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A NOTE ON THE A–SPECTRUM OF A–BOUNDED OPERATORS

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Abstract. Let A be a positive bounded operator on a Hilbert space $(\mathcal{H},\langle .,.\rangle)$ and P be the orthogonal projection on $\mathrm{cl}(\mathcal{R}(A))$. In the present paper, we prove that if A has closed range and $T\in\mathcal{B}_{A^{1/2}}(\mathcal{H})$ then

$$\sigma_A(T)=\sigma(\alpha(T)) \text{ and } \sigma_A(T)\backslash\{0\}=\sigma\left(A^{1/2}T(A^{1/2})^{\dagger}\right)\backslash\{0\}=\sigma(TP)\backslash\{0\}.$$

In particular, this allows us to prove that $r_A(T)=r(TP)=\sup\limits_{\lambda\in\sigma_A(T)}|\lambda|$, for any A-bounded operator $T\in\mathcal{B}_{A^{1/2}}(\mathcal{H})$. Moreover, we prove that if $T\in\mathcal{B}_A(\mathcal{H})$ is A-invertible and S is an A-inverse of T, then S belongs also to $\mathcal{B}_A(\mathcal{H})$. Other results are also derived.

1. Introduction

Throughout this paper, let $(\mathcal{H},\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{B}(\mathcal{H})'$ its topological dual. The identity operator on $\mathcal{H}$ is denoted by $I_{\mathcal{H}}$ (or simply by I if no ambiguity arises). If $T\in\mathcal{B}(\mathcal{H})$, then we denote by $\mathcal{R}(T)$ (resp. $\mathcal{N}(T)$) the range of T (resp. the null space of T).

Recall that given $T\in\mathcal{B}(\mathcal{H})$, the Moore-Penrose inverse of T, denoted by $T^{\dagger}$, is the unique linear mapping from $\mathcal{R}(T)\oplus\mathcal{R}(T)^{\perp}$ into $\mathcal{H}$ satisfying the "Moore-Penrose equations":

$$TXT=T,\ \ XTX=X,\ \ XT=P_{\mathrm{cl}(\mathcal{R}(T))} \text{ and } TX=P_{\mathrm{cl}(\mathcal{R}(T))}\big|_{\mathcal{R}(T)\oplus\mathcal{R}(T)^{\perp}},$$

where $P_{\mathrm{cl}(\mathcal{R}(T))}$ is the orthogonal projection on $\mathrm{cl}(\mathcal{R}(A)$, the closure of $\mathcal{R}(A)$. Or equivalently, $T^{\dagger}$ is the unique linear extension of $\widetilde{T}^{-1}$ to $D(T^{\dagger}):=\mathcal{R}(T)\oplus\mathcal{R}(T)^{\perp}$ with $\mathcal{N}(T^{\dagger})=\mathcal{R}(T)^{\perp}$. Here $\widetilde{T}$ is the isomorphism $\widetilde{T}:=T\big|_{\mathcal{N}(T)^{\perp}}:\mathcal{N}(T)^{\perp}\longrightarrow\mathcal{R}(T)$.

A linear functional $f\in\mathcal{B}(\mathcal{H})'$ is said to be positive, if $f(TT^*)\geqslant 0$ for all $T\in\mathcal{B}(\mathcal{H})$. Let $\mathcal{S}(\mathcal{B}(\mathcal{H}))$ denote the set of states on $\mathcal{B}(\mathcal{H})$ which is the set of all positive linear functionals f on $\mathcal{B}(\mathcal{H})$ such that $\|f\|=f(I)=1$.

The (algebraic) numerical range of an element $T\in\mathcal{B}(\mathcal{H})$ is

$$V(T):=\{f(T):f\in\mathcal{S}(\mathcal{B}(\mathcal{H}))\}$$

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which is a compact convex subset of $\mathbb{C}$.

Now, let A be a positive operator in $\mathcal{B}(\mathcal{H})$, and put

$$\mathcal{S}_A(\mathcal{B}(\mathcal{H})) := \left\{ \frac{f}{f(A)} : f \in \mathcal{S}(\mathcal{B}(\mathcal{H})),\, f(A) \neq 0 \right\}.$$

For an element $T \in \mathcal{B}(\mathcal{H})$, let

$$\|T\|_A := \sup \left\{ \sqrt{f(T^*AT)} : f \in \mathcal{S}_A(\mathcal{B}(\mathcal{H})) \right\}.$$

Note that $\|T\|_A = 0$ if and only if $AT = 0$ and that $\|\cdot\|_I = \|\cdot\|$. Notice also that it may happen that $\|T\|_A = \infty$ for some $T \in \mathcal{B}(\mathcal{H})$ due to the lack of compactness of $\mathcal{S}_A(\mathcal{B}(\mathcal{H}))$ (see [9, Example 3.2]). Also, it is shown in [9] that

$$\|T\|_A = \sup \left\{ \sqrt{\langle AT\xi, \xi \rangle} : \xi \in \mathcal{H},\, \|\xi\|_A = 1 \right\}. \tag{1.1}$$

Here, $\|\xi\|_A = \sqrt{\langle A\xi, \xi \rangle}$ for all $\xi \in \mathcal{H}$. In particular, we have

$$\left\{ T \in \mathcal{B}(\mathcal{H}) : \|T\|_A < \infty \right\} = \mathcal{B}_{A^{1/2}}(\mathcal{H})$$

where

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) := \left\{ T \in \mathcal{B}(\mathcal{H}) : \exists c > 0; \|T\xi\|_A \leqslant c\|\xi\|_A, \forall \xi \in \mathcal{H} \right\},$$

which is the set of operators having an $A^{1/2}$-adjoint (called also A-bounded operator) that has been studied extensively in the literature. See for instance [2, 3, 6, 11] and the references therein. It is useful to note that if $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, then $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ and $\|T\xi\|_A \leqslant \|T\|_A \|\xi\|_A$ for every $\xi \in \mathcal{H}$.

For $T \in \mathcal{B}(\mathcal{H})$, an element $Y \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of T if $AY = T^*A$. Generally, the existence of an A-adjoint operator is not guaranteed. The set of all elements in $\mathcal{B}(\mathcal{H})$ that admit A-adjoints is denoted by $\mathcal{B}_A(\mathcal{H})$. If $T \in \mathcal{B}_A(\mathcal{H})$, then the *reduced* solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which is denoted by $T^{\sharp_A}$. Note that, $T^{\sharp_A} = A^\dagger T^* A$. Also, it is well known that $\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\} \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H})$ and if $T \in \mathcal{B}_A(\mathcal{H})$, then $T^{\sharp_A} \in \mathcal{B}_A(\mathcal{H})$. Another important property is that if S and T are in $\mathcal{B}_A(\mathcal{H})$ then $ST \in \mathcal{B}_A(\mathcal{H})$ and $(ST)^{\sharp_A} = T^{\sharp_A} S^{\sharp_A}$. We refer the reader to [2] and [3] for the proof.

The A-numerical range is defined by

$$V_A(T) := \{f(AT) : f \in \mathcal{S}_A(\mathcal{B}(\mathcal{H}))\}.$$

Unlike the classical algebraic numerical range, the A-numerical range $V_A(T)$ of an element $T \in \mathcal{B}(\mathcal{H})$ may or may not be closed and/or may or may not be bounded.

Originally, these concepts were introduced and studied [1, 9] as generalization of the spatial A-numerical range for operators defined by

$$W_A(T) := \left\{ \langle AT\xi, \xi \rangle : \xi \in \mathcal{H},\, \|\xi\|_A = 1 \right\}$$

for every operator $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Observe that for T in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\mathrm{cl}(V_A(T)) = \mathrm{cl}(W_A(T)). \tag{1.2}$$

Here $\mathrm{cl}(\Gamma)$ denotes the closure of a subset Γ in $\mathbb{C}$. In particular if A has a closed range then $V_A(T) = \mathrm{cl}(W_A(T))$. For proofs and more facts about the A-numerical range of operators, we refer the reader to [9]. Some other related topics can be found in [1] and [5].

Recently, Baklouti and Namouri in [7] introduced and studied the A-spectrum of elements in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$. More precisely, they studied the following concepts.

DEFINITION 1.1. An operator $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ is said to be A-invertible in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ if it has an A-inverse $S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. That is if there exists an operator $S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ so that $A = AST = ATS$.

For an operator $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, the A-spectrum is defined by

$$\sigma_A(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not } A\text{-invertible}\}.$$

They established several permanence properties of these concepts. In particular, they investigate the A-invertibility in $\mathcal{B}(\mathcal{H})$ and spectral properties are also derived.

So it is natural to ask if we have a kind of Beurling formula for A-bounded operators. Namely if we put $r_A(T) = \max\{|\lambda| : \lambda \in \sigma_A(T)\}$. Is it true that $r_A(T) = \lim\limits_{n\to\infty} \|T^n\|_A^{\frac{1}{n}} = \inf\limits_{n\geqslant 1} \|T^n\|_A^{\frac{1}{n}}$, for every $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$?

The results of this paper are related to a notion of A-spectrum for operators in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ as considered in [7]. Our aim is not only to improve results of [7] but also to give more characterizations of A-invertibility in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$. In particular, under the assumption that A has a closed range, we show that

$$r_A(T) = r(TP) = \lim_{n\to\infty} \|T^n\|_A^{\frac{1}{n}}. \tag{1.3}$$

The backbone of the proof of this result is a characterization that tells us that

$$\sigma_A(T) = \sigma(\alpha(T)) \text{ and } \sigma_A(T)\backslash\{0\} = \sigma(TP)\backslash\{0\}. \tag{1.4}$$

Here α is the algebra homomorphism considered in [2] and [3]. Next, we prove that if $T \in \mathcal{B}_A(\mathcal{H})$ is A-invertible then any A-inverse of T is also in $\mathcal{B}_A(\mathcal{H})$. Finally, we derive some consequences.

2. Preliminaries

In this section, we review more concepts and notation. Further, we collect and establish some auxiliary lemmas needed for the proofs of the main results.

Throughout this paper, we assume that $A \in \mathcal{B}(\mathcal{H})$ is a nonzero positive operator.

In a natural way, the operator A generates a positive semi-definite semi-inner product given by $\langle \xi, \mu \rangle_A := \langle A\xi, \mu \rangle$ for all $\xi, \mu \in \mathcal{H}$. This semi-inner product $\langle\ ,\ \rangle_A$ induces on the quotient $\mathcal{H}/\mathcal{N}(A)$ an inner product which is not complete in general.

However, by [10] we know that the completion of $\mathcal{H}/\mathcal{N}(A)$ is isometrically isomorphic to $\mathcal{R}(A^{1/2})$; with the inner product

$$[A^{1/2}x, A^{1/2}y] := \langle Px, Py \rangle, \quad (x, y \in \mathcal{H}).$$

Here, P is the orthogonal projection on $\mathrm{cl}(\mathcal{R}(A))$, with $\mathrm{cl}(\mathcal{R}(A))$ denoting the norm closure of $\mathcal{R}(A)$ in $\mathcal{H}$. The Hilbert space $(\mathcal{R}(A^{1/2}), [,])$ will be denoted by $\mathbf{R}(A^{1/2})$. Note that $\mathcal{R}(A)$ is closed means that $\mathcal{R}(A) = \mathcal{R}(A^{1/2})$. For any $x, y \in \mathcal{H}$ we have $[Ax, Ay] = \langle Ax, y \rangle = \langle x, y \rangle_A$. This in particular leads to the following

$$\|Ax\|_{\mathbf{R}(A^{1/2})} = \|x\|_A, \quad (x \in \mathcal{H}). \tag{2.5}$$

For an operator $T \in \mathcal{B}(\mathcal{H})$, it was shown in [4, Propositions 3.6 & 3.9] that $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ if and only if there exists an operator $\widetilde{T} \in \mathcal{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A T = \widetilde{T} Z_A$, moreover $\widetilde{T}$ is unique. Here, $Z_A : \mathcal{H} \longrightarrow \mathbf{R}(A^{1/2})$ is defined by $Z_A x = Ax$.

For two vectors x and y in $\mathcal{H}$ (resp. $\mathbf{R}(A^{1/2})$), let $x \otimes y$ (resp. $x \otimes_A y$) stand for the operator of rank at most one on $\mathcal{H}$ (resp. $\mathbf{R}(A^{1/2})$) defined by

$$(x \otimes y)z := \langle z, y \rangle x, \quad (z \in \mathcal{H}) \quad \Big(\text{resp. } (x \otimes_A y)z := [z, y]x, \ (z \in \mathbf{R}(A^{1/2}))\Big).$$

Define

$$\widetilde{\mathcal{B}}(\mathbf{R}(A^{1/2})) := \left\{ \widetilde{T} \in \mathcal{B}(\mathbf{R}(A^{1/2})) : \mathcal{R}(\widetilde{T} Z_A) \subset \mathcal{R}(A) \right\},$$

and note that $\widetilde{\mathcal{B}}(\mathbf{R}(A^{1/2}))$ is a non closed subalgebra of $\mathcal{B}(\mathbf{R}(A^{1/2}))$ in general.

The following two mappings

$$\alpha : \mathcal{B}_{A^{1/2}}(\mathcal{H}) \longrightarrow \widetilde{\mathcal{B}}(\mathbf{R}(A^{1/2})), \quad T \longmapsto \widetilde{T},$$

and

$$\beta : \widetilde{\mathcal{B}}(\mathbf{R}(A^{1/2})) \longrightarrow \mathcal{B}_{A^{1/2}}(\mathcal{H}), \quad \widetilde{T} \longmapsto T$$

are well defined. Further by [4, Proposition 3.9], α and β are two homomorphisms and

$$\alpha \circ \beta(\widetilde{T}) = \widetilde{T}, \quad \beta \circ \alpha(T) = PTP.$$

The next lemma gives some properties of the homomorphism α. The proof may be found in [2] and [14].

LEMMA 2.1.

(i) *Let $\widetilde{T} : \mathcal{R}(A^{1/2}) \longrightarrow \mathcal{R}(A^{1/2})$ be a linear operator. Then there exists a unique linear operator $V : \mathcal{H} \longrightarrow \mathcal{H}$ such that $\mathcal{R}(V) \subset \mathrm{cl}(\mathcal{R}(A))$ and $A^{1/2}V = \widetilde{T}A^{1/2}$. Moreover, $\widetilde{T}$ is bounded in $\mathbf{R}(A^{1/2})$ if and only if V is bounded in $\mathcal{H}$ and $\|\widetilde{T}\|_{\mathbf{R}(A^{1/2})} = \|V\|$.*

(ii) *Consider $T \in \mathcal{B}(\mathcal{H})$. Then, there exists $\widetilde{T} \in \mathcal{B}(\mathbf{R}(A^{1/2}))$ such that $\widetilde{T} Z_A = Z_A T$ if and only if $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. In such case, $\widetilde{T}$ is unique*

(iii) *For every* $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, *we have* $\|T\|_A = \|\alpha(T)\|_{\mathcal{B}(\mathbf{R}(A^{1/2}))}$.

(iv) *Let* $x, y \in \mathcal{H}$. *Then,* $x \otimes Ay \in \mathcal{B}_A(\mathcal{H})$ *and* $\widetilde{x \otimes Ay} = Ax \widetilde{\otimes}_A Ay$. *Here* $\widetilde{\otimes}_A$ *is tensor product in* $\mathbf{R}(A^{1/2})$).

The proof of Theorem 3.7 uses the following lemma which is a variant of [7, Theorem 4.2].

LEMMA 2.2. *An operator* $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ *is* A*-invertible in* $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ *if and only if the following two conditions are satisfied:*

(i) *There exists* $c > 0$ *such that* $\frac{1}{c}\|x\|_A \leqslant \|Tx\|_A \leqslant c\|x\|_A$ *for any* $x \in \mathcal{H}$.

(ii) $\mathcal{R}(A^{1/2}T) = \mathcal{R}(A^{1/2})$.

Proof. By [7, Theorem 4.2], it suffices to show that $\mathcal{R}(AT) = \mathcal{R}(A)$ if and only if $\mathcal{R}(A^{1/2}T) = \mathcal{R}(A^{1/2})$. To that end, first observe that we have usually $\mathcal{R}(AT) \subset \mathcal{R}(A)$ and $\mathcal{R}(A^{1/2}T) \subset \mathcal{R}(A^{1/2})$. Now, assume that $\mathcal{R}(AT) = \mathcal{R}(A)$ and let $h = A^{1/2}x \in \mathcal{R}(A^{1/2})$. Then $A^{1/2}h = Ax = ATy$ for some $y \in \mathcal{H}$. Whence $h - A^{1/2}Ty$ is in $N(A^{1/2})$. As $h - A^{1/2}Ty \in \mathcal{R}(A^{1/2})$, then $h = A^{1/2}Ty \in \mathcal{R}(A^{1/2}T)$. Hence $\mathcal{R}(A^{1/2}T) = \mathcal{R}(A^{1/2})$. In a similar way we can show that $\mathcal{R}(AT) = \mathcal{R}(A)$ if $\mathcal{R}(A^{1/2}T) = \mathcal{R}(A^{1/2})$. $\square$

Recall that

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \exists S \in \mathcal{B}(\mathcal{H}); A^{1/2}S = T^*A^{1/2}\}.$$

S is called an $A^{1/2}$-adjoint of T. Observe that $S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ as well.

LEMMA 2.3. *Assume that* $T, R \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ *are such that* $A^{1/2}T = S^*A^{1/2}$ $A^{1/2}R = W^*A^{1/2}$ *for some* $S, W \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. *Then,* T *is* A*-invertible with* A*-inverse* R *if and only if* S *is with* A*-inverse* W. *In particular*

$$\sigma_A(T) = \overline{\sigma_A(S)} = \left\{\overline{\lambda} : \lambda \in \sigma_A(S)\right\}.$$

Proof. Assume that $ART = ATR = A$ for some $R \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Hence $A^{1/2}RT = A^{1/2}TR = A^{1/2}$ and then $W^*S^*A^{1/2} = S^*W^*A^{1/2} = A^{1/2}$. Since $A^{1/2}$ is self adjoint then $A^{1/2}SW = A^{1/2}WS = A^{1/2}$. Whence S is A-invertible with A-inverse W. The converse can be handselled similarly. $\square$

For $T \in \mathcal{B}(\mathcal{H})$, we shall call effective action of T with respect to the positive operator A, the operator on $\mathrm{cl}(\mathcal{R}(A))$ given by $T_{eff}(x) = PTP(x)$, for all $x \in \mathrm{cl}(\mathcal{R}(A))$.

The matrix of PTP in $\mathcal{H} = \mathcal{N}(A) \oplus \mathrm{cl}(\mathcal{R}(A))$ writes

$$PTP = \begin{bmatrix} 0 & 0 \\ 0 & T_{eff} \end{bmatrix}$$

in particular

$$A = PAP = \begin{bmatrix} 0 & 0 \\ 0 & A_{eff} \end{bmatrix}.$$

Also, observe that

$$(T^*)_{eff} = (T_{eff})^*.$$

We get the following lemma

LEMMA 2.4. *Let* $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. *If* T *is* A*-invertible in* $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ *then* T_{eff} *is invertible in* $\mathcal{B}(\mathrm{cl}(\mathcal{R}(A)))$. *Moreover if* S *is an* A*-inverse of* T *in* $\mathcal{B}_{A^{1/2}}(\mathcal{H})$, *then* $S_{eff} = T_{eff}^{-1}$.

Proof. Suppose that $ATS = AST = A$ for some $S \in B_{A^{1/2}}(H)$. We have $AP = A$ and $\mathcal{R}(1-P) = \mathcal{N}(A)$. As $T(\mathcal{N}(A))$ is a subset of $\mathcal{N}(A)$, then $\mathcal{R}(T(1-P)) \subseteq \mathcal{N}(A)$. Whence $PT(1-P) = 0$. Accordingly $AT = APT = APTP$. Hence $A = ATS = APTPS$. Similarly, we have $APTPS = APSPT = A$. Using the matrix representation we get $A_{eff}T_{eff}S_{eff} = A_{eff}S_{eff}T_{eff} = A_{eff}$. The injectivity of A_{eff} leads up to $T_{eff}S_{eff} = S_{eff}T_{eff} = I_{\mathrm{cl}(\mathcal{R}(A))}$. $\square$

It is worth noting that the A-inverses of T are the operators which matrices in $\mathcal{H} = \mathcal{N}(A) \oplus \mathrm{cl}(\mathcal{R}(A))$ have the form

$$S = \begin{bmatrix} S_1 & S_2 \\ 0 & T_{eff}^{-1} \end{bmatrix},$$

for some suitable operators S_1 and S_2. We also notice that the converse of the above lemma is, in general, false. Indeed, the inverse of an A-bounded invertible operator need not be A-bounded. See for instance [7].

3. Main results

We are ready now to state our first main result which gives another characterization of A-invertibility in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$.

THEOREM 3.1. *Assume that* A *has a closed range and let* $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. *Then* T *is* A*-invertible in* $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ *if and only if* $\alpha(T)$ *is invertible in* $\mathcal{B}(\mathbf{R}(A^{1/2}))$. *In particular* $\sigma_A(T) = \sigma(\alpha(T))$, *for any* $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$.

Proof. Observe first that $\alpha(A) := \widetilde{A}$ is one to one in $\mathcal{B}(\mathbf{R}(A^{1/2}))$. Now, assume that T is A-invertible in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$. Hence there exists an operator $S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ so that $A = AST = ATS$. As the map α is a homomorphism, then $\alpha(A) = \alpha(A)\alpha(T)\alpha(S) = \alpha(A)\alpha(S)\alpha(T)$. Since $\alpha(A)$ is one to one we infer that $I_{\mathcal{B}(\mathbf{R}(A^{1/2}))} = \alpha(I) = \alpha(T)\alpha(S) = \alpha(S)\alpha(T)$. Whence $\alpha(T)$ is invertible and $(\alpha(T))^{-1} = \alpha(S)$ belongs to $\widetilde{\mathcal{B}}(\mathbf{R}(A^{1/2}))$.

For the converse, let us show that if $\alpha(T)$ is invertible in $\mathcal{B}(\mathbf{R}(A^{1/2}))$ then $(\alpha(T))^{-1} \in \widetilde{\mathcal{B}}(\mathbf{R}(A^{1/2}))$. To that end, let $S \in \mathcal{B}(\mathbf{R}(A^{1/2}))$ be such that $\alpha(T)S = S\alpha(T) = \alpha(I) =$

$I_{\mathcal{B}(\mathbf{R}(A^{1/2}))}$. Lemma 2.1-(i) tells us that there exists a unique bounded linear operator $V:\mathcal{H}\longrightarrow\mathcal{H}$ such that $\mathcal{R}(V)\subset \mathrm{cl}(\mathcal{R}(A))=\mathcal{R}(A)$ and $A^{1/2}V=SA^{1/2}$. In particular $\alpha(T)A^{1/2}V=S^{-1}A^{1/2}V=A^{1/2}$ and $\mathcal{R}(SZ_A)\subset\mathcal{R}(A)$. To see why this let $h\in\mathcal{H}$ and note that

$$SZ_Ah = SAh = S\alpha(T)A^{1/2}VA^{1/2}h = A^{1/2}VA^{1/2}h$$
$$SZ_Ah = SAh = A^{1/2}VA^{1/2}h \in A^{1/2}(\mathcal{R}(A)) = \mathcal{R}(A).$$

Now, if $\alpha(I)=B\alpha(T)=\alpha(T)B$ for some $B\in\widetilde{\mathcal{B}}(\mathbf{R}(A^{1/2}))$. Based on the aforesaid, we have $B=(\alpha(T))^{-1}$ belongs to $\widetilde{\mathcal{B}}(\mathbf{R}(A^{1/2}))$. So, by applying β, we get

$$P=\beta(B)PTP=PTP\beta(B).$$

Clearly, condition (i)-of Lemma 2.2 is fulfilled. For condition (ii), we have

$$\begin{aligned}\mathcal{R}(A^{1/2}) &= \mathcal{R}(A^{1/2}P)=\mathcal{R}(A^{1/2}PTP\beta(B))\\ &= \mathcal{R}(A^{1/2}TP\beta(B))\\ &= \subset\mathcal{R}(A^{1/2}T)\subset\mathcal{R}(A^{1/2}).\end{aligned}$$

Accordingly $\mathcal{R}(A^{1/2})=\mathcal{R}(A^{1/2}T)$. This completes the proof. □

For $T\in\mathcal{B}_{A^{1/2}}(\mathcal{H})$ the operator $A^{1/2}T(A^{1/2})^{\dagger}$ is densely defined and bounded. We shall use the notation $\overline{A^{1/2}T(A^{1/2})^{\dagger}}$ for its unique continuous extension to all of $\mathcal{H}$.

The next result gives another information of $\sigma_A(T)$.

THEOREM 3.2. *Let T and S in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ be such that $A^{1/2}T=S^{*}A^{1/2}$. Then*

$$\sigma(PS^{*})\backslash\{0\}=\sigma\left(\overline{A^{1/2}T(A^{1/2})^{\dagger}}\right)\backslash\{0\}\subseteq\sigma_A(T)\backslash\{0\}. \tag{3.6}$$

If A has a closed range, then

$$\sigma_A(T)\backslash\{0\}=\sigma(PT)\backslash\{0\} \tag{3.7}$$

Proof. Pick up a non zero scalar $\lambda\in\mathbb{C}$ and suppose that $\lambda\notin\sigma_A(T)$. According to Lemma 2.2 there exists $c>0$ such that

$$\frac{1}{c}\|y\|_A\leqslant\|(\lambda I-T)y\|_A\leqslant c\|y\|_A \tag{3.8}$$

for any $x\in\mathcal{H}$ and

$$\mathcal{R}(A^{1/2}T)=\mathcal{R}(A^{1/2}). \tag{3.9}$$

The inequalities (3.8) write

$$\frac{1}{c}\|A^{1/2}y\|\leqslant\|A^{1/2}(\lambda I-T)y\|\leqslant c\|A^{1/2}y\|$$

Let $x \in \mathcal{H}$. If $x \in \mathcal{R}(A^{1/2})$ then $x = A^{1/2}y$ for some $y \in \overline{\mathcal{R}(A)}$. Therefore $y = (A^{1/2})^{\dagger}x$. Keeping in mind that $(A^{1/2})^{\dagger}A^{1/2} = P$ and $A^{1/2}(A^{1/2})^{\dagger} = P\big|_{D((A^{1/2})^{\dagger})}$, we get

$$\left\|A^{1/2}(\lambda I - T)y\right\| = \left\|A^{1/2}(\lambda I - T)(A^{1/2})^{\dagger}x\right\| = \left\|\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)x\right\|.$$

Accordingly

$$\frac{1}{c}\|x\| \leqslant \left\|\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)x\right\| \leqslant c\|x\|, \quad (x \in \mathcal{R}(A^{1/2})). \tag{3.10}$$

Now, if $x \in \mathcal{N}(A) = \mathcal{N}(A^{1/2}) = \mathcal{R}(A)^{\perp} = \mathcal{N}((A^{1/2})^{\dagger})$. Then $(A^{1/2})^{\dagger}x = 0$ and therefore

$$\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)x = \lambda x.$$

Since λ is nonzero, we infer that $x \in \mathcal{R}\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)$. Accordingly

$$\mathcal{N}(A) \subset \mathcal{R}\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right). \tag{3.11}$$

By (3.9), we have

$$\begin{aligned}
\mathcal{R}(A^{1/2}) &= \mathcal{R}\left(A^{1/2}(\lambda I - T)\right) \\
&= A^{1/2}(\lambda I - T)(\mathcal{H}) \\
&= A^{1/2}(\lambda I - T)(\overline{\mathcal{R}(A)}), \quad (\text{since } T(\mathcal{N}(A)) \subset \mathcal{N}(A)) \\
&= A^{1/2}(\lambda I - T)(A^{1/2})^{\dagger}(\mathcal{R}(A^{1/2})) \\
&= \left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)(\mathcal{R}(A^{1/2})), \ (\text{since } A^{1/2}(A^{1/2})^{\dagger} = I \text{ on } \mathcal{R}(A^{1/2})).
\end{aligned}$$

This together with (3.11) entail that

$$\mathcal{N}(A) + \mathcal{R}(A^{1/2}) \subset \mathcal{R}\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right).$$

Given that $\mathcal{N}(A) + \mathcal{R}(A^{1/2})$ is dense in $\mathcal{H}$, it follows that $\mathcal{R}\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)$ and $\mathcal{R}\left(\overline{\lambda I - A^{1/2}T(A^{1/2})^{\dagger}}\right)$ are dense in $\mathcal{H}$.

Now, let $x = x_1 + x_2 \in \mathcal{N}(A^{1/2}) \oplus^{\perp} \mathcal{R}(A)$, where $\oplus^{\perp}$ is the orthogonal sum. Based the aforesaid, we have

$$\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)x = \lambda x_1 + \left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)x_2$$

Since $x_1 \in \mathcal{N}(A)$ and $\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)x_2 \in \mathcal{R}(A^{1/2})$, by the Pythagoras's theorem

$$\left\|\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)x\right\|^2 = |\lambda|^2\ \|x_1\|^2 + \left\|\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)x_2\right\|^2.$$

The above equality together with (3.10) entail that

$$|\lambda|^2\ \|x_1\|^2 + \frac{1}{c^2}\|x_2\|^2 \leqslant \left\|\left(\lambda I - A^{1/2}T(A^{1/2})^{\dagger}\right)x\right\|^2 \leqslant |\lambda|^2\ \|x_1\|^2 + c^2\|x_2\|^2$$

Set $c_1 = \min(|\lambda|, \frac{1}{c})$ and $c_2 = \max(|\lambda|, c)$, we infer that

$$c_1\|x\| \leqslant \left\| \left(\lambda I - A^{1/2}T(A^{1/2})^\dagger\right) x \right\| \leqslant c_2\|x\| \tag{3.12}$$

for any $x \in \mathcal{N}(A^{1/2}) \oplus^\perp \mathcal{R}(A)$. We have $\mathcal{N}(A^{1/2}) \oplus^\perp \mathcal{R}(A)$ is dense in $\mathcal{H}$ and by [2, Proposition 2.3] the operator $A^{1/2}T(A^{1/2})^\dagger$ is bounded. It follows from (3.12) that

$$c_1\|x\| \leqslant \left\| \left(\overline{\lambda I - A^{1/2}T(A^{1/2})^\dagger}\right) x \right\| \leqslant c_2\|x\|, \ \ (x \in \mathcal{H}). \tag{3.13}$$

In particular $\overline{\lambda I - A^{1/2}T(A^{1/2})^\dagger}$ is one to one and $\mathcal{R}\left(\overline{\lambda I - A^{1/2}T(A^{1/2})^\dagger}\right)$ is closed. As $\mathcal{R}\left(\lambda I - A^{1/2}T(A^{1/2})^\dagger\right)$ is dense, we infer that $\overline{\lambda I - A^{1/2}T(A^{1/2})^\dagger}$ is invertible in $\mathcal{B}(\mathcal{H})$. Hence $\lambda \notin \sigma\left(\overline{A^{1/2}T(A^{1/2})^\dagger}\right)$.

So we have shown that

$$\sigma\left(\overline{A^{1/2}T(A^{1/2})^\dagger}\right) \backslash \{0\} \subseteq \sigma_A(T) \backslash \{0\}.$$

Since $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ there exists $S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ so that $A^{1/2}S = T^*A^{1/2}$. Hence

$$A^{1/2}T(A^{1/2})^\dagger = S^*A^{1/2}(A^{1/2})^\dagger = S^*P_{|D\left((A^{1/2})^\dagger\right)}.$$

The uniqueness of the extension of a bounded densely defined operator entails that $\overline{A^{1/2}T(A^{1/2})^\dagger} = S^*P$ and then

$$\sigma\left(\overline{A^{1/2}T(A^{1/2})^\dagger}\right) \backslash \{0\} = \sigma(PS^*) \backslash \{0\}.$$

This establishes (3.7). It remains to show that $\sigma_A(T) \backslash \{0\} = \sigma(PT) \backslash \{0\}$ if $\mathcal{R}(A)$ is closed. To that end, note in this case we have $\widetilde{\mathcal{B}}(\mathbf{R}(A^{1/2})) = \mathcal{B}(\mathbf{R}(A^{1/2}))$. Further, by [7, Lemma 5.5], we know that $\sigma_A(T) = \sigma_A(PT)$. Using the fact that α is a homomorphism together with Theorem 3.1, we infer that $\sigma_A(T) \subset \sigma(PT)$ for any $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. This together with (3) and (3.6) imply that

$$\sigma(PS^*) \backslash \{0\} \subseteq \sigma(PT) \backslash \{0\}. \tag{3.14}$$

Replacing T by S, a similar reasoning implies that $\sigma(PT^*) \backslash \{0\} \subseteq \sigma(PS) \backslash \{0\}$. Whence

$$\sigma(TP) \backslash \{0\} \subseteq \sigma(S^*P) \backslash \{0\}. \tag{3.15}$$

Keeping in mind that $\sigma(XY) \backslash \{0\} = \sigma(YX) \backslash \{0\}$ for any $X, Y \in \mathcal{B}(\mathcal{H})$ we infer that $\sigma_A(T) \backslash \{0\} = \sigma(TP) \backslash \{0\}$. This completes the proof. □

REMARK 3.3. Below are a few highlights of the aforementioned theorems.

(i) Observe that it may happen that $\sigma_A(T) \neq \sigma(PT)$. For instance, if the range of A is not dense in $\mathcal{H}$, then $\sigma(P) = \{0, 1\}$ but $\sigma_A(P) = \{1\}$ since $AP = AP^2 = A$.

(ii) The following example shows that the equality $\sigma_A(T) = \sigma(\alpha(T))$ in Theorem 3.1 may not remains valid if $\mathcal{R}(A)$ is not supposed to be closed.

EXAMPLE 3.4. Consider the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$ and let A be the multiplication operator defined on $\mathcal{H}$ by $A(x_n) = (a_n^2 x_n)$ where $a_n = \frac{1}{2^n}$ for any $n \geqslant 0$. Consider the unilateral weighted forward shift (resp. the unilateral weighted backward) shift operator T (resp. S) given by

$$T(x_n) = \frac{2}{5}\,(0, x_0, x_1, \cdots) \text{ (resp. } S(x_n) = \frac{1}{5}\,(x_1, x_2, \cdots))$$

for all $(x_n)_{n\geqslant 0} \in \mathcal{H}$. Straightforward computations show that $T^*A^{1/2} = A^{1/2}S$. In particular T, S are in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$. Further, since A is injective, then $P = I$ and

$$\|T\|_A = \|S\| = \frac{1}{5} \text{ and } \|S\|_A = \|T\| = \frac{2}{5}.$$

We claim that either $\sigma_A(T) \neq \sigma(\alpha(T))$ or $\sigma_A(S) \neq \sigma(\alpha(S))$. Indeed suppose that $\sigma_A(T) = \sigma(\alpha(T))$ and $\sigma_A(S) = \sigma(\alpha(S))$. It is well known that the spectrum $\sigma(\alpha(T))$ (resp. $\sigma(\alpha(S))$) of T (resp. of S) is a closed subset of the disc in the plane of centre the origin and radius $\|\alpha(T)\|_{\mathcal{B}(\mathbf{R}(A^{1/2}))}$ (resp. $\|\alpha(S)\|_{\mathcal{B}(\mathbf{R}(A^{1/2}))}$); see for instance [12, Lemma 1.2.4]. Since $\|T\|_A = \|\alpha(T)\|_{\mathcal{B}(\mathbf{R}(A^{1/2}))}$ and $\|S\|_A = \|\alpha(S)\|_{\mathcal{B}(\mathbf{R}(A^{1/2}))}$, it yields from Lemma 2.3 that

$$|\lambda| \leqslant \min(\|T\|_A, \|S\|_A) = \frac{1}{5} \tag{3.16}$$

for any $\lambda \in \sigma_A(T) = \overline{\sigma_A(S)}$. But, by Theorem 3.2, we have $\sigma(T^*) \subset \sigma_A(S)$. Keeping in mind Eq. (3.16), we get $r(T) \leqslant \frac{1}{5}$. Here $r(T)$ denotes the spectral radius of T. This is a contradiction, since $r(T) = \frac{2}{5}$; see for instance [8].

REMARK 3.5. It is worth observing that, for the operator T in Example 3.4 and using similar technique as Example 3.4, we can shows that $\sigma_A(T) \nsubseteq \sigma(\alpha(T))$ or $\sigma_A(S) \nsubseteq \sigma(\alpha(S))$.

The next Proposition tells us that if $T \in \mathcal{B}_A(\mathcal{H})$ is A-invertible then all A-inverses are also in $\mathcal{B}_A(\mathcal{H})$.

PROPOSITION 3.6. *If $T \in \mathcal{B}_A(\mathcal{H})$ is A-invertible in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ then all of the A-inverses of T are in $\mathcal{B}_A(\mathcal{H})$.*

In particular, if S is an A-inverse of T then $S \in \mathcal{B}_A(\mathcal{H})$ and $S^\sharp$ is an A inverse of $T^\sharp$.

Proof. Since $T \in \mathcal{B}_A(\mathcal{H})$, there is $X \in \mathcal{B}_A(\mathcal{H})$ such that $AX = T^*A$. It then follows that

$$A_{eff}X_{eff} = (T^*)_{eff}A_{eff}.$$

Further, easy computation shows that X is also A-invertible. Hence, by Lemma 2.4, T_{eff} and X_{eff} are both invertible. Therefore $T_{eff}^* = (T_{eff})^*$ is also invertible. It follows that

$$A_{eff}X_{eff}^{-1} = (T_{eff}^*)^{-1}A_{eff}.$$

Now, let $S = \begin{bmatrix} S_1 & S_2 \\ 0 & T_{eff}^{-1} \end{bmatrix}$ be an A-inverse of T. We get $AS = YA$ with $Y = \begin{bmatrix} 0 & 0 \\ 0 & T_{eff}^{-1} \end{bmatrix}$ which belongs to $\mathcal{B}(\mathcal{H})$. Hence $S \in \mathcal{B}_A(\mathcal{H})$.

Finally, assume that $T \in \mathcal{B}_A(\mathcal{H})$ is A-invertible and $S \in \mathcal{B}_A(\mathcal{H})$ is an A-inverse. Recall that $T^{\sharp_A} = A^\dagger T^* A$ and $S^{\sharp_A} = A^\dagger S^* A$. Therefore

$$\begin{aligned} AT^{\sharp_A}S^{\sharp_A} &= AA^\dagger T^* AA^\dagger S^* A \\ &= PT^*PS^*A \\ &= T^*S^*A, \ (\text{since } \ \mathcal{R}(S^*A) \subseteq \mathcal{R}(A) \text{ and } \mathcal{R}(S^*A) \subseteq \mathcal{R}(A)) \\ &= A, \ (\text{since } AST = A). \end{aligned}$$

Whence $AT^{\sharp_A}S^{\sharp_A} = A$. Similarly we can show that $AS^{\sharp_A}T^{\sharp_A} = A$ and the proof is thus complete. □

Observe that, when A has a closed range and using Theorem 3.7, the proofs of some results of [7] may be shortened. For instance it is easy to see that $\sigma_A(T)$ is non empty, compact subset of $\mathbb{C}$ and $|\lambda| \leqslant \|T\|_A$ for any $\lambda \in \sigma_A(T)$.

In what follows we give other applications of the above results. Some of which could be of independent interest. In the sequel we assume that A has a closed range.

The first corollary is a Beurling formula for A-bounded operators.

COROLLARY 3.7. *If T is an element of $\mathcal{B}_{A^{1/2}}(\mathcal{H})$, then*

$$r_A(T) := \sup_{\lambda \in \sigma_A(T)} |\lambda| = \lim_n \|\alpha(T)^n\|_{\mathcal{B}(\mathbf{R}(A^{1/2}))}^{\frac{1}{n}}.$$

for any $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$.

Proof. We have

$$\begin{aligned} \lim_n \|T^n\|_A^{\frac{1}{n}} &= \lim_n \|\alpha(T)^n\|_{\mathcal{B}(\mathbf{R}(A^{1/2}))}^{\frac{1}{n}}, \ \ (\text{by (2.1)-(iii)}) \\ &= \sup_{\lambda \in \sigma(\alpha(T))} |\lambda|, \ (\text{by Beurling theorem (see [12, Theorem 1.2.7])} \\ &= \sup_{\lambda \in \sigma_A(T)} |\lambda|, \ \ (\text{since } \sigma_A(T) = \sigma(\alpha(T))). \quad \square \end{aligned}$$

An element $T \in \mathcal{B}_A(\mathcal{H})$ is said to be A-normal (resp. A-self-adjoint) if $T^{\sharp_A}T = TT^{\sharp_A}$ (resp. if AT is self-adjoint. That is $T^{\sharp_A} = T$).

It is well known that $V(T)$ is a convex compact set which contains $\sigma(T)$. Also, if $T \in \mathcal{B}(\mathcal{H})$ is normal, then by [13] we have

$$V(T) = \operatorname{conv}(\sigma(T)). \tag{3.17}$$

The following corollary gives a similar result for any $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$.

COROLLARY 3.8. *The following assertions hold.*

1. $\sigma_A(T) \subseteq V(T)$ *for every* $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$.
2. *If* $T \in \mathcal{B}_A(\mathcal{H})$ *is* A*-normal, then* $V(T) = \operatorname{conv}(\sigma(T))$.

Proof. Let $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. By [1, Theorem 2.9] (see also [5, Lemma 1.]) and [13, Theorem 1], we know that $V_A(T) = V(\alpha(T))$ and $\sigma(\alpha(T)) \subset V(\alpha(T))$. Hence, Theorem 3.7 entails that $\sigma_A(T) \subseteq V_A(T)$.

Now, assume that T is A-normal and then $\alpha(T)$ in normal in $\mathcal{B}(\mathbf{R}(A^{1/2}))$, by [4, Theorem 3.11]. Whence, Theorem 8 in [13] implies $V(\alpha(T)) = \operatorname{conv}\sigma(\alpha(T))$. Theorem 3.7 and the fact that $V(\alpha(T)) = V_A(T)$ concludes the proof. $\square$

COROLLARY 3.9. *For every* $x, y \in \mathcal{H}$ *we have* $\sigma_A(x \otimes_A y) \subset \{0, [Ax, Ay]\}$.

Proof. Follows from Lemma 2.1-(iv). $\square$

COROLLARY 3.10.

1. *If* $T \in \mathcal{B}_A(\mathcal{H})$, *then* $\sigma_A(X) = \sigma_A(T^{\sharp_A}) = \sigma_A(T)^* := \left\{\overline{\lambda} : \lambda \in \sigma_A(T)\right\}$, *for any* A*-adjoint* X *of* T.
2. *If* T *is* A*-selfadjoint then* $\sigma_A(T) \subseteq \mathbb{R}$.

Proof. Since $AX = T^*A$, then by Proposition 3.10 of [4], we have $\alpha(X) = \alpha(T)^*$. Therefore

$$\sigma_A(X) = \sigma(\alpha(X)) = \sigma(\alpha(T)^*) = \sigma(\alpha(T))^* = \sigma_A(T)^*. \quad \square$$

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