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REVERSE OF FUJII–SEO TYPE LOG–MAJORIZATION AND ITS APPLICATION TO THE TSALLIS RELATIVE ENTROPIES

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Abstract. In this paper, firstly, we shall show reverse of Fujii-Seo type log-majorization, and discuss the equivalence between the log-majorization and Furuta inequality with negative power. At last, we show its application to the Tsallis relative entropies.

1. Introduction

Throughout this paper, a capital letter, such as T, means an $n\times n$ matrix. We denote $T\geqslant 0$ if T is a positive semidefinite matrix and $T>0$ if T is positive definite. For positive semidefinite A and B, let us write $A \underset{(\log)}{\prec} B$ and refer to log-majorization if

$$\prod_{i=1}^{k}\lambda_i(A)\leqslant\prod_{i=1}^{k}\lambda_i(B)\quad (k=1,2,\cdots,n-1)$$

and

$$\prod_{i=1}^{n}\lambda_i(A)=\prod_{i=1}^{n}\lambda_i(B)\quad (i.e.\quad detA=detB)$$

hold, where $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\cdots\geqslant\lambda_n(A)$ and $\lambda_1(B)\geqslant\lambda_2(B)\geqslant\cdots\geqslant\lambda_n(B)$ are the eigenvalues of A and B respectively arranged in decreasing order. There are many perfect log-majorization, see [3, 4, 6, 8, 12] for details.

Recently, in [5], M. Fujii and Y. Seo obtained the following log-majorization.

THEOREM 1.1. ([5], Fujii-Seo type log-majorization) *If $A,B>0$, then*

$$A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\prec} A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}$$

holds for all $p\geqslant q>0$ and $0<q\leqslant 1$.

In this paper, we shall show reverse of Fujii-Seo type log-majorization inspired by the idea of “reverses of log-majorization” from [11], then we show its application to the Tsallis relative entropies.

In order to prove the result, we list some lemmas first.

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LEMMA 1.1. ([7, 10], Löwner-Heinz inequality) *If $A \geqslant B \geqslant 0$, then*

$$A^{\alpha} \geqslant B^{\alpha}$$

holds for all $0 \leqslant \alpha \leqslant 1$.

LEMMA 1.2. ([9, 11, 13, 15], Furuta inequality with negative power) *If $A \geqslant B \geqslant 0$ and $A > 0$, then*

(I) $A^{1-t} \geqslant (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{1-t}{p-t}}$ holds for $1 \geqslant p > t \geqslant 0$, $p \geqslant \frac{1}{2}$;

(II) $A^{-t} \geqslant (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{-t}{p-t}}$ holds for $1 \geqslant t > p \geqslant 0$, $\frac{1}{2} \geqslant p$;

(III) $A^{2p-t} \geqslant (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{2p-t}{p-t}}$ holds for $\frac{1}{2} \geqslant p > t \geqslant 0$;

(IV) $A^{2p-t-1} \geqslant (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{2p-t-1}{p-t}}$ holds for $1 \geqslant t > p \geqslant \frac{1}{2}$.

LEMMA 1.3. ([2], Araki's inequality) *If $A,B \geqslant 0$, then*

$$A^{\frac{t}{2}}B^tA^{\frac{t}{2}} \underset{(\log)}{\prec} (A^{\frac{1}{2}}BA^{\frac{1}{2}})^t$$

holds for $0 \leqslant t \leqslant 1$.

2. Main results

In this section, we shall show reverse of Fujii-Seo type log-majorization, and the equivalence between the log-majorization and Furuta inequality with negative power.

THEOREM 2.1. *If $A,B > 0$, then*

$$A^{\frac{1-p}{2}}B^{\frac{1-p}{1-q}q}A^{\frac{1-p}{2}} \underset{(\log)}{\prec} \{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{1-p}{1-q}} \tag{2.1}$$

holds for $0 < \frac{q}{2} \leqslant p \leqslant q < 1$.

Furthermore, (2.1) is equivalent to (I) in Lemma 1.2.

Proof. We only need to prove that

$$\{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{1-p}{1-q}} \leqslant I \tag{2.2}$$

ensures that

$$A^{\frac{1-p}{2}}B^{\frac{1-p}{1-q}q}A^{\frac{1-p}{2}} \leqslant I. \tag{2.3}$$

(2.2) is equivalent to $(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}} \leqslant A^{-1}$. Let $A_1 = A^{-1}$ and $B_1 = (A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}$, we have $B_1 \leqslant A_1$, $A = A_1^{-1}$ and $B = (A^{\frac{p}{2}}B_1^{\frac{p}{q}}A^{\frac{p}{2}})^{\frac{1}{p}} = (A_1^{-\frac{p}{2}}B_1^{\frac{p}{q}}A_1^{-\frac{p}{2}})^{\frac{1}{p}}$. Let $u = \frac{p}{q}$, $v = p$. Then the assumption $0 < \frac{q}{2} \leqslant p \leqslant q < 1$ implies $0 < v < u \leqslant 1$ and $u \geqslant \frac{1}{2}$ and so it follows from (I) in Lemma 1.2 that

$$(A_1^{-\frac{v}{2}}B_1^uA_1^{-\frac{v}{2}})^{\frac{1-v}{u-v}} \leqslant A_1^{1-v}, \tag{2.4}$$

which is just (2.3).

We have proved that (2.1) can be derived from (I) in Lemma 1.2 above. Next, we shall show (I) in Lemma 1.2 can be derived from (2.1), too.

By (2.1), $\{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{1-p}{1-q}} \leqslant I$ ensures that $A^{\frac{1-q}{2}}B^{\frac{1-p}{1-q}q}A^{\frac{1-q}{2}} \leqslant I$. Therefore, $(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}} \leqslant A^{-1}$ ensures that $B^{\frac{1-p}{1-q}q} \leqslant A^{p-1}$.

Let $A_1 = A^{-1}$, $B_1 = (A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}$, $u = \frac{p}{q}$, $v = p$ above. Notice that $A = A_1^{-1}$, $B = (A_1^{-\frac{p}{2}}B_1^{\frac{p}{q}}A_1^{-\frac{p}{2}})^{\frac{1}{p}}$ and $0 < p < \frac{p}{q} \leqslant 1$, $\frac{p}{q} \geqslant \frac{1}{2}$. We have that $B_1 \leqslant A_1$ ensures (2.4) holds for $0 < v < u \leqslant 1$, $u \geqslant \frac{1}{2}$.

Hence the proof of Theorem 2.1 is completed. □

COROLLARY 2.1. *If $A, B > 0$, then $A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}$ holds for $0 < \frac{q}{2} \leqslant p \leqslant q < 1$.*

Proof. Notice that $t_1 = \frac{1-q}{1-p} \in (0,1]$. By (2.1),

$$(A^{\frac{1-p}{2}}B^{\frac{1-p}{1-q}q}A^{\frac{1-p}{2}})^{\frac{1-q}{1-p}} \underset{(\log)}{\prec} A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}. \tag{2.5}$$

By Lemma 1.3,

$$A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}} \underset{(\log)}{\prec} (A^{\frac{1-p}{2}}B^{\frac{1-p}{1-q}q}A^{\frac{1-p}{2}})^{\frac{1-q}{1-p}}. \tag{2.6}$$

Together with (2.5) and (2.6), we have

$$A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}.$$

Hence the proof of Corollary 2.1 is completed. □

THEOREM 2.2. *If $A, B > 0$, then*

$$A^{-\frac{p}{2}}B^{\frac{pq}{q-1}}A^{-\frac{p}{2}} \underset{(\log)}{\prec} \{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{p}{q-1}} \tag{2.7}$$

holds for $0 < p \leqslant 1$, $q > 1$, $q \geqslant 2p$.

Furthermore, (2.7) is equivalent to (II) in Lemma 1.2.

Proof. We only need to prove that

$$\{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{p}{q-1}} \leqslant I \tag{2.8}$$

ensures that

$$A^{-\frac{p}{2}}B^{\frac{pq}{q-1}}A^{-\frac{p}{2}} \leqslant I. \tag{2.9}$$

(2.8) is equivalent to $(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}} \leqslant A^{-1}$. Let $A_1 = A^{-1}$ and $B_1 = (A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}$, we have $B_1 \leqslant A_1$, $A = A_1^{-1}$ and $B = (A^{\frac{p}{2}}B_1^{\frac{p}{q}}A^{\frac{p}{2}})^{\frac{1}{p}} = (A_1^{-\frac{p}{2}}B_1^{\frac{p}{q}}A_1^{-\frac{p}{2}})^{\frac{1}{p}}$. Let $u = \frac{p}{q}$,

$v=p$. Then the assumption $0<p\leqslant 1$, $q>1$ and $q\geqslant 2p$ implies $1\geqslant v>u>0$ and $\frac{1}{2}\geqslant u$ and so it follows from (II) in Lemma 1.2 that

$$A_1^{-v}\geqslant (A_1^{-\frac{v}{2}}B_1^uA_1^{-\frac{v}{2}})^{\frac{-v}{u-v}}, \tag{2.10}$$

which is just (2.9).

We have proved that (2.7) can be derived from (II) in Lemma 1.2 above. Next, we shall show (II) in Lemma 1.2 can be derived from (2.7), too.

By (2.7), $\{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{p}{q-1}}\leqslant I$ ensures that $A^{-\frac{p}{2}}B^{\frac{pq}{q-1}}A^{-\frac{p}{2}}\leqslant I$. Therefore, $(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}\leqslant A^{-1}$ ensures that $B^{\frac{pq}{q-1}}\leqslant A^p$. Let $A_1=A^{-1}$, $B_1=(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}$, $u=\frac{p}{q}$, $v=p$ above. Notice that $A=A_1^{-1}$, $B=(A_1^{-\frac{p}{2}}B_1^{\frac{p}{q}}A_1^{-\frac{p}{2}})^{\frac{1}{p}}$ and $1\geqslant p>\frac{p}{q}>0$ with $\frac{1}{2}\geqslant\frac{p}{q}$. We have that $B_1\leqslant A_1$ ensures (2.10) holds for $1\geqslant v>u>0$, $\frac{1}{2}\geqslant u$. (2.10) is obvious if $u=0$.

Hence the proof of Theorem 2.2 is completed. □

COROLLARY 2.2. *If $A,B>0$, then* $A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\underset{(\log)}{\succ}A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}$ *holds for* $0<p\leqslant 1$, $q>1$, $p+1\geqslant q\geqslant 2p$.

Proof. Notice that $t_2=\frac{q-1}{p}\in(0,1]$. By (2.7),

$$(A^{-\frac{p}{2}}B^{\frac{pq}{q-1}}A^{-\frac{p}{2}})^{\frac{q-1}{p}}\underset{(\log)}{\prec}A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}. \tag{2.11}$$

By Lemma 1.3,

$$A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}\underset{(\log)}{\prec}(A^{-\frac{p}{2}}B^{\frac{pq}{q-1}}A^{-\frac{p}{2}})^{\frac{q-1}{p}}. \tag{2.12}$$

Together with (2.11) and (2.12), we have

$$A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\underset{(\log)}{\succ}A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}.$$

Hence the proof of Corollary 2.2 is completed. □

THEOREM 2.3. *If $A,B>0$, then*

$$A^{\frac{p}{q}-\frac{p}{2}}B^{\frac{2p-pq}{1-q}}A^{\frac{p}{q}-\frac{p}{2}}\underset{(\log)}{\prec}\{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{2p-pq}{q-q^2}} \tag{2.13}$$

holds for $0<2p\leqslant q<1$.

Furthermore, (2.13) *is equivalent to* (*III*) *in Lemma 1.2.*

Proof. We only need to prove that

$$\{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{2p-pq}{q-q^2}}\leqslant I \tag{2.14}$$

ensures that

$$A^{\frac{p}{q}-\frac{p}{2}}B^{\frac{2p-pq}{1-q}}A^{\frac{p}{q}-\frac{p}{2}} \leqslant I. \tag{2.15}$$

(2.14) is equivalent to $(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}} \leqslant A^{-1}$. Let $A_1 = A^{-1}$ and $B_1 = (A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}$, we have $B_1 \leqslant A_1$, $A = A_1^{-1}$ and $B = (A^{\frac{p}{2}}B_1^{\frac{p}{q}}A^{\frac{p}{2}})^{\frac{1}{p}} = (A_1^{-\frac{p}{2}}B_1^{\frac{p}{q}}A_1^{-\frac{p}{2}})^{\frac{1}{p}}$. Let $u = \frac{p}{q}$, $v = p$. Then the assumption $0 < 2p \leqslant q < 1$ implies $\frac{1}{2} \geqslant u > v > 0$ and so it follows from (III) in Lemma 1.2 that

$$A_1^{2u-v} \geqslant (A_1^{-\frac{v}{2}}B_1^uA_1^{-\frac{v}{2}})^{\frac{2u-v}{u-v}}, \tag{2.16}$$

which is just (2.15).

We have proved that (2.13) can be derived from (III) in Lemma 1.2 above. Next, we shall show (III) in Lemma 1.2 can be derived from (2.13), too.

By (2.13), $\{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{2p-pq}{q-q^2}} \leqslant I$ ensures that $A^{\frac{p}{q}-\frac{p}{2}}B^{\frac{2p-pq}{1-q}}A^{\frac{p}{q}-\frac{p}{2}} \leqslant I$. Therefore, $(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}} \leqslant A^{-1}$ ensures that $B^{\frac{2p-pq}{1-q}} \leqslant A^{p-\frac{2p}{q}}$. Let $A_1 = A^{-1}$, $B_1 = (A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}$, $u = \frac{p}{q}$, $v = p$ above. Notice that $A = A_1^{-1}$, $B = (A_1^{-\frac{p}{2}}B_1^{\frac{p}{q}}A_1^{-\frac{p}{2}})^{\frac{1}{p}}$ and $\frac{1}{2} \geqslant \frac{p}{q} > p > 0$. We have that $B_1 \leqslant A_1$ ensures (2.16) holds for $\frac{1}{2} \geqslant u > v > 0$.

Hence the proof of Theorem 2.3 is completed. □

COROLLARY 2.3. *If $A,B > 0$, then $A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}$ holds for $0 < 2p \leqslant q < 1$, $2p - pq - q + q^2 \geqslant 0$.*

Proof. Notice that $t_3 = \frac{q-q^2}{2p-pq} \in (0,1]$. By (2.13),

$$(A^{\frac{p}{q}-\frac{p}{2}}B^{\frac{2p-pq}{1-q}}A^{\frac{p}{q}-\frac{p}{2}})^{\frac{q-q^2}{2p-pq}} \underset{(\log)}{\prec} A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}. \tag{2.17}$$

By Lemma 1.3,

$$A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}} \underset{(\log)}{\prec} (A^{\frac{p}{q}-\frac{p}{2}}B^{\frac{2p-pq}{1-q}}A^{\frac{p}{q}-\frac{p}{2}})^{\frac{q-q^2}{2p-pq}}. \tag{2.18}$$

Together with (2.17) and (2.18), we have

$$A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}.$$

Hence the proof of Corollary 2.3 is completed. □

THEOREM 2.4. *If $A,B > 0$, then*

$$A^{\frac{p}{q}-\frac{p}{2}-\frac{1}{2}}B^{\frac{2p-pq-q}{1-q}}A^{\frac{p}{q}-\frac{p}{2}-\frac{1}{2}} \underset{(\log)}{\prec} \{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{2p-pq-q}{q-q^2}} \tag{2.19}$$

holds for $p \leqslant 1 < q \leqslant 2p$.

Furthermore, (2.19) *is equivalent to* (IV) *in Lemma 1.2.*

Proof. We only need to prove that

$$\{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{2p-pq-q}{q-q^2}} \leqslant I \tag{2.20}$$

ensures that

$$A^{\frac{p}{q}-\frac{p}{2}-\frac{1}{2}}B^{\frac{2p-pq-q}{1-q}}A^{\frac{p}{q}-\frac{p}{2}-\frac{1}{2}} \leqslant I. \tag{2.21}$$

(2.20) is equivalent to $(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}} \leqslant A^{-1}$. Let $A_1 = A^{-1}$ and $B_1 = (A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}$, we have $B_1 \leqslant A_1$, $A = A_1^{-1}$ and $B = (A^{\frac{p}{2}}B_1^{\frac{p}{q}}A^{\frac{p}{2}})^{\frac{1}{p}} = (A_1^{-\frac{p}{2}}B_1^{\frac{p}{q}}A_1^{-\frac{p}{2}})^{\frac{1}{p}}$. Let $u = \frac{p}{q}$, $v = p$. Then the assumption $p \leqslant 1 < q \leqslant 2p$ implies $1 \geqslant v > u \geqslant \frac{1}{2}$ and so it follows from (IV) in Lemma 1.2 that

$$(A_1^{-\frac{v}{2}}B_1^uA_1^{-\frac{v}{2}})^{\frac{2u-v-1}{u-v}} \leqslant A_1^{2u-v-1}, \tag{2.22}$$

which is just (2.21).

We have proved that (2.19) can be derived from (IV) in Lemma 1.2 above. Next, we shall show (IV) in Lemma 1.2 can be derived from (2.19), too.

By (2.19), $\{A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}\}^{\frac{2p-pq-q}{q-q^2}} \leqslant I$ ensures that $A^{\frac{p}{q}-\frac{p}{2}-\frac{1}{2}}B^{\frac{2p-pq-q}{1-q}}A^{\frac{p}{q}-\frac{p}{2}-\frac{1}{2}} \leqslant I$. Therefore, $(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}} \leqslant A^{-1}$ ensures that $B^{\frac{2p-pq-q}{1-q}} \leqslant A^{1+p-\frac{2p}{q}}$.

Let $A_1 = A^{-1}$, $B_1 = (A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}$, $u = \frac{p}{q}$, $v = p$ above. Notice that $A = A_1^{-1}$, $B = (A_1^{-\frac{p}{2}}B_1^{\frac{p}{q}}A_1^{-\frac{p}{2}})^{\frac{1}{p}}$ and $1 \geqslant p > \frac{p}{q} \geqslant \frac{1}{2}$. We have that $B_1 \leqslant A_1$ ensures (2.22) holds for $1 \geqslant v > u \geqslant \frac{1}{2}$.

Hence the proof of Theorem 2.4 is completed. □

COROLLARY 2.4. *If* $A, B > 0$, *then* $A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}$ *holds for* $p \leqslant 1 < q \leqslant 2p$.

Proof. Notice that $t_4 = \frac{(1-q)q}{2p-pq-q} \in (0,1]$. By (2.19),

$$(A^{\frac{p}{q}-\frac{p}{2}-\frac{1}{2}}B^{\frac{2p-pq-q}{1-q}}A^{\frac{p}{q}-\frac{p}{2}-\frac{1}{2}})^{\frac{(1-q)q}{2p-pq-q}} \underset{(\log)}{\prec} A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}}. \tag{2.23}$$

By Lemma 1.3,

$$A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}} \underset{(\log)}{\prec} (A^{\frac{p}{q}-\frac{p}{2}-\frac{1}{2}}B^{\frac{2p-pq-q}{1-q}}A^{\frac{p}{q}-\frac{p}{2}-\frac{1}{2}})^{\frac{(1-q)q}{2p-pq-q}}. \tag{2.24}$$

Together with (2.23) and (2.24), we have

$$A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}.$$

Hence the proof of Corollary 2.4 is completed. □

THEOREM 2.5. *If* $A, B > 0$, *then*

$$A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{1}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} B$$

holds for $0 < p \leqslant 1$.

Proof. We only need to prove that

$$A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{1}{p}}A^{\frac{1}{2}} \leqslant I \tag{2.25}$$

ensures

$$B \leqslant I. \tag{2.26}$$

(2.25) is equivalent to

$$(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{1}{p}} \leqslant A^{-1}. \tag{2.27}$$

For $0 < p \leqslant 1$, applying Löwner-Heinz inequality to (2.27), we have $A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}} \leqslant A^{-p}$, which ensures (2.26) obviously.

Hence the proof of Theorem 2.5 is completed. □

THEOREM 2.6. *If* $A, B > 0$, *then* $A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}$ *holds for* $0 < p \leqslant 1$, $p \leqslant q \leqslant p+1$ *with* $2p - pq - q + q^2 \geqslant 0$.

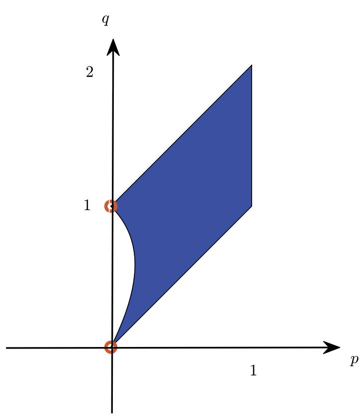

Figure 1: *Region of* $(p,\ q)$ *in Theorem 2.6*

Proof. Together with Corollary 2.1, Corollary 2.2, Corollary 2.3, Corollary 2.4 and Theorem 2.5, $A^{\frac{1}{2}}(A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1-q}{2}}B^qA^{\frac{1-q}{2}}$ holds when (p,q) satisfies one of the following:

- $0 < \frac{q}{2} \leqslant p \leqslant q < 1$,
- $0 < p \leqslant 1$, $q > 1$, $p+1 \geqslant q \geqslant 2p$,
- $0 < 2p \leqslant q < 1$ and $2p - pq - q + q^2 \geqslant 0$,
- $p \leqslant 1 < q \leqslant 2p$,
- $0 < p \leqslant 1$, $q = 1$.

Summing up these conditions we obtain Theorem 2.6. $\square$

3. Application

The famous Tsallis relative entropy ([1]) of two positive definite matrices A and B is defined by

$$D_\alpha(A|B) = \frac{\mathrm{Tr}A - \mathrm{Tr}[A^{1-\alpha}B^\alpha]}{\alpha},$$

where α belongs to (0,1]. The generalized Tsallis relative entropy of two positive definite matrices A and B is defined by

$$\hat{D}_\alpha(A|B) = \frac{\mathrm{Tr}A - \mathrm{Tr}[A^{1-\alpha}B^\alpha]}{\alpha},$$

where α belongs to $\mathbb{R}\backslash\{0\}$.

Another famous entropy, which is called Tsallis relative operator entropy ([14]) of two positive definite matrices A and B is defined by

$$T_\alpha(A|B) = \frac{A\sharp_\alpha B - A}{\alpha},$$

where $A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ and α belongs to (0,1]. The generalized Tsallis relative operator entropy of two positive definite matrices A and B is defined by

$$\hat{T}_\alpha(A|B) = \frac{A\natural_\alpha B - A}{\alpha},$$

where $A\natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ and α belongs to $\mathbb{R}\backslash\{0\}$.

In [5], Fujii and Seo obtain that

$$D_\alpha(A|B) \leqslant -\mathrm{Tr}\Big[\frac{A^{1-q}}{q}T_{\frac{\alpha}{q}}(A^q|B^q)\Big] \tag{3.1}$$

holds for $q \geqslant \alpha > 0$ and $0 < \alpha \leqslant 1$. Next, we shall show the reverse of (3.1).

By Theorem 2.6, it is easy to obtain the following theorem.

THEOREM 3.1. *If $A,B > 0$, then*

$$|||A^{\frac{1}{2}}(A^{-\frac{q}{2}}B^qA^{-\frac{q}{2}})^{\frac{\alpha}{q}}A^{\frac{1}{2}}||| \geqslant |||A^{\frac{1-\alpha}{2}}B^\alpha A^{\frac{1-\alpha}{2}}||| \tag{3.2}$$

holds for $0 < q \leqslant 1$, $q \leqslant \alpha \leqslant q+1$ with $2q - \alpha q - \alpha + \alpha^2 \geqslant 0$, where $|||\cdot|||$ stands for any unitarily invariant norm.

THEOREM 3.2. *If* $A,B>0$, *then*

$$\hat{D}_\alpha(A|B) \geqslant -\mathrm{Tr}\Big[\frac{A^{1-q}}{q}\hat{T}_{\frac{\alpha}{q}}(A^q|B^q)\Big] \tag{3.3}$$

holds for $0<q\leqslant 1$, $q\leqslant \alpha \leqslant q+1$ *with* $2q-\alpha q-\alpha+\alpha^2 \geqslant 0$.

Proof. By Theorem 3.1,

$$\mathrm{Tr}[A^{1-\alpha}B^\alpha] = \mathrm{Tr}(A^{\frac{1-\alpha}{2}}B^\alpha A^{\frac{1-\alpha}{2}}) \leqslant \mathrm{Tr}[A^{\frac{1}{2}}(A^{-\frac{q}{2}}B^qA^{-\frac{q}{2}})^{\frac{\alpha}{q}}A^{\frac{1}{2}}]$$

holds for $0<q\leqslant 1$, $q\leqslant \alpha \leqslant q+1$ with $2q-\alpha q-\alpha+\alpha^2 \geqslant 0$. It follows that

$$\begin{aligned}
\hat{D}_\alpha(A|B) &= -\mathrm{Tr}\Big[\frac{A^{1-\alpha}B^\alpha - A}{\alpha}\Big] \\
&\geqslant -\mathrm{Tr}\Big[\frac{A^{\frac{1}{2}}(A^{-\frac{q}{2}}B^qA^{-\frac{q}{2}})^{\frac{\alpha}{q}}A^{\frac{1}{2}} - A}{\alpha}\Big] \\
&= -\mathrm{Tr}\Big[\frac{A^{1-q}}{q}\Big(\frac{A^{\frac{q}{2}}(A^{-\frac{q}{2}}B^qA^{-\frac{q}{2}})^{\frac{\alpha}{q}}A^{\frac{q}{2}} - A^q}{\frac{\alpha}{q}}\Big)\Big] \\
&= -\mathrm{Tr}\Big[\frac{A^{1-q}}{q}\hat{T}_{\frac{\alpha}{q}}(A^q|B^q)\Big]. \quad \square
\end{aligned}$$

REMARK 3.1. It is still unknown that whether (3.3) holds for $0<q\leqslant 1$, $q\leqslant \alpha \leqslant q+1$ with $2q-\alpha q-\alpha+\alpha^2 < 0$.

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