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SPECTRAL REPRESENTATION OF ABSOLUTELY MINIMUM ATTAINING UNBOUNDED NORMAL OPERATORS

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Abstract. Let $T : D(T) \to H_2$ be a densely defined closed operator with domain $D(T) \subset H_1$. We say T to be absolutely minimum attaining if for every non-zero closed subspace M of H_1 with $D(T) \cap M \neq \{0\}$, the restriction operator $T|_M : D(T) \cap M \to H_2$ attains its minimum modulus $m(T|_M)$. That is, there exists $x \in D(T) \cap M$ with $\|x\| = 1$ and $\|T(x)\| = \inf\{\|T(m)\| : m \in D(T) \cap M : \|m\| = 1\}$. In this article, we prove several characterizations of this class of operators and show that every operator in this class has a nontrivial hyperinvariant subspace. One such important characterization is that an unbounded operator belongs to this class if and only if its null space is finite dimensional and its Moore-Penrose inverse is compact.

We also prove a spectral theorem for unbounded normal operators of this class. It turns out that every such operator has a compact resolvent.

1. Introduction

The class of absolutely minimum attaining unbounded operators was introduced in [12] where its basic properties and structure were described under some additional assumptions. On the other hand, similar studies in the case of bounded absolutely minimum operators were carried out in [8, 2]. There is a significant difference between the bounded absolutely minimum attaining operators and the unbounded ones. The results in the present article improve results from [12]. It is interesting to note that this class contains densely defined closed operators with finite-dimensional null space and a compact generalized inverse. A complete characterization, structure, spectral properties and hyperinvariant subspaces are studied in [8, 2, 3]. A class larger than the absolutely norm attaining operators is explored in [19, 20]. Hence in this article we exclusively study the class of absolutely minimum attaining operators in the unbounded setting. Since the methods of the bounded case do not work in this case, we adopt different methods for proving our results.

The class of minimum attaining unbounded operators has been recently studied in [12, 13]. It is proved that this class is dense in the class of densely defined closed operators with respect to the gap metric. This result can be compared with the Lindenstrauss theorem of norm attaining operators. Moreover, a quantitative version of the Lindenstrauss theorem for minimum attaining operators is discussed in [1].

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In this article we prove that a densely defined closed operator is absolutely minimum attaining if and only if it has a finite-dimensional null space and the Moore-Penrose inverse (generalized inverse) of the operator is compact. Using this characterization we study some properties of this class of operators. In particular, we prove a spectral theorem for absolutely minimum attaining unbounded normal operators. The Spectral Theorem for unbounded normal operators is already available in the literature, for example, in [22]. Since the class considered by us is a subclass of this class, we need to emphasize that the spectral theorem proved by us is of a different type and should be compared with the spectral theorem for compact normal operators. In the end, we show that every such operator has a non-trivial hyperinvariant subspace.

In the second section we provide basic results which will be used throughout the article. In the third section, we prove main results of this paper.

2. Preliminaries

In this section we give details of basic notations, definitions and results which we need to prove our main results.

Throughout we work with infinite-dimensional complex Hilbert spaces denoted by H, H_1, H_2 etc. The inner product and the induced norm on these spaces are denoted by $\langle\cdot\rangle$ and $\|.\|$, respectively. If T is a linear operator with domain $D(T)$, a subspace of H_1 and taking values in H_2, then T is said to be densely defined if $D(T)$ is dense in H_1. The graph $G(T)$ of T is defined by $G(T) := \{(x, Tx) : x \in D(T)\} \subseteq H_1 \oplus H_2$. We say T to be a closed operator if $G(T)$ is closed. Equivalently, we can say that T is closed if and only if for every sequence (x_n) in $D(T)$ such that $x_n \to x \in H_1$ and $Tx_n \to y \in H_2$, then $x \in D(T)$ and $Tx = y$.

The closed graph Theorem [22] assert that an everywhere defined closed operator is bounded. That is, the domain of an unbounded closed operator must be a proper subspace of a Hilbert space.

The space of all bounded operators between H_1 and H_2 is denoted by $\mathcal{B}(H_1, H_2)$ and the class of all closed operators between H_1 and H_2 is denoted by $\mathcal{C}(H_1, H_2)$. We write $\mathcal{B}(H)$ for $\mathcal{B}(H, H)$ and $\mathcal{C}(H)$ for $\mathcal{C}(H, H)$. We denote the space of all compact operators between H_1 and H_2 by $\mathcal{K}(H_1, H_2)$ and $\mathcal{K}(H, H)$ by $\mathcal{K}(H)$. Let us denote by $\mathcal{F}(H_1, H_2)$, the space of all finite rank bounded operators from H_1 into H_2 and by $\mathcal{F}(H)$, the space of all finite rank bounded operators on H.

The unit sphere of a subspace M of H is defined by $S_M := \{x \in M : \|x\| = 1\}$. If M is closed, then the othogonal projection of H onto M is denoted by P_M.

The null space and the range space of $T \in \mathcal{C}(H_1, H_2)$ are denoted by $N(T)$ and $R(T)$ respectively and the space $C(T) := D(T) \cap N(T)^{\perp}$ is called the carrier of T. In fact, $D(T) = N(T) \oplus^{\perp} C(T)$ [4, page 340].

If S and T are closed operators with the property that $D(T) \subseteq D(S)$ and $Tx = Sx$ for all $x \in D(T)$, then T is called the restriction of S denoted by $T \subseteq S$, and S is called an extension of T, which is denoted by $T \subseteq S$. Furthermore, $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$.

If $S,T \in C(H)$, then $S+T$ and ST are defined by

$$D(S+T)(x) := Sx+Tx, \text{ for all } x \in D(S+T) := D(S)\cap D(T),$$
$$(ST)(x) := S(Tx), \text{ for all } x \in D(ST) := \{x \in D(T) : Tx \in D(S)\},$$

respectively.

If T is a densely defined operator, then there exists a unique linear operator (in fact, a closed operator) $T^* : D(T^*) \to H_1$, with

$$D(T^*) := \{y \in H_2 : x \to \langle Tx,y\rangle \text{ for all } x \in D(T) \text{ is continuous}\} \subseteq H_2$$

satisfying $\langle Tx,y\rangle = \langle x,T^*y\rangle$ for all $x \in D(T)$ and $y \in D(T^*)$.

A densely defined operator $T \in \mathcal{C}(H)$ is said to be

1. Normal if $T^*T = TT^*$. Equivalently, T is normal if and only if $D(T) = D(T^*)$ and $\|Tx\| = \|T^*x\|$ for each $x \in D(T)$.

2. Self-adjoint if $T = T^*$.

3. Positive if $\langle Tx,x\rangle \geqslant 0$ for all $x \in D(T)$.

4. Symmetric if $T \subseteq T^*$. In other words, T is symmetric if $\langle Tx,y\rangle = \langle x,Ty\rangle$ for every $x,y \in D(T)$.

If T is positive, then there exists a unique positive operator S such that $T = S^2$. The operator S is called the square root of T and it is denoted by $S = T^{\frac{1}{2}}$.

A bounded operator $V \in \mathcal{B}(H_1,H_2)$ is said to be a partial isometry if $\|Vx\|_2 = \|x\|_1$ for all $x \in N(V)^{\perp}$. In this case $N(V)^{\perp}$ is called the initial space of V and $R(V)$ is called the final space of V.

If $T \in \mathcal{C}(H_1,H_2)$ is densely defined, then the operator $|T| := (T^*T)^{\frac{1}{2}}$ is called the modulus of T. There exists a unique partial isometry $V : H_1 \to H_2$ with the initial space $\overline{R(T^*)}$ and the final space $\overline{R(T)}$ such that $T = V|T|$. This factorization of T is called the polar factorization or the polar decomposition of T.

It can be verified that $D(|T|) = D(T)$ and $N(|T|) = N(T)$ and $\overline{R(|T|)} = \overline{R(T^*)}$.

Let $T \in \mathcal{C}(H)$ be densely defined. The resolvent of T is defined by

$$\rho(T) := \{\lambda \in \mathbb{C} : (T-\lambda I)^{-1} \in \mathcal{B}(H)\}.$$

The set $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the spectrum of T. The set

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T-\lambda I : D(T) \to H \text{ is not one-to-one}\}$$

is called the point spectrum of T.

For a self-adjoint operator $T \in \mathcal{C}(H)$, the *discrete spectrum* $\sigma_d(T)$ is defined as the set of all isolated eigenvalues of T with finite multiplicity. The set $\sigma_{\text{ess}}(T) = \sigma(T) \setminus \sigma_d(T)$ is called the *essential spectrum* of T. For more details of this concept we refer [23, Definition 8.3, page 178].

We refer [25, 10, 22, 5, 23] for the above basics of unbounded operators.

Here we recall the definition and properties of the Moore-Penrose inverse (or generalized inverse) of a densely defined closed operator that we need for our purpose.

If $T \in \mathcal{C}(H_1,H_2)$ is densely defined, then there exists a unique densely defined operator $T^\dagger \in \mathcal{C}(H_2,H_1)$ with domain $D(T^\dagger) = R(T) \oplus^\perp R(T)^\perp$ and has the following properties:

1. $TT^\dagger y = P_{\overline{R(T)}}\, y$, for all $y \in D(T^\dagger)$.

2. $T^\dagger T x = P_{N(T)^\perp}\, x$, for all $x \in D(T)$.

3. $N(T^\dagger) = R(T)^\perp$.

The operator $T^\dagger$ is called the *Moore-Penrose inverse* of T. An alternative definition of $T^\dagger$ is given below.

For every $y \in D(T^\dagger)$, let

$$L(y) := \Big\{ x \in D(T) : \|Tx - y\| \leqslant \|Tu - y\| \quad \text{for all} \quad u \in D(T) \Big\}.$$

Here any $u \in L(y)$ is called a *least square solution* of the operator equation $Tx = y$. The vector $x = T^\dagger y \in L(y)$, and $\|T^\dagger y\| \leqslant \|x\| \quad$ for all $\quad u \in L(y)$ and it is called the *least square solution of minimal norm*. A different treatment of $T^\dagger$ is given in [4, pages 314, 318–320], where the authors call this as “*the Maximal Tseng generalized Inverse*”.

Next we define minimum attaining operators and the absolutely minimum attaining operators.

DEFINITION 2.1. [4, 10] Let $T \in \mathcal{C}(H_1,H_2)$ be densely defined. Then

$$m(T) := \inf\{\|Tx\| : x \in S_{D(T)}\}$$

is called the minimum modulus of T. The operator T is said to be bounded below if and only if $m(T) > 0$.

REMARK 2.2. If $T \in \mathcal{C}(H_1,H_2)$ is densely defined, then the following holds.

(i) $m(T) > 0$ if and only if $R(T)$ is closed and T is one-to-one.

(ii) Since $D(T) = D(|T|)$ and $\|Tx\| = \||T|x\|$ for all $x \in D(T)$, we can conclude that $m(T) = m(|T|)$.

DEFINITION 2.3. Let $T \in \mathcal{C}(H_1,H_2)$ be densely defined. Then T is said to be

1. *minimum attaining* if there exists $x_0 \in S_{D(T)}$ such that $\|Tx_0\| = m(T) = \inf\{\|Tx\| : x \in S_{D(T)}\}$

2. *absolutely minimum attaining* if for every non-zero closed subspace M of H_1 with $D(T) \cap M \neq \{0\}$, the operator $T|_M : M \cap D(T) \to H_2$ is minimum attaining.

We denote the set of all densely defined closed absolutely minimum attaining operators between H_1 and H_2 by $\mathcal{AM}_c(H_1,H_2)$ and the set of all densely defined minimum attaining closed operators by $\mathcal{M}_c(H_1,H_2)$. We write $\mathcal{AM}_c(H,H)$ and $\mathcal{M}_c(H,H)$ by $\mathcal{AM}_c(H)$ and $\mathcal{M}_c(H)$, respectively.

In particular, if $T \in \mathcal{B}(H_1,H_2)$, then T is called minimum attaining if there exists $x \in H_1$ with $\|x\| = 1$ such that $\|Tx\| = m(T)$. We say T to be absolutely minimum attaining if for every closed subspace M of H_1, the restriction operator $T|_M : M \to H_2$ is minimum attaining. This class is denoted by $\mathcal{AM}(H_1,H_2)$. If $H_1 = H_2 = H$, then we denote $\mathcal{AM}(H_1,H_2)$ by $\mathcal{AM}(H)$.

Let M be a closed subspace of H and $T \in \mathcal{C}(H)$ be densely defined. Then M is said to be invariant under T, if $T(M \cap D(T)) \subseteq M$. Further, M is said to be reducing subspace for T if both M and $M^\perp$ are invariant under T.

Let $P := P_M$. If $P(D(T)) \subseteq D(T)$ and $(I-P)(D(T)) \subseteq D(T)$, then

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where $T_{ij} = P_iTP_j|M_j$ $(i,j = 1,2)$, $M_1 = P(D(T))$ and $M_2 = (I-P)(D(T))$. Here $P_1 = P$ and $P_2 = I-P$. It is known that M is invariant under T if and only if $T_{21} = 0$. Also, M reduces T if and only if $T_{21} = 0$ and $T_{12} = 0$.

3. Main results

In this section we prove our main results. First we recall a few basic results on absolutely minimum attaining and absolutely norm attaining operators, that we use frequently. Recall that $T \in \mathcal{B}(H_1,H_2)$ is norm attaining if there exists $x \in S_{H_1}$ such that $\|Tx\| = \|T\|$. We say T to be absolutely norm attaining if for every non-zero closed subspace M of H_1, the restriction operator $T|_M : M \to H_2$ is norm attaining. We denote the set of all absolutely norm attaining operators by $\mathcal{AN}(H_1,H_2)$. For more details of this class operators we refer to [18].

THEOREM 3.1. *[8, Theorem 5.14] Let $T \in \mathcal{B}(H_1,H_2)$. Then the following are equivalent;*

1. $T \in \mathcal{AM}(H_1,H_2)$.
2. $T^*T \in \mathcal{AM}(H_1)$.

THEOREM 3.2. [12, Theorem 4.6] *Let $T \in \mathcal{C}(H)$ be densely defined and $T^{-1} \in \mathcal{B}(H)$. Then $T \in \mathcal{AM}_c(H)$ if and only if $T^{-1} \in \mathcal{AN}(H)$.*

Let $T \in \mathcal{C}(H)$ be a densely defined closed operator. Then $Z_T = T(I+T^*T)^{-\frac{1}{2}}$ is called the *bounded transform* of T. Here we list a few important properties of Z_T.

THEOREM 3.3. [23, page 90] *Let $T \in \mathcal{C}(H)$ be densely defined. Then we have*

1. $\|Z_T\| \leqslant 1$.
2. $(Z_T)^* = Z_{T^*}$.
3. $Z_T^* Z_T = I - (I + T^*T)^{-1}$.

It is to be noted that in particular, $T \in \mathcal{B}(H)$ if and only $\|Z_T\| < 1$ (see [24, Corollary 2.1]). Also we can easily prove that $N(Z_T) = N(T)$ and $R(T) = R(Z_T)$ (see [17]).

THEOREM 3.4. [21, theorem VIII.3] *Let S be a densely defined closed symmetric operator. Then*

$$\|(S + iI)x\|^2 = \|Sx\|^2 + \|x\|^2, \text{ for all } x \in D(S). \tag{3.1}$$

THEOREM 3.5. *Let S be a densely defined closed symmetric operator. Then the following are true;*

1. $m(S + iI) = \sqrt{1 + m(S)^2}$.
2. $S \in \mathcal{M}_c(H)$ *if and only if* $S + iI \in \mathcal{M}_c(H)$.
3. $S \in \mathcal{AM}_c(H)$ *if and only if* $S + iI \in \mathcal{AM}_c(H)$.

Proof. All the proofs directly follow by Equation 3.1 and the definitions of the minimum modulus, minimum attaining property and the absolutely minimum attaining property, respectively. □

THEOREM 3.6. *Let T be a positive and unbounded operator. Then the following are equivalent;*

1. $T \in \mathcal{AM}_c(H)$.
2. $T^2 + I \in \mathcal{AM}_c(H)$.
3. $Z_T \in \mathcal{AM}(H)$.

Proof. Proof of (1) if and only (2): By Theorem 3.5, we know that $T \in \mathcal{AM}_c(H)$ if and only $T + iI \in \mathcal{AM}_c(H)$. Since $T + iI$ is one-to-one, by applying [12, Theorem 4.10], we can conclude that $T + iI \in \mathcal{AM}_c(H)$ if and only if $T^2 + I = (T + iI)^*(T + iI) \in \mathcal{AM}_c(H)$.

To prove the equivalence of (2) and (3), first we observe that

$$Z_T^* Z_T = T^2(I + T^2)^{-1} = I - (I + T^2)^{-1}. \tag{3.2}$$

Now, if $I+T^2 \in \mathcal{AM}_c(H)$, then $(I+T^2)^{-1}$ is absolutely norm attaining, by Theorem 3.2. Hence by [26, Theorem 2.5], there exists a compact positive operator K and positive finite rank operator F and $\alpha \geqslant 0$ such that

$$(I+T^2)^{-1} = \alpha I + K - F \tag{3.3}$$

with $KF = 0$ and $F \leqslant \alpha I$. Now, by Equation 3.2, we have that $Z_T^* Z_T = (1-\alpha)I - K + F$. Our idea is to apply [8, Theorem 5.8] and conclude $Z_T^* Z_T \in \mathcal{AM}(H)$. Then this will imply $Z_T \in \mathcal{AM}(H)$. For this purpose, we need to prove that $\|K\| \leqslant 1-\alpha$. By post multiplying Equation 3.3 and using the facts that $KF = 0$ and K commute with $(I+T^2)^{-1}$, we get

$$(I+T^2)^{-1}K = (\alpha I + K)K. \tag{3.4}$$

Since $\|(I+T^2)^{-1}\| \leqslant 1$, we have $(I+T^2)^{-1}K \leqslant K$ or $(\alpha I + K)K \leqslant K$. This implies that $K^2 \leqslant (1-\alpha)K \leqslant (1-\alpha)\|K\|I$. From this we can conclude that $\|K^2\| = \|K\|^2 \leqslant (1-\alpha)\|K\|$ or $\|K\| \leqslant 1-\alpha$.

To prove the implication (3) $\Rightarrow$ (2), let $Z_T \in \mathcal{AM}(H)$. Then $Z_T^* Z_T \in \mathcal{AM}(H)$. Hence there exists $K \in \mathcal{K}(H)_+$, $F \in \mathcal{F}(H)_+$ and $\beta \geqslant 0$ satisfying $KF = 0$ and $K \leqslant \beta I$ such that $Z_T^* Z_T = \beta I - K + F$. Hence by Equation 3.2, we have that

$$(T^2+I)^{-1} = (1-\beta)I + K - F \in \mathcal{AN}(H),$$

by [15, Theorem 5.2]. Now by Theorem 3.2, we can conclude that $I+T^2 \in \mathcal{AM}_c(H)$. □

Our next goal before proving the spectral theorem is to show that an absolutely minimum attaining closed operator can either have a finite-dimensional null space or a finite-dimensional range space. This property is helpful in deciding the spectrum of such an operator. To achieve this we prove the following results.

LEMMA 3.7. *Let $T \in \mathcal{C}(H)$ be densely defined. If Z_T is norm attaining, then $T \in \mathcal{B}(H)$.*

Proof. We know that $\|Z_T\| \leqslant 1$. If $\|Z_T\| < 1$, then clearly $T \in \mathcal{B}(H)$, by [24, Corollary 2.1]. Next assume that $\|Z_T\| = 1$. Since Z_T is norm attaining, there exists $x_0 \in S_H$ such that $Z_T^* Z_T x_0 = x_0$. That is, $(I - (I+T^*T)^{-1})x_0 = x_0$ or $(I+T^*T)^{-1}x_0 = 0$. This imply that $x_0 = 0$, a contradiction. Hence, the assumption that $\|Z_T\| = 1$ is wrong. That is, $\|Z_T\| < 1$. Now the conclusion follows by the earlier case. □

PROPOSITION 3.8. Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. If T has finite rank, then $T \in \mathcal{B}(H_1, H_2)$.

Proof. Since T is closed, $N(T)$ is closed and hence $D(T) = N(T) \oplus C(T)$, where $C(T) = D(T) \cap N(T)^{\perp}$. Note that $T_0 = T|_{C(T)}$ is a bijection from $C(T)$ onto $R(T)$. This implies that $C(T)$ is finite-dimensional and hence T_0 is bounded. Suppose its norm is M and consider any x in the domain $D(T)$ of T. Then x can be written uniquely as $x = u + v$ with $u \in N(T)$ and $v \in C(T)$. Since $D(T)$ is a subspace, $v = x - u$ is in $D(T)$. Now

$$\|T(x)\| = \|T(v)\| \leqslant M\|v\| \leqslant M\|x\|.$$

This shows that T is bounded on $D(T)$. Since $D(T)$ is an orthogonal direct sum of a closed subspace $N(T)$ and a finite-dimensional subspace $C(T)$, it is closed. As T is densely defined, we obtain that $D(T) = H$. □

COROLLARY 3.9. Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined and unbounded. Then $R(T)$ must be infinite-dimensional.

To prove the converse of Lemma 3.7, we need the following result.

THEOREM 3.10. *Let T be an unbounded positive operator $T \in \mathcal{AM}_c(H)$. Then $N(T)$ is finite-dimensional.*

Proof. It is clear that $R(T)$ is infinite dimensional by Corollary 3.9. Suppose that $N(T)$ is infinite-dimensional. Then $N(Z_T)$ is infinite-dimensional and hence $Z_T \in \mathcal{F}(H)$ by [2, Remark 3.2]. Since $R(T) = R(Z_T)$, T is a finite rank operator. Since Z_T is norm attaining, by Lemma 3.7, we can conclude that T is bounded, a contradiction. This completes the proof. □

COROLLARY 3.11. Let $T \in \mathcal{AM}_c(H_1, H_2)$ be densely defined and unbounded. Then $N(T)$ is finite-dimensional.

Proof. Since $T \in \mathcal{AM}_c(H_1, H_2)$, we have $|T| \in \mathcal{AM}_c(H_1)$ and by Theorem 3.10, we can conclude that $N(|T|) = N(T)$ is finite dimensional. □

REMARK 3.12. If $A \in \mathcal{AM}(H_1, H_2)$, then either $R(A)$ or $N(A)$ is finite-dimensional (see [8, Proposition 5.19] for details). But Corollary 3.11 says that if A is unbounded, then $N(A)$ is finite-dimensional. The reason for this difference is that the finite rank operators are included in the class of bounded absolutely minimum attaining operators.

Next, we describe the structure of unbounded positive absolutely minimum attaining operators. This result generalizes that of [12, Theorem 4.8].

THEOREM 3.13. *Let $T \in \mathcal{AM}_c(H)$ be positive and unbounded. There exists an unbounded (increasing) sequence $\{\lambda_n\}$ of eigenvalues of T with corresponding orthonormal eigenvectors $\{\phi_n\}$ such that*

1. $D(T) = \left\{ x \in H : \sum_{n=1}^{\infty} \lambda_n^2 |\langle x, \phi_n \rangle|^2 < \infty \right\}$ *and*

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n, \text{ for all } x \in D(T).$$

2. $\sigma(T) \subseteq \{\lambda_n : n \in \mathbb{N}\} \cup \{0\} = \sigma_p(T) \cup \{0\}$.

3. *If $\mu \in \sigma_p(T)$, then it is an eigenvalue with finite multiplicity.*

4. $\overline{span}\{\phi_n : n \in \mathbb{N}\} = N(T)^{\perp}$.

5. *T is diagonalizable. That is, there exists an orthonormal basis of H consisting of eigenvectors of T.*

6. *For every subset S of $\mathbb{N} \cup \{0\}$, we have*

$$\inf\{\lambda_n : n \in S\} = \min\{\lambda_n : n \in S\}.$$

Proof. If T is one-to-one then the statements (1) to (4) follow from [12, Theorem 4.8]. If T is not one-to-one, then by Corollary 3.11, $N(T)$ is finite-dimensional. That is, $0 \in \sigma_p(T)$. Since $T = 0 \oplus T|_{N(T)^{\perp}}$, applying [12, Theorem 4.8] to $T|_{N(T)^{\perp}}$, the conclusion follows.

The proof of (5) follows by taking an orthonormal basis of $N(T)$ and adjoining it to the eigenbasis of $N(T)^{\perp}$ obtained in (4). To prove (6), first note that by (5), T is diagonalizable. First assume that $N(T) \neq \{0\}$. Then $N(T)$ is finite-dimensional and hence $0 \in \sigma_d(T)$. In this case $\sigma(T) = \sigma_d(T) = \{\lambda_n : n \in \mathbb{N}\} \cup \{0\}$. We write $T = 0 \oplus T_0$, where $T_0 = T|_{N(T)^{\perp}}$. Since $T_0 \in \mathcal{AM}(N(T)^{\perp})$ and $R(T_0) = R(T)$ is closed as $T \in \mathcal{AM}_c(H)$, we conclude that T_0 has a bounded inverse and by Theorem 3.2, we have $T_0^{-1} \in \mathcal{AN}(N(T)^{\perp})$. Hence $\sigma(T_0^{-1}) = \{\lambda_n^{-1} : n \in \mathbb{N}\}$. Let $S \subseteq \mathbb{N}$. By [15, Theorem 3.8(i)], we have $\max\{\lambda_k^{-1} : k \in S\} = \sup\{\lambda_k^{-1} : k \in S\}$. From this we get that $\inf\{\lambda_k : k \in S\} = \min\{\lambda_k : k \in S\}$. Next, assume that $0 \in \sigma(T)$ and $S \subseteq \mathbb{N} \cup \{0\}$. Let us write $\lambda_0 = 0$. Then we have

$$\inf\{\lambda_k : k \in S \cup \{0\}\} = 0 = \min\{\lambda_k : k \in S \cup \{0\}\}.$$

This completes the proof. □

COROLLARY 3.14. Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined and unbounded. Then $T \in \mathcal{AM}_c(H_1, H_2)$ if and only if $D(T)$ does not contain an infinite-dimensional closed subspace.

Proof. We have that $T \in \mathcal{AM}_c(H_1, H_2)$ if and only if $|T| \in \mathcal{AM}_c(H_1)$. By Theorem 3.13, this is equivalent to the fact that $|T|$ has pure discrete spectrum. By [23, page 113, Exercise 25], this is equivalent to the fact that $D(|T|) = D(T)$ does not contain an infinite-dimensional closed subspace. □

PROPOSITION 3.15. Let $T \in \mathcal{C}(H)$ be positive and unbounded. If $T \in \mathcal{AM}_c(H)$, then $T^2 \in \mathcal{AM}_c(H)$.

Proof. Note that if T is one-to-one, then the result holds true by [12, Theorem 4.10]. So let us assume that $N(T) \neq \{0\}$. Then $T = 0 \oplus T_0$, where $T_0 = T|_{N(T)^\perp}$. By Corollary 3.11, $N(T)$ is finite-dimensional. Clearly, by definition, $T_0 \in \mathcal{AM}_c(N(T)^\perp)$ and consequently by [12, Theorem 4.10], $T_0^2 \in \mathcal{AM}_c(N(T)^\perp)$. Hence by Theorem 3.6, $Z_{T_0^2} \in \mathcal{AM}(N(T)^\perp)$. Since $N(Z_{T^2}) = N(T^2) = N(T)$ is finite-dimensional, by [2, Proposition 3.6], we can conclude that $Z_{T^2} = 0 \oplus Z_{T_0^2} \in \mathcal{AM}(H)$. Now the conclusion follows by Theorem 3.6. $\square$

PROPOSITION 3.16. If $T \in \mathcal{AM}_c(H)$ is positive, unbounded and $\alpha > 0$, then $T + \alpha I \in \mathcal{AM}_c(H)$.

Proof. First note that since $T \in \mathcal{AM}_c(H)$, it follows that $R(T)$ is closed. Also, $N(T)$ is finite-dimensional by Theorem 3.10. Since $N(T)$ is a reducing subspace for T, we have that $T = \begin{pmatrix} 0 & 0 \\ 0 & T_0 \end{pmatrix}$, where $T_0 = T|_{N(T)^\perp}$. Since T_0 is bijective and $T_0 \in \mathcal{AM}_c(N(T)^\perp)$, it follows that $T_0 + \alpha I_{N(T)^\perp} \in \mathcal{AM}_c(N(T)^\perp)$. Hence by [12, Theorem 4.8], we can conclude that $(T_0 + \alpha I_{N(T)^\perp})^{-1} \in \mathcal{K}(N(T)^\perp)$. Hence we have $(T + \alpha I)^{-1} = \begin{pmatrix} \alpha I_{N(T)} & 0 \\ 0 & (T_0 + \alpha I_{N(T)^\perp})^{-1} \end{pmatrix} \in \mathcal{K}(H)$. Now the conclusion follows by [12, Theorem 4.8]. $\square$

COROLLARY 3.17. Let $T \in \mathcal{C}(H)$ be positive and unbounded. If $T^2 \in \mathcal{AM}_c(H)$, then $T \in \mathcal{AM}_c(H)$.

Proof. If $T^2 \in \mathcal{AM}_c(H)$, then $T^2 + I \in \mathcal{AM}_c(H)$ by Proposition 3.16. But we have $T^2 + I = (T - iI)(T + iI) = (T + iI)^*(T + iI) \in \mathcal{AM}_c(H)$. Since $T + iI$ is one-to-one, by [12, Theorem 4.11], we have $T + iI \in \mathcal{AM}_c(H)$. Next, by Theorem 3.5, we can conclude that $T \in \mathcal{AM}_c(H)$. $\square$

COROLLARY 3.18. Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined and unbounded. Then $T \in \mathcal{AM}_c(H_1, H_2)$ if and only if $T^*T \in \mathcal{AM}_c(H_1)$.

Proof. We know that $T \in \mathcal{AM}_c(H_1, H_2)$ if and only if $|T| \in \mathcal{AM}_c(H_1)$. Now the result follows by Corollary 3.17 and Proposition 3.15. $\square$

THEOREM 3.19. *Let $T \in \mathcal{C}(H)$ be positive. Then $T \in \mathcal{K}(H)$ if and only if $Z_T \in \mathcal{K}(H)$.*

Proof. Assume that $T \in \mathcal{K}(H)$. Since $(I + T^2)^{-\frac{1}{2}} \in \mathcal{B}(H)$, we get that $Z_T = T(I + T^2)^{-\frac{1}{2}} \in \mathcal{K}(H)$. Next assume that $Z_T \in \mathcal{K}(H)$. That is Z_T is norm attaining. Hence by Lemma 3.7, it is clear that $T \in \mathcal{B}(H)$. Hence $(I + T^2)^{1/2} \in B(H)$. Thus $T = Z_T(I + T^2)^{1/2} \in \mathcal{K}(H)$. $\square$

The following result is proved in [12, Theorem 4.16]. Here we give a simple proof in a particular case.

PROPOSITION 3.20. Let $T \in \mathcal{C}(H)$ be positive and unbounded. If $T \in \mathcal{AM}_c(H)$ then $N(T)$ is finite-dimensional and $T^\dagger \in \mathcal{K}(H)$.

Proof. If $T \in \mathcal{AM}_c(H)$, then $N(T)$ is finite-dimensional. If T is one-to-one, then the conclusion follows by [12, Theorem 4.10]. Next, assume that $N(T) \neq \{0\}$. Then T has the representation $\begin{pmatrix} 0 & 0 \\ 0 & T_0 \end{pmatrix}$, where $T_0 = T|_{N(T)^\perp}$. Clearly, $T_0 \in \mathcal{AM}_c(N(T)^\perp)$. Hence $T_0^{-1} \in \mathcal{K}(N(T)^\perp)$, by [12, Theorem 4.8]. Hence $T^\dagger = 0 \oplus T_0^{-1}$ is compact. □

Next we illustrate the above result with an example.

EXAMPLE 3.21. Define

$$A(x_1, x_2, x_3, \ldots) = (x_1, 2x_2, 3x_3, \ldots),\ (x_n) \in D(A),$$

where $D(A) = \{(x_n) \in \ell^2 : (nx_n) \in \ell^2\}$. It can be shown that A is a densely defined closed operator. In fact, A is positive and invertible with $A^{-1} : \ell^2 \to \ell^2$ given by

$$A^{-1}(y_1, y_2, y_3, \ldots) = \left(y_1, \frac{y_2}{2}, \frac{y_3}{3}, \ldots\right),\ (y_n) \in \ell^2.$$

Let $H = \ell^2 \oplus \ell^2$. Define T on $D := \ell^2 \oplus D(A) \subset H$ by $T = 0 \oplus A$. Then $T^\dagger = 0 \oplus A^{-1}$. Clearly, $T \in \mathcal{C}(H)$ and densely defined. It can be easily seen that $N(T) = \ell^2 \oplus \{0\}$ which is infinite-dimensional. Also, $T^\dagger \in \mathcal{K}(H)$. We claim that $T \notin \mathcal{AM}_c(H)$. It is clear that $N(T)$ is a reducing subspace for T. It can be easily shown that $Z_T = 0 \oplus Z_A$. A routine computation shows that Z_A is defined on the standard orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of ℓ^2 by

$$Z_A(e_n) = \frac{n}{\sqrt{1+n^2}} e_n,\ \ n \in \mathbb{N}.$$

Note that 1 is the limit point of the point spectrum

$$\sigma_p(Z_A) = \left\{ \frac{n}{\sqrt{1+n^2}} : n \in \mathbb{N} \right\}.$$

Hence $\sigma_{\text{ess}}(Z_A) = \{1\}$ and since 0 is an eigenvalue of Z_T with infinite multiplicity, we can conclude that $0 \in \sigma_{\text{ess}}(T)$. So $\sigma_{\text{ess}}(Z_T) = \{0, 1\}$. By [2, Theorem 3.10], $Z_T \notin \mathcal{AM}(H)$. Now, by Theorem 3.6, we can conclude that $T \notin \mathcal{AM}_c(H)$.

PROPOSITION 3.22. Let $T \in \mathcal{AM}_c(H_1, H_2)$ be densely defined and unbounded. Then $N(T)$ is finite-dimensional and $T^\dagger \in \mathcal{K}(H_2, H_1)$.

Proof. It is clear that $N(T)$ is finite-dimensional by Corollary 3.11. As $T \in \mathcal{AM}_c(H_1, H_2)$, we have $|T| \in \mathcal{AM}_c(H_1)$. This implies that $|T|^\dagger = |(T^*)^\dagger| \in \mathcal{K}(H_1)$. From this we can conclude that $T^\dagger \in \mathcal{K}(H_2, H_1)$. □

The following question is asked in [12, Question 4.17].

QUESTION 3.23. Let $T \in \mathcal{C}(H)$ be densely defined, unbounded and $T^{\dagger} \in \mathcal{K}(H)$. Is it true that $T \in \mathcal{AM}_c(H)$?

The Example 3.21 shows that the answer to Question 3.23 need not be affirmative. But we have an affirmative answer in a particular case, when $N(A)$ is finite-dimensional as the following Proposition shows.

PROPOSITION 3.24. Let $T \in \mathcal{C}(H)$ be positive, unbounded and $N(T)$ be finite-dimensional. If $T^{\dagger} \in \mathcal{K}(\mathcal{H})$, then $T \in \mathcal{AM}_c(H)$.

Proof. Since $N(T)$ reduces T, with respect to the subspaces $N(T)$ and $N(T)^{\perp}$, we can write T by $T = 0 \oplus T_0$, where $T_0 = T|_{N(T)^{\perp}}$. As $N(T) = N(Z_T)$, we have that $Z_T = 0 \oplus Z_{T_0}$. As $T_0^{-1} \in \mathcal{K}(N(T)^{\perp})$, we can conclude that $Z_{T_0} \in \mathcal{K}(N(T)^{\perp})$. As $N(Z_T)$ is finite-dimensional, by [2, Proposition 3.6], it follows that $Z_T \in \mathcal{AM}(H)$. Now by Theorem 3.6, we can conclude that $T \in \mathcal{AM}_c(H)$. □

PROPOSITION 3.25. Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined, unbounded and $N(T)$ be finite-dimensional. If $T^{\dagger} \in \mathcal{K}(H_2, H_1)$, then $T \in \mathcal{AM}(H_1, H_2)$.

Proof. If $T^{\dagger} \in \mathcal{K}(H_2, H_1)$, then $|(T^{\dagger})^*| \in \mathcal{K}(H_2)$. But by [12, Proposition 3.19], we have $|T|^{\dagger} = |(T^{\dagger})^*|$. Since $N(T) = N(|T|)$, by Proposition 3.24, we have $|T| \in \mathcal{AM}_c(H_1)$. Hence $T \in \mathcal{AM}_c(H_1, H_2)$. □

By combining Propositions 3.22 and 3.25, we can state the following theorem.

THEOREM 3.26. *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined and unbounded. Then $T \in \mathcal{AM}_c(H_1, H_2)$ if and only if $N(T)$ is finite-dimensional and $T^{\dagger} \in \mathcal{K}(H_2, H_1)$.*

Next we ask whether the $\mathcal{AM}$-property of an operator implies the $\mathcal{AM}$-property of its adjoint. This question in the case of bounded operators is answered in [2, Proposition 3.11]. The unbounded case is discussed in [12] by assuming an extra condition, namely the invertibility of the operator.

THEOREM 3.27. [12, Theorem 4.11] *If $T \in \mathcal{C}(H)$ is densely defined, unbounded and has a bounded inverse, then $T \in \mathcal{AM}_c(H)$ if and only if $T^* \in \mathcal{AM}_c(H)$.*

Here we improve the above result by dropping the invertibility condition.

THEOREM 3.28. *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined and unbounded. Assume that both $N(T)$ and $N(T^*)$ are finite-dimensional. Then $T \in \mathcal{AM}_c(H_1, H_2)$ if and only if $T^* \in \mathcal{AM}_c(H_2, H_1)$.*

Proof. First assume that $T \in \mathcal{AM}_c(H_1, H_2)$. Then by Proposition 3.22, $T^{\dagger} \in \mathcal{K}(H_2, H_1)$. This implies that $(T^*)^{\dagger} = (T^{\dagger})^* \in \mathcal{K}(H_1, H_2)$. As $N(T^*)$ is finite-dimensional, by Proposition 3.24, we can conclude that $T^* \in \mathcal{AM}(H_2, H_1)$. The converse follows easily by applying the above argument to T^* and using the hypothesis that $N(T)$ is finite-dimensional. □

Next we prove the spectral theorem for unbounded self-adjoint absolutely minimum attaining operators.

THEOREM 3.29. (Spectral theorem for self-adjoint $\mathcal{AM}$-operators) *Let H be an infinite-dimensional Hilbert space and $T \in \mathcal{C}(H)$ be self-adjoint. Assume that T is unbounded. Then the following statements are equivalent.*

1. *$T \in \mathcal{AM}_c(H)$.*
2. *$T^\dagger \in \mathcal{K}(H)$ and $N(T)$ is finite-dimensional.*
3. *There exists a sequence (λ_n) of real numbers and an orthonormal subset $\{v_n : n \in \mathbb{N}\}$ of H such that $\lim\limits_{n\to\infty} |\lambda_n| \to \infty$ and $Tv_n = \lambda_n v_n$ for each $n \in \mathbb{N}$. In this case,* $D(T) = \left\{ x \in H : \sum\limits_{n=1}^{\infty} \lambda_n^2 |\langle x, v_n\rangle|^2 < \infty \right\}$ *and*
$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, v_n\rangle v_n, \text{ for all } x \in D(T).$$
4. *T is diagonalizable, that is, H has an orthonormal basis consisting of eigenvectors of T and each eigenvalue of T has finite multiplicity.*
5. *$\sigma(T) \subseteq \{0\} \cup \{\lambda_n : n \in \mathbb{N}\}$ and every spectral value is an eigenvalue with finite multiplicity, that is, $\sigma(T) = \sigma_d(T)$.*
6. *The resolvent $R_\lambda(T) = (T - \lambda I)^{-1}$ is compact for one, hence for all, $\lambda \in \rho(T)$.*
7. *The embedding $J_T : (D(T), \|\cdot\|_T) \to H$ is compact. Here $\|x\|_T = (\|x\|^2 + \|Tx\|^2)^{\frac{1}{2}}$ for all $x \in D(T)$.*

Proof. If $N(T) = \{0\}$, then the result follows from [12, Theorem 4.18]. Next assume that $N(T) \neq \{0\}$. Then the equivalence of (1) and (2) follows by Propositions 3.25 and 3.22. Since $N(T)$ is finite-dimensional and $N(T)$ reduces T, we have that $T = 0 \oplus T_0$, where $T_0 = T_{N(T)^\perp}$, equivalence of the other statements follows by applying [12, Theorem 4.18] to T_0 since T_0 is bijective and $T_0 \in \mathcal{AM}_c(N(T)^\perp)$.

To find $D(T)$ in (3), we argue as follows; $x \in D(T)$ if and only if $Tx \in H$ if and only if

$$\begin{aligned} Tx &= \sum_{n=1}^{\infty} \langle Tx, v_n\rangle v_n \\ &= \sum_{n=1}^{\infty} \langle x, Tv_n\rangle v_n \\ &= \sum_{n=1}^{\infty} \lambda_n \langle x, v_n\rangle v_n. \end{aligned}$$

It is clear that $Tx \in H$ if and only if $\sum_{n=1}^{\infty} \lambda_n^2 |\langle x, v_n \rangle|^2 < \infty$. Hence

$$D(T) = \left\{ x \in H : \sum_{n=1}^{\infty} \lambda_n^2 |\langle x, v_n \rangle|^2 < \infty \right\}$$

and $Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, v_n \rangle v_n$, for all $x \in D(T)$. $\square$

It is natural to ask whether Theorem 3.29 holds true for normal operators. Here we answer this question.

THEOREM 3.30. (Spectral theorem for unbounded normal $\mathcal{AM}$-operators) *Let H be an infinite-dimensional Hilbert space and $T \in \mathcal{C}(H)$ be normal. Assume that T is unbounded. Then the following statements are equivalent.*

1. *$T \in \mathcal{AM}_c(H)$.*

2. *$T^\dagger \in \mathcal{K}(H)$ and $N(T)$ is finite-dimensional.*

3. *There exists a sequence (λ_n) of complex numbers and an orthonormal subset $\{\varphi_n : n \in \mathbb{N}\}$ of H such that $\lim_{n\to\infty} |\lambda_n| \to \infty$ and $T\varphi_n = \lambda_n \varphi_n$ for each $n \in \mathbb{N}$. In this case, $D(T) = \left\{ x \in H : \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, \varphi_n \rangle|^2 < \infty \right\}$ and*

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \varphi_n \rangle \varphi_n, \text{ for all } x \in D(T).$$

4. *T is diagonalizable, that is, H has an orthonormal basis consisting of eigenvectors of T and each eigenvalue of T has finite multiplicity.*

5. *$\sigma(T) \subseteq \{0\} \cup \{\lambda_n : n \in \mathbb{N}\}$ and every spectral value is an eigenvalue with finite multiplicity, that is, $\sigma(T) = \sigma_d(T)$.*

6. *The resolvent $R_\lambda(T) = (T - \lambda I)^{-1}$ is compact for one, hence for all, $\lambda \in \rho(T)$.*

7. *The embedding $J_T : (D(T), \|\cdot\|_T) \to H$ is compact. Here $\|x\|_T = (\|x\|^2 + \|Tx\|^2)^{\frac{1}{2}}$ for all $x \in D(T)$.*

Proof. Equivalence of (1) and (2) follows by Theorem 3.26. Next assume that (2) is true. They by the spectral theorem for compact normal operators, there exists a sequence (μ_n) of complex numbers and an orthonormal system $\{\phi_n\}$ such that

$$T^\dagger y = \sum_{n=1}^{\infty} \mu_n \langle y, \phi_n \rangle \phi_n, \text{ for all } x \in H.$$

From the above equation it is clear that $T^{\dagger}(\phi_n) = \mu_n\phi_n$ for each $n \in \mathbb{N}$. That is, $\phi_n \in R(T^{\dagger}) = C(T)$. It is also clear that $T^{\dagger}(y) \in C(T)$. Since T is normal, it is easy to observe that $N(T) = N(T^*) = N(T^{\dagger})$. Note that $\phi_n = P_{R(T)}(\phi_n) = TT^{\dagger}(\phi_n)$ for all $n \in \mathbb{N}$. Hence we have $T\phi_n = \lambda_n\phi_n$ for all $n \in \mathbb{N}$, where

$$\lambda_n = \begin{cases} 0 & \text{if } \mu_n = 0, \\ \mu_n^{-1} & \text{if } \lambda_n \neq 0. \end{cases}$$

Since $T^{\dagger}y \in C(T)$, $TT^{\dagger}y$ is meaningful. Following the similar steps as in (3) of Theorem 3.29, we can determine $D(T)$ easily. On the other hand, following similar arguments about T and $T^{\dagger}$, we can easily prove the implication $(3) \Rightarrow (2)$.

Proof of $(3) \Leftrightarrow (4)$: By (3) it is clear that $T|_{R(T)}$ is diagonalizable. Observe that if $0 \in \sigma(T)$ and as it is an isolated point, it must be an eigenvalue of T and $R(T)$ must be closed. So we have $H = R(T) \oplus R(T)^{\perp} = R(T) \oplus N(T)$. As $N(T)$ is finite dimesnsional, we can take an orthonormal basis of it and adjoin it with the orthonormal basis $\{\phi_n : n \in \mathbb{N}\}$ of $R(T)$ so that we can obtain an orthonormal basis of H consisting of eigenvectors of T.

On the other hand, since T is not bounded, we can conclude that $\sigma(T)$ is unbounded. It is easy to establish the representation of T and determine the domain.

Proof of $(4) \Leftrightarrow (5)$: the implication $(4) \Rightarrow (5)$ is easy to prove. To prove the other way implication, let us assume that $\sigma(T) = \sigma_d(T)$. Let $\sigma(T) = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$. Write $G_n = N(T - \lambda_n I)$. It is easy to prove that each G_n reduces T. By using the Zorn's Lemma, we can show that $H = \bigoplus_{n=1}^{\infty} G_n$ from which we can conclude that T is diagonalizable.

Proof of $(5) \Leftrightarrow (6)$: Let us assume that $\sigma(T) = \sigma_d(T) = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$. Since T is unbounded, $\sigma(T)$ is unbounded closed subset of $\mathbb{C}$. Hence $Tx = \sum_{n=1}^{\infty} \lambda_n P_n x$, where P_n is an orthogonal projection onto $N(T - \lambda_n I)$ with

$$D(T) = \left\{ x \in H : \sum_{n=1}^{\infty} |\lambda_n|^2 \|P_n x\|^2 < \infty \right\}.$$

Hence if $\lambda \in \rho(T)$, then

$$(T - \lambda I)^{-1} = \sum_{n=1}^{\infty} (\lambda_n - \lambda)^{-1} P_n.$$

It is easy to see that $|(\lambda_n - \lambda)^{-1}| \to 0$ as $n \to \infty$. From this it is easy to see that $(T - \lambda I)^{-1} \in \mathcal{K}(H)$.

Proof of $(6) \Leftrightarrow (7)$: the proof of this follows along the similar lines of the self-adjoint case. We refer to [23, Theorem 5.12] for details. □

Next, we turn our attention to invariant and hyperinvariant subspaces. Recall that a closed subspace M of H is said to be invariant under a densely defined closed operator

T if $T(D(T) \cap M) \subseteq M$, and hyperinvariant if M is invariant under every $A \in \mathcal{B}(H)$ such that $AT \subseteq TA$.

If $T \in \mathcal{AM}_c(H)$ is unbounded and has a bounded inverse, then T has a nontrivial hyperinvariant subspace [12, Theorem 4.14]. Here we remove the invertibility condition and prove the same result.

PROPOSITION 3.31. If $T \in \mathcal{AM}_c(H)$ is densely defined and unbounded, then T has a non trivial hyperinvariant subspace.

Proof. If T is bijective, then $T^{-1} \in \mathcal{B}(H)$. Hence by [12, Theorem 4.14], T has a hyperinvariant subspace. So assume that T is not bijective. Note that $R(T)$ is closed by [12, Proposition 4.2]. If T is one-to-one but not onto, then $R(T)$ is a non trivial hyperinvariant subspace for T. If T is onto but not one-to-one, then $N(T)$ is a hyperinvariant subspace for T. □

COROLLARY 3.32. Assume $T \in \mathcal{C}(H)$ is densely defined, unbounded. If $R(T)$ is closed and $T^{\dagger} \in \mathcal{K}(H)$, then T has a hyperinvariant subspace.

Proof. First note that if $N(T) \neq \{0\}$, then clearly $N(T)$ is a hyperinvariant subspace for T. So assume that $N(T) = \{0\}$. By Theorem 3.25, it follows that $T \in \mathcal{AM}_c(H)$. Hence by Theorem 3.31, it is clear that T has a hyperinvariant subspace. □

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