such that $H^\infty_\Psi(\mathbb{C})$ is equal to the set of all bounded holomorphic functions on Ω and the norm $||\bullet||_\Psi$ being equal to the sup-norm. Using this class of test functions, Theorem 1.2 can be applied to characterize interpolating sequences for bounded holomorphic functions in Ω.

3. **The case of constrained algebras.** Let us denote by $A(\mathbb{D})$ the Banach algebra of holomorphic functions on $\mathbb{D}$ that are continuous upto $\overline{\mathbb{D}}$. The algebra $A(\mathbb{D})$ is called *the disk algebra*. Let B be a finite Blaschke product of degree $N \geqslant 2$ and consider the algebra $\mathscr{A}_B := \mathbb{C} + B(z)A(\mathbb{D})$. Let us denote by H^∞_B the weak-closure of $\mathscr{A}_B$. In [16], a minimal class of test functions has been constructed for the algebras H^∞_B. Therefore, one could use Theorem 1.2 to characterize interpolating sequences in $\mathbb{D}$ for the constrained algebras H^∞_B using this class of test functions.

4. **The case of symmetrized bidisc.** In this case $\Omega = \mathbb{G}_2$ where $\mathbb{G}_2$ is the symmetrized bidisc defined by $\mathbb{G}_2 := \pi_2(\mathbb{D}^2)$ where $\pi_2 : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ is the symmetrization map defined by $\pi_2(z_1, z_2) := (z_1 + z_2, z_1 z_2)$. A point in $\mathbb{G}_2$ is also denoted by a pair $(s, p) \in \mathbb{C}^2$. It is a fact due to Agler–Young (see e.g. [6]) that a point $(s, p) \in \mathbb{C}^2$ belongs to $\mathbb{G}^2$ if and only if for every $\alpha \in \overline{\mathbb{D}}$ we have

$$\frac{2\alpha p - s}{2 - \alpha s} \in \mathbb{D}.$$

Because of this one could consider the family $\Psi := \{\psi_\alpha : \psi_\alpha(s,p) = (2\alpha p - s)/(2-\alpha s) : \alpha \in \overline{\mathbb{D}}\}$, as a family of test functions for $\mathbb{G}_2$. Then it is a fact (see [9] and also [7]) that $H^\infty_\Psi(\mathbb{C}) = H^\infty(\mathbb{G}_2)$, the set of bounded holomorphic functions on $\mathbb{G}_2$ and the norm $||\bullet||_\Psi$ being equal to sup-norm. Hence one could apply Theorem 1.2 to characterize interpolating sequences for $H^\infty(\mathbb{G}_2)$. This case too has been dealt and is due to Bhattacharyya–Sau [10].

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