Operators and Matrices
Volume 17, Number 3 (2023), 613–638 doi:10.7153/oam-2023-17-41

FEEDBACK STABILIZATION OF THE LINEARIZED VISCOUS SAINT–VENANT SYSTEM BY CONSTRAINED DIRICHLET BOUNDARY CONTROL

HASSEN ARFAOUI

(*Communicated by B. Jacob*)

Abstract. In this paper, we study the stabilization of a linearized viscous Saint-Venant system by constrained Dirichlet boundary control in infinite time horizon. We proved the well posedness of the considered stabilization problem. Also, using an augmented state method, we were able to determine the optimal control (the constrained control) as a feedback control law. Moreover, thanks to the feedback control law, we proved the exponential stability of the solution to the linearized viscous Saint-Venant system, (defined by an unbounded operator). Some numerical experiments are given to illustrate the efficiency of the constrained Dirichlet boundary control.

1. Introduction

The stabilization of partial differential systems has been studied by several researchers [1, 2, 3, 4, 7, 8, 10, 11, 15, 16]. For example, a fast boundary stabilization of the wave equation has been studied in [17]. Also, in [18], the authors studied a boundary stabilization of quasilinear hyperbolic systems of balance laws: exponential decay for small source terms. Moreover, the authors in [19] studied the Neumann boundary feedback stabilization for a nonlinear wave equation by Lyapunov function, and a boundary feedbcak stabilization of the telegraph equation has been investigated in [20].

In many cases, the stabilization has been established by feedback control law. In fact, there are different methods that can be used to determine a control in the form of feedback control law. Determining a control in feedback form is a challenge that has attracted several authors [5, 10, 11, 19, 20]. In particular in [2] a Dirichlet boundary feedback control law was determined by extension method. In [4], the author determines the feedback control law by using the Kernel of a feedback operator. In [6, 7, 8], a feedback control law was determined for different partial differential systems by other different methods. The common point between the different works cited above is that the control variable is not subject to any constraints. In this case, the search for the control in the form of feedback control law is practically without major difficulties.

Mathematics subject classification (2020): 93C20, 93D15, 93B52, 65L10.
Keywords and phrases: Linearized viscous Saint-Venant system, unbounded operator, constrained boundary control, feedback control law, stabilization.

Paper OaM-17-41

Following [21], one-dimensional shallow water flows may be modeled by the viscous Saint-Venant system written in Eulerian coordinates in the form

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2) + \frac{g}{2}\partial_x(h^2) = 4\mu\partial_x(h\partial_x u) - \kappa u\left(1+\frac{\kappa h}{3\mu}\right)^{-1}, \\ h(x,0) = h_0(x), \quad u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where (h,u) denotes the (water height, velocity) of the fluid, μ is the viscosity, g the gravitation and κ and μ are the bottom friction factors [21].

We are interested in stabilizing this system by means of moving wall devices located at the extremities of the channel. Therefore the corresponding equation will be stated in a variable domain. In order to rewrite the system in a fixed domain, we will transform the system (1.1) by using the mass Lagrangian coordinates (see [1, 22, 23]). Considering the new fields

$$V(X,t) = \frac{1}{h(x,t)} \quad \text{and} \quad U(X,t) = u(x,t),$$

where (X,t) is the mass Lagrangian coordinates and (x,t) is the Eulerian coordinates. The Saint-Venant system can be reformulated as follows:

$$\begin{cases} \partial_t V - \partial_X U = 0, \\ \partial_t U + g\partial_X\left(\frac{1}{2V^2}\right) = 4\mu\partial_X\left(\frac{\partial_X U}{V^2}\right) - \frac{\kappa VU}{\left(1+\dfrac{\kappa}{3\mu V}\right)}, \\ U(X,0) = U_0(X), \quad V(X,0) = V_0(X). \end{cases} \tag{1.2}$$

For convenience, in the following we use the notation (x,t) instead of (X,t). Given that the fluid equations are considered in a fixed bounded domain $\Omega =]r,\ell[$, $(r<\ell)$, some boundary conditions must be added on the velocity U. If the channel walls are not moving, then the boundary conditions for U are

$$U(r,t) = 0, \quad U(\ell,t) = 0, \qquad t \in (0,\infty). \tag{1.3}$$

We are interested in stabilizing system (1.2) by a boundary control corresponding to moving wall devices. Then, as mentioned before, we have a time-varying domain in Eulerian coordinates, while the Lagrangian coordinates allow to work in a fixed domain. Let $q_r(t)$ be the velocity of the left side wall of the channel and $q_\ell(t)$ the velocity of the right side wall, then the homogeneous conditions (1.3) are replaced by

$$U(r,t) = q_r(t), \quad U(\ell,t) = q_\ell(t).$$

In this paper, to stabilize system (1.2) in a neighbourhood of a steady state $(V_s,U_s) \equiv (c,0)$, where $c \in \mathbb{R}_+^*$, we study the stabilization of the flow described by the linearized control system

$$\partial_t\xi - \partial_x\zeta = 0 \qquad \text{in} \quad \Omega_\infty, \tag{1.4}$$

$$\partial_t\zeta - a\partial_x\xi - \nu\partial_{xx}\zeta = 0 \qquad \text{in} \quad \Omega_\infty, \tag{1.5}$$

$$\zeta(x,0) = \rho_0(x), \quad \xi(x,0) = \sigma_0(x) \qquad \text{in} \quad \Omega, \tag{1.6}$$

$$\zeta(r,t) = q_r(t), \quad \zeta(\ell,t) = q_\ell(t) \qquad \text{in} \quad (0,+\infty). \tag{1.7}$$

The initial condition is given by

$$\rho_0(x) = U_0(x), \qquad \sigma_0(x) = V_0(x) - V_s,$$

the viscosity coefficient ν and the advection coefficient a are given by

$$a = \frac{g}{V_s^3}, \qquad \nu = \frac{4\mu}{V_s^2},$$

and q_r, q_ℓ are two boundary controls which belong to a set of admissible controls U_{ad}.

In this work, we focus on the study of the stabilization problem of the linearized viscous Saint-Venant system (1.4)–(1.7) by constrained Dirichlet boundary controls $q_r, q_\ell \in U_{ad}$ in infinite time horizon. Our objective is to find a boundary controls in feedback forms able to stabilize system (1.4)–(1.7) which is not stable but only stabilizable see [2]. The problem that arises in this work is that the controls variables are subjected to several constraints. This is obviously going to make the search for a feedback control law very difficult. To overcome this difficulty, we proposed an augmented state technique allowing to transform the initial problem with constrained controls into a new equivalent problem with unconstrained controls. The first difficulty encountered is the writing of the optimality system. In fact, we can't use the traditional techniques to obtain the optimality system, because the linearized Saint-Venant system (1.4)–(1.7) that we want to study is not stable, but only stabilizable (see [2]). Therefore, to resolve this difficulty, we determine the optimality system associated with the control problem posed in finite time horizon. Next, we use a passage to the limit, in time, in the optimality system. In fact, this technique has been used in [11]. From the optimality system thus determined, we write the optimal controls as feedback control laws using a Riccati operator who is solution of an algebraic Riccati equation ([5]).

To my acknowledgments, this kind of control problem, posed in infinite time horizon, with constraints on the control variable and the search for a feedback control law expressing the control as a function of the state of the system by Riccati operator, has never been treated in the literature.

The new results established in this paper are as follows:

- ◇ Prove the existence and uniqueness of an optimal solution of the considered control problem, (see Section 4).
- ◇ Write the optimality conditions of the considered control problem posed in infinite time horizon, (see Theorem 6.7).
- ◇ Prove the exponential stability of the optimal solution, (see Theorem 6.8).
- ◇ Write the optimal control as a feedback control law using a Riccati operator, (see Theorem 6.10).
- ◇ Give some numerical experiments that illustrate the theoretical results.

The paper is organized as follows. In section 2, we formulate the constrained optimal problem with boundary Dirichlet control. Then, we give the new boundary control problem with the new control and the augmented state. In section 3, we study the augmented system. In section 4, we prove the existence and uniqueness of an optimal solution to the control problem. The study of the adjoint system is acheved in section 5. In section 6, we give the main result of this paper where we give the feedback control law as a function of the Riccati operator and the solution of the state equations. Finally in Section 7, some numerical experiments are given that illustrate the theoretical results.

2. Setting of the control problem

In this paper, we study the stabilization of the one-dimensional linearized Saint-Venant system given by:

$$\partial_t \xi - \partial_\varkappa \zeta = 0 \qquad \text{in} \quad \Omega_\infty, \tag{2.8}$$

$$\partial_t \zeta - a\partial_\varkappa \xi - \nu \partial_{\varkappa\varkappa} \zeta = 0 \qquad \text{in} \quad \Omega_\infty, \tag{2.9}$$

$$\zeta(\varkappa,0) = \rho_0(\varkappa), \quad \xi(\varkappa,0) = \sigma_0(\varkappa) \qquad \text{in} \quad \Omega, \tag{2.10}$$

$$\zeta(0,t) = 0, \quad \zeta(\ell,t) = q(t) \qquad \text{in} \quad (0,\infty), \tag{2.11}$$

where $\Omega =]0,\ell[$ and $\Omega_\infty = \Omega \times (0,\infty)$. The parameters $\nu > 0$, a are the viscosity and the advection coefficient, respectively. The boundary data q, given at $\varkappa = \ell$, is the control.

Since we are looking for a control q stabilizing the system (2.8)–(2.11) with the exponential decay rate $e^{-\alpha t}$, it is convenient to formulate the control problem as follows: For any given $\alpha > 0$, find $q \in U_{ad}$ that minimizes the cost functional:

$$\min_{q \in U_{ad}} \left\{ j_\alpha(\xi,\zeta,q) = \frac{1}{2}\int_0^\infty \left(\|\xi(t)\|^2_{L^2(\Omega)} + \|\zeta(t)\|^2_{L^2(\Omega)} + [L(q(t))]^2 \right) e^{2\alpha t}\, dt \right\}, \tag{2.12}$$

where (ξ,ζ) is the solution to the problem (2.8)–(2.11), U_{ad} is a set of admissible controls and L is a linear invertible operator that will be determined later.

Knowing that q is the speed of displacement of the right side wall of the channel during the disturbance of the fluid, then $\int_0^T q(t)\,dt$ represents the distance traveled by the right wall during the time interval $[0,T]$ in both directions (forward/backward) with respect to its equilibrium position (where the speed is zero). Therefore, necessarily the totality of the distance traveled will be zero when the state of the system is stable, i.e.

$$\lim_{T\to\infty} \int_0^T q(t)\,dt = 0.$$

REMARK 2.1. If $(\sigma_0,\rho_0) \in H^1(\Omega) \times H^1(\Omega)$, we are looking for a control $q \in H^1(0,\infty)$ that stabilizes (2.8)–(2.11) and satisfies the constraints

$$\int_0^\infty q(t)\,dt = 0 \qquad \text{and} \qquad q(0) = \rho_0(\ell),$$

where $q(0) = \rho_0(\ell)$ is a compatibility condition between q and the initial condition $\zeta(\cdot,0)$ given by (2.10).

So, we can deduce from Remark 2.1 that the set of admissible controls U_{ad} can be defined as follows:

$$U_{ad} = \left\{ q \in H^1(0,\infty), \quad \int_0^\infty q(t)\,dt = 0, \quad q(0) = \rho_0(\ell) \right\}. \tag{2.13}$$

REMARK 2.2. It's not easy to determine the control $q \in U_{ad}$ in feedback form for the stabilization problem (2.12). Indeed, we have many constraints on the control q, (see (2.13)). To overcome this difficulty, we propose an augmented state technique that consists in introducing a new state and formulating an augmented state system for the control problem (2.8)–(2.12). For example, we consider that q is a solution of a differential equation and choose the source term of this differential equation as a new control variable. A possible choice of this differential equation is as follows: we look for a control q in the form

$$L(q(t)) = u(t), \quad \forall t \geqslant 0, \tag{2.14}$$

where the linear operator L is defined as follows:

$$\begin{aligned} L: \qquad & U_{ad} \longrightarrow L^2(0,+\infty) \\ L(q(t)) &= q'(t) + 2\frac{a}{\nu}q(t) + \frac{a^2}{\nu^2}\int_0^t q(s)\,ds, \quad \forall t \geqslant 0. \end{aligned}$$

Now, if we set

$$\theta(t) = \int_0^t q(s)\,ds \quad \Longrightarrow \quad \theta'(t) = q(t), \quad \forall t \geqslant 0 \tag{2.15}$$

and take into account equation (2.14), we can show that the state θ is the unique solution to the second order ordinary differential equation

$$L(\theta'(t)) = u(t), \quad \forall t \geqslant 0,$$

or equivalently

$$\theta'' + 2\frac{a}{\nu}\theta' + \frac{a^2}{\nu^2}\theta = u \quad \text{in} \quad (0,\infty),$$

with initial conditions

$$\theta(0) = 0, \quad \theta'(0) = \rho_0(\ell),$$

where the source term u represents the new control variable and θ is a new state.

The new stabilization problem, defined by the new control variable u and the augmented state $(\xi,\ \zeta,\ \theta)$, is now reformulated as follows: we want to determine a control u, in feedback form, who is able to stabilize the Saint-Venant system (2.8)–(2.11). So, the linear regulator problem (2.12) becomes:

$(\mathcal{Q}_\alpha)$: *For any given* $\alpha > 0$, *find* $u \in L^2(0,\infty)$ *that minimizes the cost functional*

$$\min_{u\in L^2(0,\infty)} \left\{ J_\alpha(\xi,\zeta,\theta,u) = \frac{1}{2}\int_0^\infty \left(\|\xi(t)\|^2_{L^2(\Omega)} + \|\zeta(t)\|^2_{L^2(\Omega)} + u^2(t) \right) e^{2\alpha t}\,dt \right\} \tag{2.16}$$

where the augmented state (ξ , ζ , θ) is the solution of the linear system

$$\partial_t \xi - \partial_\varkappa \zeta = 0 \qquad in \quad \Omega_\infty, \tag{2.17}$$

$$\partial_t \zeta - a\partial_\varkappa \xi - \nu \partial_{\varkappa\varkappa} \zeta = 0 \qquad in \quad \Omega_\infty, \tag{2.18}$$

$$\zeta(\varkappa,0) = \rho_0(\varkappa), \quad \xi(\varkappa,0) = \sigma_0(\varkappa) \qquad in \quad \Omega, \tag{2.19}$$

$$\zeta(0,t) = 0, \quad \zeta(\ell,t) = \theta'(t) \qquad in \quad (0,\infty), \tag{2.20}$$

$$\theta'' + 2\frac{a}{\nu}\theta' + \frac{a^2}{\nu^2}\theta = u(t) \qquad in \quad (0,\infty), \tag{2.21}$$

$$\theta(0) = 0, \quad \theta'(0) = q(0). \tag{2.22}$$

Notice that the new control variable $u \in L^2(0,\infty)$ is unconstrained. This will make the work easier later.

REMARK 2.3. Note that the control problem considered in (2.8)–(2.11) can be defined by two controls q_0 at $\varkappa = 0$ and q_1 at $\varkappa = \ell$, but the fact of using a single control has the reason of making the theoretical study easier and more readable. In fact, the study made for a single control can be extended for the case of two controls without difficulty.

2.1. Modified regulator problem

We consider the new state (σ , ρ , β) and v which are defined as follows:

$$\begin{aligned}&\sigma(\varkappa,t) = \xi(\varkappa,t)e^{\alpha t}, \quad \rho(\varkappa,t) = \zeta(\varkappa,t)e^{\alpha t} \quad \text{for all} \quad (\varkappa,t) \in \Omega_\infty,\\ &\beta(t) = \theta(t)e^{\alpha t}, \quad v(t) = u(t)e^{\alpha t} \quad \text{for all} \quad t \geqslant 0.\end{aligned} \tag{2.23}$$

Then the problem $(\mathscr{Q}_\alpha)$ becomes the following modified regulator problem:

$(\hat{\mathscr{Q}}_\alpha)$*: For any given* $\alpha > 0$*, find* $v \in L^2(0,\infty)$ *that minimizes the cost functional*

$$\min_{v \in L^2(0,\infty)} \left\{ J(\sigma,\rho,\beta,v) = \frac{1}{2}\int_0^\infty \left(\|\sigma(t)\|^2_{L^2(\Omega)} + \|\rho(t)\|^2_{L^2(\Omega)} + v^2(t) \right) dt \right\} \tag{2.24}$$

where the augmented state (σ, ρ, β) is the solution of the regulator linear system

$$\partial_t \sigma - \partial_\varkappa \rho - \alpha\sigma = 0 \qquad in \quad \Omega_\infty, \tag{2.25}$$

$$\partial_t \rho - a\partial_\varkappa \sigma - \nu \partial_{\varkappa\varkappa} \rho - \alpha\rho = 0 \qquad in \quad \Omega_\infty, \tag{2.26}$$

$$\rho(\varkappa,0) = \rho_0(\varkappa), \quad \sigma(\varkappa,0) = \sigma_0(\varkappa) \qquad in \quad \Omega, \tag{2.27}$$

$$\rho(0,t) = 0, \quad \rho(\ell,t) = \hat{q}(t) \qquad in \quad (0,\infty), \tag{2.28}$$

$$\beta'' - 2\left(\alpha - \frac{a}{\nu}\right)\beta' + \left(\alpha - \frac{a}{\nu}\right)^2 \beta = v(t) \qquad in \quad (0,\infty), \tag{2.29}$$

$$\beta(0) = 0, \quad \beta'(0) = q(0), \tag{2.30}$$

where the Dirichlet condition $\hat{q}$, (in (2.28)), is given by the relation

$$\hat{q}(t) = q(t)e^{\alpha t}, \qquad \forall t \geqslant 0.$$

In the following sections, we are interested in the study of the regulator problem $(\hat{\mathscr{Q}}_\alpha)$. More precisely, we look for a control $v \in L^2(0,\infty)$ in feedback form, able to stabilize the system (2.25)–(2.30).

2.2. Notations and assumptions

In the following, we define some spaces that will be used later.

1. The set $L^2_m(\Omega)$ is given by:

$$L^2_m(\Omega) = \Big\{ h \in L^2(\Omega) : \quad \int_\Omega h(s)\, ds = 0 \Big\}.$$

2. The set:

$$H^1_m(\Omega) = H^1(\Omega) \cap L^2_m(\Omega),$$

where $H^1(\Omega)$ is the standard Sobolev space.

3. The spaces $\boldsymbol{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$ and $\boldsymbol{L}^2_m(\Omega) = L^2_m(\Omega) \times L^2(\Omega)$ with the scalar product:

$$(\boldsymbol{W}, \boldsymbol{U})_{\boldsymbol{L}^2(\Omega)} = a \int_\Omega \sigma(s) v(s)\, ds + \int_\Omega \rho(s) w(s)\, ds,$$

where $\boldsymbol{W} = (\sigma, \rho)^T$, $\boldsymbol{U} = (v, w)^T$.

4. The set $\boldsymbol{H}^1_m(\Omega) = H^1_m(\Omega) \times H^1_0(\Omega)$ on which we define a scalar product as follows:

$$((\boldsymbol{W}, \boldsymbol{U}))_{\boldsymbol{H}^1_m(\Omega)} = a \int_\Omega \sigma'(s) v'(s)\, ds + \int_\Omega \rho'(s) w'(s)\, ds.$$

3. Study of the state equations

Let us consider the homogeneous Saint-Venant system defined by:

$$\partial_t \sigma - \partial_\varkappa \rho - \alpha \sigma = 0 \qquad \text{in} \quad \Omega_\infty, \tag{3.31}$$

$$\partial_t \rho - a \partial_\varkappa \sigma - \nu \partial_{\varkappa\varkappa} \rho - \alpha \rho = 0 \qquad \text{in} \quad \Omega_\infty, \tag{3.32}$$

$$\rho(\varkappa, 0) = \rho_0(\varkappa), \quad \sigma(\varkappa, 0) = \sigma_0(\varkappa) \qquad \text{in} \quad \Omega, \tag{3.33}$$

$$\rho(0, t) = 0, \quad \rho(\ell, t) = 0 \qquad \text{in} \quad (0, \infty), \tag{3.34}$$

If we noted $\boldsymbol{W} = (\sigma, \rho)$ and $\boldsymbol{W}_0 = (\sigma_0, \rho_0)$, then system (3.31)–(3.34) can be rewritten as follows:

$$\boldsymbol{W}'(t) + A_\alpha \boldsymbol{W}(t) = 0, \qquad \forall t > 0, \tag{3.35}$$

$$\boldsymbol{W}(0) = \boldsymbol{W}_0, \tag{3.36}$$

where A_α is an unbounded operator on $\boldsymbol{L}^2(\Omega)$ defined by

$$A_\alpha = \begin{pmatrix} -\alpha & -\frac{d}{d\varkappa} \\ -a\frac{d}{d\varkappa} & -\nu \frac{d^2}{d\varkappa^2} - \alpha \end{pmatrix}. \tag{3.37}$$

Moreover, we have

$$\mathscr{D}(A_\alpha) = \Big\{(\sigma,\rho) \in \boldsymbol{L}^2(\Omega) \mid \quad \rho \in H_0^1(\Omega), \quad a\sigma + \nu\rho' \in H^1(\Omega)\Big\}.$$

The operator $(-A_\alpha, \mathscr{D}(A_\alpha))$ generates a semigroup on $\boldsymbol{L}^2(\Omega)$ denoted by $S_\alpha(t) = e^{-tA_\alpha}$, for all $t \geqslant 0$, (see [2]).

Now, let us consider the nonhomogeneous system, with Dirichlet condition $q(t)$ at $\varkappa = \ell$, defined by the equations

$$\partial_t\sigma - \partial_\varkappa\rho - \alpha\sigma = 0 \qquad \text{in} \quad \Omega_\infty, \tag{3.38}$$

$$\partial_t\rho - a\partial_\varkappa\sigma - \nu\partial_{\varkappa\varkappa}\rho - \alpha\rho = 0 \qquad \text{in} \quad \Omega_\infty, \tag{3.39}$$

$$\rho(\varkappa,0) = \rho_0(\varkappa), \quad \sigma(\varkappa,0) = \sigma_0(\varkappa) \qquad \text{in} \quad \Omega, \tag{3.40}$$

$$\rho(0,t) = 0, \quad \rho(\ell,t) = q(t) \qquad \text{in} \quad (0,\infty), \tag{3.41}$$

REMARK 3.1. An existence and uniqueness result for the system (3.38)–(3.41) has been proved in [2, Theorem 2]: for any $\boldsymbol{W}_0 = (\sigma_0,\rho_0)^T \in H_m^1(\Omega) \times H_{\{0\}}^1(\Omega)$, (where $H_{\{0\}}^1(\Omega) = \{g \in H^1(\Omega) : g(0) = 0\}$), and any $q \in H^1(0,+\infty)$ such that $\rho_0(\ell) = q(0)$, the system (3.38)–(3.41) has a unique solution $\boldsymbol{W} \in C([0,+\infty[, \boldsymbol{H}^1(\Omega))$. Moreover, we have $\boldsymbol{W} \in L^2(0,+\infty; H^1(\Omega)) \times L^2(0,+\infty; H^1(\Omega))$.

From equations (2.29)–(2.30), we can verify that the state β has the following expression

$$\beta(t) = te^{(\alpha - \frac{a}{\nu})t}q(0) + \int_0^t (t-s)e^{(\alpha-\frac{a}{\nu})(t-s)}v(s)\,ds \qquad \text{in} \quad (0,\infty), \tag{3.42}$$

and we have the result.

LEMMA 3.1. *For all* $\alpha \in \left]0, \frac{a}{\nu}\right[$ *and* $v \in L^2(0,\infty)$, *the Cauchy problem* (2.29)–(2.30) *admits a unique solution* $\beta \in H^2(0,\infty)$ *defined by* (3.42) *and verify the estimation*

$$\|\beta\|_{H^2(0,\infty)} \leqslant C\Big(|q(0)| + \|v\|_{L^2(0,\infty)}\Big),$$

where the constant C *is dependent only on the parameters* a *and* ν.

REMARK 3.2. In the following, and until the end of this paper, the coefficient α is taken under the condition

$$\alpha \in \left]0, \frac{a}{\nu}\right[,$$

where a is the advection coefficient and ν is the viscosity coefficient, (see system (2.8)–(2.11)).

4. Existence and uniqueness of an optimal solution

In this section we prove that the minimization problem (2.24) has a unique optimal solution. Let ϑ be the space defined by:

$$\vartheta = L^2(0,+\infty;H^1(\Omega)) \times L^2(0,+\infty;H^1(\Omega)).$$

Then, we have the following result.

LEMMA 4.1. *Let $(v_n)_n$ be sequence in $L^2(0,\infty)$ which converges to v for the weak topology of $L^2(0,\infty)$. Let $(\sigma^n,\rho^n,\beta^n)_n$ be the solution of the system (2.25)–(2.30) corresponding to the sequence $(v_n)_n$. Then $(\sigma^n,\rho^n,\beta^n)_n$ converges to (σ,ρ,β) for the weak topology of $\vartheta \times H^2(0,+\infty)$ and (σ,ρ,β) is the solution to the system (2.25)–(2.30) associated with v.*

Proof. From Theorem 2 in [2], Lemma 3.1 and the definition of the control v given by the equations (2.25)–(2.30), we can deduce that the application

$$v \in L^2(0,+\infty) \longmapsto (\sigma,\rho,\beta) \in \vartheta \times H^2(0,+\infty)$$

is continuous. In addition, it is affine application. It is, therefore, weakly continuous. So, if $(v_n)_n$ weakly converges to v in $L^2(0,\infty)$, then $(\sigma^n,\rho^n,\beta^n)_n$ weakly converges to (σ,ρ,β) in $\vartheta \times H^2(0,+\infty)$ and (σ,ρ,β) is the solution to the system (2.25)–(2.30) associated with v. $\square$

THEOREM 4.1. *The minimization problem (2.24) admits a unique solution*

$$(\overline{\sigma},\overline{\rho},\overline{\beta},\overline{v}) \in \vartheta^2 \times H^2(0,\infty) \times L^2(0,\infty).$$

Proof. Let $F(v) = J(\sigma_v,\rho_v,\beta_v,v)$. The function F is coercive. Let $(v_n)_n \in L^2(0,+\infty)$ be a minimizing sequence. We have

$$\lim_{n\longrightarrow+\infty} F(v_n) = \inf_{v\in L^2(0,+\infty)} F(v).$$

Since F is coercive, then the sequence $(v_n)_n$ is bounded in $L^2(0,+\infty)$. There is therefore an extracted sequence, noted again $(v_n)_n$, which converges to v for the weak topology of $L^2(0,+\infty)$. Let $(\sigma^n,\rho^n,\beta^n)_n$ be the solution of (2.25)–(2.30) associated with $(v_n)n$. The application

$$(\sigma,\rho,\beta,v) \longmapsto \frac{1}{2}\int_0^\infty \left(\|\sigma(t)\|^2_{L^2(\Omega)} + \|\rho(t)\|^2_{L^2(\Omega)} + v^2(t)\right) dt$$

is weakly lower semi-continuous. By Lemma 4.1, we have

$$F(v) \leqslant \lim_{n\longrightarrow+\infty} \inf(F(v_n)) = \inf_{v\in L^2(0,+\infty)} F(v).$$

So, (σ,ρ,β,v) is a solution of the minimization problem (2.24). The uniqueness stems from the strict convexity of the cost function J. $\square$

5. Study of the adjoint equations

The adjoint system of the state system (3.38)–(3.41) is given as follows

$$-\partial_t\varphi + a\partial_\varkappa\psi - \alpha\varphi = f_1 \qquad \text{in } \Omega_\infty, \tag{5.43}$$
$$-\partial_t\psi + \partial_\varkappa\varphi - \nu\partial_{\varkappa\varkappa}\psi - \alpha\psi = f_2 \qquad \text{in } \Omega_\infty, \tag{5.44}$$
$$\psi(\varkappa,\infty) = \varphi(\varkappa,\infty) = 0 \qquad \text{in } \Omega, \tag{5.45}$$
$$\psi(0,t) = \psi(\ell,t) = 0 \qquad \text{in } (0,\infty). \tag{5.46}$$

If we note $\boldsymbol{\phi} = (\varphi,\psi)$ and $\boldsymbol{G} = (f_1,f_2)$, then the above system can be rewritten as

$$-\partial_t\boldsymbol{\phi} + A^*_\alpha\boldsymbol{\phi} = \boldsymbol{G} \qquad \text{in} \quad (0,\infty), \qquad \boldsymbol{\phi}(\infty) = 0, \tag{5.47}$$

where A^*_α is the adjoint operator of the unbounded operator A_α, and is completly defined on $\boldsymbol{L}^2(\Omega)$ by

$$A^*_\alpha = \begin{pmatrix} -\alpha & a\frac{d}{d\varkappa} \\ \\ \frac{d}{d\varkappa} & -\nu\frac{d^2}{d\varkappa^2} - \alpha \end{pmatrix}$$

with domain $\mathscr{D}(A^*_\alpha) \subset \boldsymbol{L}^2(\Omega)$ given by

$$\mathscr{D}(A^*_\alpha) = \Big\{(\varphi,\psi) \in \boldsymbol{L}^2(\Omega) \mid \quad \psi \in H^1_0(\Omega), \quad \varphi - \nu\psi' \in H^1(\Omega)\Big\}.$$

The unbounded operator $(-A^*_\alpha, \mathscr{D}(A^*_\alpha))$ generates a semigroup on $\boldsymbol{L}^2(\Omega)$, given by

$$S^*_\alpha(t) = e^{-tA^*_\alpha}, \quad \forall t \geqslant 0.$$

Let us mention that A_α and A^*_α have the same spectrum, then $(S_\alpha(t))_{t\geqslant 0}$ and $(S^*_\alpha(t))_{t\geqslant 0}$ have the same properties. Consequently, the solution of (5.47) is as the following form:

$$\boldsymbol{\phi}(t) = \int_t^\infty S^*_\alpha(s-t)\boldsymbol{G}(s)\,ds, \qquad \forall t \geqslant 0.$$

Let V be the space defined by:

$$V = L^2(0,\infty;L^2_m(\Omega)) \times L^2(0,\infty;L^2(\Omega)).$$

REMARK 5.1. We proved in [2, Proposition 2] that for any $\boldsymbol{G} \in V$, the solution $\boldsymbol{\phi} \in H^1(0,\infty;H^1_m(\Omega)) \times \Big[H^1(0,\infty;L^2(\Omega)) \cap L^2(0,\infty;H^2(\Omega)\cap H^1_0(\Omega))\Big]$ and satisfies the estimation:

$$\|\varphi\|_{H^1(0,\infty;H^1_m(\Omega))} + \|\psi\|_{L^2(0,\infty;H^2(\Omega))} + \|\psi\|_{H^1(0,\infty;L^2(\Omega))} \leqslant C\|\boldsymbol{G}\|_V,$$

where C is a constant.

6. Feedback control

The system (2.25)–(2.30) is equivalent to the following

$$\partial_t\sigma - \partial_\varkappa\rho - \alpha\sigma = 0 \quad \text{in} \quad \Omega_\infty, \tag{6.48}$$

$$\partial_t\rho - a\partial_\varkappa\sigma - \nu\partial_{\varkappa\varkappa}\rho - \alpha\rho = 0 \quad \text{in} \quad \Omega_\infty, \tag{6.49}$$

$$\beta' - \alpha\beta - \hat{q} = 0 \quad \text{in} \quad (0,\infty), \tag{6.50}$$

$$\hat{q}' - (\alpha - 2\frac{a}{\nu})\hat{q} + (\frac{a}{\nu})^2\beta = v(t) \quad \text{in} \quad (0,\infty), \tag{6.51}$$

$$\rho(0,t) = 0, \quad \rho(\ell,t) = \hat{q}(t) \quad \text{in} \quad (0,\infty), \tag{6.52}$$

where the initial conditions are given by

$$\rho(\varkappa,0) = \rho_0(\varkappa), \quad \sigma(\varkappa,0) = \sigma_0(\varkappa) \quad \text{in} \quad \Omega, \tag{6.53}$$

$$\beta(0) = \beta_0 = 0, \quad \hat{q}(0) = q_0 = q(0). \tag{6.54}$$

Let us introduce some spaces that will be used later:

$$\begin{aligned}
Z &= \boldsymbol{L}^2(\Omega) \times \mathbb{R}^2,\\
H_0^m(\Omega) &= H_m^1(\Omega) \times H_{\{0\}}^1(\Omega),\\
\mathscr{V}_0^m(\Omega) &= H_0^m(\Omega) \times \mathbb{R}^2,\\
\mathscr{V}(\Omega) &= H^1(\Omega) \times H^1(\Omega) \times \mathbb{R}^2,\\
\mathscr{V}(0,\infty) &= \Big(L^2(0,\infty;H^1(\Omega))\Big)^2 \times H^2(0,\infty;\mathbb{R}) \times H^1(0,\infty;\mathbb{R}),\\
\mathscr{V}(s,T) &= \Big(L^2(s,T;H^1(\Omega))\Big)^2 \times H^2(s,T;\mathbb{R}) \times H^1(s,T;\mathbb{R}),
\end{aligned}$$

for any $s \in [0,T]$.

REMARK 6.1. We proved in section 3, that the system (6.48)–(6.54) has a unique solution $(\sigma,\rho,\beta,\hat{q}) \in \mathscr{V}(0,\infty)$. Moreover, from Remark 3.1, we deduce that $(\sigma,\rho) \in C([0,+\infty[,\boldsymbol{H}^1(\Omega))$. Consequently, we get $(\sigma,\rho,\beta,\hat{q}) \in C([0,+\infty[,\boldsymbol{H}^1(\Omega)) \times C([0,+\infty[,\mathbb{R}^2)$.

6.1. Formulation of the problem by augmented state

By considering the augmented state $\boldsymbol{w} = (\sigma,\rho,\beta,\hat{q})$, the system (6.48)–(6.54) can be rewritten as an abstract form as follows:

$$\boldsymbol{w}' + \mathscr{A}_\alpha\boldsymbol{w} = Bv \quad \text{in} \quad (0,\infty), \qquad \boldsymbol{w}(0) = \boldsymbol{w}_0, \tag{6.55}$$

where the initial condition $\boldsymbol{w}_0 = (\sigma_0,\rho_0,\beta_0,q_0)$. The operators $\mathscr{A}_\alpha$ and B are defined as follows

$$\mathscr{A}_\alpha = \begin{pmatrix} A_\alpha & 0 \\ 0 & \Upsilon_\alpha \end{pmatrix}, \qquad B = (0,0,0,1)^T.$$

where A_α is given by the relation (3.37) and Υ_α is defined by:

$$\Upsilon_\alpha = \begin{pmatrix} -\alpha & -1 \\ \\ \frac{a^2}{\nu^2} & \frac{2a}{\nu} - \alpha \end{pmatrix}.$$

So, we can deduce that the operator $(-\mathscr{A}_\alpha, \mathscr{D}(\mathscr{A}_\alpha))$ generates a semigroup, $S^\alpha(t) = e^{-t\mathscr{A}_\alpha}$, for all $t > 0$, on $\boldsymbol{L}^2(\Omega) \times \mathbb{R}^2$, where the domain $\mathscr{D}(\mathscr{A}_\alpha)$ is given by $\mathscr{D}(\mathscr{A}_\alpha) = \mathscr{D}(A_\alpha) \times \mathbb{R}^2$. Moreover, this semigroup is not exponentially stable on Z and on $\mathscr{V}_0^m(\Omega)$, since the spectrum of A_α contains unstable eigenvalues (see [2]). So, the solution of the system (6.55) is not stable, but only stabilizable on the space Z. Also, we can verify that the control operator B belongs to $\mathscr{L}(\mathbb{R}, \mathscr{V}(\Omega))$. The operator $\mathscr{A}_\alpha^*$ has comparable properties to those of operator $\mathscr{A}_\alpha$ on the space Z.

Using the augmented state $\boldsymbol{w}$, we write the cost functional defined by (2.24) as follows

$$J(\boldsymbol{w}, v) = \frac{1}{2} \int_0^\infty \|C\boldsymbol{w}(t)\|_Z^2 \, dt + \frac{1}{2} \int_0^\infty v^2(t) \, dt, \tag{6.56}$$

where C is an observation operator defined by $C = \operatorname{diag}(1,1,0,0)$ and verify $C \in \mathscr{L}(\mathscr{V}(\Omega))$. In this case, we can define a linear regulator problem, posed on infinite time horizon, as follows:

$$(\mathscr{Q}_{\boldsymbol{w}_0}^\infty) \qquad \inf \Big\{ J(\boldsymbol{w}, v) \mid (\boldsymbol{w}, v) \in \mathscr{V}(0,\infty) \times L^2(0,\infty;\mathbb{R}) \text{ is the solution to } (6.55) \Big\}$$

where the cost functional $J(\cdot,\cdot)$ is given by (6.56) and the pair $(\boldsymbol{w}, v)$ verify the equation (6.55). The finite cost condition [9, p. 124] holds. Indeed, since $(-\mathscr{A}_\alpha, B)$ is stabilizable in $\mathscr{V}_0^m(\Omega)$, then for all $\boldsymbol{w}_0 \in \mathscr{V}_0^m(\Omega)$, there exists $v_{\boldsymbol{w}_0} \in L^2(0,\infty;\mathbb{R})$ such that $J(\boldsymbol{w}_{\boldsymbol{w}_0}, v_{\boldsymbol{w}_0}) < \infty$. Thus, there exists a feedback operator $\mathscr{K} \in \mathscr{L}(\mathscr{V}(\Omega), \mathbb{R})$ such that $\mathscr{A}_\alpha + B\mathscr{K}$ is stable. If we choose the control $v \in L^2(0,\infty;\mathbb{R})$ in the form:

$$v(t) = \mathscr{K}\boldsymbol{w}(t), \qquad \forall t \geqslant 0, \tag{6.57}$$

the system (6.55) becomes as follows:

$$\boldsymbol{w}' + \mathscr{A}_\alpha \boldsymbol{w} = B\mathscr{K}\boldsymbol{w} \quad \text{in} \quad (0,\infty), \qquad \boldsymbol{w}(0) = \boldsymbol{w}_0. \tag{6.58}$$

THEOREM 6.1. *There are positive constants ν_0 and M independent of t such that for all $\omega_0 \in \mathscr{V}_0^m(\Omega)$, the solution of the system (6.58) satisfies:*

$$\|\boldsymbol{w}(t)\|_{\mathscr{V}(\Omega)} \leqslant M e^{-\nu_0 t} \|\boldsymbol{w}_0\|_{\mathscr{V}_0^m(\Omega)}.$$

Proof. The proof follows from [14, Theorem 6.1] □

Knowing that $v(t) = e^{\alpha t} u(t)$ and

$$\boldsymbol{w}(\cdot,t) = e^{\alpha t} \boldsymbol{y}(\cdot,t) \tag{6.59}$$

where $\boldsymbol{y} = (\xi, \zeta, \theta, q)$ is the solution to the system (2.17)–(2.22), then we deduce from (6.57) that

$$u(t) = \mathscr{K} \mathbf{y}(t), \qquad \forall t \geqslant 0.$$

Consequently, by using relations (2.15), (2.23) and (3.42), we can deduce easily that the initial Dirichlet boundary control q can be rewritten in feedback form, as follows:

$$q(t) = \rho_0(\ell)\left(1 - \frac{a}{\nu} t\right) e^{-\frac{a}{\nu} t} + \int_0^t \left(1 - \frac{a}{\nu}(t-s)\right) e^{-\frac{a}{\nu}(t-s)} \mathscr{K} \mathbf{y}(s)\, ds, \qquad (6.60)$$

for $t \geqslant 0$ and where $\mathscr{K}$ is the feedback operator given by (6.57). Our goal, in the following, is the characterization of the operator $\mathscr{K}$, and therefore the total determination of the Dirichlet control q.

The system (2.25)–(2.28), (or the system (6.55)), is not stable, but it is only stabilizable, (see [2]). Consequently, the optimality system of the problem $(\mathscr{Q}^{\infty}_{\boldsymbol{w}_0})$ can not be obtained by the classical techniques. The steps to be followed are as follows: we approach the infinite time horizon control problem $(\mathscr{Q}^{\infty}_{\boldsymbol{w}_0})$, $(t \in (0, \infty))$, by a finite time horizon control problem posed on $[s, T]$, for any $T > 0$, $(s \in [0, T])$. Then, we pass to the limit in the finite time horizon optimality system when T tends to infinity.

6.2. The finite time horizon control problem

We consider the following optimal control problem

$$(\mathscr{Q}^{T}_{s,\xi}) \qquad \inf\left\{ J_T(\boldsymbol{w}, v) \middle| \quad (\boldsymbol{w}, v) \text{ verify (6.61)},\ v \in L^2(s, T; \mathbb{R}) \right\}$$

where $J_T(\cdot,\cdot)$ is the cost function given by

$$J_T(\boldsymbol{w}, v) = \frac{1}{2}\int_s^T \|C\boldsymbol{w}(s)\|_Z^2\, ds + \frac{1}{2}\int_s^T v^2(s)\, ds$$

and $\boldsymbol{w}$ verify the system

$$\boldsymbol{w}' + \mathscr{A}_\alpha \boldsymbol{w} = Bv \quad \text{in} \quad (s, T), \quad \boldsymbol{w}(s) = \xi. \qquad (6.61)$$

Our goal here is to determine optimality conditions for the problem $(\mathscr{Q}^{T}_{s,\xi})$. Then, we will prove that the optimal solution $(\boldsymbol{w}^s_\xi, v^s_\xi)$ is written according to the Riccati operator which will be determined later.

Characterization of the optimal solution of the problem $(\mathscr{Q}^{T}_{s,\xi})$ *by Lagrangian method:* let us consider the problem $(\hat{\mathscr{Q}}^{\alpha,T}_{s,\xi})$ defined as follows: For any given $\alpha > 0$, find $v \in L^2(s, T; \mathbb{R})$ minimizes the cost functional

$$\min_{v \in L^2(s,T;\mathbb{R})} \left\{ J_T(\sigma, \rho, \beta, v) = \frac{1}{2}\int_s^T \left(\|\sigma(\tau)\|^2_{L^2(\Omega)} + \|\rho(\tau)\|^2_{L^2(\Omega)} + v^2(\tau) \right) d\tau \right\} \quad (6.62)$$

subject to the linear control system

$$\partial_t\sigma - \partial_\varkappa\rho - \alpha\sigma = 0 \qquad \text{in} \quad \Omega\times(s,T), \tag{6.63}$$
$$\partial_t\rho - a\partial_\varkappa\sigma - \nu\partial_{\varkappa\varkappa}\rho - \alpha\rho = 0 \qquad \text{in} \quad \Omega\times(s,T), \tag{6.64}$$
$$\rho(\varkappa,s) = \xi_1(\varkappa), \quad \sigma(\varkappa,s) = \xi_2(\varkappa) \qquad \text{in} \quad \Omega, \tag{6.65}$$
$$\rho(0,t) = 0, \quad \rho(\ell,t) = \hat{q}(t) \qquad \text{in} \quad (s,T), \tag{6.66}$$
$$\hat{q}' - (\alpha - 2\frac{a}{\nu})\hat{q} + (\frac{a}{\nu})^2\beta = v(t) \qquad \text{in} \quad (s,T), \tag{6.67}$$
$$\beta' - \alpha\beta - \hat{q} = 0 \qquad \text{in} \quad (s,T), \tag{6.68}$$
$$\beta(s) = \beta_s, \quad \hat{q}(s) = q(s) = q_s. \tag{6.69}$$

We consider the linear system:

$$-\partial_t\varphi + a\partial_\varkappa\psi - \alpha\varphi = \sigma \qquad \text{in} \quad \Omega\times(s,T), \tag{6.70}$$
$$-\partial_t\psi + \partial_\varkappa\varphi - \nu\partial_{\varkappa\varkappa}\psi - \alpha\psi = \rho \qquad \text{in} \quad \Omega\times(s,T), \tag{6.71}$$
$$\psi(\varkappa,T) = \varphi(\varkappa,T) = 0 \qquad \text{in} \quad \Omega, \tag{6.72}$$
$$\psi(0,t) = \psi(\ell,t) = 0 \qquad \text{in} \quad (s,T), \tag{6.73}$$

where the pair (σ,ρ) is the solution of the system (6.63)–(6.66). Let us notice $\boldsymbol{\phi} = (\varphi,\psi)$, then the system (6.70)–(6.73) can be rewritten as

$$-\boldsymbol{\phi}' + A_\alpha^*\boldsymbol{\phi} = \boldsymbol{Y} \qquad \text{in} \quad (s,T), \qquad \boldsymbol{\phi}(T) = 0, \tag{6.74}$$

where we recall that the state $\boldsymbol{Y} = (\sigma,\rho)$. Let us mention that the equation (6.74) is well studied in section 5, (if we take $\boldsymbol{G} = \boldsymbol{Y}$).

Let us introduce the operator $\mathscr{B}$ defined on $\mathscr{D}(A_\alpha^*)$ by:

$$\langle\mathscr{B},\boldsymbol{\phi}\rangle_{\boldsymbol{L}^2(\Omega)} = (\varphi - \nu\partial_\varkappa\psi)(\ell), \qquad \forall\boldsymbol{\phi}\in\mathscr{D}(A_\alpha^*).$$

Here, the operator $\mathscr{B}$ is simply of the form:

$$\mathscr{B} = (\delta_\ell, -\nu\delta_\ell'),$$

where δ_ℓ' is the derivative of the Dirac function δ_ℓ.

Then, we have the following result.

THEOREM 6.2. *For all* $(\xi_1,\xi_2)\in H_0^m(\Omega)$ *and* $(\beta_s,q_s)\in H^2(s,T)\times H^1(s,T)$*, the unique optimal solution* $(\boldsymbol{Y},\beta,v)$ *to the problem* (6.63)–(6.69)*, is characterized by*

$$v_\xi^s = -r_\xi^s \qquad \textit{in} \quad (s,T),$$

where the function r_ξ^s *is the solution to the coupled system*

$$-g' + \left(\alpha - \frac{a}{\nu}\right)^2 r = -\alpha\langle\mathscr{B},\boldsymbol{\phi}_\xi^s\rangle_{\boldsymbol{L}^2(\Omega)} \qquad \textit{in} \quad (s,T), \qquad g(T) = 0,$$
$$-r' - 2\left(\alpha - \frac{a}{\nu}\right) r - g = -\langle\mathscr{B},\boldsymbol{\phi}_\xi^s\rangle_{\boldsymbol{L}^2(\Omega)} \qquad \textit{in} \quad (s,T)), \qquad r(T) = 0,$$

and $\boldsymbol{\phi}^s_\xi$ *is the solution to the adjoint system* (6.74).

Reciprocally, if

$$(\boldsymbol{Y}^s_\xi, \beta^s_\xi, \hat{q}^s_\xi, \boldsymbol{\phi}^s_\xi, g^s_\xi, r^s_\xi) \in \left(\mathscr{V}(s,T)\right)^2$$

is the solution to the coupled system: $\forall t \in (s,T)$

$$\begin{aligned}
\boldsymbol{Y}' + A_\alpha \boldsymbol{Y} &= \mathscr{B}(\beta' - \alpha\beta), && \boldsymbol{Y}(s) = (\xi_1, \xi_2),\\
\hat{q}' - \left(\alpha - 2\frac{a}{\nu}\right)\hat{q} + \left(\frac{a}{\nu}\right)^2 \beta &= -r, && \hat{q}(s) = q_s,\\
\beta' - \alpha\beta - \hat{q} &= 0, && \beta(s) = \beta_s,\\
-\boldsymbol{\phi}' + A^*_\alpha \boldsymbol{\phi} &= \boldsymbol{Y}, && \boldsymbol{\phi}(T) = 0, && (6.75)\\
-g' + \left(\alpha - \frac{a}{\nu}\right)^2 r &= -\alpha \langle \mathscr{B}, \boldsymbol{\phi}\rangle_{\boldsymbol{L}^2(\Omega)}, && g(T) = 0, && (6.76)\\
-r' - 2\left(\alpha - \frac{a}{\nu}\right) r - g &= -\langle \mathscr{B}, \boldsymbol{\phi}\rangle_{\boldsymbol{L}^2(\Omega)}, && r(T) = 0, && (6.77)
\end{aligned}$$

then $(\boldsymbol{Y}^s_\xi, \beta^s_\xi, v^s_\xi = -r^s_\xi)$ *is the optimal solution of* (6.62).

Proof. The proof can be established by a Lagrangien method. The regularity result on $\boldsymbol{\phi}$ can be deduced from Remark 5.1. $\square$

The optimality conditions for the control problem $(\mathscr{Q}^T_{s,\xi})$ can also be given by the following result.

THEOREM 6.3. *For all* $s \in [0,T]$ *and for all* $\xi \in H^m_0(\Omega)$, *the problem* $(\mathscr{Q}^T_{s,\xi})$ *admits a unique solution* $(\boldsymbol{w}^s_\xi, v^s_\xi)$ *and we have*

$$v^s_\xi = -B^*\boldsymbol{\Phi}^s_\xi \qquad \text{in} \quad (s,T), \tag{6.78}$$

where $\boldsymbol{\Phi}^s_\xi$ *is the solution to the system*

$$-\boldsymbol{\Phi}' = -\mathscr{A}^*_\alpha \boldsymbol{\Phi} + C\boldsymbol{w}^s_\xi \qquad \text{in} \quad (s,T), \qquad \boldsymbol{\Phi}(T) = 0.$$

Reciprocally, the system

$$\begin{aligned}
\boldsymbol{w}' &= -\mathscr{A}_\alpha \boldsymbol{w} - BB^*\boldsymbol{\Phi} && \text{in} \quad (s,T), \quad \boldsymbol{w}(s) = \xi && (6.79)\\
-\boldsymbol{\Phi}' &= -\mathscr{A}^*_\alpha \boldsymbol{\Phi} + C\boldsymbol{w} && \text{in} \quad (s,T), \quad \boldsymbol{\Phi}(T) = 0 && (6.80)
\end{aligned}$$

admits a unique solution

$$(\boldsymbol{w}^s_\xi, \boldsymbol{\Phi}^s_\xi) \in \left(\mathscr{V}(s,T)\right)^2$$

and $(\boldsymbol{w}^s_\xi, -B^*\boldsymbol{\Phi}^s_\xi)$ *is the optimal solution to the problem* $(\mathscr{Q}^T_{s,\xi})$.

Proof. The existence of a unique solution $(\boldsymbol{w}_\xi^s, v_\xi^s)$ to the problem $(\mathscr{Q}_{s,\xi}^T)$ is established in section 4. Let us set

$$l_T(v) = J_T(\boldsymbol{w}_v, v), \qquad v \in L^2(s,T;\mathbb{R}).$$

The differentiation of the function l_T gives

$$l_T'(v)u = \int_s^T \int_\Omega C\boldsymbol{w}_v \cdot \boldsymbol{z}\, dx\, dt + \int_s^T vu\, dt, \qquad \text{for all} \quad u \in L^2(s,T;\mathbb{R})$$

where $\boldsymbol{z}$ is the solution to

$$\boldsymbol{z}' = -\mathscr{A}_\alpha \boldsymbol{z} + Bu \qquad \text{in} \quad (s,T), \quad \boldsymbol{z}(s) = 0.$$

Let $\boldsymbol{\Phi}$ be the solution to the equation

$$-\boldsymbol{\Phi}' = -\mathscr{A}_\alpha^* \boldsymbol{\Phi} + C\boldsymbol{w} \qquad \text{in} \quad (s,T), \quad \boldsymbol{\Phi}(T) = 0. \tag{6.81}$$

The system (6.81) has a unique solution $\boldsymbol{\Phi} \in \mathscr{V}(s,T)$. Indeed, we can verify, without any difficulty, that the systems (6.75)–(6.77) and (6.81) have the same solution and then $\boldsymbol{\Phi} = (\boldsymbol{\phi}, g, r)$. Therefore, the regularity of $\boldsymbol{\Phi}$ follows from Remark 5.1. As a result $B^*\boldsymbol{\Phi} \in L^2(s,T;\mathbb{R})$. Then, $\boldsymbol{z}$ and $\boldsymbol{\Phi}$ satisfy the relation:

$$\int_s^T \int_\Omega C\boldsymbol{w}_v \cdot \boldsymbol{z}\, dx\, dt = \int_s^T \langle Bu, \boldsymbol{\Phi} \rangle_{(D(\mathscr{A}^*))', D(\mathscr{A}^*)}\, dt = \int_s^T uB^*\boldsymbol{\Phi}\, dt.$$

It follows then:

$$l_T'(v)u = \int_s^T uB^*\boldsymbol{\Phi}\, dt + \int_s^T vu\, dt \tag{6.82}$$

If $(\boldsymbol{w}_\xi^s, v_\xi^s)$ is the solution to the problem $(\mathscr{Q}_{s,\xi}^T)$, we have $l_T'(v)u = 0$, which gives

$$v_\xi^s = -B^*\boldsymbol{\Phi}_\xi^s \qquad \text{in} \quad (s,T).$$

Now we can deduce that $(\boldsymbol{w}_\xi^s, \boldsymbol{\Phi}_\xi^s)$ is a solution to the system (6.79)–(6.80). If $(\overline{\boldsymbol{w}}, \overline{\boldsymbol{\Phi}})$ is a solution of system (6.79)–(6.80) and if $\overline{v} = -B^*\overline{\boldsymbol{\Phi}}_\xi^s$, with (6.82) we can verify that $l_T'(\overline{v}) = 0$, which implies that $\overline{v} = v_\xi^s$. Thus $\overline{\boldsymbol{w}} = \boldsymbol{w}_\xi^s$ and $\overline{\boldsymbol{\Phi}} = \boldsymbol{\Phi}_\xi^s$. $\square$

LEMMA 6.1. *For all $s \in [0,T]$ and for all $\xi \in H_0^m(\Omega)$, the pair $(\boldsymbol{\Phi}_\xi^s, v) \in \mathscr{C}([s,T], \mathscr{V}(\Omega)) \times \mathscr{C}([s,T], \mathbb{R})$.*

Proof. Knowing that the state $\boldsymbol{\Phi}_\xi^s = (\boldsymbol{\phi}_\xi^s, g_\xi^s, r_\xi^s)$, then the regularity of $\boldsymbol{\Phi}_\xi^s$ is deduced from the regularity of $\boldsymbol{\phi}_\xi^s$ in Remark 5.1 and the regularity of (g_ξ^s, r_ξ^s) in Theorem 6.2. The regularity of v can be deduced from the relation $v = -B^*\boldsymbol{\Phi}_\xi^s$. $\square$

COROLLARY 6.1. *For all* $s \in [0,T]$ *and for all* $\xi \in H_0^m(\Omega)$, *the unique solution* $(\boldsymbol{w}_\xi^s, v_\xi^s)$ *to problem* $(\mathscr{Q}_{s,\xi}^T)$ *and the corresponding solution* $(\boldsymbol{w}_\xi^s, \boldsymbol{\Phi}_\xi^s)$ *to system* (6.79)–(6.80) *obeys*

$$J_T(\boldsymbol{w}_\xi^s, v_\xi^s) = \frac{1}{2}\int_\Omega \boldsymbol{\Phi}_\xi^s(s).\xi.$$

Proof. We multiply equation (6.61), whose solution is $(\boldsymbol{w}_\xi^s, v_\xi^s)$, by $\boldsymbol{\Phi}_\xi^s$, we integrate in space and time, using integration by parts, and considering equation (6.80) whose solution is $\boldsymbol{\Phi}_\xi^s$, we can show that:

$$\int_\Omega \boldsymbol{\Phi}_\xi^s(s).\xi = -(B^*\boldsymbol{\Phi}_\xi^s, v_\xi^s)_{L^2(s,T)} + (C\boldsymbol{w}_\xi^s, \boldsymbol{w}_\xi^s)_Z.$$

Next, using (6.78) we can establish the relation of the corollary. The proof is completed. □

Let $\Pi(s)$ be the operator defined by

$$\Pi(s) : \xi \mapsto \boldsymbol{\Phi}_\xi^s(s), \tag{6.83}$$

where $(\boldsymbol{w}_\xi^s, \boldsymbol{\Phi}_\xi^s)$ is the solution to the system (6.79)–(6.80). From Theorem 6.3, we deduce that $\Pi(s) \in \mathscr{L}(H_0^m(\Omega), \mathscr{V}(\Omega))$. By Lemma 6.1, we can show that the series of operators $(\Pi(s))_{s\in[0,T]}$ given in (6.83) belongs to $\mathscr{C}_s([0,T];\mathscr{L}(\mathscr{V}(\Omega)))$ (the set of functions Π from $[0,T]$ to $\mathscr{L}(\mathscr{V}(\Omega))$ such that: for any $\boldsymbol{g} \in \mathscr{V}(\Omega)$, $\Pi(\cdot)\boldsymbol{g}$ is continuous from $[0,T]$ to $\mathscr{V}(\Omega)$). Then, by using the equations (6.79)–(6.80), we can prove that Π is the unique solution in $\mathscr{C}_s([0,T];\mathscr{L}(\mathscr{V}(\Omega)))$ to the Riccati differential equation:

$$\begin{aligned}
&\Pi'(t) = \mathscr{A}_\alpha^*\Pi(t) + \Pi(t)\mathscr{A}_\alpha + \Pi(t)B^*B\Pi(t) - C,\\
&\Pi(T) = 0,\\
&\Pi^*(t) = \Pi(t), \quad \Pi(t) \geqslant 0, \quad \forall t \in [0,T],\\
&\|\Pi(t)\boldsymbol{g}\|_{\mathscr{V}(\Omega)} \leqslant C_0\|\boldsymbol{g}\|_{\mathscr{V}(\Omega)}, \quad \forall \boldsymbol{g} \in \mathscr{V}_0^m(\Omega) \subset \mathscr{V}(\Omega).
\end{aligned} \tag{6.84}$$

The existence and uniqueness of a solution to the Riccati equation (6.84) is given in [9, Theorem 1.2.2.1]. For more details about existence, uniqueness and regularity of the solution to the Riccati differential equation, we refer the authors to [24, 25, 26, 27, 28].

THEOREM 6.4. *The solution* $(\boldsymbol{w}, v)$ *to the problem* $(\mathscr{Q}_{0,\boldsymbol{w}_0}^T)$ *belongs to* $\mathscr{C}([0,T];\mathscr{V}(\Omega)) \times \mathscr{C}([0,T];\mathbb{R})$, *satisfies the following feedback law*

$$v(t) = -B^*\Pi(t)\boldsymbol{w}(t), \qquad \textit{for all} \quad t \in [0,T],$$

and the optimal cost is given by

$$J(\boldsymbol{w}, v) = \frac{1}{2}\Big(\Pi(0)\boldsymbol{w}_0, \boldsymbol{w}_0\Big)_{\mathscr{V}(\Omega)}.$$

Proof. From Theorem 6.3, Lemma 6.1, Corollary 6.1 and the definition of Π, we can prove this theorem. $\square$

If we set $\widehat{\Pi}(t) = \Pi(T-t)$, then $\widehat{\Pi}$ is the unique solution in $\mathscr{C}([0,T];\mathscr{L}(\mathscr{V}(\Omega)))$ to the Riccati differential equation:

$$\begin{aligned}
&\widehat{\Pi}'(t) = \mathscr{A}_\alpha^*\widehat{\Pi}(t) + \widehat{\Pi}(t)\mathscr{A}_\alpha + \widehat{\Pi}(t)B^*B\widehat{\Pi}(t) - C,\\
&\widehat{\Pi}(0) = 0,\\
&\widehat{\Pi}^*(t) = \widehat{\Pi}(t), \quad \widehat{\Pi}(t) \geqslant 0, \quad \forall t \in [0,T],\\
&\Pi(0) = \widehat{\Pi}(T),\\
&\|\widehat{\Pi}(t)\boldsymbol{g}\|_{\mathscr{V}(\Omega)} \leqslant C_0\|\boldsymbol{g}\|_{\mathscr{V}(\Omega)}, \quad \forall \boldsymbol{g} \in \mathscr{V}_0^m(\Omega).
\end{aligned}$$

where $\widehat{\Pi}^*(t) = \widehat{\Pi}(t)$ and $\widehat{\Pi}(t) \geqslant 0$, for all $t \in [0,T]$. From the definition of $\widehat{\Pi}$ it comes that $\Pi(0) = \widehat{\Pi}(T)$.

6.3. The infinite time horizon control problem

In this section, we want to study the control problem posed in infinite time horizon $(\mathscr{Q}^\infty_{\boldsymbol{w}_0})$, as well as the regularity of its solution according to that of the initial condition $\boldsymbol{w}_0$. So, the problem we consider is

$$(\mathscr{Q}^\infty_{\boldsymbol{w}_0}) \qquad \inf\Big\{J(\boldsymbol{w},v) \mid (\boldsymbol{w},v) \quad \text{satisfies} \quad (6.85)\Big\}$$

with the cost functional

$$J(\boldsymbol{w},v) = \frac{1}{2}\int_0^\infty \|C\boldsymbol{w}(t)\|_Z^2\,dt + \frac{1}{2}\int_0^\infty v^2(t)\,dt$$

and $\boldsymbol{w}$ is solution to the equation

$$\boldsymbol{w}' + \mathscr{A}_\alpha\boldsymbol{w} = Bv \qquad \text{in} \quad (0,\infty), \qquad \boldsymbol{w}(0) = \boldsymbol{w}_0. \tag{6.85}$$

Therefore, the problem $(\mathscr{Q}^k_{s,\xi})$, for $(0 \leqslant s < k < \infty)$, is defined by:

$$(\mathscr{Q}^k_{s,\xi}) \qquad \inf\Big\{J_k(s,\boldsymbol{w},v)| \quad (\boldsymbol{w},v) \text{ satisfies } (6.86),\ v \in L^2(s,k;\mathbb{R})\Big\}$$

with

$$J_k(s,\boldsymbol{w},v) = \frac{1}{2}\int_s^k \|C\boldsymbol{w}(t)\|_Z^2\,dt + \frac{1}{2}\int_s^k v^2(t)\,dt$$

and

$$\boldsymbol{w}' + \mathscr{A}_\alpha\boldsymbol{w} = Bv \qquad \text{in} \quad (s,k), \qquad \boldsymbol{w}(s) = \xi. \tag{6.86}$$

REMARK 6.2. Let us notice that the problem $(\mathscr{Q}^k_{s,\xi})$ has been studied in section 6.2. Therefore, the results obtained in section 6.2 will be used here to prove some results.

THEOREM 6.5. *For all* $\boldsymbol{w}_0 \in \mathscr{V}_0^m(\Omega)$, *the problem* $(\mathscr{Q}^\infty_{\boldsymbol{w}_0})$ *admits a unique solution* $(\boldsymbol{w}_{\boldsymbol{w}_0}, v_{\boldsymbol{w}_0})$.

Proof. In fact, we have not null controlability for the system (2.25)–(2.28), contrary to [11]. However, we showed in [2], that the system (2.25)–(2.28) is stabilizable in $H_0^m(\Omega)$. We deduce from it that there exists a control $v \in L^2(0,\infty;\mathbb{R})$ such that $(\boldsymbol{w}_v, v)$, the solution to (6.85) correspondant to v, obeys

$$J(\boldsymbol{w}_v, v) < \infty.$$

The existence of a unique solution $(\boldsymbol{w}_{\boldsymbol{w}_0}, v_{\boldsymbol{w}_0})$ to the minimization problem $(\mathscr{Q}^\infty_{\boldsymbol{w}_0})$ is done by classical arguments, (see section 4). □

THEOREM 6.6. *For all* $\boldsymbol{w}_0 \in \mathscr{V}_0^m(\Omega)$, *there exists* $\Pi \in \mathscr{L}(\mathscr{V}(\Omega))$ *satisfying* $\Pi^* = \Pi \geqslant 0$, *such that the optimal cost is given by*

$$J(\boldsymbol{w}_{\boldsymbol{w}_0}, v_{\boldsymbol{w}_0}) = \frac{1}{2}(\Pi \boldsymbol{w}_0, \boldsymbol{w}_0)_{\mathscr{V}(\Omega)},$$

where $(\boldsymbol{w}_{\boldsymbol{w}_0}, v_{\boldsymbol{w}_0})$ *is the unique optimal solution to the problem* $(\mathscr{Q}^\infty_{\boldsymbol{w}_0})$.

Proof. The dynamic programming principle shows that the mapping:

$$T \longmapsto (\widehat{\Pi}(T)\boldsymbol{w}_0, \boldsymbol{w}_0)_{\mathscr{V}(\Omega)}$$

is nondecreasing. Moreover, we have:

$$\frac{1}{2}(\widehat{\Pi}(T)\boldsymbol{w}_0, \boldsymbol{w}_0)_{\mathscr{V}(\Omega)} \leqslant J(\boldsymbol{w}_{\boldsymbol{w}_0}, v_{\boldsymbol{w}_0}) < \infty.$$

Referring to [9, 24, 25, 26, 27, 28], we can prove that there exists an operator $\Pi \in \mathscr{L}(\mathscr{V}(\Omega))$ such that $\Pi = \Pi^* \geqslant 0$ and

$$\Pi \boldsymbol{w}_0 = \lim_{T \mapsto \infty} \widehat{\Pi}(T)\boldsymbol{w}_0, \quad \text{for all} \quad \boldsymbol{w}_0 \in \mathscr{V}_0^m(\Omega).$$

In the following, we show that $J(\boldsymbol{w}_{\boldsymbol{w}_0}, v_{\boldsymbol{w}_0}) = \frac{1}{2}(\Pi \boldsymbol{w}_0, \boldsymbol{w}_0)_{\mathscr{V}(\Omega)}$. The problem $(\mathscr{Q}^k_{0,\boldsymbol{w}_0})$ has a unique solution $(\boldsymbol{w}_k, v_k)$ that satisfies the following system:

$$\begin{aligned} \boldsymbol{w}_k' &= -\mathscr{A}_\alpha \boldsymbol{w}_k + B v_k \quad \text{in} \quad (0,k), \qquad \boldsymbol{w}_k(0) = \boldsymbol{w}_0, && (6.87)\\ -\boldsymbol{\Phi}_k' &= -\mathscr{A}_\alpha^* \boldsymbol{\Phi}_k + C \boldsymbol{w}_k \quad \text{in} \quad (0,k), \qquad \boldsymbol{\Phi}_k(k) = 0, && (6.88)\\ v_k &= -B^* \boldsymbol{\Phi}_k. && (6.89) \end{aligned}$$

Let $(\tilde{\boldsymbol{w}}_k, \tilde{v}_k)$ be the extension by zero of $(\boldsymbol{w}_k, v_k)$ to the interval (k, ∞). We have:

$$\int_0^k \|C\boldsymbol{w}_k(t)\|_Z^2\,dt + \int_0^k v_k^2(t)\,dt \leqslant \int_0^\infty \|C\boldsymbol{w}_{\boldsymbol{w}_0}(t)\|_Z^2\,dt + \int_0^\infty v_{\boldsymbol{w}_0}^2(t)\,dt, \tag{6.90}$$

then the sequence $(\tilde{\boldsymbol{w}}_k)_k$ is bounded in $L^2(0,\infty;\mathscr{V}(\Omega))$ and the sequence $(\tilde{v}_k)_k$ is bounded in $L^2(0,\infty;\mathbb{R})$. Thus, there exists $(\hat{\boldsymbol{w}}, \hat{v}) \in L^2(0,\infty;\mathscr{V}(\Omega)) \times L^2(0,\infty;\mathbb{R})$ such that:

$$\begin{aligned} \tilde{\boldsymbol{w}}_k &\rightharpoonup \hat{\boldsymbol{w}} \quad \text{weakly} \quad \text{in} \quad L^2(0,\infty;\mathscr{V}(\Omega)), \\ \tilde{v}_k &\rightharpoonup \hat{v} \quad \text{weakly} \quad \text{in} \quad L^2(0,\infty;\mathbb{R}). \end{aligned}$$

By passing to the limit in the relation (6.90), we get:

$$\int_0^\infty \|C\hat{\boldsymbol{w}}(t)\|_Z^2\,dt + \int_0^\infty \hat{v}(t)\,dt \leqslant \int_0^\infty \|C\boldsymbol{w}_{\boldsymbol{w}_0}(t)\|_Z^2\,dt + \int_0^\infty v_{\boldsymbol{w}_0}^2(t)\,dt, \tag{6.91}$$

Moreover, when k tends to ∞ in equation (6.87), we get:

$$\hat{\boldsymbol{w}}' = -\mathscr{A}_\alpha \hat{\boldsymbol{w}} + B\hat{v} \qquad \text{in} \quad (0,\infty), \qquad \hat{\boldsymbol{w}}(0) = \boldsymbol{w}_0.$$

So, we deduce that $(\hat{\boldsymbol{w}}, \hat{v})$ is admissible and then we have $(\hat{\boldsymbol{w}}, \hat{v}) = (\boldsymbol{w}_{\boldsymbol{w}_0}, v_{\boldsymbol{w}_0})$, because of inequality (6.91): $J(\hat{\boldsymbol{w}}, \hat{v}) \leqslant J(\boldsymbol{w}_{\boldsymbol{w}_0}, v_{\boldsymbol{w}_0})$. Therefore, we can claim that:

$$\begin{aligned} \tilde{\boldsymbol{w}}_k &\longrightarrow \boldsymbol{w}_{\boldsymbol{w}_0} \quad \text{in} \quad L^2(0,\infty;\mathscr{V}(\Omega)), \\ \tilde{v}_k &\longrightarrow v_{\boldsymbol{w}_0} \quad \text{in} \quad L^2(0,\infty;\mathbb{R}). \end{aligned}$$

Since

$$J_k(0, \boldsymbol{w}_k, v_k) = \frac{1}{2}(\widehat{\Pi}(k)\boldsymbol{w}_0, \boldsymbol{w}_0)_{\mathscr{V}(\Omega)},$$

when k tends to ∞, we get:

$$J(\boldsymbol{w}_{\boldsymbol{w}_0}, v_{\boldsymbol{w}_0}) = \frac{1}{2}(\Pi \boldsymbol{w}_0, \boldsymbol{w}_0)_{\mathscr{V}(\Omega)}.$$

The proof is completed. □

THEOREM 6.7. *For all* $\boldsymbol{w}_0 \in \mathscr{V}_0^m(\Omega)$, *the system:*

$$\boldsymbol{w}' = -\mathscr{A}_\alpha \boldsymbol{w} - BB^*\boldsymbol{\Phi} \qquad \textit{in} \quad (0,\infty), \qquad \boldsymbol{w}(0) = \boldsymbol{w}_0, \tag{6.92}$$

$$-\boldsymbol{\Phi}' = -\mathscr{A}_\alpha^* \boldsymbol{\Phi} + C\boldsymbol{w} \qquad \textit{in} \quad (0,\infty), \qquad \boldsymbol{\Phi}(\infty) = 0, \tag{6.93}$$

$$\boldsymbol{\Phi}(t) = \Pi \boldsymbol{w}(t) \qquad \textit{for all} \quad t \in (0,\infty), \tag{6.94}$$

admits a unique solution:

$$(\boldsymbol{w}, \boldsymbol{\Phi}) \in \Big(\mathscr{V}(0,\infty)\Big)^2.$$

Moreover, we have the following estimation:

$$\|\boldsymbol{w}(t)\|_{\mathscr{V}(\Omega)} + \|\boldsymbol{\Phi}(t)\|_{\mathscr{V}(\Omega)} \leqslant C_0 \|\boldsymbol{w}_0\|_{\mathscr{V}_0^m(\Omega)}, \qquad \textit{for all} \qquad t \in (0,\infty) \tag{6.95}$$

and the pair $(\boldsymbol{w}, -B^*\boldsymbol{\Phi})$ *is the optimal solution to the problem* $(\mathscr{Q}_{\boldsymbol{w}_0}^\infty)$.

Proof. We denoted by $(\boldsymbol{w}_k^{\bar{t}}, v_k^{\bar{t}})$ the solution of $(\mathscr{Q}^k_{\bar{t},\boldsymbol{w}_k(\bar{t})})$ and $\boldsymbol{\Phi}_k^{\bar{t}}$ the adjoint state associated with $(\boldsymbol{w}_k^{\bar{t}}, v_k^{\bar{t}})$. Also, $(\boldsymbol{w}_k, v_k)$ represents the solution of $(\mathscr{Q}^k_{0,\boldsymbol{w}_0})$ that is given by the system:

$$\begin{aligned} \boldsymbol{w}_k' &= -\mathscr{A}_\alpha \boldsymbol{w}_k + Bv_k \qquad \text{in} \quad (0,k), \quad \boldsymbol{w}_k(0) = \boldsymbol{w}_0, \\ -\boldsymbol{\Phi}_k' &= -\mathscr{A}_\alpha^* \boldsymbol{\Phi}_k + C\boldsymbol{w}_k \qquad \text{in} \quad (0,k), \quad \boldsymbol{\Phi}_k(k) = 0, \\ v_k &= -B^*\boldsymbol{\Phi}_k \qquad \text{in} \quad (0,k). \end{aligned}$$

Then, by dynamic programming principle we have: $(\boldsymbol{w}_k^{\bar{t}}, v_k^{\bar{t}}, \boldsymbol{\Phi}_k^{\bar{t}})(t) = (\boldsymbol{w}_k, v_k, \boldsymbol{\Phi}_k)(t)$, $\forall t \in (\bar{t}, k)$. Now, following the idea of the proof of [11, Lemma 4.2], we can establish the proof of the theorem. $\square$

THEOREM 6.8. *For all $0 < \alpha < \frac{a}{v}$ and for all $\boldsymbol{y}_0 \in \mathscr{V}_0^m(\Omega)$, there exists a control $u \in L^2(0,\infty)$ for which the solution $\boldsymbol{y}$ to the problem* (2.16)–(2.22) *satisfies*

$$\|\boldsymbol{y}(t)\|_{\mathscr{V}(\Omega)} \leqslant C_0 \|\boldsymbol{y}_0\|_{\mathscr{V}_0^m(\Omega)} e^{-\alpha t}, \qquad \textit{for all} \qquad t \in (0,\infty). \tag{6.96}$$

Proof. From estimation (6.95), we deduce that

$$\|\boldsymbol{w}(t)\|_{\mathscr{V}(\Omega)} \leqslant C_0 \|\boldsymbol{w}_0\|_{\mathscr{V}_0^m(\Omega)}, \qquad \text{for all} \qquad t \in (0,\infty). \tag{6.97}$$

Thus, using relation (6.59) and the above estimation (6.97), we can prove the identity (6.96). $\square$

From Theorem 6.6, Theorem 6.7 and [5, 10, 26], we have the following result for the regulator problem $(\mathscr{Q}^\infty_{\boldsymbol{w}_0})$.

THEOREM 6.9. *The linear regulator problem $(\mathscr{Q}^\infty_{\boldsymbol{w}_0})$ has a unique optimal control $\overline{v} \in L^2(0,\infty;\mathbb{R})$ given by*

$$\overline{v}(t) = -B^*\Pi\overline{\boldsymbol{w}}(t), \qquad t \geqslant 0,$$

where $\overline{\boldsymbol{w}}$ is the corresponding optimal solution, B^* is the adjoint operator of B and $\Pi \in \mathscr{L}(\mathscr{V}(\Omega))$ is the unique nonnegative selt-adjoint operator satisfying the algebric Riccati equation

$$\mathscr{A}_\alpha^*\Pi + \Pi\mathscr{A}_\alpha + \Pi B^* B\Pi - C = 0. \tag{6.98}$$

The main result of this paper is given below.

THEOREM 6.10. *The optimal control $\overline{u}$ is given as a feedback control law solution to the problem* (2.16) *by:*

$$\overline{u}(t) = -B^*\Pi\overline{\boldsymbol{y}}(t), \qquad \forall t \geqslant 0.$$

Consequently, the optimal control $\overline{q}$ *of the problem* (2.8)–(2.12) *is given in feedback control law as follows:*

$$\overline{q}(t) = \rho_0(\ell)\left(1-\frac{a}{\nu}t\right)e^{-\frac{a}{\nu}t} - \int_0^t \left(1-\frac{a}{\nu}(t-s)\right)e^{-\frac{a}{\nu}(t-s)}B^*\Pi\overline{\mathbf{y}}(s)\,ds,$$

for $t \geqslant 0$ *and where the state* $\overline{\mathbf{y}} = (\overline{\xi}, \overline{\zeta}, \overline{\theta}, \overline{q})$ *is the optimal solution to the problem* (2.16)–(2.22) *and the operator* Π *is the solution to the equation* (6.98).

Proof. The proof follows from relations (2.23), (6.57), (6.59), (6.60) and Theorem 6.9. □

7. Illustrative example

In this section, we present an effective example that illustrates the theoretical results obtained previously.

Let $\Omega =]0,1[$. Consider the following system:

$$\partial_t \xi - \partial_{\varkappa}\zeta = 0, \tag{7.99}$$

$$\partial_t \zeta - a\partial_{\varkappa}\xi - \nu\partial_{\varkappa\varkappa}\zeta = 0, \tag{7.100}$$

$$\zeta(\varkappa,0) = \rho_0(\varkappa), \quad \xi(\varkappa,0) = \sigma_0(\varkappa), \tag{7.101}$$

$$\zeta(0,t) = q_0(t), \quad \zeta(1,t) = q_1(t), \tag{7.102}$$

where we consider the following data:

$$\alpha = 0.3, \quad \nu = 0.02, \quad a = 1.75.$$

The initial conditions are given by:

$$\rho_0(\varkappa) = 15\cos(9\pi\varkappa), \qquad \sigma_0(\varkappa) = 0.$$

Note that in this example we consider two controls: q_0 at $\varkappa = 0$ and q_1 at $\varkappa = 1$. The initial perturbation was plotted in Figure 1. Note that this initial condition has a very large amplitude $-15 \leqslant \rho_0(\varkappa) \leqslant 15$ for $\varkappa \in [0,1]$. Figure 2 shows that the natural stabilization (without controls) starts from $t = 6$. We have plotted the curves of the states ξ and ζ at different time $t = 1.8, 2.0$ in Figure 3 and Figure 4, respectively. It's clear that the stabilization by the two boundary controls is very fast, (see curves in the right in Figures 3 and 4), than the natural stabilization (without controls $q_0 = q_1 = 0$), (see curves in the left in Figures 3 and 4). Also, we have plotted the curves of the two controls in Figure 5. In fact, this example shows the robustness of the boundary controls that we applied for the stabilizability of the dynamical system (7.99)–(7.102). This result illustre well the exponential decay of the state (ξ, ζ) given by Theorem 6.8.

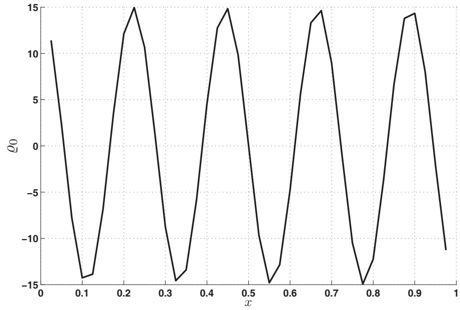


Figure 1: *The initial state* $\zeta(\varkappa,0) = \rho_0(\varkappa)$ *for* $\varkappa \in [0,1]$.

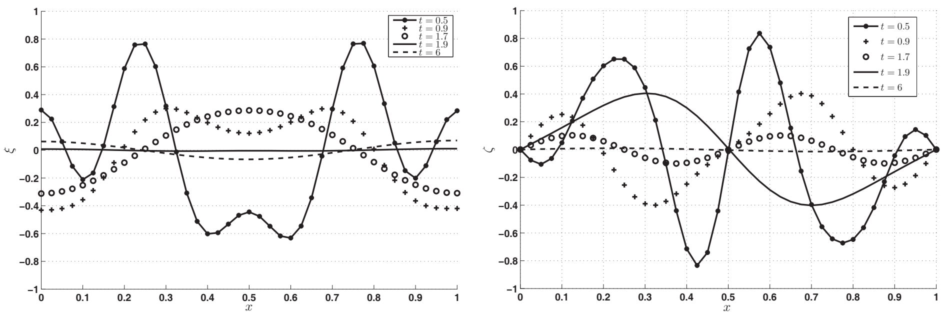


Figure 2: *The states* ξ *and* ζ *without controls.*

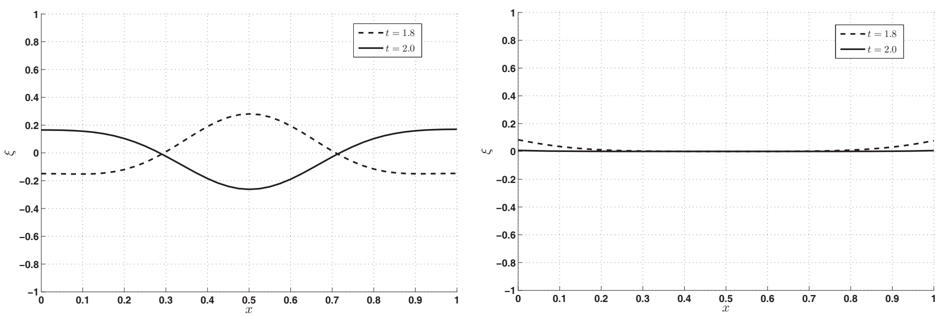
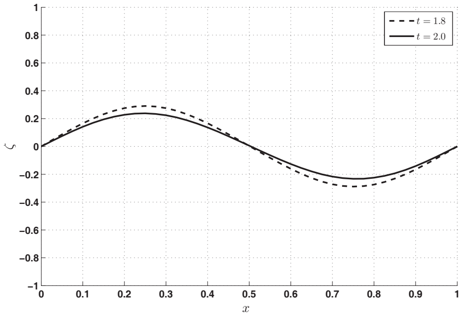


Figure 3: *The state* ξ*: the curve in the left without controls and the curve in the right with controls* q_0 *and* q_1.



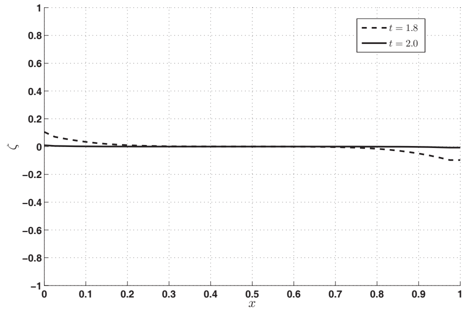


Figure 4: *The state* ζ*: the curve in the left without controls and the curve in the right with controls* q_0 *and* q_1.

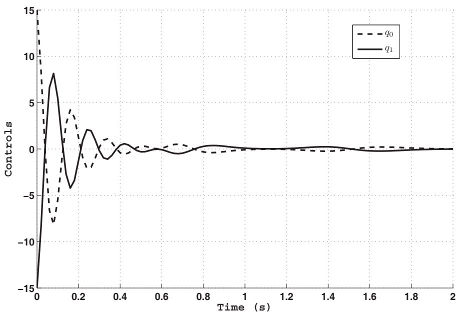


Figure 5: *The controls* $q_0(t)$ *and* $q_1(t)$ *for* $t \in [0,2]$.

8. Conclusion

In this paper, we have studied a boundary Dirichlet control problem where the control variable q is subjected to some constraints. In this case and using an augmented state technique, we succeeded in determining the control in feedback form explicitly according to the state of the system thanks to the Riccati operator which is solution of an algebraic Riccati equation, and we proved in this case that the state of the system is exponentially stable.

A great challenge in the future is to study the same problem above using fractional time derivatives such as Caputo time fractional derivative, generalized fractional derivative [12, 13]. Also, the numerical solution of the fractional control problem is a very important task.

REFERENCES

[1] H. ARFAOUI, F. BEN BELGACEM, H. EL FEKIH, P. R. RAYMOND, *Boundary stabilizability of the linearized viscous Saint-Venant system*, DCDS-B, vol. 15, Issue 3 (2011), 491–511.

[2] H. ARFAOUI, *Stabilization Method for the Saint-Venant Equations by Boundary Control*, Transactions of the Institute of Measurement and Control, vol. 42, Issue 16 (2020), 3290–3302.

[3] S. CHOWDHURY, M. RAMASWAMY, P. R. RAYMOND, *Controllability and stabilizability of the linearized compressible Navier-Stokes system in one dimension*, SIAM J. Control Optim. **50** (2012), 2959–2987.

[4] S. CHOWDHURY, D. MAITY, M. RAMASWAMY, P. R. RAYMOND, *Local stabilization of the compressible Navier-Stokes system, around null velocity, in one dimension*, J. Differential Equations **259** (2015), 371–407.

[5] F. FLANDOLI, *Algebraic Riccati equation arising in boundary control problems*, SIAM J. Control and Optimization **25**, no. 3 (1987), 612–636.

[6] A. V. FURSIKOV, *Stabilizability of quasilinear parabolic equation by feedback boundary control*, Sbornik Mathematics **192**: 4 (2001), 593–639.

[7] A. V. FURSIKOV, *Stabilizability of two-dimensional Navier-Stokes equations with help of boundary feedback control*, J. Math. Fluid Mech. **3** (2001), 259–301.

[8] A. V. FURSIKOV, *Real Processes and Realizability of a Stabilization Method forNavier-Stokes Equations by Boundary Feedback Control, Nonlinear Problems in Mathematical Physics and Related Topics II*, In Honor of Professor O. A. Ladyzhenskaya, Kluwer/Plenum Publishers, New-York, Boston, Dordrecht, London, Moscow, (2002), 137–177.

[9] I. LASIECKA, R. TRIGGIANI, *Control theory for partial differential equations: continuous and approximation theories*, Abstract parabolic systems. Cambridge Univ. Pr, January 1, vol. 1, (2000).

[10] I. LASIECKA, R. TRIGGIANI, *The regulator problem for parabolic equations with Dirichlet boundary control, Part I: Riccati's feedback synthesis and regularity of optimal solutions*, Appl. Math. Optim. vol. 16 (1987), 147–168.

[11] P. R. RAYMOND, *Feedback boundary stabilization of the two-dimensional Navier-Stokes equations*, SIAM J. Control Optim. vol. 45, no. 3, pp. 790–828, 2006.

[12] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[13] I. PODLUBANY, *Fractional Differential Equations*, Academic Press, New York, 1998.

[14] R. TRIGGIANI, *On the stabilizability problem in Banach space*, J. Math. Anal. Appl., vol. 52 (1975), 383–403.

[15] V. PERROLLAZ, L. ROSIER, *Finite-time stabilization of hyperbolic systems over a bounded interval*, In 1st IFAC workshop on Control of Systems Governed by Partial Differential Equations (CPDE2013), 239–244.

[16] V. PERROLLAZ, L. ROSIER, *Finite-time stabilization of 2×2 hyperbolic systems on tree-shaped networks*, SIAM J. Control Optim. **52**, no. 1, 143–163.

[17] V. KOMORNIK, *Rapid boundary stabilization of the wave equation*, SIAM J. Control Optim. **29**, 197–208.

[18] M. GUGAT, V. PERROLLAZ, L. ROSIER, *Boundary stabilization of quasilinear hyperbolic systems of balance laws: exponential decay for small source terms*, Journal of Evolution Equations, Springer Verlag **18** (3), pp. 1471–1500.

[19] M. GUGAT, G. LEUGERING, K. WANG, *Neumann boundary feedback stabilization for a nonlinear wave equation: A strict H2-Lyapunov function*, Math. Control Relat. Fields **7**, no. 3, 419–448.

[20] M. GUGAT, *Boundary feedbcak stabilization of the telegraph equation: Decay rates for vanishing damping term*, Systems Control Lett. **66**, 72–84.

[21] J.-F. GERBEAU AND B. PERTHAME, *Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation*, Discrete Continuous Dynam. Systems – B, **1** (2001), 89–102.

[22] S. N. ANTONTSEV, A. V. KAZHIKOV AND V. N. MONAKHOV, *Boundary values problems in mechanics of nonhomogeneous fluids*, North-Holland, Amsterdam, 1990.

[23] DAVID H. WAGNER, *Equivalence of the Euler and Lagrangian equations of a gas dynamics for weak solutions*, Journal of Differential Equations **68** (1987), 118–136.

[24] A. BENSOUSSAN, G. DA PRATO, M. C. DELFOUR, S. K. MITTER, *Representation and Control of Infinite Dimensional Systems*, vol. 1, Birkhäuser, 1992.

[25] A. BENSOUSSAN, G. DA PRATO, M. C. DELFOUR, S. K. MITTER, *Representation and Control of Infinite Dimensional Systems*, vol. 2, Birkhäuser, 1993.

[26] I. LASIECKA, R. TRIGGIANI, *Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Problems: Continuous Theory and Approximation Theory*, Springer-Verlag, 1991.

[27] I. LASIECKA, R. TRIGGIANI, *Control Theory for Partial Differential Equations*, vol. 2, Cambridge University Press, 2000.

[28] J. ZABCZYK, *Mathematical Control Theory: An Introduction*, Birkhäuser, 1992.

(Received June 9, 2022)

Hassen Arfaoui
Department of Mathematics
College of Science, Jouf University
P.O. Box: 2014, Sakaka, Saudi Arabia
e-mail: haarfaoui@ju.edu.sa