Operators
and
Matrices
Volume 17, Number 2 (2023), 485–515
doi:10.7153/oam-2023-17-32

FURTHER NEW REFINEMENTS AND REVERSES OF REAL POWER FORM FOR YOUNG–TYPE INEQUALITIES VIA FAMOUS CONSTANTS AND APPLICATIONS

DOAN THI THUY VAN AND DUONG QUOC HUY*

(*Communicated by F. Kittaneh*)

Abstract. In this paper, we propose new refinements and reverses of real power form for Young-type inequalities, which generalizes the recent inspired results by D. Q. Huy et al. [Linear Algebra Appl. **656** (2023), 368-384], and by Y. Ren et al. [J. Inequal. Appl. **2020** (2020), Paper No. 98, 13 p.]. Furthermore, the above refinements and reverses are continued to improve via the famous constants consisting of Kantorovich constant and Specht ratio. As applications, we establish operator versions, inequalities for unitarily invariant norms and inequalities for determinants of matrices.

1. Introduction

The classical Young inequality for scalars says that if $a,b>0$ and $\nu\in[0,1]$, then

$$(1-\nu)a+\nu b\geqslant a^{1-\nu}b^{\nu}, \tag{1}$$

with equality if and only if $a=b$. The inequality (1) is also called ν-weighted arithmetic-geometric mean inequality.

This inequality has been extended and generalized to many different frameworks with refinements and reverses (see, e.g., [2, 4, 6, 7, 8]). In 2017, Kórus [10] gave a refinement of the Young inequality (1) as follows

$$(1-\nu)a+\nu b\geqslant \left(1+L(\nu)\ln^2\left(\frac{a}{b}\right)\right)a^{1-\nu}b^{\nu}, \tag{2}$$

where $L(\nu)$ is a 1-periodic function given by

$$L(\nu)=\frac{\nu^2}{2}\left(\frac{1-\nu}{\nu}\right)^{2\nu} \text{ for } \nu\in(0,1] \text{ and } L(0)=0. \tag{3}$$

Besides, L is symmetric about $\frac{1}{2}$, namely, $L(\nu)=L(1-\nu)$, for all $\nu\in[0,1]$. He also applied the obtained results to the operator version.

Mathematics subject classification (2020): 15A39, 15A60, 15B48, 47A30, 47A63.

Keywords and phrases: Young inequality, logarithmic constant, Kantorovich constant, Specht ratio, Operator inequality, Positive operator, Arithmetic-Geometric mean inequality, Weak sub-majorization.

* Corresponding author.

Paper OaM-17-32

In 2019, C. Yang, Y. Gao and F. Lu [17] obtained the following refinement of inequality (1)

$$(1-\nu)a+\nu b \geqslant \left(1+\frac{L(2\nu)}{4}\ln^2\left(\frac{a}{b}\right)\right)a^{1-\nu}b^{\nu}+r_0(\sqrt{a}-\sqrt{b})^2, \tag{4}$$

where $r_0=\min\{\nu,1-\nu\}$. Similar to Kórus, Yang's group also found an application to the operator version of the resulting inequality.

Recently, Y. Ren and P. Li [13] have continued to give an improvement to the classical Young inequality of the form

$$\begin{aligned}(1-\nu)a+\nu b \geqslant{}& \left(1+\frac{L(8\nu)}{64}\ln^2\left(\frac{a}{b}\right)\right)a^{1-\nu}b^{\nu}+r_0(\sqrt{a}-\sqrt{b})^2 \\ &+r_1\left[(\sqrt{a}-\sqrt[4]{ab})^2\chi_{(0,\frac{1}{2})}(\nu)+(\sqrt{b}-\sqrt[4]{ab})^2\chi_{(\frac{1}{2},1)}(\nu)\right] \\ &+r_2\Big[(\sqrt{a}-\sqrt[8]{a^3b})^2\chi_{(0,\frac{1}{4})}(\nu)+(\sqrt[4]{ab}-\sqrt[8]{a^3b})^2\chi_{(\frac{1}{4},\frac{1}{2})}(\nu) \\ &\quad+(\sqrt[4]{ab}-\sqrt[8]{ab^3})^2\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)+(\sqrt{b}-\sqrt[8]{ab^3})^2\chi_{(\frac{3}{4},1)}(\nu)\Big],\end{aligned} \tag{5}$$

where $r_1=\min\{2r_0,1-2r_0\}$, $r_2=\min\{2r_1,1-2r_1\}$ and $\chi_I(\nu)$ is the characteristic function of an interval I, defined by $\chi_I(\nu)=1$ if $\nu\in I$ and $\chi_I(\nu)=0$ if $\nu\notin I$.

The latest relevant results are given by M. A. Ighachane and M. Akkouchi in [5], the authors proposed a multiple-term refinement and reverse of Young's inequality of the form

$$\begin{aligned}(1-\nu)a+\nu b \geqslant{}& \left(1+\frac{L\left(2^N\nu\right)}{2^{2N}}\ln^2\left(\frac{a}{b}\right)\right)a^{1-\nu}b^{\nu} \\ &+r_0(\sqrt{a}-\sqrt{b})^2+\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}f_{l,k}(a,b)\chi_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}(\nu),\end{aligned} \tag{6}$$

and

$$\begin{aligned}(1-\nu)a+\nu b \leqslant{}& \left(1+\frac{L\left(2^N(1-\nu)\right)}{2^{2N}}\ln^2\left(\frac{a}{b}\right)\right)^{-1}a^{1-\nu}b^{\nu} \\ &+R_0(\sqrt{a}-\sqrt{b})^2-\sum_{l=1}^{N-1}r_l(\nu)\sum_{k=1}^{2^l}f_{l,k}(a,b)\chi_{\left(\frac{k-1}{2^l},\frac{k}{2^l}\right)}(\nu),\end{aligned} \tag{7}$$

where N is an arbitrary positive integer, $r_l(\nu)$ and $f_{l,k}$ are defined in (8) and (9) below

$$r_l(\nu)=\begin{cases}2^l\nu-k+1, & \text{if } \frac{k-1}{2^l}\leqslant\nu\leqslant\frac{2k-1}{2^{l+1}} \\ k-2^l\nu, & \text{if } \frac{2k-1}{2^{l+1}}\leqslant\nu\leqslant\frac{k}{2^l},\end{cases} \tag{8}$$

$$f_{l,k}(a,b)=\left(\sqrt{a^{\frac{k-1}{2^l}}b^{1-\frac{k-1}{2^l}}}-\sqrt{a^{\frac{k}{2^l}}b^{1-\frac{k}{2^l}}}\right)^2, \tag{9}$$

with $l = 0,\dots,N-1$ and $k = 1,\dots,2^l$.

Observing the results of Young's inequality corresponding to $L(\nu)$, we temporarily call it the logarithmic coefficient, we find that most of the achievements are in the direction of tightening the inequality (1), there are no outstanding results for improvement exponentiation of the quantity $(1-\nu)a+\nu b$. From the above motivation, in this article we will improve and give more general results for the improvements made by mathematicians in (2), (4), (5). One of the main results of the present paper is as follows

$$\begin{aligned}
&[(1-\nu)a+\nu b]^p \\
&\geqslant \left[\left(1+\frac{L(8\nu)}{64}(\ln a-\ln b)^2\right)a^{1-\nu}b^\nu\right]^p + (2r_0)^p\left[\left(\frac{a+b}{2}\right)^p - \left(\sqrt{ab}\right)^p\right] \\
&\quad + (2r_1)^p\left(\sqrt{a}^p\chi_{(0,\frac{1}{2})}(\nu)+\sqrt{b}^p\chi_{(\frac{1}{2},1)}(\nu)\right)\left[\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^p - \left(\sqrt[4]{ab}\right)^p\right] \\
&\quad + (2r_2)^p\left(\sqrt[4]{a^3}^p\chi_{(0,\frac{1}{4})}(\nu)+\sqrt[4]{a^2b}^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)+\sqrt[4]{ab^2}^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)+\sqrt[4]{b^3}^p\chi_{(\frac{3}{4},1)}(\nu)\right) \\
&\qquad\qquad \times\left[\left(\frac{\sqrt[4]{a}+\sqrt[4]{b}}{2}\right)^p - \left(\sqrt[8]{ab}\right)^p\right],
\end{aligned}$$

and

$$\begin{aligned}
&[(1-\nu)a+\nu b]^p \\
&\leqslant \left[\left(1+\frac{L(8(1-\nu))}{64}(\ln a-\ln b)^2\right)^{-1}a^{1-\nu}b^\nu\right]^p + (2R_0)^p\left[\left(\frac{a+b}{2}\right)^p - \left(\sqrt{ab}\right)^p\right] \\
&\quad - (2r_1)^p(\sqrt{b}^p\chi_{(0,\frac{1}{2})}(\nu)+\sqrt{a}^p\chi_{(\frac{1}{2},1)})(\nu))\left[\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^p - \left(\sqrt[4]{ab}\right)^p\right] \\
&\quad - (2r_2)^p\left(\sqrt[4]{b^3}^p\chi_{(0,\frac{1}{4})}(\nu)+\sqrt[4]{ab^2}^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)+\sqrt[4]{a^2b}^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)+\sqrt[4]{a^3}^p\chi_{(\frac{3}{4},1)}(\nu)\right) \\
&\qquad\qquad \times\left[\left(\frac{\sqrt[4]{a}+\sqrt[4]{b}}{2}\right)^p - \left(\sqrt[8]{ab}\right)^p\right],
\end{aligned}$$

where $R_0 = \max\{\nu, 1-\nu\}$ and $p \geqslant 1$ is a arbitrary real number.

This paper is organized as follows. After the forgoing section, we state and prove our main results in Section 2 relying on the theory of weak sub-majorization. In section 3, we present the application of the main results to operator inequalities. Some refinements and reverses of inequalities for unitarily invariant norms are given in Section 4. Finally, Section 5 is devoted to establishing new refinements for determinants of matrices.

2. Some refinements and reverses of real power form for Young-type inequalities with famous constants

The main goal of this section is to establish further new refinements and reverses real power form for Young-type inequalities, which consist of the Kantorovich, Logarithmic constants and Specht's ratio. The method used here is based on the theory of weak sub-majorization.

2.1. Some preliminaries on the theory of weak sub-majorization and auxiliary results

We recall the definition of the weak sub-majorization. Let $x=(x_1,x_2,\dots,x_n)$ and $y=(y_1,y_2,\dots,y_n)$ are two vectors in $\mathbb{R}^n$. Then, x is called weak sub-majorization of y, denoted by $x \prec_w y$, if

$$\sum_{i=1}^{k} x_i^* \leqslant \sum_{i=1}^{k} y_i^*,$$

where $k=1,\ 2,\dots,n$ and $x_i^*,\ y_i^*,\ i=1,2,\dots,n$ respectively are components of vectors $x^*=(x_1^*,x_2^*,\dots,x_n^*)$ and $y^*=(y_1^*,y_2^*,\dots,y_n^*)$ satisfying

$$x_1^* \geqslant x_2^* \geqslant \dots \geqslant x_n^* \quad \text{and} \quad y_1^* \geqslant y_2^* \geqslant \dots \geqslant y_n^*.$$

An important feature of the theory of weak sub-majorization via continuously increasing convex functions is given in the following.

LEMMA 2.1. ([12, pp. 13]) *Let $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ be two vectors in $\mathbb{R}^n$ and $I\subset\mathbb{R}$ be an interval containing x_i,y_i for $i=1,\dots,n$. Then,*

$$x \prec_w y \text{ if and only if } \sum_{i=1}^{n} g(x_i) \leqslant \sum_{i=1}^{n} g(y_i), \tag{10}$$

for every continuously increasing convex funtion $g: I\to\mathbb{R}$.

Hereafter, we will state and prove some auxiliary results, which are used in the proofs of our main inequalities of this section.

LEMMA 2.2. *Let $x=(x_1,x_2,x_3,x_4)$, $y=(y_1,y_2,y_3,y_4)$ be two vectors in $\mathbb{R}^4$ with components*

$$x_1=\left[1+\frac{L(8\nu)}{64}(\ln a-\ln b)^2\right]a^{1-\nu}b^{\nu},$$
$$x_2=r_0(a+b),$$
$$x_3=r_1(\sqrt{a}+\sqrt{b})\left(\sqrt{a}\chi_{(0,\frac12)}(\nu)+\sqrt{b}\chi_{(\frac12,1)}(\nu)\right),$$
$$x_4=r_2(\sqrt[4]{a}+\sqrt[4]{b})\left(\sqrt[4]{a^3}\chi_{(0,\frac14)}(\nu)+\sqrt[4]{a^2b}\chi_{(\frac14,\frac12)}(\nu)+\sqrt[4]{ab^2}\chi_{(\frac12,\frac34)}(\nu)+\sqrt[4]{b^3}\chi_{(\frac34,1)}(\nu)\right),$$

and

$$y_1 = (1-\nu)a + \nu b,$$
$$y_2 = 2r_0\sqrt{ab},$$
$$y_3 = 2r_1\sqrt[4]{ab}\left(\sqrt{a}\chi_{(0,\frac{1}{2})}(\nu) + \sqrt{b}\chi_{(\frac{1}{2},1)}(\nu)\right),$$
$$y_4 = 2r_2\sqrt[8]{ab}\left(\sqrt[4]{a^3}\chi_{(0,\frac{1}{4})}(\nu) + \sqrt[4]{a^2b}\chi_{(\frac{1}{4},\frac{1}{2})}(\nu) + \sqrt[4]{ab^2}\chi_{(\frac{1}{2},\frac{3}{4})}(\nu) + \sqrt[4]{b^3}\chi_{(\frac{3}{4},1)}(\nu)\right).$$

Then, we have $x \prec_w y$.

Proof. To prove $x \prec_w y$, we clarify following inequalities

$$x_1^* + x_2^* + x_3^* + x_4^* \leqslant y_1^* + y_2^* + y_3^* + y_4^*, \tag{11}$$
$$x_1^* + x_2^* + x_3^* \leqslant y_1^* + y_2^* + y_3^*, \tag{12}$$
$$x_1^* + x_2^* \leqslant y_1^* + y_2^*, \tag{13}$$
$$x_1^* \leqslant y_1^*. \tag{14}$$

To prove inequality (11), we need to show

$$x_1 + x_2 + x_3 + x_4 \leqslant y_1 + y_2 + y_3 + y_4. \tag{15}$$

This is evidently deduced from (5).

Inequality (12) is cleared up if the following inequalities are verified

$$x_1 + x_2 + x_3 \leqslant y_1 + y_2 + y_3, \tag{16}$$
$$x_1 + x_2 + x_4 \leqslant y_1 + y_2 + y_4, \tag{17}$$
$$x_1 + x_3 + x_4 \leqslant y_1 + y_3 + y_4, \tag{18}$$
$$x_2 + x_3 + x_4 \leqslant y_1 + y_2 + y_3. \tag{19}$$

From (15) we get $x_1 + x_2 + x_3 \leqslant y_1 + y_2 + y_3 - (x_4 - y_4) \leqslant y_1 + y_2 + y_3$. The reason we have this is because $x_4 \geqslant y_4$. So (16) is true. By analogy, starting from inequality (15) and remarking that $x_3 \geqslant y_3$, $x_2 \geqslant y_2$, we can prove the inequalities (17) and (18).

However, inequality (19) is obtained in a rather complicated way, through elementary transformations, we get what we need to prove about an inequality that is always true for each particular case of ν. Specifically, we consider the following cases in turn.

- $\nu \in \left[0, \frac{1}{8}\right]$: we have $r_0 = \nu$, $r_1 = 2\nu$, $r_2 = 4\nu$, inequality (19) becomes $\nu(a+b) + 2\nu\sqrt{a}(\sqrt{a}+\sqrt{b}) + 4\nu\sqrt[4]{a^3}(\sqrt[4]{a}+\sqrt[4]{b}) \leqslant (1-\nu)a + \nu b + 2\nu\sqrt{ab} + 4\nu\sqrt[4]{ab}\sqrt{a}$. To make it short, we get $(1-8\nu)a \geqslant 0$. This is always true for ν under consideration.
- $\nu \in \left(\frac{1}{8}, \frac{1}{4}\right]$: we have $r_0 = \nu$, $r_1 = 2\nu$, $r_2 = 1-4\nu$, inequality (19) is equivalent to $\nu(a+b) + 2\nu\sqrt{a}(\sqrt{a}+\sqrt{b}) + (1-4\nu)\sqrt[4]{a^3}(\sqrt[4]{a}+\sqrt[4]{b}) \leqslant (1-\nu)a + \nu b + 2\nu\sqrt{ab} + 4\nu\sqrt[4]{ab}\sqrt{a}$. This can turn into $(8\nu-1)\sqrt[4]{a^3b} \geqslant 0$, which is always true for ν in this interval.

- $\nu \in \left(\frac{1}{4}, \frac{3}{8}\right]$: we have $r_0 = \nu$, $r_1 = 1-2\nu$, $r_2 = 4\nu - 1$, inequality (19) is specifically written as $\nu(a+b)+(1-2\nu)\sqrt{a}(\sqrt{a}+\sqrt{b})+(4\nu-1)\sqrt[4]{a^2b}(\sqrt[4]{a}+\sqrt[4]{b}) \leqslant (1-\nu)a+\nu b+2\nu\sqrt{ab}+2(1-2\nu)\sqrt[4]{ab}\sqrt{a}$, which is equivalent to $(3-8\nu)\sqrt[4]{a^3b} \geqslant 0$. This inequality is obvious in this case of ν.
- $\nu \in \left(\frac{3}{8}, \frac{1}{2}\right]$: we have $r_0 = \nu$, $r_1 = 1-2\nu$, $r_2 = 2-4\nu$. Substitute them in (19) we get $\nu(a+b)+(1-2\nu)\sqrt{a}(\sqrt{a}+\sqrt{b})+(2-4\nu)\sqrt[4]{a^2b}(\sqrt[4]{a}+\sqrt[4]{b}) \leqslant (1-\nu)a+\nu b+2\nu\sqrt{ab}+2(1-2\nu)\sqrt[4]{ab}\sqrt{a}$. The short form is as follows $(8\nu-3)\sqrt{ab} \geqslant 0$. All values of ν in this range satisfy the obtained inequality.
- $\nu \in \left(\frac{1}{2}, \frac{5}{8}\right]$: The proof can be deduced directly from the case $\nu \in \left(\frac{3}{8}, \frac{1}{2}\right]$.

Indeed, we have $r_0 = 1-\nu$, $r_1 = 2\nu - 1$, $r_2 = 4\nu - 2$ and $1-\nu \in \left[\frac{3}{8}, \frac{1}{2}\right)$, inequality (19) becomes $(1-\nu)(a+b)+(2\nu-1)\sqrt{b}(\sqrt{a}+\sqrt{b})+(4\nu-2)\sqrt[4]{ab^2}(\sqrt[4]{a}+\sqrt[4]{b}) \leqslant (1-\nu)a+\nu b+2(1-\nu)\sqrt{ab}+2(2\nu-1)\sqrt[4]{ab}\sqrt{b}$. By changing a,b and ν by b,a and $1-\nu$, respectively, we obtain the desired result exactly as the previous case $\nu \in \left(\frac{3}{8}, \frac{1}{2}\right]$. Notice that, at the endpoint $\nu = \frac{5}{8}$, then $1-\nu = \frac{3}{8}$. Therefore, when performing the above substitution technique, the inequality to be clarified becomes the case $\nu \in \left(\frac{1}{4}, \frac{3}{8}\right]$. For the remaining cases $\nu \in \left(\frac{5}{8}, \frac{3}{4}\right]$, $\nu \in \left(\frac{3}{4}, \frac{7}{8}\right]$ and $\left(\frac{7}{8}, 1\right]$, we check in the same way, so we omit the details.

Thus, we have done the inequality (12). Next we will go to test the inequality (13). Inequality (13) is checked if we have the following six minor inequalities

$$x_1 + x_2 \leqslant y_1 + y_2, \tag{20}$$
$$x_1 + x_3 \leqslant y_1 + y_3, \tag{21}$$
$$x_1 + x_4 \leqslant y_1 + y_4, \tag{22}$$
$$x_2 + x_3 \leqslant y_1 + y_2, \tag{23}$$
$$x_2 + x_4 \leqslant y_1 + y_2, \tag{24}$$
$$x_3 + x_4 \leqslant y_1 + y_3. \tag{25}$$

From (16), we get $x_1 + x_2 \leqslant y_1 + y_2 - (x_3 - y_3) \leqslant y_1 + y_2$. This is inferred from the comment $x_3 \geqslant y_3$. This means that the inequality (20) is cleared up. Also combining (16) and determining $x_2 \geqslant y_2$ we get the inequality (21). Similarly, combining (17) and $x_2 \geqslant y_2$ we get (22), combining (19) and $x_3 \geqslant y_3$ we get (24), combining (19) and $x_2 \geqslant y_2$ we get (25). The remaining inequality (23) has been clarified in [4] by D. Q. Huy et al.

Finally, our lemma is to perfect if the inequality (14) is proved. This means that we are going to unravel the following four inequalities

$$x_1 \leqslant y_1, \tag{26}$$
$$x_2 \leqslant y_1, \tag{27}$$

$$x_3 \leqslant y_1, \tag{28}$$
$$x_4 \leqslant y_1. \tag{29}$$

Since $r_0 = \min\{\nu, 1-\nu\}$, $x_2 \leqslant y_1$. This means (27) is clarified. With the same way as in the process of proving (16), combining (20) and commenting $x_2 \geqslant y_2$ we can deduce (26), also with this statement when combined with (23) we obtained (28), combined with (24) then deduce (29). This helps us not only to complete the proof (14) but also to reach the conclusion in Lemma. That means we are done with the proof of $x \prec_w y$.

In the process of proving this lemma, we have discovered something quite interesting that y_1^* is exactly y_1. Indeed, we have $y_1 \geqslant x_i$, for all $i = 1,2,3,4$, besides $x_i \geqslant y_i$, for all $i = 2,3,4$ so $y_1 \geqslant y_i$, for all $i = 2,3,4$. □

LEMMA 2.3. *Let $X = (X_1, X_2, X_3, X_4)$ and $Y = (Y_1, Y_2, Y_3, Y_4)$ be two vectors in $\mathbb{R}^4$ with components*

$$X_1 = (1-\nu)a + \nu b,\ X_2 = 2R_0\sqrt{ab},$$
$$X_3 = r_1(\sqrt{a}+\sqrt{b})\big(\sqrt{b}\chi_{(0,\frac{1}{2})}(\nu) + \sqrt{a}\chi_{(\frac{1}{2},1)}(\nu)\big),$$
$$X_4 = r_2(\sqrt[4]{a}+\sqrt[4]{b})\Big(\sqrt[4]{b^3}\chi_{(0,\frac{1}{4})}(\nu) + \sqrt[4]{b^2a}\chi_{(\frac{1}{4},\frac{1}{2})}(\nu) + \sqrt[4]{ba^2}\chi_{(\frac{1}{2},\frac{3}{4})}(\nu) + \sqrt[4]{a^3}\chi_{(\frac{3}{4},1)}(\nu)\Big),$$

and

$$Y_1 = R_0(a+b),\ Y_2 = \left[1 + \frac{L(8(1-\nu))}{64}(\ln a - \ln b)^2\right]^{-1} a^{1-\nu}b^{\nu},$$
$$Y_3 = 2r_1\sqrt[4]{ab}\big(\sqrt{b}\chi_{(0,\frac{1}{2})}(\nu) + \sqrt{a}\chi_{(\frac{1}{2},1)}(\nu)\big),$$
$$Y_4 = 2r_2\sqrt[8]{ab}\Big(\sqrt[4]{b^3}\chi_{(0,\frac{1}{4})}(\nu) + \sqrt[4]{b^2a}\chi_{(\frac{1}{4},\frac{1}{2})}(\nu) + \sqrt[4]{ba^2}\chi_{(\frac{1}{2},\frac{3}{4})}(\nu) + \sqrt[4]{a^3}\chi_{(\frac{3}{4},1)}(\nu)\Big),$$

where $R_0 = \max\{\nu, 1-\nu\}$. Then, we have $X \prec_w Y$.

Proof. The verification of $X \prec_w Y$ is not fundamentally different from $x \prec_w y$. We can summarize as follows. We need to clarify the following

$$X_1^* + X_2^* + X_3^* + X_4^* \leqslant Y_1^* + Y_2^* + Y_3^* + Y_4^*, \tag{30}$$
$$X_1^* + X_2^* + X_3^* \leqslant Y_1^* + Y_2^* + Y_3^*, \tag{31}$$
$$X_1^* + X_2^* \leqslant Y_1^* + Y_2^*, \tag{32}$$
$$X_1^* \leqslant Y_1^*. \tag{33}$$

First, inequality (30) is equivalent to $X_1 + X_2 + X_3 + X_4 \leqslant Y_1 + Y_2 + Y_3 + Y_4$. According to (7), choose $N = 3$, we get

$$(1-\nu)a + \nu b \leqslant \left[1 + \frac{L(8(1-\nu))}{64}(\ln a - \ln b)^2\right]^{-1} a^{1-\nu}b^{\nu} + R_0(\sqrt{a}-\sqrt{b})^2 - r_1\Big((\sqrt{b}-\sqrt[4]{ab})^2\chi_{(0,\frac{1}{2})}(\nu) + (\sqrt{a}-\sqrt[4]{ab})^2\chi_{(\frac{1}{2},1)}(\nu)\Big) \tag{34}$$

$$-r_2\left[(\sqrt{b}-\sqrt[8]{ab^3})^2\chi_{(0,\frac{1}{4})}(v)+(\sqrt[4]{ab}-\sqrt[8]{ab^3})^2\chi_{(\frac{1}{4},\frac{1}{2})}(v)\right.$$
$$\left.+(\sqrt[4]{ab}-\sqrt[8]{a^3b})^2\chi_{(\frac{1}{2},\frac{3}{4})}(v)+(\sqrt{a}-\sqrt[8]{a^3b})^2\chi_{(\frac{3}{4},1)}(v)\right].$$

Equivalent transformation of this result, we get $X_1+X_2+X_3+X_4 \leqslant Y_1+Y_2+Y_3+Y_4$. So (30) is complete.

Proving inequality (31) is converted to checking the correctness of the following specific inequalities

$$X_1+X_2+X_3 \leqslant Y_1+Y_2+Y_3, \tag{35}$$
$$X_1+X_2+X_4 \leqslant Y_1+Y_2+Y_4, \tag{36}$$
$$X_1+X_3+X_4 \leqslant Y_1^*+Y_2^*+Y_3^*, \tag{37}$$
$$X_2+X_3+X_4 \leqslant Y_1+Y_3+Y_4. \tag{38}$$

From (30), combined with the comments $X_4 \geqslant Y_4$, $X_3 \geqslant Y_3$ and $X_1 \geqslant Y_2$, by the same method implemented during test (16), we get clarify the inequalities (35), (36) and (38). The remaining inequality (37) is more difficult to test. We divide it into the following two cases.

- The first case, from now on we will make this case A, if $v \in \left[0,\frac{1}{2}\right]$, then $a \geqslant b$ and if $v \in \left(\frac{1}{2},1\right]$, then $a \leqslant b$. We will show that $X_3 \leqslant X_2$ so we can deduce $X_1+X_3+X_4 \leqslant X_1+X_2+X_4 \leqslant Y_1+Y_2+Y_4$. Specifically, we do the following

If $v \in \left[0,\frac{1}{4}\right]$ and $a \geqslant b$ then $X_2 = 2(1-v)\sqrt{ab}$, $X_3 = 2v\sqrt{b}(\sqrt{a}+\sqrt{b})$, so we have

$$\begin{aligned} X_2 - X_3 &= 2\sqrt{b}\left[(1-2v)\sqrt{a} - v\sqrt{b}\right] \\ &\geqslant 2v\sqrt{b}(\sqrt{a}-\sqrt{b}) \geqslant 0. \end{aligned}$$

This can be deduced $X_3 \leqslant X_2$.

If $v \in \left(\frac{1}{4},\frac{1}{2}\right]$ and $a \geqslant b$ then $X_2 = 2(1-v)\sqrt{ab}$, $X_3 = (1-2v)\sqrt{b}(\sqrt{a}+\sqrt{b})$. This leads to

$$\begin{aligned} X_2 - X_3 &= \sqrt{b}\left[\sqrt{a} - (1-2v)\sqrt{b}\right] \\ &> (1-2v)\sqrt{b}(\sqrt{a}-\sqrt{b}) \geqslant 0. \end{aligned}$$

This verifies that $X_3 \leqslant X_2$.

If $v \in \left(\frac{1}{2},1\right]$ and $a \leqslant b$ then $1-v \in \left[0,\frac{1}{2}\right)$. We also find that, if we replace a,b and v by b,a and $1-v$, respectively, as in the inequality clarification process (19), the result we need test becomes exactly the same as when considering $v \in \left[0,\frac{1}{2}\right]$ and $a \geqslant b$. So we can assert that $X_2 \geqslant X_3$ in this case.

• The second case, from now on we will make this case B, if $\nu \in \left[0,\frac{1}{2}\right]$, then $a<b$ and if $\nu \in \left(\frac{1}{2},1\right]$, then $a>b$. We will prove that $X_1+X_3+X_4 \leqslant Y_1+Y_3+Y_4$ by looking at specific cases of ν. The detailed process is as follows.

With $\nu \in \left[0,\frac{1}{8}\right]$ and $a<b$, we have $R_0=1-\nu,\ r_1=2\nu,\ r_2=4\nu$ and

$$\begin{aligned}
&Y_1+Y_3+Y_4-(X_1+X_3+X_4)\\
&\quad=(1-\nu)a+(1-\nu)b+4\nu\sqrt{b}\sqrt[4]{ab}+8\nu\sqrt[4]{b^3}\sqrt[8]{ab}\\
&\qquad-\left((1-\nu)a+\nu b+2\nu\sqrt{b}(\sqrt{a}+\sqrt{b})+4\nu\sqrt[4]{b^3}(\sqrt[4]{a}+\sqrt[4]{b})\right)\\
&\quad=(1-8\nu)b+8\nu\sqrt[8]{ab^7}-2\nu\sqrt{ab}\\
&\quad\geqslant(1-8\nu)b+2\nu\sqrt[8]{ab^4}\left(\sqrt[8]{b^3}-\sqrt[8]{a^3}\right)\geqslant 0.
\end{aligned}$$

With $\nu \in \left(\frac{1}{8},\frac{1}{4}\right]$ and $a<b$, we have $R_0=1-\nu,\ r_1=2\nu,\ r_2=1-4\nu$ and

$$\begin{aligned}
&Y_1+Y_3+Y_4-(X_1+X_3+X_4)\\
&\quad=(1-\nu)a+(1-\nu)b+4\nu\sqrt{b}\sqrt[4]{ab}+(2-8\nu)\sqrt[4]{b^3}\sqrt[8]{ab}\\
&\qquad-\left((1-\nu)a+\nu b+2\nu\sqrt{b}(\sqrt{a}+\sqrt{b})+(1-4\nu)\sqrt[4]{b^3}(\sqrt[4]{a}+\sqrt[4]{b})\right)\\
&\quad=(8\nu-1)\sqrt[4]{ab^3}+(2-8\nu)\sqrt[8]{ab^7}-2\nu\sqrt{ab}\\
&\quad\geqslant(8\nu-1)\sqrt{ab}+(2-8\nu)\sqrt{ab}-2\nu\sqrt{ab}\\
&\quad=(1-2\nu)\sqrt{ab}\geqslant 0.
\end{aligned}$$

With $\nu \in \left(\frac{1}{4},\frac{3}{8}\right]$ and $a<b$, we have $R_0=1-\nu,\ r_1=1-2\nu,\ r_2=4\nu-1$ and

$$\begin{aligned}
&Y_1+Y_3+Y_4-(X_1+X_3+X_4)\\
&\quad=(1-\nu)a+(1-\nu)b+(2-4\nu)\sqrt{b}\sqrt[4]{ab}+(8\nu-2)\sqrt[4]{ab^2}\sqrt[8]{ab}\\
&\qquad-\left((1-\nu)a+\nu b+(1-2\nu)\sqrt{b}(\sqrt{a}+\sqrt{b})+(4\nu-1)\sqrt[4]{ab^2}(\sqrt[4]{a}+\sqrt[4]{b})\right)\\
&\quad=(8\nu-2)\sqrt[8]{a^3b^5}+(3-8\nu)\sqrt[4]{ab^3}-2\nu\sqrt{ab}\\
&\quad\geqslant(8\nu-2)\sqrt{ab}+(3-8\nu)\sqrt{ab}-2\nu\sqrt{ab}\\
&\quad=(1-2\nu)\sqrt{ab}\geqslant 0.
\end{aligned}$$

With $v \in \left(\frac{3}{8}, \frac{1}{2}\right]$ and $a < b$, we have $R_0 = 1 - v$, $r_1 = 1 - 2v$, $r_2 = 2 - 4v$ and

$$\begin{aligned}
&Y_1 + Y_3 + Y_4 - (X_1 + X_3 + X_4) \\
&\quad = (1-v)a + (1-v)b + (2-4v)\sqrt{b}\sqrt[4]{ab} + (4-8v)\sqrt[4]{ab^2}\sqrt[8]{ab} \\
&\qquad - \Big((1-v)a + vb + (1-2v)\sqrt{b}(\sqrt{a}+\sqrt{b}) + (2-4v)\sqrt[4]{ab^2}(\sqrt[4]{a}+\sqrt[4]{b})\Big) \\
&\quad = (4-8v)\sqrt[8]{a^3b^5} - (3-6v)\sqrt{ab} \\
&\quad \geqslant (3-6v)\sqrt[8]{a^3b^4}(\sqrt[8]{b} - \sqrt[8]{a}) \geqslant 0.
\end{aligned}$$

When $v \in \left(\frac{1}{2}, 1\right]$ and $a > b$, we reuse the proof method of inequality (19), i.e., changing a, b and v by b, a and $1 - v$, respectively, the clarification can be obtained directly from the previous cases when $v \in \left[0, \frac{1}{2}\right]$ and $a < b$. The detailed process we do not repeat.

Next, we will clarify assertion (32) by giving truth to the following inequalities

$$X_1 + X_2 \leqslant Y_1 + Y_2, \tag{39}$$
$$X_1 + X_3 \leqslant Y_1^* + Y_2^*, \tag{40}$$
$$X_1 + X_4 \leqslant Y_1^* + Y_2^*, \tag{41}$$
$$X_2 + X_3 \leqslant Y_1 + Y_3, \tag{42}$$
$$X_2 + X_4 \leqslant Y_1 + Y_4, \tag{43}$$
$$X_3 + X_4 \leqslant Y_1 + Y_3. \tag{44}$$

With the familiar method used in the inequality proof stage (35), we will go with inequality (35) and observe that $X_3 \geqslant Y_3$ gives the result (39).

With the same method, if we combine (35) and determine $X_1 \geqslant Y_2$, we can deduce (42), combine (36) and the above statement, we get (43).

Continue the process, we clarify (40) by checking that $X_1 + X_3 \leqslant Y_1 + Y_3$ or $X_1 + X_3 \leqslant Y_1 + Y_2$. In case B, we have $X_1 + X_3 \leqslant Y_1 + Y_3$ (see [4]). In case A, reuse the result that $X_3 \leqslant X_2$ and the inequality (39), we have $X_1 + X_3 \leqslant X_1 + X_2 \leqslant Y_1 + Y_2$. Combining two cases, we have (40).

Next, we go to check the correctness of inequality (41) by proving that $X_1 + X_4 \leqslant Y_1 + Y_4$ in case B and $X_1 + X_4 \leqslant Y_1 + Y_2$ in case A. Details are as follows.

- In case B, using (37) and $X_3 \geqslant Y_3$, we get $X_1 + X_4 \leqslant Y_1 + Y_4 - (X_3 - Y_3) \leqslant Y_1 + Y_4$.
- In case A, we prove $X_4 \leqslant X_2$ from which we get $X_1 + X_4 \leqslant X_1 + X_2 \leqslant Y_1 + Y_2$. the latter part of the inequality is obtained by doing the same as in (39). Specifically, we do the following.

With $\nu \in \left[0, \frac{1}{8}\right]$ and $a \geqslant b$, we have $R_0 = 1-\nu$, $r_2 = 4\nu$ and

$$\begin{aligned} X_2 - X_4 &= 2(1-\nu)\sqrt{ab} - 4\nu\sqrt[4]{b^3}(\sqrt[4]{a}+\sqrt[4]{b}) \\ &= \left[(1-\nu)\sqrt{ab} - 4\nu\sqrt[4]{ab^3}\right] + \left[(1-\nu)\sqrt{ab} - 4\nu b\right] \\ &\geqslant 4\nu\sqrt[4]{ab^2}(\sqrt[4]{a} - \sqrt[4]{b}) + 4\nu\sqrt{b}(\sqrt{a} - \sqrt{b}) \geqslant 0. \end{aligned}$$

With $\nu \in \left(\frac{1}{8}, \frac{1}{4}\right]$ and $a \geqslant b$, we have $R_0 = 1-\nu$, $r_2 = 1-4\nu$ and

$$\begin{aligned} X_2 - X_4 &= 2(1-\nu)\sqrt{ab} - (1-4\nu)\sqrt[4]{b^3}(\sqrt[4]{a}+\sqrt[4]{b}) \\ &= \left[(1-\nu)\sqrt{ab} - (1-4\nu)\sqrt[4]{ab^3}\right] + \left[(1-\nu)\sqrt{ab} - (1-4\nu)b\right] \\ &\geqslant (1-4\nu)\sqrt[4]{ab^2}(\sqrt[4]{a} - \sqrt[4]{b}) + (1-4\nu)\sqrt{b}(\sqrt{a} - \sqrt{b}) \geqslant 0. \end{aligned}$$

With $\nu \in \left(\frac{1}{4}, \frac{3}{8}\right]$ and $a \geqslant b$, we have $R_0 = 1-\nu$, $r_2 = 4\nu - 1$ and

$$\begin{aligned} X_2 - X_4 &= 2(1-\nu)\sqrt{ab} - (4\nu-1)\sqrt[4]{ab^2}(\sqrt[4]{a}+\sqrt[4]{b}) \\ &= (3-6\nu)\sqrt[4]{a^2b^2} - (4\nu-1)\sqrt[4]{ab^3} \\ &\geqslant (4\nu-1)\sqrt[4]{ab^2}(\sqrt[4]{a} - \sqrt[4]{b}) \geqslant 0. \end{aligned}$$

With $\nu \in \left(\frac{3}{8}, \frac{1}{2}\right]$ and $a \geqslant b$, we have $R_0 = 1-\nu$, $r_2 = 2-4\nu$ and

$$\begin{aligned} X_2 - X_4 &= 2(1-\nu)\sqrt{ab} - (2-4\nu)\sqrt[4]{ab^2}(\sqrt[4]{a}+\sqrt[4]{b}) \\ &= 2\nu\sqrt[4]{a^2b^2} - (2-4\nu)\sqrt[4]{ab^3} \\ &\geqslant (2-4\nu)\sqrt[4]{ab^2}(\sqrt[4]{a} - \sqrt[4]{b}) \geqslant 0. \end{aligned}$$

With $\nu \in \left(\frac{1}{2}, 1\right]$ and $a \leqslant b$, we also comment that $1-\nu \in \left[0, \frac{1}{2}\right)$. Thus by converting the roles of a,b and ν by b,a and $1-\nu$, respectively, the inequalities to be checked $X_2 \geqslant X_4$ have been pointed out in the previous cases when $\nu \in \left[0, \frac{1}{2}\right]$ and $a \geqslant b$. We ignore the specific process.

The last operation to clarify (32) is prove (44). We accomplish this by directly computing the cases of ν, returning the above inequality to the value that is always true.

With $\nu \in \left[0, \frac{1}{8}\right]$, we have $R_0 = 1-\nu,\ r_1 = 2\nu,\ r_2 = 4\nu$ and

$$\begin{aligned} Y_1 + Y_3 - X_3 - X_4 &= (1-\nu)a + (1-\nu)b + 4\nu\sqrt{b}\sqrt[4]{ab} \\ &\quad - 2\nu\sqrt{b}(\sqrt{a}+\sqrt{b}) - 4\nu\sqrt[4]{b^3}(\sqrt[4]{a}+\sqrt[4]{b}) \\ &= (1-\nu)a + (1-7\nu)b - 2\nu\sqrt{ab} \\ &\geqslant \nu a + \nu b - 2\nu\sqrt{ab} \\ &= \nu(\sqrt{a}-\sqrt{b})^2 \geqslant 0. \end{aligned}$$

With $\nu \in \left(\frac{1}{8}, \frac{1}{4}\right]$, we have $R_0 = 1-\nu,\ r_1 = 2\nu,\ r_2 = 1-4\nu$ and

$$\begin{aligned} Y_1 + Y_3 - X_3 - X_4 &= (1-\nu)a + (1-\nu)b + 4\nu\sqrt{b}\sqrt[4]{ab} \\ &\quad - 2\nu\sqrt{b}(\sqrt{a}+\sqrt{b}) - (1-4\nu)\sqrt[4]{b^3}(\sqrt[4]{a}+\sqrt[4]{b}) \\ &= (1-\nu)a + \nu b - 2\nu\sqrt{ab} + (8\nu-1)\sqrt[4]{ab^3} \\ &\geqslant \nu(\sqrt{a}-\sqrt{b})^2 + (8\nu-1)\sqrt[4]{ab^3} \geqslant 0. \end{aligned}$$

With $\nu \in \left(\frac{1}{4}, \frac{3}{8}\right]$, we have $R_0 = 1-\nu,\ r_1 = 1-2\nu,\ r_2 = 4\nu-1$ and

$$\begin{aligned} Y_1 + Y_3 - X_3 - X_4 &= (1-\nu)a + (1-\nu)b + 2(1-2\nu)\sqrt{b}\sqrt[4]{ab} \\ &\quad - (1-2\nu)\sqrt{b}(\sqrt{a}+\sqrt{b}) - (4\nu-1)\sqrt[4]{ab^2}(\sqrt[4]{a}+\sqrt[4]{b}) \\ &= (1-\nu)a + \nu b - 2\nu\sqrt{ab} + (3-8\nu)\sqrt[4]{ab^3} \\ &\geqslant \nu(\sqrt{a}-\sqrt{b})^2 + (3-8\nu)\sqrt[4]{ab^3} \geqslant 0. \end{aligned}$$

With $\nu \in \left(\frac{3}{8}, \frac{1}{2}\right]$, we have $R_0 = 1-\nu,\ r_1 = 1-2\nu,\ r_2 = 2-4\nu$ and

$$\begin{aligned} Y_1 + Y_3 - X_3 - X_4 &= (1-\nu)a + (1-\nu)b + 2(1-2\nu)\sqrt{b}\sqrt[4]{ab} \\ &\quad - (1-2\nu)\sqrt{b}(\sqrt{a}+\sqrt{b}) - (2-4\nu)\sqrt[4]{ab^2}(\sqrt[4]{a}+\sqrt[4]{b}) \\ &= (1-\nu)a + \nu b - (3-6\nu)\sqrt{ab} \\ &\geqslant \nu a + \nu b - 2\nu\sqrt{ab} \\ &= \nu(\sqrt{a}-\sqrt{b})^2 \geqslant 0. \end{aligned}$$

For $\nu \in \left(\frac{1}{2}, 1\right]$, by the same manner as used while checking for inequalities (19), (37) or (41), we also get $X_3 + X_4 \leqslant Y_1 + Y_3$ satisfying for any ν value in this range.

To finish the proof of the lemma, we need to clarify inequality (33), which is equivalent to check the following four minor inequalities:

$$X_1 \leqslant Y_1, \tag{45}$$

$$X_2 \leqslant Y_1, \tag{46}$$

$$X_3 \leqslant Y_1, \tag{47}$$

$$X_4 \leqslant Y_1. \tag{48}$$

Since $R_0 = \max\{\nu, 1-\nu\}$ and applying the classical Young's inequality we easily get (45) and (46). Besides, we got (47) (see in [4]). Finally, according to (44) and using the property $X_3 \geqslant Y_3$, we have $X_4 \geqslant Y_1 - (X_3 - Y_3) \geqslant Y_1$. That means (48) is proven and we end the lemma. $\square$

2.2. Main results for scalars

Based on (10), we can prove the following important result. This improvement will extend the class of Young's inequality refinements with logarithmic coefficient. Details are as in the following theorem.

THEOREM 2.4. *If $a,b > 0$ and $0 \leqslant \nu \leqslant 1$, then for all real number $p \geqslant 1$, we have*

$$\begin{aligned}[(1-\nu)a+\nu b]^p \geqslant & \left[\left(1+\frac{L(8\nu)}{64}(\ln a - \ln b)^2\right)a^{1-\nu}b^\nu\right]^p + (2r_0)^p S_0 \\ & + (2r_1)^p\left(\sqrt{a}^p \chi_{(0,\frac{1}{2})}(\nu) + \sqrt{b}^p \chi_{(\frac{1}{2},1)}(\nu)\right) S_1 \\ & + (2r_2)^p\left(\sqrt[4]{a^3}^p \chi_{(0,\frac{1}{4})}(\nu) + \sqrt[4]{a^2b}^p \chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\right. \\ & \left. + \sqrt[4]{ab^2}^p \chi_{(\frac{1}{2},\frac{3}{4})}(\nu) + \sqrt[4]{b^3}^p \chi_{(\frac{3}{4},1)}(\nu)\right) S_2, \end{aligned} \tag{49}$$

and

$$\begin{aligned}[(1-\nu)a+\nu b]^p \leqslant & \left[\left(1+\frac{L(8(1-\nu))}{64}(\ln a - \ln b)^2\right)^{-1} a^{1-\nu}b^\nu\right]^p + (2R_0)^p S_0 \\ & - (2r_1)^p(\sqrt{b}^p \chi_{(0,\frac{1}{2})}(\nu) + \sqrt{a}^p \chi_{(\frac{1}{2},1)}(\nu))S_1 \\ & - (2r_2)^p\left(\sqrt[4]{b^3}^p \chi_{(0,\frac{1}{4})}(\nu) + \sqrt[4]{ab^2}^p \chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\right. \\ & \left. + \sqrt[4]{a^2b}^p \chi_{(\frac{1}{2},\frac{3}{4})}(\nu) + \sqrt[4]{a^3}^p \chi_{(\frac{3}{4},1)}(\nu)\right) S_2, \end{aligned} \tag{50}$$

where

$$S_0 := \left(\frac{a+b}{2}\right)^p - \left(\sqrt{ab}\right)^p, \quad S_1 := \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^p - \left(\sqrt[4]{ab}\right)^p$$

and

$$S_2 := \left(\frac{\sqrt[4]{a}+\sqrt[4]{b}}{2}\right)^p - \left(\sqrt[8]{ab}\right)^p.$$

Proof. Consider

$$g_0 : [0,+\infty) \to [0,+\infty)$$
$$t \mapsto t^p,$$

then g_0 is continuous increasing convex function. Apply Lemma 2.1 to g_0 and two vectors x, y is defined as in Lemma 2.2, we get

$$x_1^p + x_2^p + x_3^p + x_4^p \leqslant y_1^p + y_2^p + y_3^p + y_4^p,$$

or

$$\begin{aligned}
&\left[\left(1+\frac{L(8\nu)}{64}(\ln a-\ln b)^2\right)a^{1-\nu}b^{\nu}\right]^p + r_0^p(a+b)^p \\
&+ r_1^p(\sqrt{a}+\sqrt{b})^p\left(\sqrt{a}\chi_{(0,\frac{1}{2})}(\nu)+\sqrt{b}\chi_{(\frac{1}{2},1)}(\nu)\right)^p \\
&+ r_2^p(\sqrt[4]{a}+\sqrt[4]{b})^p\left(\sqrt[4]{a^3}\chi_{(0,\frac{1}{4})}(\nu)+\sqrt[4]{a^2b}\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\right. \\
&\qquad \left.+\sqrt[4]{ab^2}\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)+\sqrt[4]{b^3}\chi_{(\frac{3}{4},1)}(\nu)\right)^p \\
\leqslant\ & [(1-\nu)a+\nu b]^p + (2r_0)^p\sqrt{ab}^p \\
&+ (2r_1)^p\sqrt[4]{ab}^p\left(\sqrt{a}\chi_{(0,\frac{1}{2})}(\nu)+\sqrt{b}\chi_{(\frac{1}{2},1)}(\nu)\right)^p \\
&+ (2r_2)^p\sqrt[8]{ab}^p\left(\sqrt[4]{a^3}\chi_{(0,\frac{1}{4})}(\nu)+\sqrt[4]{a^2b}\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\right. \\
&\qquad \left.+\sqrt[4]{ab^2}\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)+\sqrt[4]{b^3}\chi_{(\frac{3}{4},1)}(\nu)\right)^p.
\end{aligned}$$

This inequality can be rewritten as

$$\begin{aligned}
[(1-\nu)a+\nu b]^p \geqslant & \left[\left(1+\frac{L(8\nu)}{64}(\ln a-\ln b)^2\right)a^{1-\nu}b^{\nu}\right]^p \\
&+ (2r_0)^p\left[\left(\frac{a+b}{2}\right)^p-\left(\sqrt{ab}\right)^p\right] \\
&+ (2r_1)^p\left(\sqrt{a}^p\chi_{(0,\frac{1}{2})}(\nu)+\sqrt{b}^p\chi_{(\frac{1}{2},1)}(\nu)\right)\left[\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^p-\left(\sqrt[4]{ab}\right)^p\right] \\
&+ (2r_2)^p\left(\sqrt[4]{a^3}^p\chi_{(0,\frac{1}{4})}(\nu)+\sqrt[4]{a^2b}^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)+\sqrt[4]{ab^2}^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)+\sqrt[4]{b^3}^p\chi_{(\frac{3}{4},1)}(\nu)\right) \\
&\times\left[\left(\frac{\sqrt[4]{a}+\sqrt[4]{b}}{2}\right)^p-\left(\sqrt[8]{ab}\right)^p\right].
\end{aligned}$$

Let $S_0 := \left(\frac{a+b}{2}\right)^p-\left(\sqrt{ab}\right)^p$, $S_1 := \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^p-\left(\sqrt[4]{ab}\right)^p$, $S_2 := \left(\frac{\sqrt[4]{a}+\sqrt[4]{b}}{2}\right)^p-\left(\sqrt[8]{ab}\right)^p$, we have (49).

Inequality (50) is similarly proved by applying using Lemma 2.1 to the function g_0 and the two vectors X,Y showed as in Lemma 2.3. $\square$

Using the same approach as Theorem 2.4, we can obtain the following important result for Young's inequality with Kantorovich constants. Recall that the Kantorovich constant has the form

$$K(h) := \frac{(h+1)^2}{4h}, \quad h > 0.$$

Clearly, the function K is strictly decreasing on $(0,1)$ and strictly increasing on $[1,+\infty)$, with $K(h) > 1$ for all $h \neq 1$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

THEOREM 2.5. *If $a,\ b > 0$ and $0 \leqslant \nu \leqslant 1$, then for all real number $p \geqslant 1$, we have*

$$\begin{aligned}[(1-\nu)a+\nu b]^p \geqslant & \left(K_3^{r_3} a^{1-\nu} b^\nu\right)^p + (2r_0)^p S_0 \\ & + (2r_1)^p \left(\sqrt{a}^p \chi_{(0,\frac{1}{2})}(\nu) + \sqrt{b}^p \chi_{(\frac{1}{2},1)}(\nu)\right) S_1 \\ & + (2r_2)^p \left(\sqrt[4]{a^3}\chi_{(0,\frac{1}{4})}(\nu) + \sqrt[4]{a^2 b}\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\right. \\ & \left. + \sqrt[4]{ab^2}\chi_{(\frac{1}{2},\frac{3}{4})}(\nu) + \sqrt[4]{b^3}\chi_{(\frac{3}{4},1)}(\nu)\right) S_2, \end{aligned} \tag{51}$$

and

$$\begin{aligned}[(1-\nu)a+\nu b]^p \leqslant & \left(K_3^{-r_3} a^{1-\nu} b^\nu\right)^p + (2R_0)^p S_0 \\ & - (2r_1)^p \left(\sqrt{b}^p \chi_{(0,\frac{1}{2})}(\nu) + \sqrt{a}^p \chi_{(\frac{1}{2},1)}(\nu)\right) S_1 \\ & - (2r_2)^p \left(\sqrt[4]{b^3}\chi_{(0,\frac{1}{4})}(\nu) + \sqrt[4]{ab^2}\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\right. \\ & \left. + \sqrt[4]{a^2 b}\chi_{(\frac{1}{2},\frac{3}{4})}(\nu) + \sqrt[4]{a^3}\chi_{(\frac{3}{4},1)}(\nu)\right) S_2, \end{aligned} \tag{52}$$

where $S_0,\ S_1,\ S_2$ are defined as in Theorem 2.4, $r_3 = \min\{2r_2, 1-2r_2\}$ and $K_3 := K\left(\sqrt[8]{\frac{b}{a}}\right) = \frac{(\sqrt[8]{a}+\sqrt[8]{b})^2}{4\sqrt[8]{ab}}$.

Proof. Comparing the similarity between the components of the terms in the Inequalities (49) and (51), we find that Inequality (51) will be cleared up if we can show that

$$\begin{aligned}(1-\nu)a+\nu b \geqslant & K_3^{r_3} a^{1-\nu} b^\nu + r_0(\sqrt{a}-\sqrt{b})^2 \\ & + r_1\left[(\sqrt{a}-\sqrt[4]{ab})^2\chi_{(0,\frac{1}{2})}(\nu) + (\sqrt{b}-\sqrt[4]{ab})^2\chi_{(\frac{1}{2},1)}(\nu)\right] \\ & + r_2\left[(\sqrt{a}-\sqrt[8]{a^3 b})^2\chi_{(0,\frac{1}{4})}(\nu) + (\sqrt[4]{ab}-\sqrt[8]{a^3 b})^2\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\right. \\ & \left. + (\sqrt[4]{ab}-\sqrt[8]{ab^3})^2\chi_{(\frac{1}{2},\frac{3}{4})}(\nu) + (\sqrt{b}-\sqrt[8]{a^3 b})^2\chi_{(\frac{3}{4},1)}(\nu)\right].\end{aligned}$$

This refinement was obtained by the authors M. Sababheh and M.S. Moslehian given in Theorem 2.11 in [15] when choosing the corresponding $N=3$.

To prove (52), we also use the same method as when we prove (50). We also remark that the two inequalities (52) and (50) only differ in two quantities, $K_3^{-r_3}a^{1-\nu}b^{\nu}$ and $\left(1+\frac{L(8(1-\nu))}{64}(\ln a-\ln b)^2\right)^{-1}a^{1-\nu}b^{\nu}$. Therefore, to clarify (52), we will give a proof that

$$\begin{aligned}(1-\nu)a+\nu b\leqslant K_3^{-r_3}a^{1-\nu}b^{\nu}&+R_0(\sqrt{a}-\sqrt{b})^2\\&-r_1\left((\sqrt{b}-\sqrt[4]{ab})^2\chi_{(0,\frac{1}{2})}(\nu)+(\sqrt{a}-\sqrt[4]{ab})^2\chi_{(\frac{1}{2},1)}(\nu)\right)\\&-r_2\Big[(\sqrt{b}-\sqrt[8]{ab^3})^2\chi_{(0,\frac{1}{4})}(\nu)+(\sqrt[4]{ab}-\sqrt[8]{ab^3})^2\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\\&\qquad+(\sqrt[4]{ab}-\sqrt[8]{a^3b})^2\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)+(\sqrt{a}-\sqrt[8]{a^3b})^2\chi_{(\frac{3}{4},1)}(\nu)\Big].\end{aligned}$$

We get this by applying Theorem 2.1 in [17], choosing $N=3$ respectively. □

Next, we mention another ratio that is also interested by many mathematicians, which is Specht's ratio, defined in [16] by

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h\in(0,1)\cup(1,\infty),\\ 1 & \text{if } h=1.\end{cases}\tag{53}$$

It is well known that $\lim_{h\to1}S(h)=1$ and $S(h)=S\left(\frac{1}{h}\right)>1$, for $h>0$, $h\neq1$. The function S is strictly decreasing on $(0,1)$ and strictly increasing on $(1,\infty)$. Moreover, the authors in [18] clarified that

$$K^r(h)\geqslant S(h^r)\quad\text{for } h>0 \text{ and } r\in\left[0,\frac{1}{2}\right].\tag{54}$$

This interesting relationship makes it easy to construct Young's inequality for Specht's ratio based on the available results with Kantorovich constants. Naturally, this is combined with the statement $r_3\in\left[0,\frac{1}{2}\right]$ helps us get the following achievement about Specht's ratio.

THEOREM 2.6. *If a, $b>0$ and $0\leqslant\nu\leqslant1$, then for all real number $p\geqslant1$, we have*

$$\begin{aligned}[(1-\nu)a+\nu b]^p\geqslant\left(S(h^{r_3})a^{1-\nu}b^{\nu}\right)^p&+(2r_0)^pS_0\\&+(2r_1)^p\left(\sqrt{a}^p\chi_{(0,\frac{1}{2})}(\nu)+\sqrt{b}^p\chi_{(\frac{1}{2},1)}(\nu)\right)S_1\\&+(2r_2)^p\Big(\sqrt[4]{a^3}^p\chi_{(0,\frac{1}{4})}(\nu)+\sqrt[4]{a^2b}^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\\&\qquad+\sqrt[4]{ab^2}^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)+\sqrt[4]{b^3}^p\chi_{(\frac{3}{4},1)}(\nu)\Big)S_2,\end{aligned}\tag{55}$$

and

$$\begin{aligned}[(1-\nu)a+\nu b]^p \leqslant \left(S^{-1}(h^{r_3})a^{1-\nu}b^{\nu}\right)^p &+ (2R_0)^p S_0 \\ &- (2r_1)^p\left(\sqrt{b}^p\chi_{(0,\frac{1}{2})}(\nu)+\sqrt{a}^p\chi_{(\frac{1}{2},1)}(\nu)\right)S_1 \\ &- (2r_2)^p\left(\sqrt[4]{b^3}^p\chi_{(0,\frac{1}{4})}(\nu)+\sqrt[4]{ab^2}^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\right. \\ &\left.+\sqrt[4]{a^2b}^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)+\sqrt[4]{a^3}^p\chi_{(\frac{3}{4},1)}(\nu)\right)S_2,\end{aligned} \tag{56}$$

where S_0, S_1, S_2 are defined as in Theorem 2.4 and $h=\sqrt[8]{\frac{b}{a}}$.

COROLLARY 2.7. *If $a,\ b>0$ and $0\leqslant\nu\leqslant1$, then for all real number $p\geqslant1$, we have*

$$\begin{aligned}[(1-\nu)a+\nu b]^p \geqslant \left(a^{1-\nu}b^{\nu}\right)^p &+ (2r_0)^p S_0 \\ &+ (2r_1)^p\left(\sqrt{a}^p\chi_{(0,\frac{1}{2})}(\nu)+\sqrt{b}^p\chi_{(\frac{1}{2},1)}(\nu)\right)S_1 \\ &+ (2r_2)^p\left(\sqrt[4]{a^3}^p\chi_{(0,\frac{1}{4})}(\nu)+\sqrt[4]{a^2b}^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\right. \\ &\left.+\sqrt[4]{ab^2}^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)+\sqrt[4]{b^3}^p\chi_{(\frac{3}{4},1)}(\nu)\right)S_2,\end{aligned} \tag{57}$$

and

$$\begin{aligned}[(1-\nu)a+\nu b]^p \leqslant \left(a^{1-\nu}b^{\nu}\right)^p &+ (2R_0)^p S_0 \\ &- (2r_1)^p\left(\sqrt{b}^p\chi_{(0,\frac{1}{2})}(\nu)+\sqrt{a}^p\chi_{(\frac{1}{2},1)}(\nu)\right)S_1 \\ &- (2r_2)^p\left(\sqrt[4]{b^3}^p\chi_{(0,\frac{1}{4})}(\nu)+\sqrt[4]{ab^2}^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\right. \\ &\left.+\sqrt[4]{a^2b}^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)+\sqrt[4]{a^3}^p\chi_{(\frac{3}{4},1)}(\nu)\right)S_2,\end{aligned} \tag{58}$$

where S_0, S_1, S_2 are defined as in Theorem 2.4.

Proof. Because of $1+\frac{L(8\nu)}{64}(\ln a-\ln b)^2\geqslant1$, for all $a,b>0$ and $\nu\in[0,1]$, we have

$$\begin{cases}\left(1+\frac{L(8\nu)}{64}(\ln a-\ln b)^2\right)a^{1-\nu}b^{\nu} & \geqslant a^{1-\nu}b^{\nu},\\ \left(1+\frac{L(8(1-\nu))}{64}(\ln a-\ln b)^2\right)^{-1}a^{1-\nu}b^{\nu} & \leqslant a^{1-\nu}b^{\nu}.\end{cases}$$

So, from (49), we get (57) and from (50), we receive (58). □

REMARK 2.8. The performances obtained in Corollary 2.7 provide one refining term for the recent appropriate results given in [4] by Huy, Van and Xinh. Furthermore, the results given in Theorems 2.5 and 2.6 are better than those in Corollary 2.7.

3. Operator versions for the generalizations of Young-type inequalities with famous constants

Our main goal in this section is to use versions of Young-type inequalities with some famous constants such as logarithmic, Kantorovich, Spechts to establish their operator forms.

On a complex Hilbert space H, we denote invertible positive operators by capital letters and the identity operator by I. In addition, we also use the following notations

- $A \geqslant 0$ $(A > 0)$ if A is a positive (invertible positive) operator;
- $A \geqslant B$ $(A > B)$ if $A - B$ is a positive (invertible positive) operator.

For $A, B > 0$ and $\nu \in (0,1)$ the ν-weighted arithmetic and geometric means of A and B are defined respectively by

$$A \nabla_\nu B = (1-\nu)A + \nu B,$$

$$A \sharp_\nu B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu} A^{1/2}.$$

We also write $A\nabla B$ and $A\sharp B$ instead of $A\nabla_{\frac{1}{2}}B$ and $A_{\sharp_{\frac{1}{2}}}B$, respectively. We also use the same symbol as geometric mean for $\nu \in \mathbb{R}$.

The main idea for showing operator inequalities corresponding to their scalar versions is to use the operator monotonicity of continuous functions in the following.

LEMMA 3.1. *Let X be an arbitrary self-adjoint operator. If f and g are continuous real-valued functions on the spectrum $\mathrm{Sp}(X)$ satisfying that $f(t) \geqslant g(t)$ for all $t \in \mathrm{Sp}(X)$, we then have an operator inequality $f(X) \geqslant g(X)$.*

Based on Theorems 2.4, 2.5, 2.6 and Corollary 2.7, we have the following results for the operator version.

THEOREM 3.2. *Let $0 < \nu < 1$ and $A, B > 0$ satisfy one of the following conditions*

(i) $0 < mI \leqslant A \leqslant \gamma I < \Gamma I \leqslant B \leqslant MI$,

(ii) $0 < mI \leqslant B \leqslant \gamma I < \Gamma I \leqslant A \leqslant MI$,

where $0 < M, m, \Gamma, \gamma < +\infty$ are scalars. Then for all a real number $p \geqslant 1$, we have

$$\begin{aligned} A\sharp_p(A\nabla_\nu B) \geqslant Q(\nu)^p A\sharp_{\nu p}B &+ (2r_0)^p[A\sharp_p(A\nabla B) - A\sharp_{\frac{p}{2}}B] \\ &+ r_1^p\chi_{(0,\frac{1}{2})}(\nu)[A\sharp_p(A + A\sharp B) - 2^p A\sharp_{\frac{p}{4}}B] \\ &+ r_1^p\chi_{(\frac{1}{2},1)}(\nu)[A\sharp_p(A\sharp B + B) - 2^p A\sharp_{\frac{3p}{4}}B] \\ &+ r_2^p\chi_{(0,\frac{1}{4})}(\nu)[A\sharp_p(A + A\sharp_{\frac{1}{4}}B) - 2^p A\sharp_{\frac{p}{8}}B] \\ &+ r_2^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)[A\sharp_p(A\sharp_{\frac{1}{4}}B + A\sharp B) - 2^p A\sharp_{\frac{3p}{8}}B] \\ &+ r_2^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)[A\sharp_p(A\sharp B + A\sharp_{\frac{3}{4}}B) - 2^p A\sharp_{\frac{5p}{8}}B] \\ &+ r_2^p\chi_{(\frac{3}{4},1)}(\nu)[A\sharp_p(A\sharp_{\frac{3}{4}}B + B) - 2^p A\sharp_{\frac{7p}{8}}B], \end{aligned} \tag{59}$$

and

$$
\begin{aligned}
A\sharp_p(A\nabla_\nu B) \leqslant Q(1-\nu)^{-p}A\sharp_{\nu p}B + (2R_0)^p[A\sharp_p(A\nabla B) - A\sharp_{\frac{p}{2}}B] \\
- r_1^p\chi_{(0,\frac{1}{2})}(\nu)[A\sharp_p(A\sharp B + B) - 2^pA\sharp_{\frac{3p}{4}}B] \\
- r_1^p\chi_{(\frac{1}{2},1)}(\nu)[A\sharp_p(A + A\sharp B) - 2^pA\sharp_{\frac{p}{4}}B] \\
- r_2^p\chi_{(0,\frac{1}{4})}(\nu)[A\sharp_p(B + A\sharp_{\frac{3}{4}}B) - 2^pA\sharp_{\frac{7p}{8}}B] \\
- r_2^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)[A\sharp_p(A\sharp_{\frac{3}{4}}B + A\sharp B) - 2^pA\sharp_{\frac{5p}{8}}B] \\
- r_2^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)[A\sharp_p(A\sharp B + A\sharp_{\frac{1}{4}}B) - 2^pA\sharp_{\frac{3p}{8}}B] \\
- r_2^p\chi_{(\frac{3}{4},1)}(\nu)[A\sharp_p(A\sharp_{\frac{1}{4}}B + A) - 2^pA\sharp_{\frac{p}{8}}B],
\end{aligned} \tag{60}
$$

where $Q(\nu) = 1 + \frac{L(8\nu)}{64}\ln^2\left(\frac{\Gamma}{\gamma}\right)$ *and* $L(\nu)$ *is given in (3).*

Proof. Notice that, $\frac{m}{M} \leqslant \frac{\gamma}{\Gamma} < 1 < \frac{\Gamma}{\gamma} \leqslant \frac{M}{m}$.

Firstly, we suppose that the operators A, B satisfy the condition (i). Utilizing the inequality (49) and the increase of the function $Q(\nu)(x) = 1 + \frac{L(8\nu)}{64}\ln^2(x)$ on $[1, +\infty)$, we have, for all $x \in \left[\frac{\Gamma}{\gamma}, \frac{M}{m}\right] \subset \left[\frac{m}{M}, \frac{M}{m}\right]$,

$$
\begin{aligned}
[(1-\nu)+\nu x]^p \geqslant (Q(\nu)(x)x^\nu)^p + (2r_0)^p\left[\left(\frac{1+x}{2}\right)^p - x^{p/2}\right] \\
+ r_1^p\chi_{(0,\frac{1}{2})}(\nu)\left[(1+x^{1/2})^p - 2^px^{p/4}\right] \\
+ r_1^p\chi_{(\frac{1}{2},1)}(\nu)\left[(x^{1/2}+x)^p - 2^px^{3p/4}\right] \\
+ r_2^p\chi_{(0,\frac{1}{4})}(\nu)\left[(1+x^{1/4})^p - 2^px^{p/8}\right] \\
+ r_2^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\left[(x^{1/4}+x^{1/2})^p - 2^px^{3p/8}\right] \\
+ r_2^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)\left[(x^{1/2}+x^{3/4})^p - 2^px^{5p/8}\right] \\
+ r_2^p\chi_{(\frac{3}{4},1)}(\nu)\left[(x^{3/4}+x)^p - 2^px^{7p/8}\right] \\
\geqslant \min_{h\leqslant x\leqslant h'}(Q(\nu)(x))^p x^{\nu p} + (2r_0)^p\left[\left(\frac{1+x}{2}\right)^p - x^{p/2}\right] \\
+ r_1^p\chi_{(0,\frac{1}{2})}(\nu)\left[(1+x^{1/2})^p - 2^px^{p/4}\right] \\
+ r_1^p\chi_{(\frac{1}{2},1)}(\nu)\left[(x^{1/2}+x)^p - 2^px^{3p/4}\right] \\
+ r_2^p\chi_{(0,\frac{1}{4})}(\nu)\left[(1+x^{1/4})^p - 2^px^{p/8}\right] \\
+ r_2^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\left[(x^{1/4}+x^{1/2})^p - 2^px^{3p/8}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ r_2^p \chi_{(\frac{1}{2},\frac{3}{4})}(\nu)\left[(x^{1/2}+x^{3/4})^p - 2^p x^{5p/8}\right]\\
&+ r_2^p \chi_{(\frac{3}{4},1)}(\nu)\left[(x^{3/4}+x)^p - 2^p x^{7p/8}\right]\\
= (Q(\nu)(h))^p x^{\nu p} &+ (2r_0)^p \left[\left(\frac{1+x}{2}\right)^p - x^{p/2}\right]\\
&+ r_1^p \chi_{(0,\frac{1}{2})}(\nu)\left[(1+x^{1/2})^p - 2^p x^{p/4}\right]\\
&+ r_1^p \chi_{(\frac{1}{2},1)}(\nu)\left[(x^{1/2}+x)^p - 2^p x^{3p/4}\right]\\
&+ r_2^p \chi_{(0,\frac{1}{4})}(\nu)\left[(1+x^{1/4})^p - 2^p x^{p/8}\right]\\
&+ r_2^p \chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\left[(x^{1/4}+x^{1/2})^p - 2^p x^{3p/8}\right]\\
&+ r_2^p \chi_{(\frac{1}{2},\frac{3}{4})}(\nu)\left[(x^{1/2}+x^{3/4})^p - 2^p x^{5p/8}\right]\\
&+ r_2^p \chi_{(\frac{3}{4},1)}(\nu)\left[(x^{3/4}+x)^p - 2^p x^{7p/8}\right],
\end{aligned}
$$

where $h = \dfrac{\Gamma}{\gamma}$ and $h' = \dfrac{M}{m}$. This, together with Lemma 3.1, implies that, for every positive operator X with its spectrum in $[h,h']$,

$$
\begin{aligned}
[(1-\nu)I+\nu X]^p \geqslant (Q(\nu)(h))^p X^{\nu p} &+ (2r_0)^p \left[\left(\tfrac{1}{2}I+\tfrac{1}{2}X\right)^p - X^{p/2}\right]\\
&+ r_1^p \chi_{(0,\frac{1}{2})}(\nu)\left[(I+X^{1/2})^p - 2^p X^{p/4}\right]\\
&+ r_1^p \chi_{(\frac{1}{2},1)}(\nu)\left[(X^{1/2}+X)^p - 2^p X^{3p/4}\right]\\
&+ r_2^p \chi_{(0,\frac{1}{4})}(\nu)\left[(I+X^{1/4})^p - 2^p X^{p/8}\right]\\
&+ r_2^p \chi_{(\frac{1}{4},\frac{1}{2})}(\nu)\left[(X^{1/4}+X^{1/2})^p - 2^p X^{3p/8}\right]\\
&+ r_2^p \chi_{(\frac{1}{2},\frac{3}{4})}(\nu)\left[(X^{1/2}+X^{3/4})^p - 2^p X^{5p/8}\right]\\
&+ r_2^p \chi_{(\frac{3}{4},1)}(\nu)\left[(X^{3/4}+X)^p - 2^p X^{7p/8}\right].
\end{aligned}
$$

On the other hand, by the condition (i), the spectrum $\mathrm{Sp}(A^{-1/2}BA^{-1/2})$ of the operator $A^{-1/2}BA^{-1/2}$ is in $\left[\dfrac{\Gamma}{\gamma}, \dfrac{M}{m}\right]$. Thus, replacing X in the above inequality with $A^{-1/2}BA^{-1/2}$, we have

$$
\begin{aligned}
&\left[(1-\nu)I+\nu A^{-1/2}BA^{-1/2}\right]^p \geqslant (Q(\nu)(h))^p \left(A^{-1/2}BA^{-1/2}\right)^{\nu p}\\
&+ (2r_0)^p \left[\left(\tfrac{1}{2}I+\tfrac{1}{2}A^{-1/2}BA^{-1/2}\right)^p - \left(A^{-1/2}BA^{-1/2}\right)^{p/2}\right]\\
&+ r_1^p \chi_{(0,\frac{1}{2})}(\nu)\left\{\left[I+\left(A^{-1/2}BA^{-1/2}\right)^{1/2}\right]^p - 2^p\left(A^{-1/2}BA^{-1/2}\right)^{p/4}\right\}
\end{aligned}
$$

$$+ r_1^p \chi_{(\frac{1}{2},1)}(\nu) \left\{ \left[\left(A^{-1/2}BA^{-1/2}\right)^{1/2} + A^{-1/2}BA^{-1/2} \right]^p - 2^p \left(A^{-1/2}BA^{-1/2}\right)^{3p/4} \right\}$$

$$+ r_2^p \chi_{(0,\frac{1}{4})}(\nu) \left\{ \left[I + \left(A^{-1/2}BA^{-1/2}\right)^{1/4} \right]^p - 2^p \left(A^{-1/2}BA^{-1/2}\right)^{p/8} \right\}$$

$$+ r_2^p \chi_{(\frac{1}{4},\frac{1}{2})}(\nu) \left\{ \left[\left(A^{-1/2}BA^{-1/2}\right)^{1/4} + (A^{-1/2}BA^{-1/2})^{1/2} \right]^p - 2^p \left(A^{-1/2}BA^{-1/2}\right)^{3p/8} \right\}$$

$$+ r_2^p \chi_{(\frac{1}{2},\frac{3}{4})}(\nu) \left\{ \left[\left(A^{-1/2}BA^{-1/2}\right)^{1/2} + (A^{-1/2}BA^{-1/2})^{3/4} \right]^p - 2^p \left(A^{-1/2}BA^{-1/2}\right)^{5p/8} \right\}$$

$$+ r_2^p \chi_{(\frac{3}{4},1)}(\nu) \left\{ \left[\left(A^{-1/2}BA^{-1/2}\right)^{3/4} + A^{-1/2}BA^{-1/2} \right]^p - 2^p \left(A^{-1/2}BA^{-1/2}\right)^{7p/8} \right\}.$$

Multiplying both sides of above inequality by $A^{1/2}$, we get

$$A^{1/2} \left\{ A^{-1/2} [(1-\nu)A + \nu B] A^{-1/2} \right\}^p A^{1/2}$$

$$\geqslant (Q(\nu)(h))^p A^{1/2} \left(A^{-1/2}BA^{-1/2}\right)^{\nu p} A^{1/2}$$

$$+ (2r_0)^p \left\{ A^{1/2} \left[A^{-1/2} \left(\tfrac{1}{2}I + \tfrac{1}{2}B\right) A^{-1/2} \right]^p A^{1/2} - A^{1/2} \left(A^{-1/2}BA^{-1/2}\right)^{p/2} A^{1/2} \right\}$$

$$+ r_1^p \chi_{(0,\frac{1}{2})}(\nu) \left\{ A^{1/2} \left[A^{-1/2} \left(A + A^{1/2} \left(A^{-1/2}BA^{-1/2}\right)^{1/2} A^{1/2} \right) A^{-1/2} \right]^p A^{1/2} - 2^p A^{1/2} \left(A^{-1/2}BA^{-1/2}\right)^{p/4} A^{1/2} \right\}$$

$$+ r_1^p \chi_{(\frac{1}{2},1)}(\nu) \left\{ A^{1/2} \left[A^{-1/2} \left(A^{1/2} \left(A^{-1/2}BA^{-1/2}\right)^{1/2} A^{1/2} + B \right) A^{-1/2} \right]^p A^{1/2} - 2^p A^{1/2} \left(A^{-1/2}BA^{-1/2}\right)^{3p/4} A^{1/2} \right\}$$

$$+ r_2^p \chi_{(0,\frac{1}{4})}(\nu) \left\{ A^{1/2} \left[A^{-1/2} \left(A + A^{1/2} \left(A^{-1/2}BA^{-1/2}\right)^{1/4} A^{1/2} \right) A^{-1/2} \right]^p A^{1/2} - 2^p A^{1/2} \left(A^{-1/2}BA^{-1/2}\right)^{p/8} A^{1/2} \right\}$$

$$
\begin{aligned}
&+r_2^p\chi_{(\frac14,\frac12)}(\nu)\Big\{A^{1/2}\Big[A^{-1/2}\Big(A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{1/4}A^{1/2}\\
&\qquad\qquad +A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{1/2}A^{1/2}\Big)A^{-1/2}\Big]^pA^{1/2}\\
&\qquad\qquad -2^pA^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{3p/8}A^{1/2}\Big\}\\
&+r_2^p\chi_{(\frac12,\frac34)}(\nu)\Big\{A^{1/2}\Big[A^{-1/2}\Big(A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{1/2}A^{1/2}\\
&\qquad\qquad +A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{3/4}A^{1/2}\Big)A^{-1/2}\Big]^pA^{1/2}\\
&\qquad\qquad -2^pA^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{5p/8}A^{1/2}\Big\}\\
&+r_2^p\chi_{(\frac34,1)}(\nu)\Big\{A^{1/2}\Big[A^{-1/2}\Big(A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{3/4}A^{1/2}+B\Big)A^{-1/2}\Big]^pA^{1/2}\\
&\qquad\qquad -2^pA^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{7p/8}A^{1/2}\Big\}.
\end{aligned}
$$

It is easy to see that this is equivalent to the inequality to be proved (59). Inequality (60) is tested in a similar way, so we omit the details. □

Using the achievement of the numerical version in Theorems 2.5, 2.6 and Corollary 2.7 with the proof method for Theorem 3.2, we have the following results for the operator version of Young's inequality in the fields: Kantorovich constants, Specht's ratio and coefficientless case, respectively.

THEOREM 3.3. *Under the hypotheses and notations of Theorem 3.2, we have*

$$
\begin{aligned}
A\sharp_p(A\nabla_\nu B) \geqslant K_3(h)^{r_3p}A\sharp_{\nu p}B &+ (2r_0)^p[A\sharp_p(A\nabla B)-A\sharp_{\frac p2}B]\\
&+r_1^p\chi_{(0,\frac12)}(\nu)[A\sharp_p(A+A\sharp B)-2^pA\sharp_{\frac p4}B]\\
&+r_1^p\chi_{(\frac12,1)}(\nu)[A\sharp_p(A\sharp B+B)-2^pA\sharp_{\frac{3p}4}B]\\
&+r_2^p\chi_{(0,\frac14)}(\nu)[A\sharp_p(A+A\sharp_{\frac14}B)-2^pA\sharp_{\frac p8}B]\\
&+r_2^p\chi_{(\frac14,\frac12)}(\nu)[A\sharp_p(A\sharp_{\frac14}B+A\sharp B)-2^pA\sharp_{\frac{3p}8}B]\\
&+r_2^p\chi_{(\frac12,\frac34)}(\nu)[A\sharp_p(A\sharp B+A\sharp_{\frac34}B)-2^pA\sharp_{\frac{5p}8}B]\\
&+r_2^p\chi_{(\frac34,1)}(\nu)[A\sharp_p(A\sharp_{\frac34}B+B)-2^pA\sharp_{\frac{7p}8}B],
\end{aligned}
\tag{61}
$$

and

$$
\begin{aligned}
A\sharp_p(A\nabla_\nu B) \leqslant K_3(h)^{-r_3p}A\sharp_{\nu p}B &+ (2R_0)^p[A\sharp_p(A\nabla B)-A\sharp_{\frac p2}B]\\
&-r_1^p\chi_{(0,\frac12)}(\nu)[A\sharp_p(A\sharp B+B)-2^pA\sharp_{\frac{3p}4}B]\\
&-r_1^p\chi_{(\frac12,1)}(\nu)[A\sharp_p(A+A\sharp B)-2^pA\sharp_{\frac p4}B]
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
&- r_2^p\chi_{(0,\frac{1}{4})}(\nu)[A\sharp_p(B+A\sharp_{\frac{3}{4}}B)-2^pA\sharp_{\frac{7p}{8}}B]\\
&- r_2^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)[A\sharp_p(A\sharp_{\frac{3}{4}}B+A\sharp B)-2^pA\sharp_{\frac{5p}{8}}B]\\
&- r_2^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)[A\sharp_p(A\sharp B+A\sharp_{\frac{1}{4}}B)-2^pA\sharp_{\frac{3p}{8}}B]\\
&- r_2^p\chi_{(\frac{3}{4},1)}(\nu)[A\sharp_p(A\sharp_{\frac{1}{4}}B+A)-2^pA\sharp_{\frac{p}{8}}B],
\end{aligned}
$$

where $K_3 := K(\sqrt[8]{h})$ *with* $h=\frac{\Gamma}{\gamma}$.

THEOREM 3.4. *Under the hypotheses and notations as in Theorem 3.2, we have*

$$
\begin{aligned}
A\sharp_p(A\nabla_\nu B) \geqslant S(h^{r_3})^pA\sharp_{\nu p}B &+ (2r_0)^p[A\sharp_p(A\nabla B)-A\sharp_{\frac{p}{2}}B] \qquad (63)\\
&+ r_1^p\chi_{(0,\frac{1}{2})}(\nu)[A\sharp_p(A+A\sharp B)-2^pA\sharp_{\frac{p}{4}}B]\\
&+ r_1^p\chi_{(\frac{1}{2},1)}(\nu)[A\sharp_p(A\sharp B+B)-2^pA\sharp_{\frac{3p}{4}}B]\\
&+ r_2^p\chi_{(0,\frac{1}{4})}(\nu)[A\sharp_p(A+A\sharp_{\frac{1}{4}}B)-2^pA\sharp_{\frac{p}{8}}B]\\
&+ r_2^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)[A\sharp_p(A\sharp_{\frac{1}{4}}B+A\sharp B)-2^pA\sharp_{\frac{3p}{8}}B]\\
&+ r_2^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)[A\sharp_p(A\sharp B+A\sharp_{\frac{3}{4}}B)-2^pA\sharp_{\frac{5p}{8}}B]\\
&+ r_2^p\chi_{(\frac{3}{4},1)}(\nu)[A\sharp_p(A\sharp_{\frac{3}{4}}B+B)-2^pA\sharp_{\frac{7p}{8}}B],
\end{aligned}
$$

and

$$
\begin{aligned}
A\sharp_p(A\nabla_\nu B) \leqslant (S^{-1}(h^{r_3}))^pA\sharp_{\nu p}B &+ (2R_0)^p[A\sharp_p(A\nabla B)-A\sharp_{\frac{p}{2}}B] \qquad (64)\\
&- r_1^p\chi_{(0,\frac{1}{2})}(\nu)[A\sharp_p(A\sharp B+B)-2^pA\sharp_{\frac{3p}{4}}B]\\
&- r_1^p\chi_{(\frac{1}{2},1)}(\nu)[A\sharp_p(A+A\sharp B)-2^pA\sharp_{\frac{p}{4}}B]\\
&- r_2^p\chi_{(0,\frac{1}{4})}(\nu)[A\sharp_p(B+A\sharp_{\frac{3}{4}}B)-2^pA\sharp_{\frac{7p}{8}}B]\\
&- r_2^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)[A\sharp_p(A\sharp_{\frac{3}{4}}B+A\sharp B)-2^pA\sharp_{\frac{5p}{8}}B]\\
&- r_2^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)[A\sharp_p(A\sharp B+A\sharp_{\frac{1}{4}}B)-2^pA\sharp_{\frac{3p}{8}}B]\\
&- r_2^p\chi_{(\frac{3}{4},1)}(\nu)[A\sharp_p(A\sharp_{\frac{1}{4}}B+A)-2^pA\sharp_{\frac{p}{8}}B],
\end{aligned}
$$

where $S(h)$ *is Specht's ratio with* $h=\frac{\Gamma}{\gamma}$.

THEOREM 3.5. *Let* A,B *be two positive invertible operators. We have, for every real number* $p\geqslant 1$*:*

$$
\begin{aligned}
A\sharp_p(A\nabla_\nu B) \geqslant A\sharp_{\nu p}B &+ (2r_0)^p[A\sharp_p(A\nabla B)-A\sharp_{\frac{p}{2}}B] \qquad (65)\\
&+ r_1^p\chi_{(0,\frac{1}{2})}(\nu)[A\sharp_p(A+A\sharp B)-2^pA\sharp_{\frac{p}{4}}B]\\
&+ r_1^p\chi_{(\frac{1}{2},1)}(\nu)[A\sharp_p(A\sharp B+B)-2^pA\sharp_{\frac{3p}{4}}B]
\end{aligned}
$$

$$+r_2^p\chi_{(0,\frac{1}{4})}(\nu)[A\sharp_p(A+A\sharp_{\frac{1}{4}}B)-2^pA\sharp_{\frac{p}{8}}B]$$
$$+r_2^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)[A\sharp_p(A\sharp_{\frac{1}{4}}B+A\sharp B)-2^pA\sharp_{\frac{3p}{8}}B]$$
$$+r_2^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)[A\sharp_p(A\sharp B+A\sharp_{\frac{3}{4}}B)-2^pA\sharp_{\frac{5p}{8}}B]$$
$$+r_2^p\chi_{(\frac{3}{4},1)}(\nu)[A\sharp_p(A\sharp_{\frac{3}{4}}B+B)-2^pA\sharp_{\frac{7p}{8}}B],$$

and

$$\begin{aligned} A\sharp_p(A\nabla_\nu B)\leqslant A\sharp_{\nu p}B &+(2R_0)^p[A\sharp_p(A\nabla B)-A\sharp_{\frac{p}{2}}B] \\ &-r_1^p\chi_{(0,\frac{1}{2})}(\nu)[A\sharp_p(A\sharp B+B)-2^pA\sharp_{\frac{3p}{4}}B] \\ &-r_1^p\chi_{(\frac{1}{2},1)}(\nu)[A\sharp_p(A+A\sharp B)-2^pA\sharp_{\frac{p}{4}}B] \\ &-r_2^p\chi_{(0,\frac{1}{4})}(\nu)[A\sharp_p(B+A\sharp_{\frac{3}{4}}B)-2^pA\sharp_{\frac{7p}{8}}B] \\ &-r_2^p\chi_{(\frac{1}{4},\frac{1}{2})}(\nu)[A\sharp_p(A\sharp_{\frac{3}{4}}B+A\sharp B)-2^pA\sharp_{\frac{5p}{8}}B] \\ &-r_2^p\chi_{(\frac{1}{2},\frac{3}{4})}(\nu)[A\sharp_p(A\sharp B+A\sharp_{\frac{1}{4}}B)-2^pA\sharp_{\frac{3p}{8}}B] \\ &-r_2^p\chi_{(\frac{3}{4},1)}(\nu)[A\sharp_p(A\sharp_{\frac{1}{4}}B+A)-2^pA\sharp_{\frac{p}{8}}B]. \end{aligned} \tag{66}$$

4. Inequalities for unitarily invariant norms

Let M_n be the algebra of all $n\times n$ complex matrices. A norm $|||\cdot|||$ on M_n is said to be *unitarily invariant* if $|||UAV|||=|||A|||$ for all $A\in M_n$ and for all unitary matrices $U,V\in M_n$. Let $s_1(A)\geqslant s_2(A)\geqslant\cdots\geqslant s_n(A)$ be the singular values of $A\in M_n$. The Schatten p-norms, $p\in[1,\infty)$, written $\|\cdot\|_p$, and defined by

$$\|A\|_p=\left(\sum_{i=1}^n s_i^p(A)\right)^{1/p},$$

are typical examples of unitarily invariant norms. The trace norm of $A\in M_n$, as usually expressed as $\|A\|_1=\operatorname{tr}|A|$, is defined as the Schatten 1-norm of A, where $|A|:=(A^*A)^{1/2}$ is a positive semidefinite matrix.

It is known (see [1]) that if $X\in M_n$ and $A,B\in M_n$ are positive semidefinite, the inequality $|||A^{1-\nu}XB^\nu|||\leqslant|||(1-\nu)AX+\nu XB|||$ is not valid for $\nu\in[0,1]$. However, Kosaki [9] showed that

$$|||A^{1-\nu}XB^\nu|||\leqslant(1-\nu)\,|||AX|||+\nu\,|||XB|||.$$

This inequality can be regarded as a unitarily invariant norm inequality form of Young's inequality (1) for matrices. It has been generalized to different frameworks and one of the most remarkable results is given in the following.

THEOREM 4.1. ([4, Theorem 4.1]) *Let $\nu\in[0,1]$, $X\in M_n$ and $A,B\in M_n$ be two positive semidefinite matrices. Then, for all real numbers $p\geqslant 1$, the following inequalities hold*

$$[(1-\nu)\,|||AX|||+\nu\,|||XB|||]^p\geqslant|||A^{1-\nu}XB^\nu|||^p+(2r_0(\nu))^p\mathfrak{N}_0$$

$$+(2r_1(\nu))^p(\chi_{(0,\frac{1}{2})}(\nu)\,|||\,AX\,|||^{p/2} +\chi_{(\frac{1}{2},1)}(\nu)\,|||\,XB|||^{p/2})\mathfrak{N}_1,$$

and

$$\begin{aligned}[(1-\nu)\,|||\,AX\,|||+\nu\,|||\,XB|||]^p \leqslant{}& [|||AX\,|||^{1-\nu}\,|||XB|||^{\nu}]^p+(2R_0(\nu))^p\mathfrak{N}_0\\ &-(2r_1(\nu))^p(\chi_{(0,\frac{1}{2})}(\nu)\,|||\,XB\,|||^{p/2}\\ &+\chi_{(\frac{1}{2},1)}(\nu)\,|||\,AX|||^{p/2})\mathfrak{N}_1,\end{aligned}$$

where

$$\mathfrak{N}_0=\left(\frac{|||AX\,|||+|||\,XB|||}{2}\right)^p-\sqrt{|||AX\,|||\;|||XB|||}^{\,p}$$

and

$$\mathfrak{N}_1=\left(\frac{\sqrt{|||AX|||}+\sqrt{|||XB|||}}{2}\right)^p-\sqrt[4]{|||AX\,|||\;|||XB|||}^{\,p}.$$

Our main result in this section provides further new refinements of the above results, which is stated as follows.

THEOREM 4.2. *Let $0\leqslant\nu\leqslant1$ and $A,B\in M_n$ be two positive semidefinite matrices. Let $X\in M_n$ be such that $|||AX|||>0$ and $|||XB|||>0$. Then, for every real number $p\geqslant1$, we have*

$$\begin{aligned}(|||AX\,|||\,\nabla_\nu\,|||\,XB|||)^p &\geqslant \mathscr{M}(A,B)^p\,|||\,A^{1-\nu}XB^{\nu}\,|||^p+\sum_{i=0}^{2}(2r_i)^p\mathscr{N}_i(A,B)\mathscr{S}_i(A,B)\\ &\geqslant S(h^{r_3})^p\,|||\,A^{1-\nu}XB^{\nu}\,|||^p+\sum_{i=0}^{2}(2r_i)^p\mathscr{N}_i(A,B)\mathscr{S}_i(A,B)\\ &\geqslant |||A^{1-\nu}XB^{\nu}\,|||^p+\sum_{i=0}^{2}(2r_i)^p\mathscr{N}_i(A,B)\mathscr{S}_i(A,B)\end{aligned}\tag{67}$$

and

$$\begin{aligned}(|||AX|||\nabla_\nu\,|||\,XB|||)^p \leqslant{}& m(A,B)^p(|||AX\,|||\,\sharp_\nu\,|||\,XB|||)^p+(2R_0)^p\mathscr{S}_0(A,B)\\ &-(2r_1)^p\mathscr{N}_1(B,A)\mathscr{S}_1(A,B)-(2r_2)^p\mathscr{N}_2(B,A)S_2(A,B)\\ \leqslant{}& S^{-p}(h^{r_3})(|||AX\,|||\,\sharp_\nu\,|||\,XB|||)^p+(2R_0)^p\mathscr{S}_0(A,B)\\ &-(2r_1)^p\mathscr{N}_1(B,A)\mathscr{S}_1(A,B)-(2r_2)^p\mathscr{N}_2(B,A)\mathscr{S}_2(A,B)\\ \leqslant{}& (|||AX\,|||\,\sharp_\nu\,|||\,XB|||)^p+(2R_0)^p\mathscr{S}_0(A,B)\\ &-(2r_1)^p\mathscr{N}_1(B,A)\mathscr{S}_1(A,B)-(2r_2)^p\mathscr{N}_2(B,A)\mathscr{S}_2(A,B),\end{aligned}\tag{68}$$

where S is Specht's ratio given in (53) with $h = \left(\frac{|||XB|||}{|||AX|||}\right)^{\frac{1}{8}}$ and $\mathcal{N}_0(A,B) = 1$,

$$\mathcal{N}_1(A,B) = |||AX|||^{\frac{p}{2}}\chi_{(0,\frac{1}{2})}(\nu) + |||XB|||^{\frac{p}{2}}\chi_{(\frac{1}{2},1)}(\nu),$$

$$\mathcal{N}_2(A,B) = \sum_{i=0}^{3}(|||A^{1-\frac{i}{3}}XB^{\frac{i}{3}}|||)^{\frac{3p}{4}}\chi_{(\frac{i}{4},\frac{i+1}{4})}(\nu),$$

$$\mathcal{S}_i(A,B) = \left(\frac{|||AX|||^{\frac{1}{2^i}} + |||XB|||^{\frac{1}{2^i}}}{2}\right)^p - (|||AX|||\,|||XB|||)^{\frac{p}{2^{i+1}}}, \quad i = 0,1,2,$$

$$\mathcal{M}(A,B) = \max\left\{\left(\frac{(|||AX|||^{\frac{1}{8}} + |||XB|||^{\frac{1}{8}})^2}{4(|||AX|||\,|||XB|||)^{\frac{1}{8}}}\right)^{r_3}, 1 + \frac{L(8\nu)}{64}\ln^2\left(\frac{|||AX|||}{|||XB|||}\right)\right\},$$

$$m(A,B) = \min\left\{\left(\frac{(|||AX|||^{\frac{1}{8}} + |||XB|||^{\frac{1}{8}})^2}{4(|||AX|||\,|||XB|||)^{\frac{1}{8}}}\right)^{-r_3}, \left(1 + \frac{L(8(1-\nu))}{64}\ln^2\left(\frac{|||AX|||}{|||XB|||}\right)\right)^{-1}\right\}.$$

REMARK 4.3. (i) By using Corollary 2.7, we can prove that the inequalities between the first and last terms in the inequalities (67) and (68) are valid without the conditions $|||AX||| > 0$ and $|||XB||| > 0$.

(ii) By substituting the unitarily invariant norm $|||\cdot|||$ by the trace norm $\operatorname{tr}(\cdot)$ and taking $X = I$ in Theorem 4.2, we will obtain further new refinements of [4, Corollary 4.2].

To prove this theorem, we need to recall the following lemma, which is a Heinz-Kato type inequality for unitarily invariant norms (see [11] for details).

LEMMA 4.4. ([11]) *Let $A,B,X \in M_n$ such that A and B are positive semidefinite. If $0 \leqslant \nu \leqslant 1$, then*

$$|||A^{1-\nu}XB^{\nu}||| \leqslant |||AX|||^{1-\nu}\,|||XB|||^{\nu}.$$

Proof. [Proof of Theorem 4.2] First of all, by Lemma 4.4, we find that

$$\begin{aligned} N_2(A,B) &:= \sum_{i=0}^{3}(|||AX|||^{1-\frac{i}{3}}\,|||BX|||^{\frac{i}{3}})^{\frac{3p}{4}}\chi_{(\frac{i}{4},\frac{i+1}{4})}(\nu) \\ &\geqslant \sum_{i=0}^{3}(|||A^{1-\frac{i}{3}}XB^{\frac{i}{3}})^{\frac{3p}{4}}\chi_{(\frac{i}{4},\frac{i+1}{4})}(\nu) \\ &= \mathcal{N}_2(A,B). \end{aligned} \tag{69}$$

On the one hand, it follows from the first inequalities in Theorems 2.4 and 2.5 that

$$[(1-\nu)a+\nu b]^p \geqslant M(a,b)^p(a^{1-\nu}b^{\nu})^p + \sum_{i=0}^{2}(2r_i)^p N_i(a,b)S_i(a,b), \tag{70}$$

where $M(a,b) = \max\left\{\left(\frac{(a^{\frac{1}{8}}+b^{\frac{1}{8}})^2}{4(ab)^{\frac{1}{8}}}\right)^{r_3}, 1 + \frac{L(8\nu)}{64}\ln^2\left(\frac{a}{b}\right)\right\}$, $N_0(a,b) = 1$ and

$$N_1(a,b) = \sqrt{a}^p\chi_{(0,\frac{1}{2})}(\nu) + \sqrt{b}^p\chi_{(\frac{1}{2},1)}(\nu),$$

$$N_2(a,b) = \sqrt[4]{a^3}^p \chi_{(0,\frac{1}{4})}(\nu) + \sqrt[4]{a^2b}^p \chi_{(\frac{1}{4},\frac{1}{2})}(\nu) + \sqrt[4]{ab^2}^p \chi_{(\frac{1}{2},\frac{3}{4})}(\nu) + \sqrt[4]{b^3}^p \chi_{(\frac{3}{4},1)}(\nu),$$

$$S_i(a,b) = \Big(\frac{a^{\frac{1}{2^i}} + b^{\frac{1}{2^i}}}{2}\Big)^p - (ab)^{\frac{p}{2^{i+1}}}, \quad i = 0,1,2.$$

On the other hand, by taking $a = |||AX|||,\ b = |||XB|||$ in the inequality (70), combining the inequalities (54) and (69) with Lemma 4.4, we infer that

$$\begin{aligned}(|||AX|||\,\nabla_\nu\,|||XB|||)^p &\geqslant \mathscr{M}(A,B)^p(|||AX|||^{1-\nu}\,|||XB|||^{\nu})^p \\ &\quad + \sum_{i=0}^{2}(2r_i)^p N_i(A,B)S_i(A,B) \\ &\geqslant \mathscr{M}(A,B)^p\,|||A^{1-\nu}XB^{\nu}|||^p + \sum_{i=0}^{2}(2r_i)^p \mathscr{N}_i(A,B)S_i(A,B) \\ &\geqslant S(h^{r_3})^p\,|||A^{1-\nu}XB^{\nu}|||^p + \sum_{i=0}^{2}(2r_i)^p \mathscr{N}_i(A,B)S_i(A,B) \\ &\geqslant |||A^{1-\nu}XB^{\nu}|||^p + \sum_{i=0}^{2}(2r_i)^p \mathscr{N}_i(A,B)S_i(A,B),\end{aligned}$$

where S is Specht's ratio given in (53) with $h = \big(\frac{|||XB|||}{|||AX|||}\big)^{\frac{1}{8}}$. This also finishes the proof of (67).

Similarly, we deduce from the second inequalities in Theorems 2.4 and 2.5 that

$$[(1-\nu)a+\nu b]^p \leqslant m(a,b)^p(a^{1-\nu}b^{\nu})^p + (2R_0)^p S_0(a,b) - \sum_{i=1}^{2}(2r_i)^p N_i(b,a)S_i(a,b),$$

where N_i, S_i for $i = 0,1,2$ are as above and

$$m(a,b) = \min\Big\{\Big(\frac{(a^{\frac{1}{8}}+b^{\frac{1}{8}})^2}{4(ab)^{\frac{1}{8}}}\Big)^{-r_3}, \Big(1+\frac{L(8(1-\nu))}{64}\ln^2\Big(\frac{a}{b}\Big)\Big)^{-1}\Big\}.$$

By substituting $a = |||AX|||,\ b = |||XB|||$ in this inequality and combining the inequalities (54) and (70) with Lemma 4.4, we gain

$$\begin{aligned}(|||AX|||\,\nabla_\nu\,|||XB|||)^p &\leqslant m(A,B)^p(|||AX|||\,\sharp_\nu\,|||XB|||)^p + (2R_0)^p\mathscr{S}_0(A,B) \\ &\quad - (2r_1)^p\mathscr{N}_1(B,A)\mathscr{S}_1(A,B) - (2r_2)^p N_2(B,A)\mathscr{S}_2(A,B) \\ &\leqslant m(A,B)^p(|||AX|||\,\sharp_\nu\,|||XB|||)^p + (2R_0)^p\mathscr{S}_0(A,B) \\ &\quad - (2r_1)^p\mathscr{N}_1(B,A)S_1(A,B) - (2r_2)^p\mathscr{N}_2(B,A)S_2(A,B) \\ &\leqslant S^{-p}(h^{r_3})(|||AX|||\,\sharp_\nu\,|||XB|||)^p + (2R_0)^p S_0(A,B) \\ &\quad - (2r_1)^p\mathscr{N}_1(B,A)S_1(A,B) - (2r_2)^p\mathscr{N}_2(B,A)S_2(A,B) \\ &\leqslant (|||AX|||\,\sharp_\nu\,|||XB|||)^p + (2R_0)^p S_0(A,B) \\ &\quad - (2r_1)^p\mathscr{N}_1(B,A)S_1(A,B) - (2r_2)^p\mathscr{N}_2(B,A)S_2(A,B),\end{aligned}$$

where S is Specht's ratio given in (53) with $h = \big(\frac{|||XB|||}{|||AX|||}\big)^{\frac{1}{8}}$, which finishes the proof. □

5. Inequalities for determinants of matrices

In this section, we present a refinement of Young-type inequality for determinants of positive definite matrices due to Theorems 2.4, 2.5, 2.6 and Corollary 2.7. The matrix version of Young inequality (1) says that (see [3, p. 467])

$$\det(A\nabla_\nu B) \geqslant \det(A\sharp_\nu B),$$

where $\nu \in [0,1]$, the matrices $A,B \in M_n$ are positive definite. This inequality was refined by Huy, Van and Xinh in [4] as follows.

THEOREM 5.1. ([4, Theorem 5.1]) *Let $\nu \in [0,1]$ and $A,B \in M_n$ be positive definite matrices. Then, for all real numbers $p \geqslant 1$, we have*

$$[\det(A\nabla_\nu B)]^p \geqslant [\det(A\sharp_\nu B)]^p + (2r_0(\nu))^{np} D_0 + (2r_1(\nu))^{np}\left[\chi_{(0,\frac{1}{2})}(\nu)(\det A)^{\frac{p}{2}} + \chi_{(\frac{1}{2},1)}(\det B)^{\frac{p}{2}}\right] D_1,$$

where

$$D_0 = \left(\frac{(\det A)^{1/n} + (\det B)^{1/n}}{2}\right)^{np} - (\det(AB))^{p/2},$$

and

$$D_1 = \left(\frac{(\det A)^{1/2n} + (\det B)^{1/2n}}{2}\right)^{np} - (\det(AB))^{p/4}.$$

We will propose further refinements of this theorem in the following.

THEOREM 5.2. *Let $0 \leqslant \nu \leqslant 1$ and N be a positive integer. Then, for all positive definite matrices $A,B \in M_n$ and every positive real number $p \geqslant 1$, we have*

$$\begin{aligned}
\det(A\nabla_\nu B)^p &\geqslant M(A,B)^{np}\det(A\sharp_\nu B)^p + \sum_{i=0}^{2}(2r_i)^{np} N_i(A,B)S_i(A,B) \\
&\geqslant S(h^{r_3})^{np}\det(A\sharp_\nu B)^p + \sum_{i=0}^{2}(2r_i)^{np} N_i(A,B)S_i(A,B) \\
&\geqslant \det(A\sharp_\nu B)^p + \sum_{i=0}^{2}(2r_i)^{np} N_i(A,B)S_i(A,B),
\end{aligned}$$

where S is Specht's ratio given in (53) with $h = \left(\frac{\det B}{\det A}\right)^{\frac{1}{8n}}$ and $N_0(A,B) = 1$,

$$N_1(A,B) = \det(A)^{\frac{p}{2}}\chi_{(0,\frac{1}{2})}(\nu) + \det(B)^{\frac{p}{2}}\chi_{(\frac{1}{2},1)}(\nu),$$

$$N_2(A,B) = \sum_{i=0}^{3}\det(A^{3-i}B^i)^{\frac{p}{4}}\chi_{(\frac{i}{4},\frac{i+1}{4})}(\nu),$$

$$M(A,B) = \max\left\{\left(\frac{((\det A)^{\frac{1}{8n}} + (\det B)^{\frac{1}{8n}})^2}{4(\det(AB))^{\frac{1}{8n}}}\right)^{r_3}, 1 + \frac{L(8\nu)}{64n^2}\ln^2\left(\frac{\det A}{\det B}\right)\right\},$$

$$S_i(A,B) = \left(\frac{(\det A)^{\frac{1}{2^i n}} + (\det B)^{\frac{1}{2^i n}}}{2}\right)^{np} - (\det(AB))^{\frac{p}{2^{i+1}}}, \quad \textit{for } i = 0,1,2.$$

To prove this theorem, we need the following important lemma, which is known as Minkowski's inequality for determinants.

LEMMA 5.3. ([11, Lemma 5]) *Let $A,B \in M_n$ be positive definite. Then*

$$[\det(A+B)]^{1/n} \geqslant (\det A)^{1/n} + (\det B)^{1/n}.$$

Proof of Theorem 5.2. First of all, from Theorems 2.4, 2.5, 2.6 and the inequality (54), we deduce that

$$\begin{aligned}
[(1-\nu)a+\nu b]^{np} &\geqslant M(a,b)^{np}(a^{1-\nu}b^{\nu})^{np} + \sum_{i=0}^{2}(2r_i)^{np}N_i(a,b)S_i(a,b) \\
&\geqslant S(h^{r_3})^{np}(a^{1-\nu}b^{\nu})^{np} + \sum_{i=0}^{2}(2r_i)^{np}N_i(a,b)S_i(a,b) \qquad (71)\\
&\geqslant (a^{1-\nu}b^{\nu})^{np} + \sum_{i=0}^{2}(2r_i)^{np}N_i(a,b)S_i(a,b),
\end{aligned}$$

where $M(a,b) = \max\left\{\left(\frac{(a^{\frac{1}{8}}+b^{\frac{1}{8}})^2}{4(ab)^{\frac{1}{8}}}\right)^{r_3}, 1 + \frac{L(8\nu)}{64}\ln^2\left(\frac{a}{b}\right)\right\}$, $N_0(a,b) = 1$ and

$$N_1(a,b) = \sqrt{a}^{np}\chi_{(0,\frac{1}{2})}(\nu) + \sqrt{b}^{np}\chi_{(\frac{1}{2},1)}(\nu),$$
$$N_2(a,b) = \sqrt[4]{a^3}^{np}\chi_{(0,\frac{1}{4})}(\nu) + \sqrt[4]{a^2b}^{np}\chi_{(\frac{1}{4},\frac{1}{2})}(\nu) + \sqrt[4]{ab^2}^{np}\chi_{(\frac{1}{2},\frac{3}{4})}(\nu) + \sqrt[4]{b^3}^{np}\chi_{(\frac{3}{4},1)}(\nu),$$
$$S_i(a,b) = \left(\frac{a^{\frac{1}{2^i}} + b^{\frac{1}{2^i}}}{2}\right)^{np} - (ab)^{\frac{np}{2^{i+1}}}, \quad i = 0,1,2.$$

Now, on the one hand, by using the substitution of variables $a = (\det A)^{\frac{1}{n}}$ and $b = (\det B)^{\frac{1}{n}}$ in the inequality (71), we obtain the following series of inequalities

$$\begin{aligned}
&[(1-\nu)(\det A)^{1/n} + \nu(\det B)^{1/n}]^{np} \\
&\geqslant M(A,B)^{np}\det(A\sharp_\nu B)^p + \sum_{i=0}^{2}(2r_i)^{np}N_i(A,B)S_i(A,B) \\
&\geqslant S(h^{r_3})^{np}\det(A\sharp_\nu B)^p + \sum_{i=0}^{2}(2r_i)^{np}N_i(A,B)S_i(A,B) \qquad (72)\\
&\geqslant \det(A\sharp_\nu B)^p + \sum_{i=0}^{2}(2r_i)^{np}N_i(A,B)S_i(A,B).
\end{aligned}$$

On the other hand, it follows from Lemma 5.3 that

$$\begin{aligned}[\det(A\nabla_\nu B)]^p &= \left\{[\det((1-\nu)A+\nu B)]^{\frac{1}{n}}\right\}^{np}\\ &\geqslant \left\{[\det((1-\nu)A)]^{\frac{1}{n}}+[\det(\nu B)]^{\frac{1}{n}}\right\}^{np}\\ &= \left[(1-\nu)(\det A)^{\frac{1}{n}}+\nu(\det B)^{\frac{1}{n}}\right]^{np}.\end{aligned}$$

Combining inequalities (72) and (73), we gain

$$\begin{aligned}\det(A\nabla_\nu B)^p &\geqslant M(A,B)^{np}\det(A\sharp_\nu B)^p+\sum_{i=0}^{2}(2r_i)^{np}N_i(A,B)S_i(A,B)\\ &\geqslant S(h^{r_3})^{np}\det(A\sharp_\nu B)^p+\sum_{i=0}^{2}(2r_i)^{np}N_i(A,B)S_i(A,B)\\ &\geqslant \det(A\sharp_\nu B)^p+\sum_{i=0}^{2}(2r_i)^{np}N_i(A,B)S_i(A,B),\end{aligned}$$

this also finishes the proof. □

Acknowledgement. The authors wish to express our gratitude to Prof. Fuad Kittaneh and the anonymous reviewer(s) for insightful suggestions and comments which makes the proofs more concise in Lemmas 2.2 and 2.3.

REFERENCES

[1] T. ANDO, *Matrix Young inequality*, Operator Theory: Advances and Applications **75** (1995), 33–38.
[2] D. CHOI, M. KRNIĆ AND J. PEĆARIĆ, *Improved Jensen-type inequalities via linear interpolation and applications*, Journal of Mathematical Inequalities **11**(2) (2017), 301–322.
[3] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, New York (1985).
[4] D. Q. HUY, D. T. T. VAN AND D. T. XINH, *Some generalizations of real power form for Young-type inequalities and their applications*, Linear Algebra and its Applications **656** (2023), 368–384.
[5] M. A. IGHACHANE AND M. AKKOUCHI, *Further refinements of Young's type inequality for positive linear maps*, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM **115** (2021), no. 2, Paper no. 94, 19 p.
[6] M. A. IGHACHANE AND M. BOUCHANGOUR, *Some refinements of real power form inequalities for convex functions via weak sub-majorization*, Operator and Matrices **17** (1) (2023), 7383–7399.
[7] M. A. IGHACHANE, Z. TAKI AND M. BOUCHANGOUR, *An improvement of Alzer-Fonseca-Kovačec's type inequalities with applications*, Filomat. **37** (22) (2023), 213–233.
[8] F. KITTANEH AND Y. MANASRAH, *Improved Young and Heinz inequalities for matrices*, Journal of Mathematical Analysis and Applications, **361** (1) (2010), 262–269.
[9] H. KOSAKI, *Arithmetic-geometric mean and related inequalities for operators*, Journal of Functional Analysis **156** (1998), 429–451.
[10] P. KÓRUS, *A refinement of Young's inequality*, Acta Mathematica Hungarica **153** (2017), 430–435.
[11] Y. MANASRAH AND F. KITTANEH, *A generalization of two refined Young inequalities*, Positivity **19** (2015), no. 4, 757–768.
[12] A. W. MARSHALL, I. OLKIN AND B. C. ARNOLD, *Inequalities: theory of majorization and its applications*, second edition, Springer Series in Statistics, Springer, New York, 2011.
[13] Y. REN AND P. LI, *Further refinements of reversed AM-GM operator inequalities*, Journal of Inequalities and Applications **2020** (2020), Paper no. 98, 13 p.

[14] M. SABABHEH AND D. CHOI, *A complete refinement of Young's inequality*, Journal of Mathematical Analysis and Applications **440** (2016), no. 1, 379–393.
[15] M. SABABHEH AND M. S. MOSLEHIAN, *Advanced refinements of Young and Heinz inequalities*, Journal of Number Theory **172** (2017), 178–199.
[16] W. SPECHT, *Zer Theorie der elementaren Mittel*, Journal of Mathematical Inequalities **74** (1960), 91–98.
[17] C. YANG, Y. GAO AND F. LU, *Some refinements of Young type inequality for positive linear map*, Mathematica Slovaca **69** (2019), no. 4, 919–930.
[18] H. ZUO, G. SHI AND M. FUJII, *Refined Young inequality with Kantorovich constant*, Journal of Mathematical Inequalities **5** (2011), no. 4, 551–556.

(Received April 12, 2023)

Doan Thi Thuy Van
Department of Mathematics
Tay Nguyen University
567 Le Duan, Buon Ma Thuot, Dak Lak, Vietnam
e-mail: doanthithuyvan@ttn.edu.vn

Duong Quoc Huy
Department of Mathematics
Tay Nguyen University
567 Le Duan, Buon Ma Thuot, Dak Lak, Vietnam
e-mail: duongquochuy@ttn.edu.vn