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PROJECTIONS AND PROPER INFINITENESS FOR CORONA ALGEBRAS

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Abstract. Let $\mathscr{B}$ be a nonunital separable simple Jiang–Su-stable C*-algebra with stable rank one. We show that $\mathscr{M}(\mathscr{B})$ is the closed linear span of its projections, which implies Property I for $\mathscr{B}$.

We also show that the corona algebra $\mathscr{C}(\mathscr{B})$ is properly infinite if and only if $T(\mathscr{B})$ is weak* compact. We also provide a number of other equivalent characterizations.

1. Introduction

Let $\mathscr{B}$ be a separable, stable C*-algebra. It is an elementary fact from operator theory that

$$1_{\mathscr{M}(\mathscr{B})} \sim 1_{\mathscr{M}(\mathscr{B})} \oplus 1_{\mathscr{M}(\mathscr{B})},$$

where $\mathscr{M}(\mathscr{B})$ is the multiplier algebra of $\mathscr{B}$ and $\sim$ here is Murray–von Neumann equivalence of projections in $\mathbb{M}_2 \otimes \mathscr{M}(\mathscr{B})$. This is the basic observation underlying the *Brown–Douglas–Fillmore (BDF) sum* which led to the extension semigroup $Ext(\mathscr{A}, \mathscr{B})$, which is a group when $\mathscr{A}$ is separable and nuclear. When $\mathscr{B} = \mathscr{K}$ and $\mathscr{A} = C(X)$ for X a compact subset of the plane, BDF used the functorial properties of this object in their outstanding classification of all essentially normal operators via Fredholm indices ([3]).

Perhaps, as witnessed above, one of the reasons for the success of the BDF theory is that their multiplier algebra $\mathscr{M}(\mathscr{K}) = \mathbb{B}(l_2)$ and corona algebra $\mathbb{B}(l_2)/\mathscr{K}$ have particularly nice structure. For example, the BDF–Voiculescu absorption theorem, which roughly says that all essential extensions are absorbing, would not be true if the Calkin algebra $\mathbb{B}(l_2)/\mathscr{K}$ were not simple ([38]).

Thus, structural properties of multiplier and corona algebras are indispensible for the advancement of extension theory and the associated operator theory beyond the small number of successful classical cases. This idea was well understood by previous researchers, and has had its most successful realizations in the definitive work of Lin (see, e.g., [15], [17], [18], [19], [20], [23]). One problem of the current moment is the case where the canonical ideal need not be stable. One of the insights of previous researchers is that, while this nonstable case is very interesting in itself, it is also indispensible for progress in the classical case of stable canonical ideals (see, e.g., [17],

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[20], [23]). For instance, in the classical stable case, under a nuclearity hypothesis, Kasparov's KK^1 only classifies the absorbing extensions – a very thin class, and thus misses many relevant essential extensions. To delve further, even in the classical stable case, requires finer examination of the structure of the corona algebras and more delicate nonstable absorption theory. Following in the footsteps of previous researchers, this has been the program that we have been pursuing (e.g., [14], [27], [29], [32]).

A unital C*-algebra $\mathscr{C}$ is said to be properly infinite if

$$1_{\mathscr{C}} \succeq 1_{\mathscr{C}} \oplus 1_{\mathscr{C}},$$

where here $\preceq$ is Murray–von Neumann subequivalence of projections in $\mathbb{M}_2 \otimes \mathscr{C}$. It was observed in [8] that when a corona algebra $\mathscr{C}(\mathscr{B}) =_{\mathrm{def}} \mathscr{M}(\mathscr{B})/\mathscr{B}$ was properly infinite, there is a generalized BDF sum on the class of extensions which may serve the needs of extension theory even for nonstable $\mathscr{B}$. This anticipated later works (e.g., [17], [19], [20], [29]) where definitive nonstable generalizations of the BDF index theorem were achieved.

We note that aside from connections to extension theory, proper infiniteness of a C*-algebra (especially a corona algebra) is in itself an interesting and fundamental structural property, which is connected to many other interesting properties. For example, it is an open question whether every properly infinite unital C*-algebra is K_1-injective ([2]). Among other things, K_1-injectivity of the Paschke dual algebras (which are properly infinite) imply interesting uniqueness theorems and generalizations of the BDF essential codimension result (e.g., see [25]).

In [33], it was proven that for $\mathscr{B}$ a separable simple nonunital $\mathscr{Z}$-stable C*-algebra with an approximate unit consisting projections and Property I, $\mathscr{C}(\mathscr{B})$ is properly infinite if and only if $T(\mathscr{B})$ is weak*-compact. In fact, for $\mathscr{B}$ with hypotheses as in the previous statement, we proved that proper infiniteness of $\mathscr{C}(\mathscr{B})$ is equivalent to a number of other interesting statements, generalizing the main result of [8] which was for the case where $\mathscr{B}$ was a simple nonelementary AF algebra. In this paper, we remove the strong restriction that $\mathscr{B}$ has an approximate unit consisting of projections, and we show that $\mathscr{B}$ always has Property I (see Theorem 2.14 and Theorem 3.5). In a related direction, we prove the interesting property that when $\mathscr{B}$ is a nonunital separable simple $\mathscr{Z}$-stable C*-algebra, $\mathscr{M}(\mathscr{B})$ is the (norm) closed linear span of its projections (see Theorem 2.12).

Theorem 2.12 is an intriguing result in many ways, with connections to multiple interesting phenomena in operator theory and K theory. Firstly, this result is true even for the multiplier algebra of a stably projectionless C*-algebra. But also, it immediately leads to the question of whether, for any nonunital separable simple $\mathscr{Z}$-stable C*-algebra $\mathscr{B}$, the corona algebra $\mathscr{C}(\mathscr{B})$ has real rank zero. This question generalizes some conjectures of Brown and Pedersen, which ask whether, for a nonunital separable simple real rank zero C*-algebra $\mathscr{B}$, i. $\mathscr{C}(\mathscr{B})$ has real rank zero, and ii. if, in addition, $K_1(\mathscr{B}) = 0$ then $\mathscr{M}(\mathscr{B})$ has real rank zero (see [4]). These conjectures were proven for the case where $\mathscr{B}$ is simple purely infinite by Zhang (see [42] Corollary 2.6) and the case where $\mathscr{B}$ has, additionally, stable rank one by Lin (see [16] and [18]).

Perhaps one reason for the interest in the Brown–Pedersen Conjectures, was the result, due to Zhang, that when $\mathscr{B}$ is a separable real rank zero C*-algebra, real rank zero for $\mathscr{M}(\mathscr{B})$ is equivalent to a number of interesting properties, including a Weyl–von Neumann theorem for self adjoint operators in $\mathscr{M}(\mathscr{B})$ (see [40] and [41]). Indeed, the real rank zero property has been implicitly present in the subject since the beginning, even though the terminology "real rank zero" was introduced before the original BDF paper – for instance, the original BDF proof of the uniqueness of the neutral element was essentially the Weyl–von Neumann–Berg theorem, and this phenomenon reoccurs all over the place. In another direction, the Kasparov technical lemma implies that the corona algebra of a σ-unital C*-algebra is an SAW* algebra – a property with formal similarities and interesting connections to real rank zero.

Finally, we note that when $\mathscr{B}$ is a separable nonunital simple $\mathscr{Z}$-stable C*-algebra with quasicontinuous scale, then $\mathscr{C}(\mathscr{B})$ has real rank zero ([30]). For such $\mathscr{B}$, $\mathscr{C}(\mathscr{B})$ is purely infinite and $T(\mathscr{B})$ is weak*-compact, giving independent confirmation of our characterization of properly infinite corona algebras. Quasicontinuity of the scale (like continuity of the scale) was developed to provide a "nice setting" for generalizing BDF theory beyond the classical stable cases. (E.g., see [17], [28] and [32].)

We end this introduction by introducing some notations that are to be used in this paper. This paper uses only elementary techniques and should be accessible to a reader with basic knowledge of C*-algebra theory – modulo knowing about multiplier algebras, strict topology, Choquet simplexes, lower semicontinuous affine functions on compact convex sets, and basic notions and regularity properties (like AF-algebras, irrational rotation algebras, real rank zero, strict comparison, stable rank one) from the current theory of simple C*-algebras. We recall some notation here, and recall others in later parts of the paper.

For a nonunital C*-algebra $\mathscr{B}$, $\mathscr{M}(\mathscr{B})$ and $\mathscr{C}(\mathscr{B}) =_{\text{def}} \mathscr{M}(\mathscr{B})/\mathscr{B}$ denote the multiplier and corona algebras (resp.) of $\mathscr{B}$. Recall that the multiplier algebra $\mathscr{M}(\mathscr{B})$, of $\mathscr{B}$, is roughly speaking, the largest unital C*-algebra containing $\mathscr{B}$ as an essential ideal. Good references for multiplier algebras, corona algebras, strict topology and associated subjects are [21] and [39].

For a compact convex set K, let $Aff(K)$ denote the vector space of all affine continuous functions from K to $\mathbb{R}$. Note that, with the uniform norm, $Aff(K)$ is a Banach space. $LAff(K)$ denotes the semigroup of all lower semicontinuous, affine functions from K to $(-\infty,\infty]$. $Aff(K)_+$ (resp. $LAff(K)_+$) denotes all $f \in Aff(K)$ (resp. $LAff(K)$) such that $f \geqslant 0$. $Aff(K)_{++}$ (resp. $LAff(K)_{++}$) denotes all $f \in Aff(K)_+$ (resp. $LAff(K)_+$) such that $f(x) > 0$ for all $x \in K$. References for the above material are [1], [12], [13], [14], [33] and the references therein.

For a C*-algebra $\mathscr{D}$ (unital or nonunital), we let $T(\mathscr{D})$ denote the tracial state space of $\mathscr{D}$, given the weak* topology. We will be interested in $T(\mathscr{B})$, $T(\mathscr{M}(\mathscr{B}))$ and $T(\mathscr{C}(\mathscr{B}))$ (some or all of which could be empty) for some nonunital $\mathscr{B}$. Note that when $\mathscr{D}$ is unital, then $T(\mathscr{D})$ is a compact convex set – in fact, if $\mathscr{D}$ is additionally separable, then $T(\mathscr{D})$ is a metrizable Choquet simplex. Suppose that $\mathscr{D}$ is additionally separable. For an element $e \in Ped(\mathscr{D})_+ - \{0\}$, we let $T_e(\mathscr{D})$ denote all densely defined, norm-lower semicontinuous traces $\mathscr{D}_+ \to [0,\infty]$ which are normalized at e. Recall that $Ped(\mathscr{D})$ denotes the Pedersen ideal of $\mathscr{D}$; and when $\mathscr{D}$ is separable, then $T_e(\mathscr{D})$, with the topology of pointwise convergence on $Ped(\mathscr{D})$, is a metrizable Choquet simplex. Recall also that any densely defined, norm lower semicontinuous trace τ on $\mathscr{D}$ has a unique extension to a strictly lower semicontinuous trace on $\mathscr{M}(\mathscr{D})_+$. Unless otherwise specified, we will also denote this extension trace by "τ". We also recall that if $\tau \in T(\mathscr{D})$ then this extension trace is actually an element of $T(\mathscr{M}(\mathscr{D}))$.

For any element $A \in \mathscr{M}(\mathscr{D})_+ - \{0\}$, A induces an element $\widehat{A} \in LAff(T_e(\mathscr{D}))_{++}$ via

$$\widehat{A}(\tau) =_{\text{def}} \tau(A)$$

for all $\tau \in T_e(\mathscr{D})$. In a similar manner, A induces elements in $Aff(T(\mathscr{M}(\mathscr{D})))_+$ and $Aff(T(\mathscr{D}))_+$, which we will also denote by $\widehat{A}$.

Also, for all $\tau \in T_e(\mathscr{D})$ (or $T(\mathscr{D})$ or $T(\mathscr{M}(\mathscr{D}))$), recall that the dimension function d_τ is defined by

$$d_\tau(A) =_{\text{def}} \lim_{n\to\infty} \tau(A^{1/n}),$$

for all $A \in \mathscr{M}(\mathscr{D})_+$.

For any C*-algebra $\mathscr{C}$, and $a, b \in \mathscr{C}_+$, $a \preceq b$ if there exists a sequence $\{x_n\}$ in $\mathscr{C}$ such that $x_n b x_n^* \to a$. Note that when a and b are projections, the above $\preceq$ is the same as Murray–von Neumann subequivalence of projections. Also, $a \sim_c b$ if $a \preceq b$ and $b \preceq a$. Note that when a and b are projections, $\sim_c$ *need not* be the same as Murray–von Neumann equivalence of projections which (following longstanding convention) we denote by $\sim$.

References for the above material are again [12], [13], [14], [33] and the references therein.

We will assume that, in all relevant places, all our simple, separable C*-algebras have the property that every quasitrace is a trace.

We caution the reader that in this paper, we use one terminology different from what is in the papers [12], [13], [14], *and other works: In* [12], [13], *and* [14], $T(\mathscr{D})$ *means* $T_e(\mathscr{D})$ *for some* $e \in Ped(\mathscr{D})_+ - \{0\}$, *but that is **NOT** the case in this paper.*

We note that in this paper, when we write "$T_e(\mathscr{D})$", we just mean the aforementioned object with some element $e \in Ped(\mathscr{D})_+ - \{0\}$. For our results, it will not matter which positive nonzero element e of the Pedersen ideal is used.

Good basic references for the theory of simple C*-algebras are [7] and [21].

Finally, many of the ideas of this paper, are generalizations of those from the paper [8], though we need the comparison theory for multiplier algebras as from [14], [24] and the references therein.

The above give the basic references required for understanding the contents of this paper. To understand, for example, the connections with KK theory, extension theory and operator theory, which requires a bit more work, we recommend beginning with the basics in [3], [11], [17], [20], [21], [22], [38], and moving on to the more advanced theory from later references.

2. Projections and Property I

Throughout this section, $\mathscr{B}$ is a nonunital, separable, simple, finite, $\mathscr{Z}$-stable (see paragraph after Definition 2.9), stable rank one C-algebra for which every quasitrace is a trace.* We call the aforementioned the *standing assumptions* on or *standing properties* of $\mathscr{B}$.

DEFINITION 2.1. Let K be a compact convex set, and let $f, g \in LAff(K)_{++}$. f is said to be *complemented under* g if there exists an $h \in LAff(K)_{++}$ such that $f + h = g$.

With the above, we also say that h *complements* f *under* g.

LEMMA 2.2.

1. *$\mathscr{B}$ has projection surjectivity. I.e., suppose that $f \in LAff(T_e(\mathscr{B}))_{++}$ is complemented under $\widehat{1_{\mathscr{M}(\mathscr{B})}}$. Then there exists a projection $P \in \mathscr{M}(\mathscr{B}) - \mathscr{B}$ such that $\widehat{P} = f$.*

2. *$\mathscr{B}$ has projection injectivity. I.e., suppose that $P, Q \in \mathscr{M}(\mathscr{B}) - \mathscr{B}$ are projections for which $\widehat{P} = \widehat{Q}$ on $T_e(\mathscr{B})$. Then P is Murray–von Neumann equivalent to Q in $\mathscr{M}(\mathscr{B})$.*

3. *$\mathscr{B}$ has stable projection surjectivity. I.e., suppose that $f \in LAff(T_e(\mathscr{B}))_{++}$. Then there exists a projection $P \in \mathscr{M}(\mathscr{B} \otimes \mathscr{K}) - (\mathscr{B} \otimes \mathscr{K})$ such that $\widehat{P} = f$.*

4. *$\mathscr{B}$ has stable projection injectivity. I.e., suppose that $P, Q \in \mathscr{M}(\mathscr{B} \otimes \mathscr{K}) - (\mathscr{B} \otimes \mathscr{K})$ are projections for which $\widehat{P} = \widehat{Q}$ on $T_e(\mathscr{B})$. Then P is Murray–von Neumann equivalent to Q in $\mathscr{M}(\mathscr{B} \otimes \mathscr{K})$.*

Proof. Statements (3) and (4) follow from [24] Proposition 4.2 and Corollary 4.6 (see also [5] and [35] 6.2.3).

Let $\{e_{j,k}\}_{1 \leqslant j,k < \infty}$ be a system of matrix units for $\mathscr{K}$. We identify $\mathscr{M}(\mathscr{B})$ with the hereditary C*-subalgebra $\mathscr{M}(\mathscr{B}) \otimes e_{1,1} = (1_{\mathscr{M}(\mathscr{B})} \otimes e_{1,1})\mathscr{M}(\mathscr{B} \otimes \mathscr{K})(1_{\mathscr{M}(B)} \otimes e_{1,1}) \subseteq \mathscr{M}(\mathscr{B} \otimes \mathscr{K})$. Then (2) follows immediately from (4).

We now prove (1). Suppose that $f, g \in LAff(T_e(\mathscr{B}))_{++}$ with $f + g = \widehat{1_{\mathscr{M}(\mathscr{B})}}$. Again identifying $\mathscr{M}(\mathscr{B})$ with the hereditary C*-subalgebra $M(\mathscr{B}) \otimes e_{1,1} \subset \mathscr{M}(\mathscr{B} \otimes \mathscr{K})$, we have that $f + g = (1_{\mathscr{M}(\mathscr{B}) \otimes e_{1,1}})\widehat{\ }$. By (3), we can find projections $P', Q' \in \mathscr{M}(\mathscr{B} \otimes \mathscr{K}) - (\mathscr{B} \otimes \mathscr{K})$ such that $\widehat{P'} = f$ and $\widehat{Q'} = g$. Moreover, since $\mathscr{M}(\mathscr{B} \otimes \mathscr{K})$ is properly infinite, replacing P' and Q' with Murray–von Neumann equivalent projections if necessary, we may assume that $P' \perp Q'$. Now $(P' \oplus Q')\widehat{\ } = (1_{\mathscr{M}(\mathscr{B}) \otimes e_{1,1}})\widehat{\ }$. Hence, by (4), we must have that $P' \oplus Q' \sim 1_{\mathscr{M}(\mathscr{B}) \otimes e_{1,1}}$. Hence, there must be pairwise orthogonal projections $P, Q \in \mathscr{M}(\mathscr{B}) \otimes e_{1,1} - \mathscr{B} \otimes e_{1,1}$ such that $P \sim P'$, $Q \sim Q'$ and $P + Q = 1_{\mathscr{M}(\mathscr{B}) \otimes e_{1,1}}$. Hence, $\widehat{P} = f$. $\square$

REMARK 2.3. The proof of Lemma 2.2 actually shows that for all $n \geqslant 1$, for all $f \in LAff(T_e(\mathscr{B}))_{++}$ which is complemented under $\widehat{1_{\mathscr{M}(\mathbb{M}_n(\mathscr{B}))}}$, there exists a projection $P \in \mathscr{M}(\mathbb{M}_n(\mathscr{B})) - \mathbb{M}_n(\mathscr{B})$ such that $\widehat{P} = f$.

This latter property is sometimes called *n projection surjectivity*. .

Recall that, by [15] Remark 2.9, there exists a (necessarily unique) smallest ideal I_{min} of $\mathscr{M}(\mathscr{B})$ such that I_{min} properly contains $\mathscr{B}$. (See also [13] Theorem 4.7, Proposition 5.4 and Theorem 5.6.)

LEMMA 2.4. *Let $A \in \mathscr{I}_{min+} - \mathscr{B}$ be such that $\|A\| \leqslant 1$ and $\widehat{A}$ is continuous on $T_e(\mathscr{B})$.*

Then there exists a projection $P \in \overline{A\mathscr{M}(\mathscr{B})A} - \mathscr{B}$ for which $\widehat{P} = \widehat{A}$ on $T_e(\mathscr{B})$.

Proof. If A is a projection, then we can take $P = A$.

Hence, we may assume that A is not a projection, and hence, $sp(A)$ contains a point in $(0,1)$. Hence, since $\mathscr{B}$ is simple, $d_\tau(A) > \tau(A)$ for all $\tau \in T_e(\mathscr{B})$. Since $\mathscr{B}$ has projection surjectivity, and since $\widehat{1_{\mathscr{M}(\mathscr{B})}} - A$ complements $\widehat{A}$ under $\widehat{1_{\mathscr{M}(\mathscr{B})}}$, let $Q \in \mathscr{M}(\mathscr{B}) - \mathscr{B}$ be a projection for which $\widehat{Q} = \widehat{A}$ on $T_e(\mathscr{B})$. Hence, by [13] Theorem 5.6, $Q \in \mathscr{I}_{min} - \mathscr{B}$. Also, $d_\tau(A) > \tau(Q) = d_\tau(Q)$ for all $\tau \in T_e(\mathscr{B})$. Hence, by [13] Theorem 6.4, $Q \preceq A$. Hence, there exists a projection $P \in \overline{A\mathscr{M}(\mathscr{B})A} - \mathscr{B} = \overline{AI_{min}A} - \mathscr{B}$ such that P is Murray–von Neumann equivalent to Q in $\mathscr{M}(\mathscr{B})$. Hence, $\widehat{P} = \widehat{A}$ on $T_e(\mathscr{B})$. $\square$

Recall that for any C*-algebra $\mathscr{C}$ and for any $x, y \in \mathscr{C}$, the commutator $[x,y]$ is defined by $[x,y] =_{\text{def}} xy - yx$.

THEOREM 2.5. *Let $\mathscr{D}$ be a σ-unital simple purely infinite C*-algebra. Then for every self-adjoint element $a \in \mathscr{D}$, there exist $x_j \in \mathscr{D}$ $(1 \leqslant j \leqslant 5)$ such that*

$$a = \sum_{j=1}^{5} [x_j, x_j^*]$$

and

$$\|x_j\| \leqslant 3\|a\|^{1/2}$$

for all $1 \leqslant j \leqslant 5$.

As a consequence, for all $x \in \mathscr{D}$, there exist $x_j, y_j \in \mathscr{D}$ $(1 \leqslant j \leqslant 10)$ such that

$$x = \sum_{j=1}^{10} [x_j, y_j]$$

and

$$\|x_j\|, \|y_j\| \leqslant 3\|x\|^{1/2}$$

for all $1 \leqslant j \leqslant 10$.

Proof. Recall, by [42], that $\mathscr{D}$ is either unital or stable.
The result follows from the arguments of [9] Theorems 1.1 and 2.1. □

Recall that for any C*-algebra $\mathscr{C}$, $[\mathscr{C},\mathscr{C}] \subseteq \mathscr{C}$ is the linear subspace spanned by the commutators of $\mathscr{C}$, i.e., $[\mathscr{C},\mathscr{C}] =_{\text{def}} Span\{[x,y] : x,y \in \mathscr{C}\}$. Recall also that

$$\overline{[\mathscr{C},\mathscr{C}]} = \bigcap \{ker(\tau) : \tau \in T(\mathscr{C})\}. \tag{2.1}$$

(See [6] Theorem 2.9; see also the proof of [36] Lemma 3.1.)

THEOREM 2.6. *There exists a universal constant C (that applies to all C*-algebras with the standing properties of $\mathscr{B}$) such that for all $x \in \mathscr{B}$, if $\tau(x) = 0$ for all $\tau \in T(\mathscr{B})$, then there exist $x_j, y_j \in \mathscr{B}$ $(1 \leqslant j \leqslant 7)$ such that*

$$x = \sum_{j=1}^{7} [x_j, y_j]$$

and

$$\|x_j\|, \|y_j\| \leqslant C\|x\|^{1/2}$$

for all $1 \leqslant j \leqslant 7$.

Proof. This follows from [31] Theorem 1.1. (See also (2.1).) □

Recall that an element x in a C*-algebra $\mathscr{C}$ has a *local unit* $a \in \mathscr{C}_+$ if $ax = xa = x$.
Recall that every element τ of $T(\mathscr{B})$ extends to an element of $T(\mathscr{M}(\mathscr{B}))$ (which we also denote by "τ"), and this is the unique (strict) lower semicontinuous extension.

LEMMA 2.7. *Let C be the universal constant from Theorem 2.6.*
Let $A \in \mathscr{I}_{min+} - \mathscr{B}$ be such that $\|A\| \leqslant 1$. Suppose that $P \in \overline{A\mathscr{M}(\mathscr{B})A} - \mathscr{B}$ is a projection for which $\widehat{P} = \widehat{A}$ on $T(\mathscr{B})$.
Then there exist $X_j, Y_j \in \overline{A\mathscr{M}(\mathscr{B})A}$ $(1 \leqslant j \leqslant 12)$ such that

$$A - P = \sum_{j=1}^{12} [X_j, Y_j]$$

and

$$\|X_j\|, \|Y_j\| \leqslant \max\{\sqrt{161}C, 4\}\|A - P\|^{1/2}$$

for all $1 \leqslant j \leqslant 12$.

Proof. Recall that $\pi : \mathscr{M}(\mathscr{B}) \to \mathscr{C}(\mathscr{B})$ is the quotient map. Then, by [13] Theorem 4.8 (see also [15]), $\pi(\mathscr{I}_{min})$ is a simple purely infinite C*-algebra.
So $\pi(A - P) \in \pi(\overline{A\mathscr{I}_{min}A})$, which is a σ-unital, simple purely infinite C*-algebra. So by Theorem 2.5, let $\widetilde{X}_j, \widetilde{Y}_j \in \pi(\overline{A\mathscr{I}_{min}A})$ $(1 \leqslant j \leqslant 5)$ be such that

$$\pi(A - P) = \sum_{j=1}^{5} [\widetilde{X}_j, \widetilde{Y}_j]$$

and

$$\|\widetilde{X}_j\|, \|\widetilde{Y}_j\| \leqslant 3\|A-P\|^{1/2}$$

for all $1 \leqslant j \leqslant 5$.

Let $X_j, Y_j \in \overline{A\mathscr{M}(\mathscr{B})A}$ be such that $\pi(X_j) = \widetilde{X}_j$ and $\pi(Y_j) = \widetilde{Y}_j$ and

$$\|X_j\|, \|Y_j\| \leqslant 4\|A-P\|^{1/2}$$

for all $1 \leqslant j \leqslant 5$.

Let $b \in \overline{A\mathscr{B}A}$ be such that $A - P = \sum_{j=1}^{5}[X_j, Y_j] + b$. Therefore, $\tau(b) = \tau(A - P) - \sum_{j=1}^{5}\tau([X_j, Y_j]) = 0$ for all $\tau \in T(\overline{A\mathscr{B}A})$.

Let us next find an upper bound for the norm of b. Firstly, for all $1 \leqslant j \leqslant 5$, $\|[X_j, Y_j]\| \leqslant 2\|X_j\|\|Y_j\| \leqslant 32\|A-P\|$. So

$$\begin{aligned} \|b\| &= \|A - P - \sum_{j=1}^{5}[X_j, Y_j]\| \\ &\leqslant \|A-P\| + 160\|A-P\| \\ &= 161\|A-P\|. \end{aligned}$$

So by Theorem 2.6, there exist $X_j, Y_j \in \overline{A\mathscr{B}A}$, for $6 \leqslant j \leqslant 12$, such that

$$b = \sum_{j=6}^{12}[X_j, Y_j]$$

and

$$\|X_j\|, \|Y_j\| \leqslant C\|b\|^{1/2} \leqslant C\sqrt{161}\|A-P\|^{1/2}$$

for all $6 \leqslant j \leqslant 12$. $\square$

LEMMA 2.8. *Let C be the universal constant from Theorem 2.6.*

Let $a \in \mathscr{B}_+$ be such that $\|a\| \leqslant 1$. Suppose that $P \in \mathscr{I}_{min+} - \mathscr{B}$ is a projection for which $\widehat{P} = \widehat{a}$ on $T(\mathscr{B})$.

Then there exist $X_j, Y_j \in \mathscr{I}_{min}$ $(1 \leqslant j \leqslant 12)$ such that

$$a - P = \sum_{j=1}^{12}[X_j, Y_j]$$

and

$$\|X_j\|, \|Y_j\| \leqslant \max\{\sqrt{161}C, 4\}\|a-P\|^{1/2}$$

for all $1 \leqslant j \leqslant 12$.

Proof. The proof is essentially the same as that of Lemma 2.7, except that we replace $\pi(\overline{A\mathscr{I}_{min}A})$ with $\pi(P\mathscr{I}_{min}P)$. $\square$

We will use a definition of bidiagonal decomposition that is slightly stronger than those in [12] and [13]. However, the existence of such a stronger decomposition is actually proven in these papers.

DEFINITION 2.9. Let $\mathscr{D}$ be a nonunital C*-algebra. An element $A \in \mathscr{M}(\mathscr{D})_+$ is said to *be bidiagonal* or *have a bidiagonal decomposition* if there exist a sequential approximate unit $\{e_n\}$ for $\mathscr{D}$, a bounded sequence $\{a_l\}$ in $\mathscr{D}_+$ and positive integers $n_{k+1} < m_k < n_{k+2} < m_{k+1}$ for all k such that the following statements are true:

1. $e_{n+1}e_n = e_n$ for all n,
2. $a_k \in \overline{(e_{m_k} - e_{n_k})\mathscr{D}(e_{m_k} - e_{n_k})}$ for all $k \geqslant 1$, and
3. $A = \sum_{n=1}^{\infty} a_n$, where the sum converges strictly in $\mathscr{M}(\mathscr{D})$.

The Jiang–Su algebra $\mathscr{Z}$ [10] is the unique simple unital nonelementary inductive limit of dimension drop algebras with K theory invariant being the same as that for the complex numbers $\mathbb{C}$, i.e.,

$$(K_0(\mathscr{Z}), K_0(\mathscr{Z})_+, K_1(\mathscr{Z}), T(\mathscr{Z})) = (\mathbb{Z}, \mathbb{Z}_+, 0, \{pt\}) = (K_0(\mathbb{C}), K_0(\mathbb{C})_+, K_1(\mathbb{C}), T(\mathbb{C})).$$

We let $\tau_{\mathscr{Z}}$ denote the unique tracial state of $\mathscr{Z}$. A C*-algebra $\mathscr{D}$ is said to be *$\mathscr{Z}$-stable* (or *Jiang–Su-stable*) if $\mathscr{D} \cong \mathscr{D} \otimes \mathscr{Z}$. $\mathscr{Z}$-stability is a regularity property which is an axiom in the classification program for simple amenable C*-algebras.

LEMMA 2.10. *Let $\mathscr{D}$ be a $\mathscr{Z}$-stable C*-algebra and let $\iota : \mathscr{D} \to \mathscr{D} \otimes \mathscr{Z}$ be the *-embedding given by*

$$\iota : d \mapsto d \otimes 1_{\mathscr{Z}}.$$

*Then there exists a *-isomorphism $\Phi : \mathscr{D} \to \mathscr{D} \otimes \mathscr{Z}$ such that ι and Φ are approximately unitarily equivalent, i.e., there exists a sequence $\{u_n\}$ of unitaries in $\mathscr{M}(\mathscr{D}) \otimes \mathscr{Z} \subset \mathscr{M}(\mathscr{D} \otimes \mathscr{Z})$ for which*

$$u_n \iota(a) u_n^* \to \Phi(a)$$

for all $a \in \mathscr{D}$.

Proof. This follows from [10] Theorems 7.6 and 8.7. □

LEMMA 2.11. *Let $A \in \mathscr{M}(\mathscr{B})_+ - \mathscr{B}$ have a bidiagonal decomposition. Then for every $\varepsilon > 0$, there exist 4 bounded sequences $\{A_{k,n}\}_{n=1}^{\infty}$ in $\mathscr{I}_{min+} - \mathscr{B}$ $(1 \leqslant k \leqslant 4)$ such that the following statements are true:*

1. *For all $1 \leqslant k \leqslant 4$, for all $n \geqslant 1$, $\|A_{k,n}\| \leqslant \|A\|$.*
2. *For all $1 \leqslant k \leqslant 4$, for all $n \geqslant 1$, $\widehat{A_{k,n}}$ is continuous on $T_e(\mathscr{B})$.*
3. *For all $1 \leqslant k \leqslant 4$, for all $n \neq n'$, $A_{k,n} \perp A_{k,n'}$.*
4. *For all $1 \leqslant k \leqslant 4$, if $\{X_n\}$ is a bounded sequence in $\mathscr{M}(\mathscr{B})$ such that $X_n \in \overline{A_{k,n}\mathscr{M}(\mathscr{B})A_{k,n}} = \overline{A_{k,n}\mathscr{I}_{min}A_{k,n}}$ for all n, then $\sum_{n=1}^{\infty} X_n$ converges strictly in $\mathscr{M}(\mathscr{B})$.*

5. $A-\sum_{k=1}^{4}\sum_{n=1}^{\infty}A_{k,n}\in\mathscr{B}$.

6. $\|A-\sum_{k=1}^{4}\sum_{n=1}^{\infty}A_{k,n}\|<\varepsilon$.

Proof. Let $\{\varepsilon_l\}_{l=1}^{\infty}$ be a sequence of strictly positive real numbers such that $\sum_{l=1}^{\infty}\varepsilon_l < \frac{\varepsilon}{2}$.

We may assume that $\|A\|\leqslant 1$. Since A is bidiagonal, we can choose a bounded sequence $\{a_l\}$ in $\mathscr{B}_+$, as in Definition 2.9, such that the statements in Definition 2.9 are true. In particular, $A=\sum_{l=1}^{\infty}a_l$, where the sum converges strictly in $\mathscr{M}(\mathscr{B})$. Note also that the statements in Definition 2.9 imply that whenever $\{x_l\}$ is a bounded sequence in $\mathscr{B}$ such that $x_l\in\overline{a_l\mathscr{B}a_l}$ for all l, then $\sum_{l=1}^{\infty}x_l$ converges strictly in $\mathscr{M}(\mathscr{B})$. Replacing appropriate terms a_l with appropriate finite sums $\sum_{j=l_1}^{l_2}a_j$ if necessary, we may also assume that $\liminf_{l\to\infty}\|a_{2l}\|>0$ and $\liminf_{l\to\infty}\|a_{2l+1}\|>0$. And hence, $\sum_{l=1}^{\infty}a_{2l}$ and $\sum_{l=0}^{\infty}a_{2l+1}$ are both outside of $\mathscr{B}$. Also, for all l, a_l has a local unit (which in turn has a local unit) in $\mathscr{B}$. Hence, for all l,

$$\sup_{\tau\in T_e(\mathscr{B})} d_\tau(a_l)<\infty.$$

Hence, for all l, let $n_l\geqslant 1$ be an integer such that

$$d_\tau(a_l)<n_l \text{ for all } \tau\in T_e(\mathscr{B}).$$

For each l, let $\{e_{l,j}\}_{j=1}^{2^{l+1}n_l}$ be finitely many norm one, positive elements in $\mathscr{Z}$ such that the following statements are true:

1. For all j, $e_{l,j}$ has a local unit in $\mathscr{Z}$, which itself has a local unit in $\mathscr{Z}$; i.e., there exist contractive positive elements $e'_{l,j}, e''_{l,j}\in\mathscr{Z}$ for which $e'_{l,j}e_{l,j}=e_{l,j}$ and $e''_{l,j}e'_{l,j}=e'_{l,j}$.

2. $1_{\mathscr{Z}}=\sum_{j=1}^{2^{l+1}n_l}e_{l,j}$.

3. $e''_{l,j}\perp e''_{l,j'}$ for $|j-j'|\geqslant 2$.

4. $d_{\tau_{\mathscr{Z}}}(e''_{l,j})<\frac{1}{2^l n_l}$, for all j. (Recall that $\tau_{\mathscr{Z}}$ is the unique tracial state of $\mathscr{Z}$.)

To simplify the notation in our proof, for all $j>2^{l+1}n_l$, we let

$$e_{l,j}=_{\text{def}}0.$$

For all l, let $\mathscr{B}_l=_{\text{def}}\overline{a_l\mathscr{B}a_l}$. Recalling that every hereditary C*-subalgebra of a separable, $\mathscr{Z}$-stable C*-algebra is $\mathscr{Z}$-stable ([37]), for each l, we may work with $\mathscr{B}_l\otimes\mathscr{Z}$ instead of $\mathscr{B}_l$ and identify a_l with $a'_l\in(\mathscr{B}_l\otimes\mathscr{Z})_+$. (So $\mathscr{B}_l\otimes\mathscr{Z}=\overline{a'_l\mathscr{B}a'_l}$.) By Lemma 2.10, for every l, there exists an element $d_l\in(\mathscr{B}_l)_+$, with $\|d_l\|=\|a'_l\|=\|a_l\|\leqslant 1$, and a unitary $u_l\in\mathscr{M}(\mathscr{B}_l)\otimes\mathscr{Z}$ such that $\|u_l(d_l\otimes 1_{\mathscr{Z}})u_l^*-a'_l\|<\varepsilon_l$. In fact, we require that $d_l\otimes 1_{\mathscr{Z}}$ and a'_l be approximately unitarily equivalent in $\mathscr{M}(\mathscr{B}_l)\otimes\mathscr{Z}$.

Let $\{A'_j\}_{j=1}^{\infty}$ and $\{A''_j\}_{j=1}^{\infty}$ be the two contractive sequences in $\mathscr{M}(\mathscr{B})_+$ given by

$$A'_j =_{\text{def}} \sum_{l=1}^{\infty} u_{2l}(d_{2l} \otimes e_{2l,j})u_{2l}^* \text{ and } A''_j =_{\text{def}} \sum_{l=0}^{\infty} u_{2l+1}(d_{2l+1} \otimes e_{2l+1,j})u_{2l+1}^*,$$

for all j, where the sums converge strictly in $\mathscr{M}(\mathscr{B})$.

Then it follows that $A'_j, A''_j \in \mathscr{I}_{min+} - \mathscr{B}$, $\widehat{A'_j}$ and $\widehat{A''_j}$ are both continuous on $T_e(\mathscr{B})$, and $A'_j \perp A'_{j_1}$ and $A''_j \perp A''_{j_1}$ for all $|j - j_1| \geqslant 2$. Moreover, if $\{X'_j\}_{j=1}^{\infty}$ and $\{X''_j\}_{j=1}^{\infty}$ are bounded sequences such that $X'_j \in \overline{A'_j \mathscr{M}(\mathscr{B}) A'_j}$ and $X''_j \in \overline{A''_j \mathscr{M}(\mathscr{B}) A''_j}$ for all j, then both $\sum_j X'_j$ and $\sum_j X''_j$ converge strictly in $\mathscr{M}(\mathscr{B})$.

Finally, $A - \sum_j A'_j - \sum_j A''_j \in \mathscr{B}$ and $\|A - \sum_j A'_j - \sum_j A''_j\| < \varepsilon$.

Define $A_{1,j} =_{\text{def}} A'_{2j}$, $A_{2,j} =_{\text{def}} A'_{2j+1}$, $A_{3,j} =_{\text{def}} A''_{2j}$ and $A_{4,j} =_{\text{def}} A''_{2j+1}$, for all j. $\square$

THEOREM 2.12. *$\mathscr{M}(\mathscr{B})$ is the (norm) closed linear span of its projections.*

Proof. It suffices to prove that every positive element in $\mathscr{M}(\mathscr{B})$ is the norm limit of finite linear combinations of projections in $\mathscr{M}(\mathscr{B})$.

Suppose that $A \in \mathscr{M}(\mathscr{B})_+$. We may assume that $\|A\| = 1$.

Case 1: Suppose that $A \notin \mathscr{B}$. By [12] Theorem 4.2 and its proof, A is the norm limit of bidiagonal operators as in Definition 2.9. Hence, we may assume that A is bidiagonal.

Let $\varepsilon > 0$ be arbitrary. Plug the given operator A and the given ε into Lemma 2.11 to get four bounded sequences $\{A_{l,n}\}_{n=1}^{\infty}$ $(1 \leqslant l \leqslant 4)$ in $\mathscr{I}_{min+} - \mathscr{B}$.

Let $1 \leqslant l \leqslant 4$ and $n \geqslant 1$ be arbitrary. If $A_{l,n}$ is a projection, then we let $P_{l,n} =_{\text{def}} A_{l,n}$.

Suppose that $A_{l,n}$ is not a projection. $(1_{\mathscr{M}(\mathscr{B})} - A_{l,n})\widehat{}$ complements $\widehat{A_{l,n}}$ under $\widehat{1_{\mathscr{M}(\mathscr{B})}}$. By Lemma 2.2, let $Q_{l,n} \in \mathscr{M}(\mathscr{B}) - \mathscr{B}$ be a projection such that $\widehat{Q_{l,n}} = \widehat{A_{l,n}}$ on $T_e(\mathscr{B})$. We have that $\widehat{A_{l,n}}$ is a continuous function on $T_e(\mathscr{B})$. Hence $\widehat{Q_{l,n}}$ is a continuous function on $T_e(\mathscr{B})$. Hence, by [13] Theorem 5.6, $Q_{l,n} \in \mathscr{I}_{min} - \mathscr{B}$. Note that, by the definition of $A_{l,n}$ in Lemma 2.11, $\|A_{l,n}\| \leqslant 1$. Hence, since $A_{l,n}$ is not a projection, $d_\tau(A_{l,n}) > \tau(A_{l,n}) = \tau(Q_{l,n})$ for all $\tau \in T_e(\mathscr{B})$. Hence, by [13] Theorem 6.4, let $P_{l,n} \in \overline{A_{l,n}\mathscr{M}(\mathscr{B})A_{l,n}} - \mathscr{B}$ be a projection such that $P_{l,n} \sim Q_{l,n}$.

Hence, for all $1 \leqslant l \leqslant 4$ and all $n \geqslant 1$, we have a projection $P_{l,n} \in \overline{A_{l,n}\mathscr{M}(\mathscr{B})A_{l,n}} - \mathscr{B} = \overline{A_{l,n}\mathscr{I}_{min}A_{l,n}} - \mathscr{B}$ such that $\widehat{P_{l,n}} = \widehat{A_{l,n}}$ on $T_e(\mathscr{B})$, and hence, $\widehat{P_{l,n}} = \widehat{A_{l,n}}$ on $T(\mathscr{B})$. Hence, by Lemma 2.7, for all l, n, let $X_{l,n,j}, Y_{l,n,j} \in \overline{A_{l,n}\mathscr{M}(\mathscr{B})A_{l,n}}$ $(1 \leqslant j \leqslant 12)$ be such that

$$A_{l,n} - P_{l,n} = \sum_{j=1}^{12} [X_{l,n,j}, Y_{l,n,j}]$$

and

$$\|X_{l,n,j}\|, \|Y_{l,n,j}\| \leqslant \max\{\sqrt{161}C, 4\}\|A_{l,n} - P_{l,n}\|^{1/2} \leqslant \sqrt{2}\max\{\sqrt{161}C, 4\}.$$

Here, C is the universal constant from Lemma 2.7.

By the definition of the $A_{l,n}$ s (in Lemma 2.11), for all l, for all $n \neq n'$, $A_{l,n} \perp A_{l,n'}$, and also for all l, j, the sums $X_{l,j} =_{\text{def}} \sum_{n=1}^{\infty} X_{l,n,j}$ and $Y_{l,j} =_{\text{def}} \sum_{n=1}^{\infty} Y_{l,n,j}$ converge strictly in $\mathscr{M}(\mathscr{B})$. Also, for all $1 \leqslant l \leqslant 4$, let $P_l \in \mathscr{M}(\mathscr{B})$ be the projection given by $P_l =_{\text{def}} \sum_{n=1}^{\infty} P_{l,n}$, where the sum converges strictly in $\mathscr{M}(\mathscr{B})$.

Hence,

$$
\begin{aligned}
&\sum_{l=1}^{4}\sum_{j=1}^{12}[X_{l,j}, Y_{l,j}] \\
&= \sum_{l=1}^{4}\sum_{j=1}^{12}\left[\sum_{n=1}^{\infty} X_{l,n,j}, \sum_{m=1}^{\infty} Y_{l,m,j}\right] \\
&= \sum_{l=1}^{4}\sum_{j=1}^{12}\sum_{n=1}^{\infty}[X_{l,n,j}, Y_{l,n,j}] \text{ (since } A_{l,m} \perp A_{l,n} \text{ for } m \neq n \text{)} \\
&= \sum_{l=1}^{4}\sum_{n=1}^{\infty}\sum_{j=1}^{12}[X_{l,n,j}, Y_{l,n,j}] \\
&= \sum_{l=1}^{4}\sum_{n=1}^{\infty}(A_{l,n} - P_{l,n}) \\
&= \sum_{l=1}^{4}\sum_{n=1}^{\infty} A_{l,n} - \sum_{l=1}^{4} P_l \\
&\approx_{\varepsilon} A - \sum_{l=1}^{4} P_l.
\end{aligned}
$$

By Lemma 2.2, we can find pairwise orthogonal projections $R_1, R_2, R_3 \in \mathscr{M}(\mathscr{B})$ with $R_j \sim R_k$ for all j, k and $R_1 + R_2 + R_3 = 1_{\mathscr{M}(\mathscr{B})}$. Hence, by [26] Theorem 3.8, $\sum_{l=1}^{4}\sum_{j=1}^{12}[X_{l,j}, Y_{l,j}]$ is a finite linear span of projections. Hence, A is norm within ε of a finite linear span of projections. Since ε was arbitrary, A is in the (norm) closed linear span of projections in $\mathscr{M}(\mathscr{B})$.

Case 2: Suppose that $A \in \mathscr{B}$. Since A is the norm limit $A = \lim_{\delta \to 0+}(A - \delta)_+$, we may assume that A has a local unit in $\mathscr{B}$. Now $\widehat{(1_{\mathscr{M}(\mathscr{B})} - A)}$ complements $\widehat{A}$ under $\widehat{1_{\mathscr{M}(\mathscr{B})}}$. By Lemma 2.2, we can find a projection $P \in \mathscr{M}(\mathscr{B}) - \mathscr{B}$ for which $\widehat{A} = \widehat{P}$ on $T_e(\mathscr{B})$. Since $A \in \mathscr{B}_+$ has a local unit, $\widehat{A}$ is continuous on $T_e(\mathscr{B})$. Thus, $\widehat{P}$ is continuous on $T_e(\mathscr{B})$. Thus, by [13] Theorem 5.6, $P \in \mathscr{I}_{min} - \mathscr{B}$. Since $\widehat{A} = \widehat{P}$ on $T_e(\mathscr{B})$, $\widehat{A} = \widehat{P}$ on $T(\mathscr{B})$. By Lemma 2.8 and by [26] Theorem 3.8 (using the same decomposition of the unit $R_1 + R_2 + R_3 = 1_{\mathscr{M}(\mathscr{B})}$ as in Case 1), we have that $A - P$ is a finite linear combination of projections. Hence, A is a finite linear combination of projections. $\square$

REMARK 2.13. In Case 1 of the proof Theorem 2.12, by [26] Theorem 3.8, $\sum_{l=1}^{4}\sum_{j=1}^{12}[X_{l,j}, Y_{l,j}]$ can be expressed as a linear combination of $(4)(12)(84) = 4032$

projections. Hence, A can be approximated arbitrarily close, in norm, by a linear combination of 4036 projections.

In Case 2 of the proof of Theorem 2.12, if A has a local unit in $\mathscr{B}$, then by Lemma 2.8, $A-P$ can be expressed as the sum of 12 commutators. Hence, by [26] Theorem 3.8, $A-P$ can be expressed as a linear combination of $(12)(84) = 1008$ projections. Hence, A can be expressed as a linear combination of 1009 projections.

Recall that every element, of a C*-algebra, is a linear combination of 4 positive elements. Hence, in summary, if $X \in \mathscr{M}(\mathscr{B}) - \mathscr{B}$, then X can be approximated arbitrarily (norm) close by a linear combination of $(4036)(4) = 16144$ projections. And if $X \in \mathscr{B}$, then X can be approximated arbitrarily (norm) close by a linear combination of $(1009)(4) = 4036$ projections. We do not believe that these numbers are optimal.

THEOREM 2.14. *$\mathscr{B}$ has Property I. I.e., the usual map*

$$K_0(\mathscr{M}(\mathscr{B})) \to Aff(T(\mathscr{M}(\mathscr{B}))) : [p]-[q] \mapsto \widehat{p}-\widehat{q}$$

has image which separates the points of $T(\mathscr{M}(\mathscr{B}))$.

Proof. This follows immediately from Theorem 2.12.

Note that by convention, if $T(\mathscr{M}(\mathscr{B})) = \emptyset$, then $\mathscr{B}$ trivially has Property I. In [33], see Remarks 2.6, 2.14 and the comment after Definition 2.12. $\square$

REMARK 2.15. In the this section, the only places, where the stable rank one standing assumption on $\mathscr{B}$ is used, are in Lemma 2.2 (specifically for proving (stable) projection injectivity in Lemma 2.2) and all places in this section which appeals to this lemma.

3. Properly infinite corona algebras

The next lemma should be known, but we exhibit the short computation for the convenience of the reader.

LEMMA 3.1. *Let $\mathscr{D}$ be a C*-algebra and suppose that $a,b,c \in \mathscr{D}$ are contractive positive elements, $x \in \mathscr{D}$ and $0 < \delta < 1$ such that $ab = b$ and $\|xbx^* - c\| < \delta$.*

Then there exists a $y \in \mathscr{D}$ with $\|y\| \leqslant 2$ such that $\|yay^* - c\| < \delta$.

Proof. Let $y =_{\text{def}} xb^{1/2}$. Then $\|y\|^2 = \|xbx^*\| \leqslant \|c\| + \delta \leqslant 1+\delta < 2$.
Moreover, $yay^* = xb^{1/2}ab^{1/2}x^* = xbx^* \approx_\delta c$, as required. $\square$

Let $\mathscr{C}$ be a C*-algebra and $x \in \mathscr{C}$. Recall that x is *full* in $\mathscr{C}$ means that the (C*-) ideal $Ideal(x) =_{\text{def}} \overline{\mathscr{C}x\mathscr{C} + \mathscr{C}x^*\mathscr{C}} = \mathscr{C}$.

Recall that a simple C*-algebra $\mathscr{D}$ has *strict comparison* or *strict comparison for positive elements* if for all $a,b \in (\mathscr{D}\otimes\mathscr{K})_+$, $d_\tau(a) < d_\tau(b)$ or $d_\tau(b) = \infty$ for all $\tau \in T_e(\mathscr{D})$ implies that $a \preceq b$ in $\mathscr{D}\otimes\mathscr{K}$.

LEMMA 3.2. *Let $\mathscr{D}$ be a nonunital, separable, simple, stably finite C*-algebra with strict comparison for positive elements.*

Suppose that $A \in \mathscr{M}(\mathscr{D})_+$ is a full element and $d \in \mathscr{D}_+$ satisfies that $d_\tau(d) < d_\tau(A)$ for all $\tau \in T(\mathscr{D})$.

Then $d \preceq A$ in $\mathscr{M}(\mathscr{D})$.

Proof. Let $a \in \overline{A\mathscr{D}A}_+$ be a strictly positive element. Then $d_\tau(a) = d_\tau(A)$ for all $\tau \in T_e(\mathscr{D})$. Thus, since A is full in $\mathscr{M}(\mathscr{D})_+$, for all $\tau \in T_e(\mathscr{D})$, *either* $\tau|_{\mathscr{D}}$ induces a nonzero bounded trace on $\mathscr{D}$ and $d_\tau(a) = d_\tau(A) > d_\tau(d)$ *or* $d_\tau(a) = d_\tau(A) = \infty$. Hence, since $\mathscr{D}$ has strict comparison for positive elements, $d \preceq a$ in $\mathscr{D} \otimes \mathscr{K}$ (and thus in $\mathscr{D}$). Since $a \preceq A$ in $\mathscr{M}(\mathscr{D})$, we have that $d \preceq A$ in $\mathscr{M}(\mathscr{D})$. □

LEMMA 3.3. *Let $\mathscr{D}$ be a nonunital, separable, simple C*-algebra. Let $\{e_n\}$ be an approximate unit for $\mathscr{D}$ such that $e_n \neq e_{n+1}$ and $e_{n+1}e_n = e_n$ for all $n \geqslant 1$.*

Then for all $N \geqslant 1$, $1_{\mathscr{M}(\mathscr{D})} - e_N$ is a full element of $\mathscr{M}(\mathscr{D})$.

Proof. Note that $1_{\mathscr{M}(\mathscr{D})} - e_{N+1} = \sum_{n=N+1}^{\infty}(e_{n+1} - e_n)$ where the sum converges strictly in $\mathscr{M}(\mathscr{D})$. Since $\mathscr{D}$ is a simple C*-algebra, we can find $L \geqslant 1$ and $x_1, \ldots, x_L \in \mathscr{D}$ such that

$$\sum_{l=1}^{L} x_l(e_{N+2} - e_{N+1})x_l^* \approx_{\frac{1}{10}} e_{N+1}.$$

Hence,

$$e_N \preceq \left(e_{N+1} - \frac{1}{10}\right)_+ \preceq \bigoplus^{L}(e_{N+2} - e_{N+1}).$$

Hence,

$$1_{\mathscr{M}(\mathscr{D})} \preceq \bigoplus^{L+1}(1_{\mathscr{M}(\mathscr{D})} - e_N).$$

Hence, $1_{\mathscr{M}(\mathscr{D})} - e_N$ is full in $\mathscr{M}(\mathscr{D})$. □

LEMMA 3.4. *Let $\mathscr{D}$ be a nonunital, separable, simple, stably finite C*-algebra with strict comparison for positive elements and $T(\mathscr{D})$ being weak* compact. Let $\{e_n\}$ be an approximate unit for $\mathscr{D}$ for which $e_{n+1}e_n = e_n$ and $e_{n+1} \neq e_n$ for all n.*

Then there exists an $N \geqslant 1$ such that

$$(1_{\mathscr{M}(\mathscr{D})} - e_N) \oplus (1_{\mathscr{M}(\mathscr{D})} - e_N) \preceq 1_{\mathscr{M}(\mathscr{D})}.$$

in $\mathbb{M}_2 \otimes \mathscr{M}(\mathscr{D})$.

Proof. Note that $\mathbb{M}_2 \otimes \mathscr{D}$ is also nonunital, separable, simple, stably finite and has strict comparison; so we may apply Lemma 3.2 to $\mathbb{M}_2 \otimes \mathscr{D}$ and $\mathscr{M}(\mathbb{M}_2 \otimes \mathscr{D}) \cong \mathbb{M}_2 \otimes \mathscr{M}(\mathscr{D})$.

Note that $1_{\mathscr{M}(\mathscr{D})} = \sum_{n=1}^{\infty}(e_n - e_{n-1})$, where $e_0 =_{\mathrm{def}} 0$ and where the series converges strictly in $\mathscr{M}(\mathscr{D})$. Moreover, for all $n \geqslant 1$, $\widehat{e_n}$ is a continuous function on

$T(\mathscr{D})$ and $1 = \widehat{1_{\mathscr{M}(\mathscr{D})}} = \sum_{n=1}^{\infty} \widehat{(e_n - e_{n-1})}$ on $T(\mathscr{D})$, where by Dini's theorem, the series converges uniformly on the compact set $T(\mathscr{D})$. In particular, $(1_{\mathscr{M}(\mathscr{D})} - e_{m-1})\widehat{\ } = \sum_{n=m}^{\infty} \widehat{(e_n - e_{n-1})} \to 0$ as $m \to \infty$ uniformly on $T(\mathscr{D})$. Thus, we can choose an $N \geqslant 1$ so that for all $\tau \in T(\mathscr{D})$,

$$1 > 2 \sum_{n=N}^{\infty} \tau(e_n - e_{n-1}) = 2\tau(1_{\mathscr{M}(\mathscr{D})} - e_{N-1}).$$

Let $\varepsilon > 0$ be given. Choose a sequence $\{\varepsilon_k\}$ in $(0,1)$ such that $\sum_{k=1}^{\infty} \varepsilon_k < \varepsilon$.

We now construct, by induction, three subsequences $\{L_k\}_{k=1}^{\infty}$, $\{M_k\}_{k=1}^{\infty}$ and $\{M'_k\}_{k=1}^{\infty}$ of the positive integers, a sequence $\{\alpha_k\}_{k=1}^{\infty}$ in $(0,\infty)$, and sequence $\{x_k\}_{k=1}^{\infty}$ of elements in $\mathbb{M}_2 \otimes \mathscr{D}$ with norm at most 2.

Basis step $k = 1$. Since, by [33] Lemma 2.10, $(1_{\mathscr{M}(\mathscr{D})} - e_{N-1})\widehat{\ }$ is continuous on the compact set $T(\mathscr{D})$, we can find $\alpha_1 > 0$ so that for all $\tau \in T(\mathscr{D})$,

$$1 > \alpha_1 + 2 \sum_{n=N}^{\infty} \tau(e_n - e_{n-1}) = \alpha_1 + 2\tau(1_{\mathscr{M}(\mathscr{D})} - e_{N-1}).$$

Since $(1_{\mathscr{M}(\mathscr{D})} - e_{m-1})\widehat{\ } \to 0$ uniformly on $T(\mathscr{D})$, choose $L_1 \geqslant 6$ so that

$$\tau(e_{L_1+5} - e_{L_1-5}) < \alpha_1$$

for all $\tau \in T(\mathscr{D})$. Hence,

$$\tau(1_{\mathscr{M}(\mathscr{D})} - e_{L_1+5} + e_{L_1-5}) > 2 \sum_{n=N}^{\infty} \tau(e_n - e_{n-1}) = 2\tau(1_{\mathscr{M}(\mathscr{D})} - e_{N-1})$$

for all $\tau \in T(\mathscr{D})$.

Choose $M_1 > N + 5$ so that

$$\tau(e_{L_1} - e_{L_1-1}) > 2 \sum_{n=M_1-1}^{\infty} \tau(e_n - e_{n-1}) = 2\tau(1_{\mathscr{M}(\mathscr{D})} - e_{M_1-2})$$

for all $\tau \in T(\mathscr{D})$. So

$$d_\tau(1_{\mathscr{M}(\mathscr{D})} - e_{L_1+5} + e_{L_1-5}) \geqslant \tau(1_{\mathscr{M}(\mathscr{D})} - e_{L_1+5} + e_{L_1-5}) > 2 d_\tau\left(\sum_{n=N+1}^{M_1} e_n - e_{n-1}\right)$$

for all $\tau \in T(\mathscr{D})$.

By Lemma 3.3, $(1_{\mathscr{M}(\mathscr{D})} - e_{L_1+5} + e_{L_1-5}) \oplus 0$ is full in $\mathbb{M}_2 \otimes \mathscr{M}(\mathscr{D})$. Hence, by Lemma 3.2,

$$1_{\mathscr{M}(\mathscr{D})} - e_{L_1+5} + e_{L_1-5} \succeq (e_{M_1} - e_N) \oplus (e_{M_1} - e_N)$$

in $\mathbb{M}_2 \otimes \mathscr{M}(\mathscr{D})$. Therefore, by Lemma 3.1, choose $M'_1 \geqslant L_1 + 10$ and an element $x_1 \in \overline{((e_{M_1} - e_N) \oplus (e_{M_1} - e_N))(\mathbb{M}_2 \otimes \mathscr{D})((e_{M'_1} - e_{L_1+4} + e_{L_1-4}) \oplus 0)}$ with $\|x_1\| \leqslant 2$ such that

$$x_1 x_1^* \approx_{\varepsilon_1} (e_{M_1} - e_N) \oplus (e_{M_1} - e_N).$$

Induction step: Suppose that we have chosen L_k, M_k, M'_k, α_k and x_k. We now choose L_{k+1}, M_{k+1}, M'_{k+1}, α_{k+1} and x_{k+1}.

By the induction hypothesis, for all $\tau \in T(\mathscr{D})$,

$$\tau((e_{L_k} - e_{L_k-1}) + (1_{\mathscr{M}(\mathscr{D})} - e_{M'_k+5})) > 2\tau(1_{\mathscr{M}(D)} - e_{M_k-2}).$$

Hence, since $((e_{L_k} - e_{L_k-1}) + (1_{\mathscr{M}(\mathscr{D})} - e_{M'_k+5}))\widehat{\ }$ and $(1_{\mathscr{M}(D)} - e_{M_k-2})\widehat{\ }$ are continuous on the compact set $T(\mathscr{D})$, choose $\alpha_{k+1} > 0$ so that for all $\tau \in T(\mathscr{D})$,

$$\tau((e_{L_k} - e_{L_k-1}) + (1_{\mathscr{M}(\mathscr{D})} - e_{M'_k+5})) > \alpha_{k+1} + 2\tau(1_{\mathscr{M}(D)} - e_{M_k-2}).$$

Since $(1_{\mathscr{M}(\mathscr{D})} - e_{m-1})\widehat{\ } \to 0$ as $m \to \infty$ uniformly on $T(\mathscr{D})$, choose $L_{k+1} \geqslant M'_k + 20$ such that for all $\tau \in T(\mathscr{D})$,

$$\tau(e_{L_{k+1}+5} - e_{L_{k+1}-5}) < \alpha_{k+1}.$$

Hence, for all $\tau \in T(\mathscr{D})$,

$$\tau((e_{L_k} - e_{L_k-1}) + (1_{\mathscr{M}(\mathscr{D})} - e_{L_{k+1}+5} + e_{L_{k+1}-5} - e_{M'_k+5})) > 2\tau(1_{\mathscr{M}(\mathscr{D})} - e_{M_k-2}).$$

So for all $\tau \in T(\mathscr{D})$,

$$d_\tau((e_{L_k} - e_{L_k-1}) + (1_{\mathscr{M}(\mathscr{D})} - e_{L_{k+1}+5} + e_{L_{k+1}-5} - e_{M'_k+5})) > 2d_\tau(1_{\mathscr{M}(\mathscr{D})} - e_{M_k-1}).$$

Since $(1_{\mathscr{M}(\mathscr{D})} - e_{m-1})\widehat{\ } \to 0$ uniformly on $T(\mathscr{D})$, choose $M_{k+1} > M_k + 10$ so that for all $\tau \in T(\mathscr{D})$,

$$\tau(e_{L_{k+1}} - e_{L_{k+1}-1}) > 2\tau(1_{\mathscr{M}(\mathscr{D})} - e_{M_{k+1}-2}).$$

By Lemma 3.3, $(1_{\mathscr{M}(\mathscr{D})} - e_{L_{k+1}+5} + e_{L_{k+1}-5} - e_{M'_k+5}) \oplus 0$ is full in $\mathbb{M}_2 \otimes \mathscr{M}(\mathscr{D})$.

Hence, by Lemma 3.2,

$$(e_{L_k} - e_{L_k-1}) + (1_{\mathscr{M}(\mathscr{D})} - e_{L_{k+1}+5} + e_{L_{k+1}-5} - e_{M'_k+5}) \succeq (e_{M_{k+1}} - e_{M_k}) \oplus (e_{M_{k+1}} - e_{M_k})$$

in $\mathbb{M}_2 \otimes \mathscr{M}(\mathscr{D})$. Hence, by Lemma 3.1, choose $M'_{k+1} > L_{k+1} + 10$ and $x_{k+1} \in \mathbb{M}_2 \otimes \mathscr{D}$ with

$$x_{k+1}x^*_{k+1} \in Her_{\mathbb{M}_2 \otimes \mathscr{D}}((e_{M_{k+1}} - e_{M_k}) \oplus (e_{M_{k+1}} - e_{M_k}))$$

and

$$x^*_{k+1}x_{k+1} \in Her_{\mathbb{M}_2 \otimes \mathscr{D}}((((e_{L_k+1} - e_{L_k-2}) + (e_{M'_{k+1}} - e_{L_{k+1}+4} + e_{L_{k+1}-4} - e_{M'_k+4})) \oplus 0)$$

and with $\|x_{k+1}\| \leqslant 2$ such that

$$x_{k+1}x^*_{k+1} \approx_{\varepsilon_{k+1}} (e_{M_{k+1}} - e_{M_k}) \oplus (e_{M_{k+1}} - e_{M_k}).$$

This completes the inductive construction.

Let $X =_{\text{def}} \sum_{k=1}^{\infty} x_k$ where the sum converges strictly in $\mathbb{M}_2 \otimes \mathscr{M}(\mathscr{D})$.

Then
$$X1_{\mathscr{M}(\mathscr{D})}X^* \approx_\varepsilon (1_{\mathscr{M}(\mathscr{D})} - e_N) \oplus (1_{\mathscr{M}(\mathscr{D})} - e_N).$$
Since $\varepsilon > 0$ was arbitrary, $(1_{\mathscr{M}(\mathscr{D})} - e_N) \oplus (1_{\mathscr{M}(\mathscr{D})} - e_N) \preceq 1_{\mathscr{M}(\mathscr{D})}$. $\square$

We recall some more terminology from the theory of simple C*-algebras, some of which have already been reviewed in the introduction. Let $\mathscr{D}$ be a separable C*-algebra. For all $n \geqslant 1$, we have a *-embedding
$$\mathbb{M}_n \otimes \mathscr{D} \hookrightarrow \mathbb{M}_{n+1} \otimes \mathscr{D}$$
given by
$$b \mapsto diag(b, 0).$$
We let $\mathbb{M}_\infty(\mathscr{D})$ denote the *-algebra
$$\mathbb{M}_\infty(\mathscr{D}) =_{\text{def}} \bigcup_{n=1}^{\infty} \mathbb{M}_n \otimes \mathscr{D}.$$

Recall the subequivalence relation $\preceq$ on positive elements, which generalizes Murray-von Neumann subequivalence of projections, that is given as follows: For all $a, b \in \mathbb{M}_\infty(\mathscr{D})_+$,
$$a \preceq b$$
means that there exists an $N \geqslant 1$ with $a, b \in \mathbb{M}_N \otimes \mathscr{D}$ and a sequence $\{x_k\}$ in $\mathbb{M}_N \otimes \mathscr{D}$ such that
$$x_k b x_k^* \to a.$$

Recall that $a \sim_c b$ means $a \preceq b$ and $b \preceq a$.

Recall also that, when a and b are projections, $a \sim_c b$, as above defined, is *not* the same as Murray–von Neumann equivalence of projections (which we denote by $\sim$, following standard convention). In fact, in any simple purely infinite C*-algebra (e.g., O_∞), any two nonzero positive elements a, b will satisfy $a \sim_c b$, as above defined – this includes the case where a, b are nonzero projections that are not Murray–von Neumann equivalent.

For all $a \in \mathbb{M}_\infty(\mathscr{D})_+$, we let $[a]$ be the equivalence class of a under $\sim_c$ in $\mathbb{M}_\infty(\mathscr{D})$.

We let
$$W(\mathscr{D}) =_{\text{def}} \{[a] : a \in \mathbb{M}_\infty(\mathscr{D})_+\}.$$

$W(\mathscr{D})$ is a partially ordered semigroup under the order induced by $\preceq$ and with addition given by
$$[a] + [b] =_{\text{def}} [diag(a, b)].$$

Note that $W(\mathscr{D})$ is analogous to the Murray–von Neumann semigroup (which consists of $\sim$-equivalence classes of projections in $\mathbb{M}_\infty(\mathscr{D})$).

Suppose that $\mathscr{D}$ is, additionally, simple. Then for all $[a] \in W(\mathscr{D}) - \{0\}$, $[a]$ induces an element
$$\widehat{[a]} \in LAff(T_e(\mathscr{D}))_{++}$$

given by

$$\widehat{[a]}(\tau) =_{\text{def}} d_\tau(a)$$

where

$$d_\tau(a) =_{\text{def}} \lim_{n\to\infty} \tau(a^{1/n})$$

for all $\tau \in T_e(\mathscr{D})$. By the same procedure, $[a]$ also induces elements in $LAff(T(\mathscr{D}))_+$ and $LAff(T(\mathscr{M}(\mathscr{D})))_+$ which we also denote by $\widehat{[a]}$.

Suppose that $\mathscr{D}_0 \subseteq \mathscr{D}$ is a hereditary C*-subalgebra. Let

$$\widehat{\mathbb{M}_\infty(\mathscr{D}_0)}_+ =_{\text{def}} \{\widehat{a} \in LAff(T(\mathscr{M}(\mathscr{D})))_{++} \cup \{0\} : a \in (\mathbb{M}_n \otimes \mathscr{D}_0)_+ \text{ and } n \geqslant 1\}.$$

Suppose, in addition, that $\mathscr{D}$ is unital. We let $S(W(\mathscr{D}))$ denote the collection of all order preserving, semigroup maps

$$\rho : W(\mathscr{D}) \to [0,\infty)$$

such that

$$\rho([1]) = 1.$$

Most of the proof of the next argument is contained in the proof of [33] Theorem 2.15. Thus, in the proof of the next argument, we mainly work on the part that is different from that of [33] Theorem 2.15.

THEOREM 3.5. *Let $\mathscr{B}$ be a separable, simple, nonunital, $\mathscr{Z}$-stable C*-algebra with stable rank one. Then the following statements are equivalent:*

1. *$\mathscr{C}(\mathscr{B})$ is properly infinite.*

2. *$T(\mathscr{B})$ is weak*-compact.*

3. *The image of $T(\mathscr{B})$ in $T(\mathscr{M}(\mathscr{B}))$ is weak*-compact. (For the definition of the map $T(\mathscr{B}) \to T(\mathscr{M}(\mathscr{B}))$ see the previous paper Lemma 2.10 and the paragraph before it.)*

4. *$T(\mathscr{M}(\mathscr{B})) = T(\mathscr{B})$*

5. *$T(\mathscr{C}(\mathscr{B})) = \emptyset$.*

6. *For every nonzero nonunital hereditary C*-subalgebra $\mathscr{D} \subseteq \mathscr{B}$, $\widehat{\mathbb{M}_\infty(\mathscr{D})}_+$ is uniformly dense in $Aff(T(\mathscr{M}(\mathscr{B})))_{++} \cup \{0\}$.*

7. *$D_W(\mathscr{C}(\mathscr{B})) =_{\text{def}} \{[a] \in W(\mathscr{C}(\mathscr{B})) : a \in \mathscr{C}(\mathscr{B})_+\}$ is a semigroup.*

8. *$S(W(\mathscr{C}(\mathscr{B}))) = \emptyset$.*

Sketch of proof. Firstly, we note, for the convenience of the reader, that the degenerate case where $T(\mathscr{B}) = \emptyset$ is covered in [33] Remark 2.14.

The above statement is the similar (modulo minor modifications) to that of [33] Theorem 2.15, with the main differences being that [33] Theorem 2.15 assumes that $\mathscr{B}$ has Property I and $\mathscr{B}$ has an approximate unit consisting of projetions.

The requirement that $\mathscr{B}$ has Property I is now redundant by (this paper) Theorem 2.14.

By [33] Remark 2.16, the only place, in the proof of [33] Theorem 2.15, where $\mathscr{B}$ having an approximate unit consisting of projections is needed, is in the proof of the implication (2) $\Rightarrow$ (1). Thus, to complete the proof, we need only prove this direction:

(2) $\Rightarrow$ (1): Suppose that $T(\mathscr{B})$ is weak* compact. Since $\mathscr{B}$ is separable, let $\{e_n\}$ be a sequential approximate unit for $\mathscr{B}$ such that $e_{n+1}e_n = e_n$ for all n. (Note that $\{e_n\}$ need NOT be a sequence of projections.)

By Lemma 3.4, since $T(\mathscr{B})$ is weak* compact, we can find an integer $N \geqslant 1$ so that

$$(1_{\mathscr{M}(\mathscr{B})} - e_N) \oplus (1_{\mathscr{M}(\mathscr{B})} - e_N) \preceq 1_{\mathscr{M}(\mathscr{B})}$$

in $\mathbb{M}_2 \otimes \mathscr{M}(\mathscr{B})$.

Hence,

$$\pi(1_{\mathscr{M}(\mathscr{B})}) \oplus \pi(1_{\mathscr{M}(\mathscr{B})}) \preceq \pi(1_{\mathscr{M}(\mathscr{B})})$$

in $\mathbb{M}_2 \otimes \mathscr{C}(\mathscr{B})$. I.e., $\mathscr{C}(\mathscr{B})$ is properly infinite. $\square$

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