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COWEN SETS FOR TOEPLITZ OPERATORS

HYOUNG JOON KIM AND JAEHUI PARK*

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Abstract. The Cowen set for $\varphi \in L^\infty$ is defined by

$$\mathscr{E}(\varphi) = \left\{k \in H^\infty : \|k\|_\infty \leqslant 1,\, \varphi - k\overline{\varphi} \in H^\infty\right\}.$$

It is known that the Toeplitz operator T_φ is hyponormal if and only if $\mathscr{E}(\varphi)$ is nonempty. In this paper, we study various properties of Cowen sets. Especially, we investigate in detail the case when $\mathscr{E}(\varphi)$ contains a constant and find a condition where $\mathscr{E}(\varphi)$ consists of exactly one constant.

1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane, and $\mathbb{T} = \partial\mathbb{D}$ the unit circle. Let $L^p = L^p(\mathbb{T})$ be the Lebesgue space on $\mathbb{T}$ and let $H^p = H^p(\mathbb{T})$ be the corresponding Hardy space, for $1 \leqslant p \leqslant \infty$. We also use the notation H_0^p for $zH^p = \{f \in H^2 : \hat{f}(0) = 0\}$, and $(H^\infty)_1$ the closed unit ball H^∞. For $\varphi \in L^\infty$, the Toeplitz operator T_φ with symbol φ is the bounded linear operator on H^2 defined by

$$T_\varphi f = P(\varphi f) \qquad (f \in H^2),$$

where P is the orthogonal projection of L^2 onto H^2. The Hankel operator with symbol φ is the operator H_φ on H^2 defined by $H_\varphi f = J(I-P)(\varphi f)$ for $f \in H^2$, where $J : L^2 \to L^2$ is given by $Jf(z) = \overline{z}f(\overline{z})$. Note that $J^*J = J^2 = I$, and J maps $(H^2)^\perp$ onto H^2.

Brown and Halmos [2] studied various algebraic properties of Toeplitz operators, and in particular showed that the Toeplitz operator T_φ is normal (i.e., $T_\varphi^*T_\varphi = T_\varphi T_\varphi^*$) if and only if $\varphi = \alpha + \beta\rho$ where α and β are complex numbers and ρ is a real-valued function in L^∞.

The next topic to be studied in this direction was subnormality and hyponormality of Toeplitz operators. A bounded linear operator T is hyponormal if its self-commutator $[T^*, T] = T^*T - TT^*$ is positive semidefinite. A subnormal operator is the restriction of a normal operator N to an invariant subspace for N. A Toeplitz operator T_φ is said to be analytic if its symbol φ is analytic. Every analytic Toeplitz operator is subnormal, because it is the rectriction of the multiplication by its symbol on L^2 to

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* Corresponding author.

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H^2, and every multiplication operator on L^2 is normal. On the other hand, every normal Toeplitz operator is subnormal. Therefore, the following question raised by Halmos [9] was a natural one:

Is every subnormal Toeplitz operator either analytic or normal?

A function on $\mathbb{T}$ is said to be of bounded type if it can be written as the quotient of two H^∞-functions. Abrahamse [1] proved that if φ or $\overline{\varphi}$ is of bounded type, the answer to Halmos's question is yes. The class of functions of bounded type is quite large, so the result of Abrahamse is a fairly general sufficient condition for the answer to Halmos's question to be yes. Thus, the answer to Halmos's question seemed almost true. But surprisingly, Cowen and Long [6] answered negatively Halmos's question by giving a concrete example of a subnormal Toeplitz operator that is neither analytic nor normal.

Subsequent studies on subnormality and hyponormality of Toeplitz operators can be found in [3], [4], and [5]. One of the notable results is the following characterization of symbols of hyponormal Toeplitz operators demonstrated by Cowen [5]: For $\varphi = f+\overline{g} \in L^\infty$ with $f,g \in H^2$, the Toeplitz operator T_φ is hyponormal if and only if there exists a function k_0 in the closed unit ball $(H^\infty)_1$ such that

$$H_{\overline{g}} = T^*_{k_0} H_{\overline{f}}.$$

Note that $T^*_{k_0} H_{\overline{f}} = H_{\widetilde{k}_0 \overline{f}}$, where $\widetilde{k}_0(z) = \overline{k_0(\overline{z})}$. Hence $H_{\overline{g}} = T^*_{k_0} H_{\overline{f}}$ if and only if $\overline{g} - \widetilde{k}_0\overline{f}$ is analytic, or equivalently, $\varphi - \widetilde{k}_0\overline{\varphi}$ is analytic. This shows the following theorem:

COWEN'S THEOREM. [5], [11] *For $\varphi \in L^\infty$, T_φ is hyponormal if and only if the set*

$$\mathscr{E}(\varphi) = \left\{ k \in (H^\infty)_1 : \varphi - k\overline{\varphi} \in H^\infty \right\}$$

is nonempty.

The set $\mathscr{E}(\varphi)$ was introduced by Nakazi and Takahashi [11]. We call $\mathscr{E}(\varphi)$ the *Cowen set* for φ. Lee [10] has given a complete description on the Cowen set $\mathscr{E}(\varphi)$ when the selfcommutator of T_φ has finite rank. The purpose of this paper is to study the various properties of Cowen sets. In particular, we study the shape of the Cowen set $\mathscr{E}(\varphi)$ in detail when it contains a constant function. There are two extreme cases: (i) $0 \in \mathscr{E}(\varphi)$ and (ii) $e^{it} \in \mathscr{E}(\varphi)$. These correspond to the cases that T_φ is analytic and that T_φ is normal, respectively.

Before we begin, we introduce one notation we will use throughout this paper. Note that an L^∞-function φ is of bounded type if and only if $\ker H_\varphi \neq \{0\}$ [1]. Since $\ker H_\varphi$ is a shift-invariant subspace of H^2, it follows from the Beurling's theorem that

$$\ker H_\varphi = \theta H^2 \tag{1}$$

for some inner function θ. In this case we have $H_\varphi \theta = 0$, which implies that $\varphi\theta \in H^\infty$. Thus

$$\varphi = a\overline{\theta} \quad \text{a.e. on } \mathbb{T}$$

for some $a \in H^\infty$. Moreover, the functions a and θ are coprime, i.e., the only inner functions that divide both a and θ are constants. Indeed, if θ' is an inner function that divides both a and θ, then $\varphi(\theta/\theta') = a/\theta' \in H^\infty$, and hence $\theta/\theta' \in \ker H_\varphi = \theta H^2$, which implies that θ' is a constant. The inner function θ in (1) is uniquely determined up to a constant multiple. We denote it by θ_φ. Note that θ_φ is constant if and only if φ is analytic. For convenience, if φ is not of bounded type, we define $\theta_\varphi = 0$. Then the equation (1) can be rewritten as

$$\ker H_\varphi = \theta_\varphi H^2. \tag{2}$$

The relationship between φ and θ_φ is one of the interesting research subjects in view of the interaction between function theory and operator theory. A related study in the case where φ is a matrix-valued function can be found in [8].

In this paper we describe the shape of the Cowen set $\mathscr{E}(\varphi)$ by using θ_φ. In section 2, we consider properties of the Cowen sets and the interpolating sets. In section 3, we consider the Cowen sets for the cases of containing a constant function.

2. Cowen sets and interpolaing sets

In this section we study several properties of the Cowen sets. To investigate Cowen sets in more detail, we introduce the interpolaing set

$$\mathscr{C}(\varphi) = \{k \in H^2 : \varphi - k\overline{\varphi} \in H^2\},$$

for $\varphi \in L^\infty$. Then

$$\mathscr{E}(\varphi) = \mathscr{C}(\varphi) \cap (H^\infty)_1. \tag{3}$$

Thus information about $\mathscr{C}(\varphi)$ will tell us what $\mathscr{E}(\varphi)$ is.

We begin with examples of Cowen sets for relatively simple functions.

EXAMPLE 2.1. (a) Let $\varphi = z^2 + \overline{z}$. For $k \in H^2$, $k \in \mathscr{C}(\varphi)$ if and only if $\overline{z} - k\overline{z}^2 \in H^2$, that is $k = z + z^2 h$ for some $h \in H^2$. It follows that

$$\mathscr{C}(\varphi) = z + z^2 H^2.$$

If $k = z + z^2 h$ belongs to $\mathscr{E}(\varphi)$, then $1 \leqslant \|k\|_2 \leqslant \|k\|_\infty \leqslant 1$, and hence $h = 0$, i.e., $k = z$. Thus we have

$$\mathscr{E}(\varphi) = \{z\}.$$

(b) Let us look at an example of the Cowen set for a slightly more general function. Suppose that

$$\varphi = z^m + c\overline{z}^n,$$

where $m, n \geqslant 1$ and $c \in \mathbb{C}$. As above we can show that for $k \in H^2$, $k \in \mathscr{C}(\varphi)$ if and only if $k = cz^{m-n} + z^m h$ for some $h \in H^2$. Thus if $m < n$ then $\mathscr{C}(\varphi)$ is empty and so is $\mathscr{E}(\varphi)$. If $m \geqslant n$, then

$$\mathscr{C}(\varphi) = cz^{m-n} + z^m H^2.$$

In this case, $\|k\|_2 \geqslant |c|$ for every $k \in \mathscr{C}(\varphi)$. Thus $\mathscr{E}(\varphi)$ is nonempty if and only if $|c| \leqslant 1$. If $|c| = 1$, then $\mathscr{E}(\varphi) = \{cz^{m-n}\}$. On the other hand, if $|c| < 1$, then $\mathscr{E}(\varphi)$ contains infinitely many functions, e.g., $cz^{m-n} + dz^{m+k}$ for any $k \geqslant 0$ and $d \in \mathbb{C}$ with $|c| + |d| \leqslant 1$.

By Cowen's theorem, T_φ is hyponormal if and only if $m \geqslant n$ and $|c| \leqslant 1$.

Now we investigate some basic properties that will simplify our discussion.

LEMMA 2.2. *Let $\varphi \in L^\infty$.*

(a) *If $\alpha, \beta \in \mathbb{C}$ and $\beta \neq 0$, then $\mathscr{C}(\alpha + \beta\varphi) = (\beta/\overline{\beta})\mathscr{C}(\varphi)$.*

(b) *If k_1 and k_2 are two functions contained in $\mathscr{C}(\varphi)$, then $k_1 - k_2 \in \ker H_{\overline{\varphi}}$. Thus if $k_0 \in \mathscr{C}(\varphi)$, then*

$$\mathscr{C}(\varphi) = k_0 + \ker H_{\overline{\varphi}} \equiv \{k_0 + h : h \in \ker H_{\overline{\varphi}}\}. \tag{4}$$

(c) *If $k_1, k_2 \in \mathscr{C}(\varphi)$ and $k_1 - k_2$ is a nonzero constant, then φ is constant.*

Proof. (a) Observe that for $k \in H^2$,

$$(\alpha + \beta\varphi) - k\overline{(\alpha + \beta\varphi)} = (\alpha - \overline{\alpha}k) + \beta(\varphi - (\overline{\beta}/\beta)k\overline{\varphi}).$$

Thus $k \in \mathscr{C}(\alpha + \beta\varphi)$ if and only if $(\overline{\beta}/\beta)k \in \mathscr{C}(\varphi)$. This implies that $\mathscr{C}(\alpha + \beta\varphi) = (\beta/\overline{\beta})\mathscr{C}(\varphi)$.

(b) Suppose that k_1 and k_2 are two functions in $\mathscr{C}(\varphi)$. Then both $\varphi - k_1\overline{\varphi}$ and $\varphi - k_2\overline{\varphi}$ are analytic, and so is their difference $(k_1 - k_2)\overline{\varphi}$. It follows that

$$H_{\overline{\varphi}}(k_1 - k_2) = J(I - P)((k_1 - k_2)\overline{\varphi}) = 0.$$

Thus $k_1 - k_2 \in \ker H_{\overline{\varphi}}$.

Suppose now that $k_0 \in \mathscr{C}(\varphi)$. If $k \in \mathscr{C}(\varphi)$, then $k - k_0 \in \ker H_{\overline{\varphi}}$ by the preceding paragraph. Thus $k = k_0 + (k - k_0) \in k_0 + \ker H_{\overline{\varphi}}$. This implies that

$$\mathscr{C}(\varphi) \subseteq k_0 + \ker H_{\overline{\varphi}}.$$

Conversely, if $k \in k_0 + \ker H_{\overline{\varphi}}$, then $H_{\overline{\varphi}}(k - k_0) = 0$, and hence $(k - k_0)\overline{\varphi}$ is analytic. It follows that $\varphi - k\overline{\varphi} = (\varphi - k_0\overline{\varphi}) - (k - k_0)\overline{\varphi} \in H^2$, and so $k \in \mathscr{C}(\varphi)$. This gives the reverse inclusion, and completes the proof of (4).

(c) Let $c = k_1 - k_2$. Then c is a nonzero constant belonging to $\ker H_{\overline{\varphi}}$, by (b). Thus $(I - P)(c\overline{\varphi}) = 0$, which implies that $\overline{\varphi}$ is analytic. Since $k_1 \in \mathscr{C}(\varphi) \subseteq H^2$, both $\varphi - k_1\overline{\varphi}$ and $k_1\overline{\varphi}$ are analytic. It follows that φ is also analytic. Therefore φ is constant, and the proof is complete. $\square$

Then we have:

THEOREM 2.3. *Let* $\varphi \in L^\infty$.

(a) *If* $\alpha, \beta \in \mathbb{C}$ *and* $\beta \neq 0$, *then*

$$\mathscr{E}(\alpha + \beta\varphi) = (\beta/\overline{\beta})\mathscr{E}(\varphi).$$

(b) ([11]) *If* $k_0 \in \mathscr{E}(\varphi)$, *then*

$$\mathscr{E}(\varphi) = (k_0 + \theta_{\overline{\varphi}} H^2) \cap (H^\infty)_1.$$

(c) *If* $k_1, k_2 \in \mathscr{E}(\varphi)$ *and* $k_1 - k_2$ *is a nonzero constant, then* φ *is constant. In particular, if* $\mathscr{E}(\varphi)$ *contains two distinct constant functions, then* φ *is constant.*

Proof. (a) Note that $(H^\infty)_1 = (\beta/\overline{\beta})(H^\infty)_1$. Hence (3) and Lemma 2.2(a) imply that

$$\mathscr{E}(\alpha + \beta\varphi) = \mathscr{C}(\alpha + \beta\varphi) \cap (H^\infty)_1 = (\beta/\overline{\beta})(\mathscr{C}(\varphi) \cap (H^\infty)_1) = (\beta/\overline{\beta})\mathscr{E}(\varphi).$$

This proves (a).

(b) Recall that $\theta_{\overline{\varphi}}$ is the function such that $\ker H_{\overline{\varphi}} = \theta_{\overline{\varphi}} H^2$. Thus the result follows from (3) and (4) in Lemma 2.2(b).

(c) It is an immediate consequence of Lemma 2.2(c). $\square$

3. Cowen sets containing a constant function

It is clear that if φ is constant, then $\mathscr{C}(\varphi) = H^2$ and $\mathscr{E}(\varphi) = (H^\infty)_1$. Notice that the converses are also true, which follows from Lemma 2.2(c) and Theorem 2.3(c), respectively. From now on, we assume that φ is a nonconstant function in L^∞. Then $\mathscr{E}(\varphi)$ contains at most one constant function unless φ. On the other hand, Example 2.1 shows that a Cowen set may not contain a constant even though it is nonempty. Our first objective is to look at the Cowen sets containing a constant.

THEOREM 3.1. *Let* $\varphi \in L^\infty$ *be a nonconstant function, and let* λ *be a complex number with* $|\lambda| \leqslant 1$. *Then the Cowen set* $\mathscr{E}(\varphi)$ *contains* λ *if and only if there exists a function* $f \in H_0^2$ *and a number* $c \in \mathbb{C}$ *such that*

$$\varphi = f + \lambda\overline{f} + c. \tag{5}$$

In this case, we have

$$\mathscr{E}(\varphi) = (\lambda + \theta_{\overline{\varphi}} H^2) \cap (H^\infty)_1. \tag{6}$$

In particular, if $\overline{\varphi}$ *is not of bounded type, then* $\mathscr{E}(\varphi) = \{\lambda\}$.

Proof. Without loss of generality we may assume that $\widehat{\varphi}(0)=0$ since $\mathscr{E}(\varphi)=\mathscr{E}(\varphi-c)$ for any $c\in\mathbb{C}$ by Theorem 2.3(a). Write $\varphi=f+\overline{g}$, where $f,g\in H_0^2$.

Suppose that $\lambda\in\mathscr{E}(\varphi)$. We first prove that (5) holds when λ is nonnegative. Since $\lambda\in\mathscr{E}(\varphi)$, it follows that

$$\varphi-\lambda\overline{\varphi}=(f-\lambda g)+(\overline{g}-\lambda\overline{f})$$

belongs to H^2. This shows that the function $g-\lambda f\in H_0^2$ is also co-analytic. Thus we have $g-\lambda f=0$, i.e., $g=\lambda f$. Therefore $\varphi=f+\lambda\overline{f}$, which proves (5) for the case when $0\leqslant\lambda\leqslant 1$.

Now suppose that $\lambda=re^{it}$ belongs to $\mathscr{E}(\varphi)$, where $0\leqslant r\leqslant 1$ and $t\in\mathbb{R}$. Then $r\in e^{-it}\mathscr{E}(\varphi)$. By Theorem 2.3(a), $e^{-it}\mathscr{E}(\varphi)=\mathscr{E}(e^{-it/2}\varphi)$. Hence $e^{-it/2}\varphi=f_0+r\overline{f}_0$ for some $f_0\in H^2$, by the preceding paragraph. If we put $f=e^{it/2}f_0$, then $f\in H^2$ and we have

$$\varphi=e^{it/2}f_0+re^{it}e^{-it/2}\overline{f}_0=f+\lambda\overline{f}.$$

Thus (5) holds for the general case.

Conversely, suppose that $\varphi=f+\lambda\overline{f}$ for some $f\in H^2$. Then

$$\varphi-\lambda\overline{\varphi}=(f+\lambda\overline{f})-\lambda(\overline{f}+\overline{\lambda}f)=(1-|\lambda|^2)f. \tag{7}$$

Thus $\varphi-\lambda\overline{\varphi}\in H^2$, and hence $\lambda\in\mathscr{C}(\varphi)$. This proves the first assertion of the theorem.

Since $\lambda\in\mathscr{E}(\varphi)$, (6) now follows from Theorem 2.3(b). If $\overline{\varphi}$ is not of bounded type, then $\theta_{\overline{\varphi}}=0$ by definition, and hence $\mathscr{E}(\varphi)=\{\lambda\}$. □

REMARK 3.2. (a) The function in (7) is in fact bounded. Thus the analytic function f in (5) is bounded whenever $r\neq 1$. If $r=1$, however, the boundedness of f is not guaranteed. For example, the analytic function

$$f(z)=i\sum_{n=1}^{\infty}\frac{z^n}{n}\qquad(z\in\mathbb{D})$$

is not bounded on $\mathbb{D}$, although the function $\varphi=f+\overline{f}$ on $\mathbb{T}$ is bounded.

(b) Note that if φ and f are related as in (5) with $r\neq 0$, then $H_{\overline{\varphi}}=H_{\overline{f}}=H_{\varphi}$.

For a record, we state the two extreme cases separately as follows.

COROLLARY 3.3. *Let $\varphi\in L^\infty$ be a nonconstant function.*

(a) *T_φ is analytic if and only if $0\in\mathscr{E}(\varphi)$, in which case*

$$\mathscr{E}(\varphi)=\theta_{\overline{\varphi}}H^2\cap(H^\infty)_1=\theta_{\overline{\varphi}}(H^\infty)_1.$$

(b) *T_φ is normal if and only if $e^{it}\in\mathscr{E}(\varphi)$ for some t, in which case*

$$\mathscr{E}(\varphi)=(e^{it_0}+\theta_{\overline{\varphi}}H^2)\cap(H^\infty)_1.$$

EXAMPLE 3.4. Let $\varphi = a_0 + a_1 z + \cdots + a_n z^n$ be an analytic polynomial of degree n. Then $\ker H_{\overline{\varphi}} = z^n H^2$, and so $\theta_{\overline{\varphi}} = z^n$. Thus we have

$$\mathscr{E}(\varphi) = z^n (H^\infty)_1.$$

and

$$\mathscr{E}(\varphi + \overline{\varphi}) = (1 + z^n H^2) \cap (H^\infty)_1 = \{1\}.$$

If the Toeplitz operator T_φ is subnormal, what can we say about the Cowen set $\mathscr{E}(\varphi)$? First, $\mathscr{E}(\varphi)$ is nonempty because every subnormal operator is hyponormal. By Abrahamse's result [1], if φ or $\overline{\varphi}$ is of bounded type, then T_φ is analytic or normal. Thus the Cowen set $\mathscr{E}(\varphi)$ is given by Corollary 3.3. Let us assume that T_φ is subnormal but not analytic or normal. Then neither φ nor $\overline{\varphi}$ is of bounded type. In this case, $\theta_{\overline{\varphi}} = 0$, and so $\mathscr{E}(\varphi)$ is a singleton by Theorem 2.3(b).

The following explicit example of subnormal Toeplitz operators was found by Cowen and Long [6]: *For* $0 < \alpha < 1$, *let* $\psi = \psi_\alpha$ *be a conformal mapping of the unit disk onto the interior of the ellipse with vertices* $\pm i(1-\alpha)^{-1}$ *and passing through* $\pm(1+\alpha)^{-1}$. *Then* $T_{\psi + \alpha\overline{\psi}}$ *is a subnormal weighted shift that is neither analytic nor normal.* For the function $\varphi = \psi + \alpha\overline{\psi}$, the Cowen set is $\mathscr{E}(\varphi) = \{\alpha\}$, by Theorem 3.1. Of course, the fact that $\mathscr{E}(\varphi)$ consists of one constant does not imply that T_φ is subnormal: If $|\lambda| \leqslant 1$ and $\varphi = \psi_\alpha + \lambda\overline{\psi}_\alpha$, then $\mathscr{E}(\varphi) = \{\lambda\}$, but T_φ is subnormal if and only if $\lambda = \alpha$ or $\lambda = (\alpha^k e^{i\theta} + \alpha)(1 + \alpha^{k+1} e^{i\theta})^{-1}$ for some $k = 0, 1, 2, \ldots$ and $\theta \in \mathbb{R}$ [3]. We raise two questions at this point:

QUESTION 3.5. (1) *If* f *is an arbitrary function in* H^∞ *such that* $\overline{f}$ *is not of bounded type, does there exists a constant* λ *such that* $T_{f + \lambda\overline{f}}$ *is subnormal?*

(2) *If* T_φ *is subnormal, does the Cowen set* $\mathscr{E}(\varphi)$ *always contain a constant?*

Our next concern is the shape of the Cowen set for φ when T_φ is normal. Observe that if $\mathscr{E}(\varphi)$ contains a constant λ with $|\lambda| < 1$ and if $\theta_{\overline{\varphi}} \neq 0$, then $\mathscr{E}(\varphi)$ contains infinitely many functions, for example, it contains the infinite set

$$\lambda + (1-\lambda)\theta_{\overline{\varphi}}(H^\infty)_1.$$

However, if $|\lambda| = 1$, then $\mathscr{E}(\varphi)$ could be a singleton even if $\theta_{\overline{\varphi}} \neq 0$ as Example 3.4 shows. Hence we may ask the following question.

QUESTION 3.6. *If* $\varphi \in L^\infty$ *is a nonconstant function such that* T_φ *is normal, does it follows that* $\mathscr{E}(\varphi) = \{e^{it}\}$ *for some* t*?*

If the answer to this question is affirmative, then we obtain the following characterization for normal Toeplitz operators:

CONJECTURE 3.7. *Let* $\varphi \in L^\infty$ *be a nonconstant function. Then* T_φ *is normal if and only if* $\mathscr{E}(\varphi)$ *consists of one point which is a constant of unit modulus.*

We have shown the sufficiency, but it does not seem easy to prove the necessity. We give a partial answer to this question.

THEOREM 3.8. *Let $\varphi \in L^\infty$ be a nonconstant function such that the Toeplitz operator T_φ is normal. If the function $\theta_{\overline{\varphi}}$ has a zero in $\mathbb{D}$, then $\mathscr{E}(\varphi)$ consists of one point which is a constant of unit modulus.*

Proof. By Corollary 3.3(b), $\mathscr{E}(\varphi) = (e^{it_0} + \theta_{\overline{\varphi}} H^2) \cap (H^\infty)_1$ for some t_0. If $\theta_{\overline{\varphi}} = 0$, it is clear that $\mathscr{E}(\varphi) = \{e^{it_0}\}$. Thus we assume that $\theta_{\overline{\varphi}}$ is a (nonzero) inner function, that is, $\overline{\varphi}$ is of bounded type. Suppose that $k \in \mathscr{E}(\varphi)$. Then

$$k = e^{it_0} + \theta_{\overline{\varphi}} f$$

for some $f \in H^2$, and $\|k\|_\infty \leqslant 1$. We show that $f = 0$ if $\theta_{\overline{\varphi}}$ has a zero in $\mathbb{D}$.

Let λ a zero of $\theta_{\overline{\varphi}}$ in $\mathbb{D}$. Assume first that $\lambda = 0$, i.e., $\theta_{\overline{\varphi}}$ has a zero at the origin. Then e^{it_0} and $\theta_{\overline{\varphi}}$ are orthogonal functions in H^2. Thus

$$\|e^{it_0} + \theta_{\overline{\varphi}} f\|_\infty^2 \geqslant \|e^{it_0} + \theta_{\overline{\varphi}} f\|_2^2 = \|e^{it_0}\|_2^2 + \|\theta_{\overline{\varphi}} f\|_2^2 \geqslant 1.$$

But $\|e^{it_0} + \theta_{\overline{\varphi}} f\|_\infty = \|k\|_\infty \leqslant 1$, which implies that $\|\theta_{\overline{\varphi}} f\|_2^2 = 0$. Since $|\theta_{\overline{\varphi}}| = 1$ on $\mathbb{T}$, it follows that $\|f\|_2 = 0$, and therefore $f = 0$.

For the general case, consider the function

$$b_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z} \qquad (z \in \mathbb{D}), \tag{8}$$

and put $h_\lambda = h \circ b_\lambda$ for $h \in H^\infty$. Note that for every $h \in H^\infty$ we have $h_\lambda \in H^\infty$ and $h_\lambda(0) = h(\lambda)$. Since b_λ is a one-to-one mapping of $\mathbb{D}$ onto $\mathbb{D}$, it follows that

$$\begin{aligned}
\|e^{it_0} - \theta_{\overline{\varphi}} f\|_\infty &= \sup_{z \in \mathbb{D}} |e^{it_0} - \theta_{\overline{\varphi}}(z) f(z)| \\
&= \sup_{z \in \mathbb{D}} |e^{it_0} - \theta_{\overline{\varphi}}(b_\lambda(z)) f(b_\lambda(z))| \\
&= \|e^{it_0} - (\theta_{\overline{\varphi}})_\lambda f_\lambda\|_\infty.
\end{aligned}$$

Thus $\|e^{it_0} - (\theta_{\overline{\varphi}})_\lambda f_\lambda\|_\infty \leqslant 1$. Also, $(\theta_{\overline{\varphi}})_\lambda$ has a zero at the origin, because $(\theta_{\overline{\varphi}})_\lambda(0) = \theta_{\overline{\varphi}}(\lambda) = 0$. The preceding paragraph shows that $f_\lambda = 0$. Thus $f = f_\lambda \circ b_\lambda = 0$, and hence $k = e^{it_0}$.

Therefore $\mathscr{E}(\varphi) = \{e^{it_0}\}$, and the proof is complete. □

The function in (8) is called a Blaschke factor. For $\lambda \in \mathbb{D}$, the function

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda} z} \qquad (z \in \mathbb{D})$$

is a kernel function for the Hardy space H^2, because $\langle f, k_\lambda \rangle = f(\lambda)$ for all $f \in H^2$.

COROLLARY 3.9. *Let* $\varphi \in L^\infty$ *be a nonconstant function such that* T_φ *is normal. If the closure of* $\operatorname{ran} H_{\overline{\varphi}}$ *contains a kernel function* k_λ, *then* $\mathscr{E}(\varphi)$ *consists of one point which is a constant of unit modulus.*

Proof. Note that $\varphi = \alpha + \beta\rho$, where $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$ and $\rho \in L^\infty$ is real-valued. By Theorem 2.3(a), $\mathscr{E}(\varphi) = (\beta/\overline{\beta})\mathscr{E}(\rho)$. Also, $\operatorname{ran} H_{\overline{\varphi}} = \operatorname{ran} H_\rho$. Thus we may assume, without loss of generality, that φ is real-valued. Then $H_{\overline{\varphi}}$ is self-adjoint. Since $\ker H_{\overline{\varphi}} = \theta_{\overline{\varphi}} H^2$, it follows that the closure of $\operatorname{ran} H_{\overline{\varphi}}$ is equal to

$$\mathscr{H}(\theta_{\overline{\varphi}}) := H^2 \ominus \theta_{\overline{\varphi}} H^2.$$

Now suppose that k_λ belongs to $\overline{\operatorname{ran}} H_{\overline{\varphi}}$. Since $\mathscr{H}(b_\lambda)$ is the one-dimensional space generated by k_λ, it follows that $\mathscr{H}(b_\lambda) \subseteq \mathscr{H}(\theta_{\overline{\varphi}})$. This implies that b_λ devides $\theta_{\overline{\varphi}}$. Then $\theta_{\overline{\varphi}}$ has a zero at λ. Thus the result follows from Theorem 3.8. $\square$

REMARK 3.10. Suppose that $e^{it_0} \in \mathscr{E}(\varphi)$. To prove Conjecture 3.7, we must show that

$$(e^{it_0} + \theta_{\overline{\varphi}} H^2) \cap (H^\infty)_1 = \{e^{it_0}\}. \tag{9}$$

This is equivalent to that

$$g \in H^\infty \text{ and } \|\lambda + \theta_{\overline{\varphi}} g\|_\infty \leqslant 1 \text{ implies } g = 0.$$

In other words, $\|\overline{\theta}_{\overline{\varphi}} - g\|_\infty = \|\lambda + \theta_{\overline{\varphi}}(-\lambda g)\|_\infty > 1$ for all $g \in H^\infty \setminus \{0\}$. Hence (9) holds if and only if

$$\operatorname{dist}(\overline{\theta}_{\overline{\varphi}}, H^\infty) = \inf_{h \in H^\infty} \|\overline{\theta}_{\overline{\varphi}} - h\|_\infty = 1 \tag{10}$$

and $h = 0$ is the unique function satisfying $\|\overline{\theta}_{\overline{\varphi}} - h\|_\infty = \operatorname{dist}(\overline{\theta}_{\overline{\varphi}}, H^\infty)$, i.e., the zero function is the unique best approximation to $\overline{\theta}_{\overline{\varphi}}$ in H^∞. We have proven that this is true whenever the inner function $\theta_{\overline{\varphi}}$ has a zero in $\mathbb{D}$.

In fact, (10) holds for arbitrary nonconstant inner functions.

THEOREM 3.11. *If* θ *is a nonconstant inner function, then*

$$\operatorname{dist}(\overline{\theta}, H^\infty) = 1. \tag{11}$$

Proof. Since $\|\overline{\theta} - 0\|_\infty = 1$, it follows that $\operatorname{dist}(\overline{\theta}, H^\infty) \leqslant 1$. To show the equality, assume to the contrary that $\operatorname{dist}(\overline{\theta}, H^\infty) < 1$. Then

$$\overline{\theta} = F/|F|$$

for some $F \in H^1$ (cf. [7, p. 172]). Thus $|F| = F\theta$ on $\mathbb{T}$. Since $F\theta \in H^1$, it follows that $F\theta$ is constant, say c. Then $\overline{\theta} = c^{-1}F \in H^1$, which implies that θ is a constant, a contradiction. This shows that (11) holds for all nonconstant inner functions θ. $\square$

Note that the zero function is one best approximation to $\overline{\theta}$: $\|\overline{\theta}-0\|_\infty = 1 = \operatorname{dist}(\overline{\theta}, H^\infty)$. Is $h=0$ the unique best approximation? The answer to this question is yes if and only if Conjecture 3.7 is true.

We now turn our attention to the geometric properties of Cowen sets. For $\varphi \in L^\infty$, $\mathscr{C}(\varphi)$ is a weakly closed convex subset of H^2. Consequently, $\mathscr{E}(\varphi)$ is a closed convex subset of $(H^\infty)_1$. A convex C is absolutely convex if it is balanced, i.e., $\alpha x \in C$ whenever $x \in C$, $|\alpha| \leqslant 1$. The Cowen set $\mathscr{E}(\varphi)$ cannot be balanced unless φ is analytic. Indeed, if $k, \alpha k \in \mathscr{E}(\varphi)$, where $k \neq 0$ and $\alpha \neq 1$, then $\varphi - k\overline{\varphi} \in H^\infty$ and $\varphi - \alpha k\overline{\varphi} \in H^\infty$. Thus $(1-\alpha)k\overline{\varphi} \in H^\infty$ and $(1-\alpha)\varphi \in H^\infty$. It follows that $\varphi \in H^\infty$ and $\overline{\varphi}$ is of bounded type.

Since $\mathscr{E}(\varphi)$ is closed and convex, it is natural to ask about the extreme point of it. Every extreme point of $(H^\infty)_1$ that is contained in $\mathscr{E}(\varphi)$ is an extreme point of $\mathscr{E}(\varphi)$. That is,

$$\operatorname{ext}(H^\infty)_1 \cap \mathscr{E}(\varphi) \subseteq \operatorname{ext}\mathscr{E}(\varphi).$$

Recall that the extreme point of $(H^\infty)_1$ is the functions φ such that

$$\int_{\mathbb{T}} \log(1-|\varphi|)\, dm = -\infty.$$

Every inner function is an extreme point of $(H^\infty)_1$. It is not difficult to find $\operatorname{ext}\mathscr{E}(\varphi)$ when φ is analytic.

COROLLARY 3.12. *Let $\varphi \in H^\infty$ be such that $\overline{\varphi}$ is of bounded type. Then*

$$\operatorname{ext}\mathscr{E}(\varphi) = \theta_{\overline{\varphi}} \operatorname{ext}(H^\infty)_1 = \operatorname{ext}(H^\infty)_1 \cap \theta_{\overline{\varphi}}(H^\infty)_1.$$

Proof. By Corollary 3.3(a),

$$\mathscr{E}(\varphi) = \theta_{\overline{\varphi}}(H^\infty)_1.$$

Suppose that $k = \theta_{\overline{\varphi}} f \in \operatorname{ext}\mathscr{E}(\varphi)$, where $f \in (H^\infty)_1$. To show that $f \in \operatorname{ext}(H^\infty)_1$, let $f = \frac{1}{2}(f_1+f_2)$, $f_1, f_2 \in (H^\infty)_1$. Then $\theta_{\overline{\varphi}} f = \frac{1}{2}(\theta_{\overline{\varphi}} f_1 + \theta_{\overline{\varphi}} f_2)$. Since $\theta_{\overline{\varphi}} f$ is an extreme point of $\mathscr{E}(\varphi)$ and $\theta_{\overline{\varphi}} f_1, \theta_{\overline{\varphi}} f_2 \in \mathscr{E}(\varphi)$, it follows that $\theta_{\overline{\varphi}} f = \theta_{\overline{\varphi}} f_1 = \theta_{\overline{\varphi}} f_2$. Hence $f = f_1 = f_2$, which shows that $f \in \operatorname{ext}(H^\infty)_1$. Thus $\operatorname{ext}\mathscr{E}(\varphi) \subseteq \theta_{\overline{\varphi}} \operatorname{ext}(H^\infty)_1$.

Suppose that $f \in \operatorname{ext}(H^\infty)_1$ and that $\theta_{\overline{\varphi}} f = \frac{1}{2}(\theta_{\overline{\varphi}} h_1 + \theta_{\overline{\varphi}} h_2)$, where $h_1, h_2 \in (H^\infty)_1$. Then $f = \frac{1}{2}(h_1+h_2)$, and so $f = h_1 = h_2$. Thus $\theta_{\overline{\varphi}} f \in \operatorname{ext}\mathscr{E}(\varphi)$. Therefore $\theta_{\overline{\varphi}} \operatorname{ext}(H^\infty)_1 \subseteq \operatorname{ext}\mathscr{E}(\varphi)$. Thus the first equality holds.

For $f \in (H^\infty)_1$, $f \in \operatorname{ext}(H^\infty)_1$ if and only if $\theta_{\overline{\varphi}} f \in \operatorname{ext}(H^\infty)_1$. The second equality follows immediately from this. $\square$

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Hyoung Joon Kim
Department of Mathematical Sciences
Seoul National University
Seoul 08826, South Korea
e-mail: hjkim76@snu.ac.kr

Jaehui Park
Research Institute of Mathematics
Seoul National University
Seoul 08826, South Korea
e-mail: hiems1855@gmail.com