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# ON ZEROS OF MATRIX–VALUED ANALYTIC FUNCTIONS

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*Abstract.* We extend a result proved by Dirr and Wimmer [IEEE Trans. Automat. Control 52(2007)] for polynomials to the matrix valued analytic functions and thereby obtain generalizations of some well-known results concerning the zero free regions of a class of analytic functions.

## 1. Introduction and statement of results

Let $\Omega \subseteq \mathbb{C}$ be an open set, $\mathbb{M}_n$ be the set of $n\times n$ matrices, $n \geqslant 1$, with entries in $\mathbb{C}$ and $\|\cdot\|$ denote the operator norm, induced by the Euclidean norm on $\mathbb{C}^n$. Then a matrix-valued function $F:\Omega \to \mathbb{M}_n$ is said to be analytic in $\Omega$, if for each $z_0 \in \mathbb{M}_n$, there is a member of $\mathbb{M}_n$, denoted by $F'(z_0)$, such that $\left\|\frac{F(z)-F(z_0)}{z-z_0} - F'(z_0)\right\| \to 0$ as $z \to z_0$. A number $\lambda \in \mathbb{C}$ is said to be a zero of $F(z)$ if there exists a vector $x \in \mathbb{C}^n \setminus \{0\}$ such that $F(\lambda)x = 0$. In other words $\lambda$ is a zero of $F(z)$, if $F(\lambda)$ is less than full rank. Some authors also refer to $\lambda$ (see e.g. [2]), as an eigenvalue of $F(z)$.

Many differential equations in science and engineering lead to the consideration of the matrix-valued analytic functions. For instance, the standard model of an RLC circuit, gives rise to the formulation of the problem, $x'(t) = Ax(t)$, where $A \in \mathbb{M}_n$ and $x(t)$ is a vector valued function. Its solution is of the form of $e^{At} = \sum_{j=0}^{\infty} \frac{A^j t^j}{k!}$ and the decay of these solutions is controlled by the operator norm $\left\|e^{At}\right\|$. Matrix-valued functions also play an important role in the spectral analysis of a matrix $A \in \mathbb{M}_n$. After all, $\lambda \in \mathbb{C}$, is an eigenvalue of a matrix $A$ if and only if the resolvent function defined by $z \to (A-zI)^{-1}$ has a singularity at $z=\lambda$, that is, $A-\lambda I$ is not invertible. Here $I$ represents the identity matrix.

Analytic matrix-valued functions also appear in many other areas such as harmonic analysis of an operator on a Hilbert space, for example, finite-rank perturbation of self-adjoint and unitary operator. As a result they also arise in mathematical physics, for example, Schrödinger operators. Practically, problems related to spectral properties of an operator are generally solved with the help of matrix-valued analytic functions defined on the upper-half plane, called characteristic functions.

For matrices $A, B \in \mathbb{M}_n$, we write $A \geqslant 0$ or $A > 0$, if $A$ is positive semi-definite or positive definite respectively. Similarly $A \geqslant B$, means $A-B \geqslant 0$ and $A > B$ implies $A-$

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$B > 0$. Also, $A^*$ and $tr(A)$ denote the transpose conjugate and trace of $A$, respectively. In the same way, $x^*$ denotes the conjugate transpose of a vector $x \in \mathbb{C}^n$. A vector $u$ is unit vector if $\|u\| := \sqrt{u^*u} = 1$. It should also be noted that every matrix $A$ can be uniquely expressed as $A = H + iK$, where $H = \frac{A+A^*}{2}$ and $K = \frac{A-A^*}{2i}$ are Hermitian. We call $H$ and $K$ the real and imaginary parts of $A$ and write $\Re(A) = H$ and $\Im(A) = K$. Also $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum of all the eigenvalues of a Hermitian matrix $A$, respectively.

For an inner product $\langle\cdot,\cdot\rangle$ on a vector space $V$ over the field $\mathbb{F}$ $(\mathbb{F} = \mathbb{C}$ or $\mathbb{R})$, $\measuredangle(x,y) := \cos^{-1}\Re\left(\frac{\langle x,y\rangle}{\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}}}\right)$ defines an angle between vectors $x, y \in V \setminus \{0\}$. We also note that for the vector space $V = \mathbb{M}_n$, over $\mathbb{C}$, the function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ given by $\langle A, B\rangle = tr(B^*A)$, defines an inner product called Frobenius inner product and the corresponding induced norm, denoted by $\|\cdot\|_F$, is called the Frobenius norm.

We must also mark down that any matrix-valued function $F(z)$ analytic in $|z| \leqslant 1$ can be expressed as a power series $F(z) = \sum_{j=0}^{\infty} A_j z^j$, $A_j \in \mathbb{M}_n$, $|z| \leqslant 1$ (for ref. see [13]).

The following theorem of Eneström and Kakeya is well-known in the theory of distribution of zeros of a polynomial.

THEOREM A. *If* $p(z) := \sum_{j=0}^{n} a_j z^j$ *is a polynomial with real coefficients such that* $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_0 > 0$. *Then all the zeros of* $p(z)$ *lie in* $|z| \leqslant 1$.

Theorem A was first proved by Gustov Eneström [3], while he was studying a problem in the theory of pension funds. Kakeya [10] independently proved the following more general result and published it in English.

THEOREM B. *Let* $p(z) := \sum_{j=0}^{n} a_j z^j$, *be a polynomial with real and positive coefficients, then all the zeros of* $p(z)$ *lie in the annulus* $R_1 \leqslant |z| \leqslant R_2$, *where* $R_1 = \min_{j=0,\ldots,n-1} \left\{\frac{a_j}{a_{j+1}}\right\}$, $R_2 = \max_{j=0,\ldots,n-1} \left\{\frac{a_j}{a_{j+1}}\right\}$.

Eneström [4] later published a French translation of his earlier proof and it is due to these reasons that the result is known as Eneström-Kakeya theorem. For a detailed survey of the result and its generalizations, see [8, 12].

Joyal, Labelle and Rahman [9] generalized Theorem A by dropping the condition of non-negativity and maintaining the condition of monotonicity. They proved:

THEOREM C. *If* $p(z) := \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* $n$ *such that* $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1 \geqslant a_0$. *Then all the zeros of* $p(z)$ *lie in* $|z| \leqslant \frac{1}{|a_n|}(a_n - a_0 + |a_0|)$.

Theorem C like the Eneström-Kakeya theorem is only applicable to the polynomials with real coefficients. Govil and Rahman [6] proved the following result for polynomials with complex coefficients.

THEOREM D. *Let* $p(z) := \sum_{j=0}^{n} a_j z^j$, *be a polynomial of degree* $n$ *with complex coefficients such that for some real* $\beta$, $|\arg a_j - \beta| \leqslant \alpha \leqslant \frac{\pi}{2}$, *for* $0 \leqslant j \leqslant n$ *and* $|a_n| \geqslant |a_{n-1}| \geqslant \cdots \geqslant |a_1| \geqslant |a_0|$. *Then all zeros of* $p(z)$ *lie in the disk* $|z| \leqslant (\sin\alpha + \cos\alpha) + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1} |a_j|$.

Govil and Rahman [6], in the same paper, extended the above result to complex valued analytic functions with similar conditions on the angles and moduli of the coefficients, appearing in their series representation. They proved

THEOREM E. *Let* $f(z) := \sum_{j=0}^{\infty} a_j z^j$, *be analytic in* $|z| \leqslant 1$, *such that for some real* $\beta$, $|\arg a_j - \beta| \leqslant \alpha \leqslant \frac{\pi}{2}$, *for* $j = 0, 1, 2, \ldots$ *and* $|a_0| \geqslant |a_1| \geqslant |a_2| \geqslant \cdots$. *Then* $f(z)$ *does not vanish in the disk* $|z| \leqslant \left(\sin\alpha + \cos\alpha + \frac{2\sin\alpha}{|a_0|}\sum_{j=0}^{n-1} |a_j|\right)^{-1}$.

They also [6] proved a different result for polynomials with complex coefficients while imposing a non-negative and monotone condition on the real parts of the coefficients of a polynomial, as follows:

THEOREM F. *Let* $p(z) := \sum_{j=0}^{n} a_j z^j$, *be a polynomial of degree* $n$ *with complex coefficients such that* $\Re(a_j) = \alpha_j$ *and* $\Im a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$ *satisfying* $\alpha_n \geqslant \alpha_{n-1} \geqslant \cdots \geqslant \alpha_1 \geqslant 0$, $\alpha_n \neq 0$. *Then all the zeros of* $p(z)$ *lie in* $|z| \leqslant 1 + \frac{2}{\alpha_n}\sum_{j=0}^{n} |\beta_j|$.

Dirr and Wimmer [2] extended Theorem A to matrix polynomials and proved the following result concerning the bound estimate of the zeros of a matrix polynomial.

THEOREM G. *Let* $P(z) := \sum_{j=0}^{n} A_j z^j$, $A_j \in \mathbb{M}_k$, $k > 0$, $0 \leqslant j \leqslant n$ *be a matrix polynomial of degree* $n$ *such that*

$$A_n \geqslant A_{n-1} \geqslant \cdots \geqslant A_0 \geqslant 0,\ A_n > 0. \tag{1.1}$$

*Then the zeros of* $P(z)$ *lie in the closed unit disk* $|z| \leqslant 1$.

Le, Du and Nguyen [11] extended Theorem B to matrix polynomials as follows.

THEOREM H. *Let* $P(z) := \sum_{j=0}^{n} A_j z^j$, *where* $A_j \in \mathbb{M}_k$, $k > 0$ *are positive-definite, be a matrix polynomial of degree* $n$ *. Then the eigenvalues of* $P(z)$ *lie in the annulus* $R_1' \leqslant |z| \leqslant R_2'$, *where* $R_1' = \min_{j=0,\dots,n-1} \left\{ \frac{\lambda_{\min}(A_j)}{\lambda_{\max}(A_{j+1})} \right\}$ *and* $R_2' = \max_{j=0,\dots,n-1} \left\{ \frac{\lambda_{\max}(A_j)}{\lambda_{\min}(A_{j+1})} \right\}$.

In this paper we extend Theorem G to matrix valued analytic functions by associating a monotone condition of the form of (1.1), on coefficients of the Taylor series expansion of a matrix valued analytic function. We further extend Theorem E and Theorem F by firstly restricting the angle and then binding the real parts of the coefficients of a matrix valued analytic function. We first prove:

THEOREM 1. *Let* $F(z) := \sum_{j=0}^{\infty} A_j z^j$, $A_j \in \mathbb{M}_n$, $j = 0, 1, \dots$, *be analytic in* $|z| \leqslant 1$. *Assume* $A_0 \geqslant A_1 \geqslant \cdots$, $A_0 > 0$ . *Then the zeros of* $F(z)$ *lie outside the disk* $|z| < 1$.

We next prove:

THEOREM 2. *Let* $F(z) := \sum_{j=0}^{\infty} A_j z^j$, $det(A_0) \neq 0$, $A_j \in \mathbb{M}_n$, $j = 0, 1, \dots$, *be analytic in* $|z| \leqslant 1$. *Assume* $\|A_0\|_F \geqslant \|A_1\|_F \geqslant \cdots$, *and* $\measuredangle(A_j, C) \leqslant \alpha \leqslant \frac{\pi}{2}$, $j = 0, 1, \dots$, *for some non-zero matrix* $C \in \mathbb{M}_n$. *Then the zeros of* $F(z)$ *lie outside the disk*

$$|z| < \frac{1}{\|A_0^{-1}\|_F} \left\{ \|A_0\|_F (\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=1}^{\infty} \|A_j\|_F \right\}^{-1}. \tag{1.2}$$

Finally we prove:

THEOREM 3. *Let* $F(z) := \sum_{j=0}^{\infty} A_j z^j$, $A_j \in \mathbb{M}_n$, $j = 0, 1, \dots$, *be analytic in* $|z| \leqslant 1$. *Let* $\Re(A_j) = B_j$ *and* $\Im(A_j) = C_j$, $j = 0, 1, \dots$, *and assume* $B_0 \geqslant B_1 \geqslant \cdots$, $B_0 > 0$. *Then the zeros of* $F(z)$ *lie outside the disk*

$$|z| < \frac{1}{1 + \frac{2}{\lambda_{\min}(B_0)} \sum_{j=0}^{\infty} |r(C_j)|}. \tag{1.3}$$

*where, for a matrix* $A \in \mathbb{M}_n$, $r(A) = \max\{|u^* A u|; \|u\| = 1\}$.

For $C_j = 0$, Theorem 3 reduces to Theorem 1. We also note that for a matrix $A$, $r(A)$ is called the numerical radius of $A$.

## 2. Lemmas and proofs of theorems

For the proofs of these theorems, we need the following lemmas.

LEMMA 1. *Let $A,B \in \mathbb{M}_n$, be such that $\|A\|_F \geqslant \|B\|_F$ and $\measuredangle(A,B) = \theta \leqslant 2\alpha \leqslant \pi$, then*

$$\|A-B\|_F \leqslant (\|A\|_F - \|B\|_F)\cos\alpha + (\|A\|_F + \|B\|_F)\sin\alpha. \tag{2.1}$$

*Proof.*

$$\begin{aligned}\|A-B\|_F^2 &= \|A\|_F^2 + \|B\|_F^2 - 2\|A\|_F\|B\|_F\cos\theta \\ &\leqslant \|A\|_F^2 + \|B\|_F^2 - 2\|A\|_F\|B\|_F\cos(2\alpha) \\ &= (\|A\|_F - \|B\|_F)^2\cos^2\alpha + (\|A\|_F + \|B\|_F)^2\sin^2\alpha \\ &\leqslant ((\|A\|_F - \|B\|_F)\cos\alpha + (\|A\|_F + \|B\|_F)\sin\alpha)^2.\end{aligned}$$

Thus

$$\|A-B\|_F \leqslant (\|A\|_F - \|B\|_F)\cos\alpha + (\|A\|_F + \|B\|_F)\sin\alpha.$$

This proves Lemma 1. □

LEMMA 2. *Let $\langle\cdot,\cdot\rangle$ be an inner product on a vector space $V$ over $\mathbb{F}$. Let $a,b,c \in V\setminus\{0\}$ such that $\measuredangle(a,b) = \theta_1 \leqslant \frac{\pi}{2}$, $\measuredangle(b,c) = \theta_2 \leqslant \frac{\pi}{2}$, then*

$$\measuredangle(a,b) = \theta \leqslant \theta_1 + \theta_2.$$

*Proof.* Without loss of generality we assume $\langle a,a\rangle = \langle b,b\rangle = \langle c,c\rangle = 1$. Since $\Re\langle x,y\rangle$, $x,y \in V$ defines an inner product on $V$ and determinant of a gram matrix is non-negative, therefore

$$\begin{vmatrix} \Re\langle a,a\rangle & \Re\langle a,b\rangle & \Re\langle a,c\rangle \\ \Re\langle b,a\rangle & \Re\langle b,b\rangle & \Re\langle b,c\rangle \\ \Re\langle c,a\rangle & \Re\langle c,b\rangle & \Re\langle c,c\rangle \end{vmatrix} \geqslant 0.$$

That is

$$\begin{vmatrix} 1 & \cos\theta_1 & \cos\theta \\ \cos\theta_1 & 1 & \cos\theta_2 \\ \cos\theta & \cos\theta_2 & 1 \end{vmatrix} \geqslant 0.$$

This gives

$$1 - \cos^2\theta_1 - \cos^2\theta_2 - \cos^2\theta + 2\cos\theta_1\cos\theta_2\cos\theta \geqslant 0.$$

That is

$$(1-\cos^2\theta_1)(1-\cos^2\theta_2) - (\cos\theta - \cos\theta_1\cos\theta_2)^2 \geqslant 0,$$

or

$$|\cos\theta - \cos\theta_1\cos\theta_2| \leqslant \sin\theta_1\sin\theta_2.$$

Equivalently

$$\cos\theta \geqslant \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \cos(\theta_1+\theta_2).$$

This gives

$$\theta \leqslant \theta_1+\theta_2.$$

This proves Lemma 2. □

We also need following lemmas (for ref. see [7]) for the proof of the theorems.

LEMMA 3. *Let* $A \in \mathbb{M}_n$, *then*

$$r(A) \leqslant \|A\| \leqslant \|A\|_F,$$

LEMMA 4. *Let* $A \in M_n$, *be a Hermitian matrix, then*

$$\lambda_{\min}(A) = \min_{\|u\|=1}\{u^*Au\} \leqslant \max_{\|u\|=1}\{u^*Au\} = \lambda_{\max}(A).$$

*Proof of Theorem* 1. Let $u$ be a unit vector and define $F_u(z) = u^*(1-z)F(z)u = \sum_{j=0}^{\infty}(1-z)u^*A_juz^j$. Since $F(z)$ is analytic in $|z| \leqslant 1$, therefore $F_u(z)$ is analytic in $|z| \leqslant 1$. Also $A_j \geqslant A_{j+1}$, $A_0 > 0$, therefore

$$u^*A_ju \geqslant u^*A_{j+1}u,\ \ u^*A_0u > 0,\ \ j = 0,1,\dots. \tag{2.2}$$

Now for $|z| \leqslant 1$, we have

$$\begin{aligned}|F_u(z)| &= |u^*(1-z)F(z)u|\\ &= |u^*A_0u + z\sum_{j=0}^{\infty}u^*(A_{j+1}-A_j)uz^j|\\ &\geqslant |u^*A_0u| - |z|\sum_{j=0}^{\infty}|u^*(A_{j+1}-A_j)u|.\end{aligned}$$

This gives on using (2.2),

$$\begin{aligned}|F_u(z)| &\geqslant u^*A_0u(1-|z|)\\ &> 0, \text{ if } |z| < 1.\end{aligned}$$

This shows that the zeros of $F_u(z)$ and therefore, the zeros of $F(z)$ lie outside the disk $|z| < 1$. □

*Proof of Theorem* 2. Since $\measuredangle(A_j, C) \leqslant \alpha \leqslant \frac{\pi}{2}$, $j = 0,1,\dots$, therefore by Lemma 2, $\measuredangle(A_{j-1}, A_j) \leqslant 2\alpha \leqslant \pi$, $j = 1,2,\dots$. Define $G(z) = (1-z)F(z)$ and let $u$ be a unit vector, then we have for $|z| \leqslant 1$,

$$\begin{aligned} \|G(z)u\| &= \|(1-z)F(z)u\| \\ &= \left\| A_0 u - \sum_{j=0}^{\infty} (A_{j+1} - A_j) u z^{j+1} \right\| \\ &\geqslant \|A_0 u\| - \left\| \sum_{j=1}^{\infty} (A_{j+1} - A_j) u z^{j+1} \right\| \\ &\geqslant \left\|A_0^{-1}\right\|^{-1} - |z| \left\| \sum_{j=0}^{\infty} (A_{j+1} - A_j) z^j \right\| \\ &\geqslant \left\|A_0^{-1}\right\|^{-1} - |z| \sum_{j=0}^{\infty} \left\|A_{j+1} - A_j\right\|. \end{aligned}$$

This gives with the help of Lemma 3,

$$\|G(z)u\| \geqslant \left\|A_0^{-1}\right\|_F^{-1} - |z| \sum_{j=0}^{\infty} \left\|A_{j+1} - A_j\right\|_F.$$

Therefore by using Lemma 1, we get

$$\begin{aligned} \|G(z)u\| &\geqslant \left\|A_0^{-1}\right\|_F^{-1} - |z| \sum_{j=0}^{\infty} \left\{ (\left\|A_j\right\|_F - \left\|A_{j+1}\right\|_F) \cos\alpha + (\left\|A_j\right\|_F + \left\|A_{j+1}\right\|_F) \sin\alpha \right\} \\ &= \left\|A_0^{-1}\right\|_F^{-1} - |z| \left\{ \|A_0\|_F (\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=1}^{\infty} \left\|A_j\right\|_F \right\} \\ &> 0, \end{aligned}$$

if

$$|z| < \frac{1}{\left\|A_0^{-1}\right\|_F} \left( \|A_0\|_F (\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=1}^{\infty} \left\|A_j\right\|_F \right)^{-1} \leqslant 1.$$

Since Frobenius norm is submultiplicative and $0 \leqslant \alpha \leqslant \frac{\pi}{2}$, therefore the inequality on the right is true. This shows that all the zeros of $G(z)$, and therefore of $F(z)$ lie in

$$|z| \geqslant \frac{1}{\left\|A_0^{-1}\right\|_F} \left( \|A_0\|_F (\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=1}^{\infty} \left\|A_j\right\|_F \right)^{-1}. \quad \square$$

Since $A \geqslant B$, implies $\|A\|_F \geqslant \|B\|_F$, therefore on using Lemma 3 and Lemma 4 we have the following:

COROLLARY 1. *Let* $F(z) := \sum_{j=0}^{\infty} A_j z^j$, $A_j \in \mathbb{M}_n$, $j = 0, 1, \ldots$, *be analytic in* $|z| \leqslant 1$. *Assume* $A_0 \geqslant A_1 \geqslant \ldots$, *and* $\measuredangle(A_j, C) \leqslant \alpha \leqslant \dfrac{\pi}{2}$, $j = 0, 1, \ldots$, *for some non-zero* *matrix* $C \in \mathbb{M}_n$. *Then the zeros of* $F(z)$ *lie outside the disk*

$$|z| < \frac{1}{\lambda_{\max}(A_0^{-1})}\left\{\lambda_{\max}(A_0)(\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=1}^{\infty}\lambda_{\max}(A_j)\right\}^{-1}. \tag{2.3}$$

*Proof of Theorem* 3. Let $u$ be a unit vector. Define $F_u(z) = u^*(1-z)F(z)u$, then $F_u(z)$ is a complex function analytic in $|z| \leqslant 1$. Now for $|z| \leqslant 1$, we have

$$\begin{aligned}|F_u(z)| &= |u^*A_0u - z\sum_{j=0}^{\infty}u^*(A_{j+1}-A_j)uz^j|\\ &\geqslant |u^*A_0u| - |z|\sum_{j=0}^{\infty}|u^*(B_{j+1}-B_j)u + iu^*(C_{j+1}-C_j)u|\\ &\geqslant u^*B_0u - |z|(u^*B_0u - |u^*C_0u| + 2\sum_{j=0}^{\infty}|u^*C_ju|)\\ &\geqslant u^*B_0u - |z|(u^*B_0u + 2\sum_{j=0}^{\infty}|u^*C_ju|)\\ &> 0,\end{aligned}$$

if

$$|z| < \frac{1}{1+\frac{2}{|u^*B_0u|}\sum_{j=0}^{\infty}|u^*C_ju|}.$$

Using Lemma 3 and Lemma 4, this is possible if

$$|z| < \frac{1}{1+\frac{2}{\lambda_{\min}(B_0)}\sum_{j=0}^{\infty}|r(C_j)|}.$$

This shows that all the zeros of $F_u(z)$ and therefore the zeros of $F(z)$ lie outside the disk

$$|z| < \frac{1}{1+\frac{2}{\lambda_{\min}(B_0)}\sum_{j=0}^{\infty}|r(C_j)|}. \quad \square$$

The following can be easily obtained from the above theorem by using Lemma 3 and Lemma 4:

COROLLARY 2. *Let* $F(z) := \sum_{j=0}^{\infty}A_jz^j$, $A_j \in \mathbb{M}_n$, $j = 0,1,\ldots$ *be analytic in* $|z| \leqslant 1$. *Let* $\Re(A_j) = B_j$ *and* $\Im(A_j) = C_j$, $j = 0,1,\ldots$ *and assume* $B_0 \geqslant B_1 \geqslant \ldots$, $B_0 > 0$. *Then the zeros of* $F(z)$ *lie outside the disk*

$$|z| < \frac{1}{1+\frac{2}{\|B_0^{-1}\|^{-1}}\sum_{j=0}^{\infty}\|C_j\|}. \tag{2.4}$$

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