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INEQUALITIES FOR THE WEIGHTED A–NUMERICAL RADIUS OF SEMI–HILBERTIAN SPACE OPERATORS

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Abstract. In this paper, we introduce the weighted A-numerical radius $\omega_{(A,\nu)}(\cdot)$ for semi-Hilbertian space operators. Further we obtain some basic properties and inequalities for $\omega_{(A,\nu)}(\cdot)$, which will be matched with earlier results about $\omega_A(\cdot)$. Moreover, we provide a refinement and generalization for inequalities obtained in [6, 16].

1. Introduction

In this article, we introduce the weighted A-numerical radius $\omega_{(A,\nu)}(\cdot)$, which generalizes the A-numerical radius and numerical radius. We present some interesting properties of $\omega_{(A,\nu)}(\cdot)$. Meanwhile, we derive upper and lower bounds for this numerical radius. Some inequalities obtained for $\omega_{(A,\nu)}(\cdot)$ will be matched with known inequalities for $\omega_A(\cdot)$. We first introduce the notions and terminologies.

Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space equipped with the norm $\|\cdot\|$, and let $B(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. We assume A is a positive operator on $\mathscr{H}$. The positive operator A induces semi-inner product $\langle x,y\rangle_A = \langle Ax,y\rangle$ for all $x,y \in \mathscr{H}$. Let $\|\cdot\|_A$ denote seminorm on $\mathscr{H}$, that is, $\|x\|_A = \sqrt{\langle x,x\rangle_A}$. About more, we refer readers to see [8, 11, 16].

For $T \in B(\mathscr{H})$, the null space of every operator T is denoted by $N(T)$, its range by $R(T)$. By $\overline{R(T)}$ we denote the norm closure of $R(T)$ in $\mathscr{H}$. For $T \in B(\mathscr{H})$, A-operator seminorm of T, denoted by $\|T\|_A$, is defined as

$$\|T\|_A = \sup_{x\in\overline{R(A)}, x\neq 0} \frac{\|Tx\|_A}{\|x\|_A}.$$

Here, given $T \in B(\mathscr{H})$, if there exists $c > 0$ such that $\|Tx\|_A \leqslant c\|x\|_A$ for all $x \in \overline{R(A)}$, then $\|T\|_A < \infty$. We set $B^A(\mathscr{H}) = \{T \in B(\mathscr{H}) : \|T\|_A < \infty\}$. Let $T \in B(\mathscr{H})$, an operator $R \in B(\mathscr{H})$ is called an A-adjoint of T if $\langle Tx,y\rangle_A = \langle x,Ry\rangle_A$ for every $x,y \in \mathscr{H}$, that is $AR = T^*A$. An operator $T \in B(\mathscr{H})$ is said to be A-selfadjoint if AT is selfadjoint, that is $AT = T^*A$, where T^* is the adjoint of T.

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The existence of an A-adjoint of T is not guaranteed. In fact, an operator $T \in B(\mathscr{H})$ may admit none, one or many A-adjoints. The set of all operators that admit A-adjoints is denoted by $B_A(\mathscr{H})$. By Douglas theorem [7], it follows that

$$B_A(\mathscr{H}) = \{T \in B(\mathscr{H}) : R(T^*A) \subseteq R(A)\}.$$

If $T \in B_A(\mathscr{H})$, then the operator equation $AX = T^*A$ has a unique solution, denoted by $T^{\sharp_A}$, satisfying $R(T^{\sharp_A}) \subseteq \overline{R(A)}$. Note that $T^{\sharp_A} = A^{\dagger}T^*A$, where $A^{\dagger}$ is the Moore-Penrose inverse of A and the A-adjoint operator $T^{\sharp_A}$ verifies

$$AT^{\sharp_A} = T^*A,\ R(T^{\sharp_A}) \subseteq \overline{R(A)} \text{ and } N(T^{\sharp_A}) = N(T^*A).$$

Notice that if $T \in B_A(\mathscr{H})$, then $T^{\sharp_A} \in B_A(\mathscr{H})$, $(T^{\sharp_A})^{\sharp_A} = PTP$ and $((T^{\sharp_A})^{\sharp_A})^{\sharp_A} = T^{\sharp_A}$, the P is the orthogonal projection onto $\overline{R(A)}$. For $T \in B_A(\mathscr{H})$, $TT^{\sharp_A}$ and $T^{\sharp_A}T$ are A-selfadjoint and A-positive, so we have

$$\|TT^{\sharp_A}\|_A = \|T^{\sharp_A}T\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2.$$

For more about this class of operators, we refer the interested readers to see [11, 16].

DEFINITION 1.1. Let $0 \leqslant \nu \leqslant 1$ and $T \in B_A(\mathscr{H})$. The weighted real and imaginary parts of T are defined as

$$\mathfrak{R}_{(A,\nu)}(T) = \nu T + (1-\nu)T^{\sharp_A} \text{ and } \mathfrak{I}_{(A,\nu)}(T) = \nu(-iT) + (1-\nu)iT^{\sharp_A},$$

respectively. When $\nu = \frac{1}{2}$, we can see that $\mathfrak{R}_{(A,\nu)}(T) = \mathfrak{R}_A(T)$ and $\mathfrak{I}_{(A,\nu)}(T) = \mathfrak{I}_A(T)$.

Some interesting relationships about $\mathfrak{R}_{(A,\nu)}(T)$, $\mathfrak{I}_{(A,\nu)}(T)$ and $\mathfrak{R}_A(T)$, $\mathfrak{I}_A(T)$ are as follows.

PROPOSITION 1.2. *Let $0 \leqslant \nu \leqslant 1$ and $T \in B_A(\mathscr{H})$. Then*

$$\mathfrak{R}_{(A,\nu)}(T) = \mathfrak{R}_A(T) + i(2\nu-1)\mathfrak{I}_A(T), \tag{1.1}$$

$$\mathfrak{I}_{(A,\nu)}(T) = \mathfrak{I}_A(T) - i(2\nu-1)\mathfrak{R}_A(T) \tag{1.2}$$

and

$$\mathfrak{R}_{(A,\nu)}(T) + i\mathfrak{I}_{(A,\nu)}(T) = 2\nu T. \tag{1.3}$$

The A-numerical radius and the A-Crawford number of $T \in B(\mathscr{H})$ are defined by

$$\omega_A(T) = \sup\{|\langle Tx,x\rangle_A| : x \in \mathscr{H}, \|x\|_A = 1\}$$

and

$$c_A(T) = \inf\{|\langle Tx,x\rangle_A| : x \in \mathscr{H}, \|x\|_A = 1\}.$$

For $T \in B_A(\mathscr{H})$, it is well-known that A-numerical radius of T is equivalent to A-operator seminorm of T, (see [16]), satisfying the following inequality:

$$\frac{1}{2}\|T\|_A \leqslant \omega_A(T) \leqslant \|T\|_A. \tag{1.4}$$

In [16], Zamani proved that

$$\omega_A(T) = \sup_{\theta\in\mathbb{R}} \|\Re_A(e^{i\theta}T)\|_A. \tag{1.5}$$

Recently, this identity (1.5) has been widely used, some novel A-numerical radius inequalities improved were found. There are some refinements of the inequalities (1.4) in references [6, 8, 11, 12, 16]. For example, in [16], let $T \in B_A(\mathscr{H})$, Zamani proved that

$$\omega_A(T) \leqslant \frac{1}{2}\sqrt{\|TT^{\sharp_A} + T^{\sharp_A}T\|_A + 2\omega_A(T^2)} \leqslant \|T\|_A. \tag{1.6}$$

Another refinement of the inequalities (1.4) has been established in [6], Bhunia computed some inequalities for $\mathbb{A}$-numerical radius of 2×2 operator matrices, where $\mathbb{A} = \begin{pmatrix} A & O \\ O & A \end{pmatrix}$. Let $X, Y \in B_A(\mathscr{H})$, some results are as follows,

$$\omega_{\mathbb{A}}^2\begin{pmatrix} O & X \\ Y & 0 \end{pmatrix} \geqslant \frac{1}{4}\max\{\|XX^{\sharp_A} + Y^{\sharp_A}Y\|_A, \|X^{\sharp_A}X + YY^{\sharp_A}\|_A\} \tag{1.7}$$

and

$$\omega_{\mathbb{A}}^2\begin{pmatrix} O & X \\ Y & 0 \end{pmatrix} \leqslant \frac{1}{2}\max\{\|XX^{\sharp_A} + Y^{\sharp_A}Y\|_A, \|X^{\sharp_A}X + YY^{\sharp_A}\|_A\}. \tag{1.8}$$

It should be remarked that many mathematicians have developed various inequalities about A-numerical radius and other results on numerical radius inequalities of 2×2 operator matrices, see, e.g., [1, 2, 3, 4, 9, 15, 17].

In this paper, motivated by [14], we define the weighted A-numerical radius, develop and generalize inequalities for the A-numerical radius of operators in $B_A(\mathscr{H})$. We obtain a generalization for inequality (1.6). Further, we derive the weighted $\mathbb{A}$-numerical radius inequalities of 2×2 operator matrices that refine the inequalities (1.7) and (1.8).

2. Weighted A-numerical radius inequalities

DEFINITION 2.1. Let $0 \leqslant \nu \leqslant 1$ and $T \in B_A(\mathscr{H})$. The weighted A-numerical radius of T denoted by $\omega_{(A,\nu)}(T)$, is defined as

$$\omega_{(A,\nu)}(T) = \sup_{\theta\in\mathbb{R}} \|\Re_{(A,\nu)}(e^{i\theta}T)\|_A.$$

For $T \in B_A(\mathscr{H})$, obviously, $\omega_{(A,\frac{1}{2})}(T) = \omega_A(T)$, and $\omega_{(A,0)}(T) = \omega_{A(1)}(T) = \|T\|_A$.

PROPOSITION 2.2. *Let* $0 \leqslant \nu \leqslant 1$ *and* $T \in B_A(\mathscr{H})$. *Then*

$$\omega_{(A,\nu)}(T) = \sup_{\theta\in\mathbb{R}} \|\mathfrak{I}_{(A,\nu)}(e^{i\theta}T)\|_A.$$

Proof. To prove this, we first prove that $\omega_{(A,\nu)}(\lambda T) = |\lambda|\,\omega_{(A,\nu)}(T)$ for all $\lambda \in \mathbb{C}$. For every nonzero $\lambda \in \mathbb{C}$, there exists $\varphi \in \mathbb{R}$ such that $\lambda = |\lambda|\,e^{i\varphi}$, we have

$$\begin{aligned}\omega_{(A,\nu)}(\lambda T) &= \sup_{\theta\in\mathbb{R}} \|\nu(e^{i\theta}\lambda T) + (1-\nu)(e^{-i\theta})\overline{\lambda}T^{\sharp_A})\|_A \\ &= \sup_{\theta\in\mathbb{R}} \|\nu e^{i\theta}|\lambda|e^{i\varphi}T + (1-\nu)e^{-i\theta}|\lambda|e^{-i\varphi}T^{\sharp_A})\|_A \\ &= |\lambda|\sup_{\theta\in\mathbb{R}} \|\nu(e^{i(\theta+\varphi)}T) + (1-\nu)(e^{-i(\theta+\varphi)})T^{\sharp_A})\|_A \\ &= |\lambda|\,\omega_{(A,\nu)}(T).\end{aligned}$$

Then by replacing T by iT in $\omega_{(A,\nu)}(T) = \sup_{\theta\in\mathbb{R}} \|\mathfrak{R}_{(A,\nu)}(e^{i\theta}T)\|_A$, we have

$$\begin{aligned}\omega_{(A,\nu)}(T) &= \sup_{\theta\in\mathbb{R}} \|\nu e^{i\theta}iT - (1-\nu)e^{-i\theta}iT^{\sharp_A}\|_A \\ &= \sup_{\theta\in\mathbb{R}} \|\mathfrak{I}_{(A,\nu)}(e^{i\theta}T)\|_A. \quad \square\end{aligned}$$

THEOREM 2.3. *Let* $0 \leqslant \nu \leqslant 1$ *and* $T \in B_A(\mathscr{H})$. *Then for* $\alpha,\beta \in \mathbb{R}$,

$$\omega_{(A,\nu)}(T) = \sup_{\alpha^2+\beta^2=1} \|\alpha\mathfrak{R}_{(A,\nu)}(T) + \beta\mathfrak{I}_{(A,\nu)}(T)\|_A.$$

Proof. Let $\theta \in \mathbb{R}$. Put $\alpha = \cos\theta$ and $\beta = \sin\theta$. Then

$$\begin{aligned}\nu e^{i\theta}T + (1-\nu)e^{-i\theta}T^{\sharp_A} &= \nu(\cos\theta + i\sin\theta)T + (1-\nu)(\cos\theta - i\sin\theta)T^{\sharp_A} \\ &= \cos\theta(\nu T + (1-\nu)T^{\sharp_A}) - \sin\theta(-\nu iT + (1-\nu)iT^{\sharp_A}) \\ &= \cos\theta\mathfrak{R}_{(A,\nu)}(T) - \sin\theta\mathfrak{I}_{(A,\nu)}(T).\end{aligned}$$

Therefore,

$$\sup_{\theta\in\mathbb{R}} \|\nu e^{i\theta}T + (1-\nu)e^{-i\theta}T^{\sharp_A}\|_A = \sup_{\alpha^2+\beta^2=1} \|\alpha\mathfrak{R}_{(A,\nu)}(T) + \beta\mathfrak{I}_{(A,\nu)}(T)\|_A.$$

Hence, we obtain the desired result by Definition 2.1. $\square$

PROPOSITION 2.4. *Let* $T \in B_A(\mathscr{H})$, $0 \leqslant \nu \leqslant 1$ *and* $\gamma = \max\{\nu, 1-\nu\}$. *Then*
(*a*) $\omega_{(A,\nu)}(T) = \omega_{(A,\nu)}(T^{\sharp_A})$,
(*b*) $\omega_{(A,\nu)}(T) = \omega_{(A,1-\nu)}(T)$,
(*c*) $\gamma\,\|T\|_A \leqslant \omega_{(A,\nu)}(T) \leqslant \|T\|_A$,
(*d*) $\omega_A(T) \leqslant \omega_{(A,\nu)}(T) \leqslant 2\gamma\,\omega_A(T)$.

Proof. (a) and (b) can be easily obtained by definition and the properties of the A-seminorm.

Next, we prove (c), by Definition 2.1 and the triangle inequality, then $\omega_{(A,\nu)}(T) \leqslant \|T\|_A$. Another, we have

$$\omega_{(A,\nu)}(T) = \sup_{\alpha^2+\beta^2=1} \|\alpha \Re_{(A,\nu)}(T) + \beta \Im_{(A,\nu)}(T)\|_A.$$

Letting $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$, we get that

$$\omega_{(A,\nu)}(T) \geqslant \|\Re_{(A,\nu)}(T)\|_A \text{ and } \omega_{(A,\nu)}(T) \geqslant \|\Im_{(A,\nu)}(T)\|_A.$$

By adding two inequalities, and the triangle inequality, we obtain

$$2\omega_{(A,\nu)}(T) \geqslant \|\Re_{(A,\nu)}(T)\|_A + \|\Im_{(A,\nu)}(T)\|_A \geqslant 2\nu\|T\|_A.$$

Then by replacing ν by $(1-\nu)$, we get

$$\omega_{(A,\nu)}(T) \geqslant (1-\nu)\|T\|_A.$$

We start proving (d), for $\theta \in \mathbb{R}$, let $x \in \mathscr{H}$ with $\|x\|_A = 1$ and $\frac{1}{2} \leqslant \nu \leqslant 1$, then

$$\begin{aligned}
&\|\big(\Re_A(e^{i\theta}T) + i(2\nu-1)\Im_A(e^{i\theta}T)\big)x\|_A^2 \\
&= \|\Re_A(e^{i\theta}T)x\|_A^2 + (2\nu-1)^2\|\Im_A(e^{i\theta}T)x\|_A^2 + 2(2\nu-1)\Re\Big(\Big\langle \Re_A(e^{i\theta}T)x, i\Im_A(e^{i\theta}T)x\Big\rangle_A\Big) \\
&\leqslant \|\Re_A(e^{i\theta}T)x\|_A^2 + (2\nu-1)^2\|\Im_A(e^{i\theta}T)x\|_A^2 + 2(2\nu-1)\Big|\Big\langle \Re_A(e^{i\theta}T)x, \Im_A(e^{i\theta}T)x\Big\rangle_A\Big| \\
&\leqslant \|\Re_A(e^{i\theta}T)\|_A^2 + (2\nu-1)^2\|\Im_A(e^{i\theta}T)\|_A^2 + 2(2\nu-1)\|\Re_A(e^{i\theta}T)\|_A\|\Im_A(e^{i\theta}T)\|_A \\
&\leqslant (2\nu\omega_A(T))^2.
\end{aligned}$$

Taking the supremum over all $\|x\|_A = 1$ and $\theta \in \mathbb{R}$, together with (1.1), we get

$$\omega_{(A,\nu)}(T) \leqslant 2\nu\omega_A(T).$$

If $0 \leqslant \nu \leqslant \frac{1}{2}$, then, $\frac{1}{2} \leqslant 1-\nu \leqslant 1$, we can get $\omega_{(A,\nu)}(T) \leqslant 2(1-\nu)\omega_A(T)$. Therefore, $\omega_{(A,\nu)}(T) \leqslant 2\gamma\,\omega_A(T)$.

On the other hand, similar to the method in the literature [14], first we prove that $f(\nu) = \omega_{(A,\nu)}(T)$ is a convex continuous function on $[0,1]$. For $0 \leqslant \nu_1, \nu_2, \lambda \leqslant 1$. Then

$$\begin{aligned}
f(\lambda\nu_1 + (1-\lambda)\nu_2) &= \omega_{(A,\lambda\nu_1+(1-\lambda)\nu_2)}(T) \\
&= \sup_{\theta\in\mathbb{R}} \|(\lambda\nu_1 + (1-\lambda)\nu_2)e^{i\theta}T + (1-\lambda\nu_1 - \nu_2 + \lambda\nu_2)e^{-i\theta}T^{\sharp_A}\|_A \\
&\leqslant \lambda \sup_{\theta\in\mathbb{R}} \|\nu_1 e^{i\theta}T + (1-\nu_1)e^{-i\theta}T^{\sharp_A}\|_A + (1-\lambda)\sup_{\theta\in\mathbb{R}} \|\nu_2 e^{i\theta}T + (1-\nu_2)e^{-i\theta}T^{\sharp_A}\|_A \\
&= \lambda\,\omega_{(A,\nu_1)}(T) + (1-\lambda)\omega_{(A,\nu_2)}(T) \\
&= \lambda f(\nu_1) + (1-\lambda)f(\nu_2).
\end{aligned}$$

Therefore, f is convex on $[0,1]$. By the property of convex function, f is continuous on $(0,1)$. According to (c), we have

$$0 \leqslant \|T\|_A - \omega_{(A,\nu)}(T) \leqslant \|T\|_A(1-\gamma).$$

Hence f is continuous at $\nu = 0$ and $\nu = 1$, i.e. it is continuous on $[0,1]$. Also we know $f(\nu) = f(1-\nu)$, f is symmetric about $\nu = \frac{1}{2}$. It means that f is decreasing on $[0,\frac{1}{2}]$ and increasing on $[\frac{1}{2},1]$. Therefore, we get that the minimum of f is $f(\frac{1}{2})$ and the maximum of f is $f(1)$ and $f(0)$. It shows that $\omega_A(T) \leqslant \omega_{(A,\nu)}(T)$. This completes the proof. □

THEOREM 2.5. *Let $T \in B_A(\mathscr{H})$, $0 \leqslant \nu \leqslant 1$ and $\gamma = \max\{\nu, 1-\nu\}$. Then we have*

$$\omega_{(A,\nu)}(T) \geqslant \gamma\|T\|_A + \sup_{\theta\in\mathbb{R}} \frac{\left|\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A - \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A\right|}{2}.$$

Proof. Let $\theta \in \mathbb{R}$. Then

$$\begin{aligned}\omega_{(A,\nu)}(T) &\geqslant \max\{\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A, \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A\} \\ &= \frac{\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A + \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A}{2} + \frac{\left|\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A - \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A\right|}{2} \\ &\geqslant \nu\|T\|_A + \frac{\left|\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A - \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A\right|}{2}.\end{aligned}$$

Furthermore, by replacing ν by $(1-\nu)$, we get

$$\omega_{(A,\nu)}(T) \geqslant (1-\nu)\|T\|_A + \frac{\left|\|\Re_{A(1-\nu)}(e^{i\theta}T)\|_A - \|\Im_{A(1-\nu)}(e^{i\theta}T)\|_A\right|}{2}.$$

We also have

$$\|\Re_{(A,1-\nu)}(e^{i\theta}T)\|_A = \|\Re_{(A,\nu)}(e^{i\theta}T)\|_A \quad \text{and} \quad \|\Im_{(A,1-\nu)}(e^{i\theta}T)\|_A = \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A.$$

Therefore,

$$\omega_{(A,\nu)}(T) \geqslant \max\{\nu, (1-\nu)\}\|T\|_A + \frac{\left|\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A - \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A\right|}{2}.$$

This completes the proof. □

COROLLARY 2.6. *Let $T \in B_A(\mathscr{H})$, $0 \leqslant \nu \leqslant 1$ and $\gamma = \max\{\nu, 1-\nu\}$. Then $\omega_{(A,\nu)}(T) = \gamma\|T\|_A$ if and only if $\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A = \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A = \gamma\|T\|_A$ for all $\theta \in \mathbb{R}$.*

Proof. According to Definition 2.1, the sufficient part is trivial, we only prove the necessary part. For $\nu \in [\frac{1}{2}, 1]$, by Theorem 2.5, $\omega_{(A,\nu)}(T) = \gamma \|T\|_A$, we get $\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A = \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A$, also

$$\begin{aligned}\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A &\leqslant \omega_{(A,\nu)}(T) = \nu\|T\|_A = \|\nu e^{i\theta}T\|_A \\ &= \left\|\frac{\Re_{(A,\nu)}(e^{i\theta}T) + i\Im_{(A,\nu)}(e^{i\theta}T)}{2}\right\|_A \leqslant \|\Re_{(A,\nu)}(e^{i\theta}T)\|_A\end{aligned}$$

for all $\theta \in \mathbb{R}$. For $\nu \in [0, \frac{1}{2}]$, $1-\nu \in [\frac{1}{2}, 1]$, similarly, we can prove the conclusion. $\square$

THEOREM 2.7. *Let $T \in B_A(\mathscr{H})$ and $0 \leqslant \nu \leqslant 1$. Then we have*

$$\omega_{(A,\nu)}(T) \leqslant \inf_{\theta\in\mathbb{R}} \sqrt{\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A^2 + \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A^2}.$$

Proof. For $\alpha, \beta \in \mathbb{R}$, replacing T by $e^{i\theta}T$ in Theorem 2.3, then

$$\begin{aligned}\omega_{(A,\nu)}(T) &= \sup_{\alpha^2+\beta^2=1} \|\alpha\Re_{(A,\nu)}(e^{i\theta}T) + \beta\Im_{(A,\nu)}(e^{i\theta}T)\|_A \\ &\leqslant \sup_{\alpha^2+\beta^2=1} \left(|\alpha|\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A + |\beta|\|\Im_{(A,\nu)}(e^{i\theta}T)\|_A\right).\end{aligned}$$

By the Cauchy-Schwarz inequality, we get that

$$\omega_{(A,\nu)}(T) \leqslant \sup_{\alpha^2+\beta^2=1} \left(\sqrt{|\alpha|^2+|\beta|^2}\sqrt{\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A^2 + \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A^2}\right).$$

Thus,

$$\omega_{(A,\nu)}(T) \leqslant \inf_{\theta\in\mathbb{R}} \sqrt{\|\Re_{(A,\nu)}(e^{i\theta}T)\|_A^2 + \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A^2}. \quad \square$$

THEOREM 2.8. *Let $T \in B_A(\mathscr{H})$ and $0 \leqslant \nu \leqslant 1$. Then we have*

$$\begin{aligned}&\omega_{(A,\nu)}^2(T) \\ &\geqslant \frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A + \frac{(2\nu-1)^2[c_A^2(\Im_A(e^{i\theta}T)) + c_A^2(\Re_A(e^{i\theta}T))]}{2} \\ &\quad + \frac{\left|\|\Re_A(e^{i\theta}T)\|_A^2 - \|\Im_A(e^{i\theta}T)\|_A^2 + (2\nu-1)^2[c_A^2(\Im_A(e^{i\theta}T)) - c_A^2(\Re_A(e^{i\theta}T))]\right|}{2}\end{aligned}$$

for all $\theta \in \mathbb{R}$.

Proof. Let $\theta \in \mathbb{R}$, we first prove the following two inequalities:

$$\begin{aligned}\omega_{(A,\nu)}^2(T) &\geqslant \|\Re_A(e^{i\theta}T)\|_A^2 + (2\nu-1)^2 c_A^2(\Im_A(e^{i\theta}T)), \\ \omega_{(A,\nu)}^2(T) &\geqslant \|\Im_A(e^{i\theta}T)\|_A^2 + (2\nu-1)^2 c_A^2(\Re_A(e^{i\theta}T)).\end{aligned}$$

Let $x \in \mathscr{H}$ with $\|x\|_A = 1$. Then we have

$$\begin{aligned}\sup_{\theta\in\mathbb{R}} \|\Re_{(A,\nu)}(e^{i\theta}T)\|_A^2 &\geqslant \left\|\left(Re_A(e^{i\theta}T) + i(2\nu-1)\Im_A(e^{i\theta}T)\right)x\right\|_A^2 \\ &\geqslant \left|\left\langle \left(\Re_A(e^{i\theta}T) + i(2\nu-1)\Im_A(e^{i\theta}T)\right)x, x\right\rangle_A\right|^2 \\ &= \left|\left\langle \Re_A(e^{i\theta}T)x, x\right\rangle_A\right|^2 + (2\nu-1)^2\left|\left\langle \Im_A(e^{i\theta}T)x, x\right\rangle_A\right|^2 \\ &\geqslant \left|\left\langle \Re_A(e^{i\theta}T)x, x\right\rangle_A\right|^2 + (2\nu-1)^2 c_A^2(\Im_A(e^{i\theta}T)).\end{aligned}$$

By taking the supremum over $x \in \mathscr{H}$ with $\|x\|_A = 1$, implies that

$$\omega_{(A,\nu)}^2(T) \geqslant \|\Re_A(e^{i\theta}T)\|_A^2 + (2\nu-1)^2 c_A^2(\Im_A(e^{i\theta}T)) = a.$$

Similarly, by using (1.2), we get that

$$\sup_{\theta\in\mathbb{R}} \|\Im_{(A,\nu)}(e^{i\theta}T)\|_A \geqslant \left\|\left(\Im_A(e^{i\theta}T) - i(2\nu-1)\Re_A(e^{i\theta}T)\right)x\right\|_A.$$

Then

$$\omega_{(A,\nu)}^2(T) \geqslant \|\Im_A(e^{i\theta}T)\|_A^2 + (2\nu-1)^2 c_A^2(\Re_A(e^{i\theta}T)) = b.$$

Therefore, we have

$$\begin{aligned}&\omega_{(A,\nu)}^2(T) \\ &\geqslant \max\{a,b\} \\ &= \frac{\|\Re_A(e^{i\theta}T)\|_A^2 + \|\Im_A(e^{i\theta}T)\|_A^2}{2} + \frac{(2\nu-1)^2[c_A^2(\Im_A(e^{i\theta}T)) + c_A^2(\Re_A(e^{i\theta}T))]}{2} \\ &\quad + \frac{\left|\|\Re_A(e^{i\theta}T)\|_A^2 - \|\Im_A(e^{i\theta}T)\|_A^2 + (2\nu-1)^2[c_A^2(\Im_A(e^{i\theta}T)) - c_A^2(\Re_A(e^{i\theta}T))]\right|}{2} \\ &\geqslant \frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A + \frac{(2\nu-1)^2[c_A^2(\Im_A(e^{i\theta}T)) + c_A^2(\Re_A(e^{i\theta}T))]}{2} \\ &\quad + \frac{\left|\|\Re_A(e^{i\theta}T)\|_A^2 - \|\Im_A(e^{i\theta}T)\|_A^2 + (2\nu-1)^2[c_A^2(\Im_A(e^{i\theta}T)) - c_A^2(\Re_A(e^{i\theta}T))]\right|}{2}.\end{aligned}$$

This completes the proof. $\square$

COROLLARY 2.9. *Let $T \in B_A(\mathscr{H})$. Then $\omega_A^2(T) = \frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A$ if and only if $\|\Re_A(e^{i\theta}T)\|_A^2 = \|\Im_A(e^{i\theta}T)\|_A^2 = \frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A$ for all $\theta \in \mathbb{R}$.*

Proof. The sufficient part is trivial, we only prove the necessary part. Taking $\nu = \frac{1}{2}$ in Theorem 2.8, if $\omega_A^2(T) = \frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A$, then $\|\Re_A(e^{i\theta}T)\|_A^2 = \|\Im_A(e^{i\theta}T)\|_A^2$

for all $\theta \in \mathbb{R}$. Also, we have $(\Re_A(e^{i\theta}T))^2 + (\Im_A(e^{i\theta}T))^2 = \frac{T^{\sharp_A}T + TT^{\sharp_A}}{2}$. Then

$$\begin{aligned}\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A &= \frac{1}{2}\|(\Re_A(e^{i\theta}T))^2 + (\Im_A(e^{i\theta}T))^2\|_A \\ &\leqslant \frac{1}{2}(\|\Re_A(e^{i\theta}T)\|_A^2 + \|\Im_A(e^{i\theta}T)\|_A^2) \\ &\leqslant \frac{1}{2}(\omega_A^2(T) + \omega_A^2(T)) \\ &= \frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A.\end{aligned}$$

Thus, $\|\Re_A(e^{i\theta}T)\|_A^2 = \|\Im_A(e^{i\theta}T)\|_A^2 = \frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A$ for all $\theta \in \mathbb{R}$. $\square$

REMARK 2.10. Very recently, as our work in progress, for an arbitrary norm $N(\cdot)$ on $B(H)$ and $0 \leqslant \nu \leqslant 1$, Zamani [18] defined the $w_{(N,\nu)}(\cdot)$ as a generalization of the weighted numerical radius. Mabrouk and Zamani [10] introduced an extension of the a-numerical radius on C^*-algebra. Theorem 2.5 (i) in [10] is an extension of Theorem 2.3. Theorem 2.6 in [18] and [10] are extensions of Proposition 2.4, respectively. Our approach here is different from theirs.

3. Weighted A-numerical radius inequalities for 2×2 operator matrices

To prove our results, we begin with the following results.

LEMMA 3.1. [5, 6] *Let $T, S, X, Y \in B_A(\mathscr{H})$. Then*

$$(i) \qquad \begin{pmatrix} T & X \\ Y & S \end{pmatrix}^{\sharp_{\mathbb{A}}} = \begin{pmatrix} T^{\sharp_A} & Y^{\sharp_A} \\ X^{\sharp_A} & S^{\sharp_A} \end{pmatrix}.$$

$$(ii) \qquad \left\|\begin{pmatrix} T & O \\ O & S \end{pmatrix}\right\|_{\mathbb{A}} = \left\|\begin{pmatrix} O & T \\ S & O \end{pmatrix}\right\|_{\mathbb{A}} = \max\{\|T\|_A, \|S\|_A\}.$$

LEMMA 3.2. [13] *Let $X, Y \in B_A(\mathscr{H})$. Then*

$$(i) \qquad \omega_{\mathbb{A}}\begin{pmatrix} O & X \\ Y & O \end{pmatrix} = \omega_{\mathbb{A}}\begin{pmatrix} O & Y \\ X & O \end{pmatrix}.$$

$$(ii) \qquad \omega_{\mathbb{A}}\begin{pmatrix} X & Y \\ Y & X \end{pmatrix} = \max\{\omega_A(X+Y),\ \omega_A(X-Y)\}.$$

In particular $\omega_{\mathbb{A}}\begin{pmatrix} O & Y \\ Y & O \end{pmatrix} = \omega_A(Y)$.

THEOREM 3.3. *Let $X, Y \in B_A(\mathscr{H})$. Then we have*

$$\begin{aligned}\omega_{(\mathbb{A},\nu)}^2\begin{pmatrix} O & X \\ Y & O \end{pmatrix} \leqslant \max\big\{&\|\nu^2 XX^{\sharp_A} + (1-\nu)^2 Y^{\sharp_A}Y\|_A + 2\nu(1-\nu)\omega_A(XY), \\ &\|\nu^2 YY^{\sharp_A} + (1-\nu)^2 X^{\sharp_A}X\|_A + 2\nu(1-\nu)\omega_A(YX)\big\}.\end{aligned}$$

In particular, if $\nu = \frac{1}{2}$, then

$$\omega_{\mathbb{A}}^2 \begin{pmatrix} O & X \\ Y & O \end{pmatrix} \leqslant \max\Big\{\frac{1}{4}\|XX^{\sharp_A} + Y^{\sharp_A}Y\|_A + \frac{1}{2}\omega_A(XY), \tag{3.1}$$
$$\frac{1}{4}\|X^{\sharp_A}X + YY^{\sharp_A}\|_A + \frac{1}{2}\omega_A(YX)\Big\}. \tag{3.2}$$

Proof. Let $T = \begin{pmatrix} O & X \\ Y & O \end{pmatrix}$. By Lemma 3.1, we have

$$\begin{aligned}\omega_{(\mathbb{A},\nu)}^2 \begin{pmatrix} O & X \\ Y & O \end{pmatrix} &= \sup_{\theta\in\mathbb{R}} \left\| \begin{pmatrix} O & \nu e^{i\theta}X + (1-\nu)e^{-i\theta}Y^{\sharp_A} \\ \nu e^{i\theta}Y + (1-\nu)e^{-i\theta}X^{\sharp_A} & O \end{pmatrix} \right\|_A^2 \\ &= \sup_{\theta\in\mathbb{R}} \left\| \begin{pmatrix} P & O \\ O & Q \end{pmatrix} \right\|_A,\end{aligned}$$

where

$$\begin{aligned} P &= \nu^2 XX^{\sharp_A} + (1-\nu)^2 Y^{\sharp_A}(Y^{\sharp_A})^{\sharp_A} + 2\nu(1-\nu)\Re_A(e^{2i\theta}X(Y^{\sharp_A})^{\sharp_A}), \\ Q &= \nu^2 YY^{\sharp_A} + (1-\nu)^2 X^{\sharp_A}(X^{\sharp_A})^{\sharp_A} + 2\nu(1-\nu)\Re_A(e^{2i\theta}Y(X^{\sharp_A})^{\sharp_A}). \end{aligned}$$

Then, we can get

$$\begin{aligned} \|P\|_A &\leqslant \|\nu^2 XX^{\sharp_A} + (1-\nu)^2 Y^{\sharp_A}Y\|_A + 2\nu(1-\nu)\|\Re_A(e^{2i\theta}X(Y^{\sharp_A})^{\sharp_A})\|_A \\ &\leqslant \|\nu^2 XX^{\sharp_A} + (1-\nu)^2 Y^{\sharp_A}Y\|_A + 2\nu(1-\nu)\omega_A(XY). \end{aligned}$$

$$\begin{aligned} \|Q\|_A &\leqslant \|\nu^2 YY^{\sharp_A} + (1-\nu)^2 X^{\sharp_A}X\|_A + 2\nu(1-\nu)\|\Re_A(e^{2i\theta}Y(X^{\sharp_A})^{\sharp_A})\|_A \\ &\leqslant \|\nu^2 YY^{\sharp_A} + (1-\nu)^2 X^{\sharp_A}X\|_A + 2\nu(1-\nu)\omega_A(YX). \end{aligned}$$

By using Lemma 3.1, we have

$$\omega_{(\mathbb{A},\nu)}^2 \begin{pmatrix} O & X \\ Y & O \end{pmatrix} = \sup_{\theta\in\mathbb{R}} \max\{\|P\|_A, \|Q\|_A\}.$$

In conclusion, we obtain the desired inequality. □

REMARK 3.4. Letting $\nu = \frac{1}{2}$ and $X = Y = T$ in Theorem 3.3, and by Lemma 3.2, we get the inequality (1.6) proved by Zamani in [16]. Together with (3.1) and (3.2), we can see that the bound provided in Theorem 3.3 is sharper than (1.8) given in [6].

THEOREM 3.5. *Let $X, Y \in B_A(\mathscr{H})$. Then we have*

$$\begin{aligned}\omega_{(\mathbb{A},\nu)}^2 \begin{pmatrix} O & X \\ Y & O \end{pmatrix} \geqslant \max\big\{&\|\nu^2 XX^{\sharp_A} + (1-\nu)^2 Y^{\sharp_A}Y\|_A + 2\nu(1-\nu)c_A(XY), \\ &\|\nu^2 YY^{\sharp_A} + (1-\nu)^2 X^{\sharp_A}X\|_A + 2\nu(1-\nu)c_A(YX)\big\}.\end{aligned}$$

In particular, if $\nu = \frac{1}{2}$, *then*

$$\omega_{\mathbb{A}}^2 \begin{pmatrix} O & X \\ Y & O \end{pmatrix} \geqslant \max\Big\{ \frac{1}{4}\|XX^{\sharp_A} + Y^{\sharp_A}Y\|_A + \frac{1}{2}c_A(XY), \tag{3.3}$$

$$\frac{1}{4}\|X^{\sharp_A}X + YY^{\sharp_A}\|_A + \frac{1}{2}c_A(YX) \Big\}. \tag{3.4}$$

Proof. From the proof of Theorem 3.3 we know that

$$\omega_{(\mathbb{A},\nu)}^2 \begin{pmatrix} O & X \\ Y & O \end{pmatrix} = \sup_{\theta\in\mathbb{R}} \max\{\|P\|_A, \|Q\|_A\}.$$

Here, P and Q are the same as Theorem 3.3.

Let $x \in \mathscr{H}$ with $\|x\|_A = 1$. For all $\theta \in \mathbb{R}$, we have that

$$\begin{aligned}\|P\|_A &\geqslant |\langle Px, x\rangle_A| \\ &= \left|\left\langle \left(\nu^2 XX^{\sharp_A} + (1-\nu)^2 Y^{\sharp_A}(Y^{\sharp_A})^{\sharp_A} + 2\nu(1-\nu)\Re_A(e^{2i\theta}X(Y^{\sharp_A})^{\sharp_A})\right)x, x\right\rangle_A\right|.\end{aligned}$$

We assume that

$$\left\langle e^{2i\theta_0}X(Y^{\sharp_A})^{\sharp_A}x, x\right\rangle_A = \left|\left\langle X(Y^{\sharp_A})^{\sharp_A}x, x\right\rangle_A\right|.$$

Thus, by replacing θ by θ_0 in the above formula, we get

$$\begin{aligned}\|P\|_A &\geqslant \left|\left\langle \nu^2 XX^{\sharp_A} + (1-\nu)^2 Y^{\sharp_A}(Y^{\sharp_A})^{\sharp_A}x, x\right\rangle_A\right| + 2\nu(1-\nu)\left|\left\langle X(Y^{\sharp_A})^{\sharp_A}x, x\right\rangle_A\right| \\ &\geqslant \left|\left\langle \nu^2 XX^{\sharp_A} + (1-\nu)^2 Y^{\sharp_A}(Y^{\sharp_A})^{\sharp_A}x, x\right\rangle_A\right| + 2\nu(1-\nu)c_A(YX).\end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\|_A = 1$ in the above inequality, we get

$$\|P\|_A \geqslant \|\nu^2 XX^{\sharp_A} + (1-\nu)^2 Y^{\sharp_A}Y\|_A + 2\nu(1-\nu)c_A(XY).$$

In a similar way, we can get

$$\|Q\|_A \geqslant \|\nu^2 YY^{\sharp_A} + (1-\nu)^2 X^{\sharp_A}X\|_A + 2\nu(1-\nu)c_A(YX).$$

Thus, we complete the proof. $\square$

REMARK 3.6. Taking $\nu = \frac{1}{2}$ in Theorem 3.5, the inequalities (3.3) and (3.4) improve the inequality (1.7) obtained in [6].

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