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SOME REFINEMENTS OF REAL POWER FORM INEQUALITIES FOR (p,h)–CONVEX FUNCTIONS VIA WEAK SUB–MAJORIZATION

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Abstract. The main goal of this paper, is to develop a general method for improving some new real power inequalities for (p,h)-convex and (p,h)-log-convex functions, which extends and unifies two recent and important results due to M. A. Ighachane and M. Bouchangour, (Filomat, **37** (16), (2023), 5259–5271) and (Operators and Matrices, **17** (1), (2023), 213–233). As applications of our results, we present further inequalities for the symmetric norms for τ-measurable operators.

1. Introduction and preliminaries

During the last years, various types and generalizations of convex functions have been introduced and studied by many mathematicians because of their important role in the theory of inequalities, see for instance [2, 4, 10, 14, 18] and the references therein. Among these types are the functions (p,h)-convex and (p,h)-log-convex (e.g., see [4, 14]), which are respectively generalizations of the well-known concepts of convexity and log-convexity. Before giving the definitions of these functions, let us first recall some notations and terminology that will be used in the following.

Along this work, $p \in \mathbb{R} \setminus \{0\}$ and $h : J \longrightarrow (0,+\infty)$ is a function defined on an interval J of $\mathbb{R}^+$. Recall that the function h is said to be super-multiplicative if for all $x, y \in J$, we have

$$xy \in J \text{ and } h(x)h(y) \leqslant h(xy). \tag{1}$$

If the inequality sign in (1) is reversed, then h is called a sub-multiplicative function. If the equality holds in (1), then h is said to be a multiplicative function. Similary, h is said to be super-additive function, if we have

$$x+y \in J \text{ and } h(x)+h(y) \leqslant h(x+y) \text{ for all } x,y \in J. \tag{2}$$

If the inequality sign in (2) is reversed, we say that h is a sub-additive function. If the equality (2) holds, we say that h is an additive function. It should be notice that if h is super-additive function then $h(x-y) \leqslant h(x)-h(y)$ whenever $x-y \in J$, and if h is

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super-multiplicative then $h\left(\frac{x}{y}\right) \leqslant \frac{h(x)}{h(y)}$ whenever $\frac{x}{y} \in J$ and $h(y) \neq 0$, in particular, $h\left(\frac{1}{x}\right) \leqslant \frac{1}{h(x)}$ for all $x \in J$ such that $h(x) \neq 0$.

EXAMPLE 1. ([7])

1. Let $h : (0,+\infty) \longrightarrow (0,+\infty)$ be a function given by $h(x) = x^k$. Then h is
   (a) additive if $k = 1$,
   (b) sub-additive if $k \in (-\infty,-1] \cup [0,1)$,
   (c) super-additive if $k \in (-1,0) \cup (1,\infty)$.
2. Let $h : [1,+\infty) \longrightarrow \mathbb{R}^+$ defined by $h(x) = x^3 - x^2 + x$. We have
   (a) $h(xy) - h(x)h(y) = xy(x+y)(1-x)(1-y) \geqslant 0$
   (b) $h(x+y) - h(x) - h(y) = xy(x+y+(x-1)+(y-1)) \geqslant 0$.

   Then h is a super-multiplicative and super-additive function.
3. If $h : [0,+\infty) \longrightarrow [0,+\infty)$ is a convex function such that $h(0) = 0$, then h is a super-additive function. In particular, for each $k > 0$, the following function $h(x) = \exp(x^k) - 1$ for $x > 0$ is super-additive.

From now on, we assume that the interval J contains $[0,1]$. Let $a,b > 0$ and $\alpha \in [0,1]$, the $h(\alpha)$-arithmetic and power means of a and b are respectively given by:

$$a\nabla^{h(\alpha)}b = h(1-\alpha)a + h(\alpha)b \text{ and } a\sharp_{p,\alpha}b = \left[(1-\alpha)a^p + \alpha b^p\right]^{\frac{1}{p}}.$$

Note that $b\nabla^{h(1-\alpha)}a = a\nabla^{h(\alpha)}b$ and $b\sharp_{p,1-\alpha}a = a\sharp_{p,\alpha}b$. A subset I of $\mathbb{R}^+$ is said to be p-convex if $a\sharp_{p,\alpha}b \in I$ for all $a,b \in I$ and all $\alpha \in [0,1]$. It is clear that I is a 1-convex set if and only if it is a convex set. For more informations about the p-convex sets, we refer the reader to [20].

In all what follows, I is a p-convex set of $\mathbb{R}^+$ and $f : I \longrightarrow [0,+\infty)$ is a function defined on I. The function f is said to be (p,h)-convex if it satisfies the following inequality

$$f\left(\left[(1-\alpha)a^p + \alpha b^p\right]^{\frac{1}{p}}\right) \leqslant h(1-\alpha)f(a) + h(\alpha)f(b), \tag{3}$$

for all $a,b \in I$ and $\alpha \in [0,1]$. In other word, f is (p,h)-convex if and only if the following inequality holds

$$f(a\sharp_{p,\alpha}b) \leqslant f(a)\nabla^{h(\alpha)}f(b) \quad (a,b \in I \text{ and } 0 \leqslant \alpha \leqslant 1).$$

Clearly, if $h = id$ (id stands for the identity function) in (3) then we get the definition of the p-convexity [20], in addition if we take $p = 1$ (resp. $p = -1$) then we get the usual definition of the convexity (resp. harmonic convexity [10]):

$$f\big((1-\alpha)a + \alpha b\big) \leqslant (1-\alpha)f(a) + \alpha f(b),$$

$$\left(\text{resp. } f\left(\frac{ab}{(1-\alpha)a+\alpha b}\right) \leqslant (1-\alpha)f(a)+\alpha f(b)\right).$$

The function f is called (p,h)-log-convex if the function $\log\circ f$ is (p,h)-convex, where log stands for the logarithmic function. This definition is equivalent to

$$f(a\sharp_{p,\alpha}b) \leqslant f^{h(1-\alpha)}(a)f^{h(\alpha)}(b),$$

for all $a,b\in I$ and $\alpha\in[0,1]$.

EXAMPLE 2. For $r\in[0,1]$, we define the function $h(t)=t^r$ for $0\leqslant t\leqslant 1$. A straightforward calculation shows that $h(t)\geqslant t$ for all $t\in[0,1]$. On the other hand, the function $f:[0,+\infty)\longrightarrow(0,+\infty)$ given by $f(x)=e^{x^p}$, where $p>0$, is (p,h)-convex and also (p,h)-log-convex. In fact, for $a,b\geqslant 0$ we have

$$\begin{aligned}
f\left(\left(1-\alpha)a^p+\alpha b^p\right]^{\frac{1}{p}}\right) &= e^{(1-\alpha)a^p+\alpha b^p}\\
&\leqslant (1-\alpha)e^{a^p}+\alpha e^{b^p} \quad [\text{by the convexity of } t\mapsto e^t]\\
&\leqslant h(1-\alpha)f(a)+h(\alpha)f(b),
\end{aligned}$$

and

$$\begin{aligned}
\log f\left(\left(1-\alpha)a^p+\alpha b^p\right]^{\frac{1}{p}}\right) &= (1-\alpha)a^p+\alpha b^p\\
&\leqslant h(1-\alpha)\log f(a)+h(\alpha)\log f(b),
\end{aligned}$$

EXAMPLE 3. By mimicking the argument of Example 2, we can show that if $g:[0,+\infty)\longrightarrow\mathbb{R}$ is a convex function and $h:J\longrightarrow[0,+\infty)$ is a function such that $h(t)\geqslant t$ for all $t\in[0,1]$, then the function $f:[0,+\infty)\longrightarrow\mathbb{R}$ given by $f(x)=g(x^p)$, where $p>0$, is (p,h)-convex. In particular, if we take $g(x)=x$ for $x\geqslant 0$ and $h(t)=\sqrt{t}$ for $t\geqslant 0$, then the function f given by $f(x)=x^p$ $(x\geqslant 0)$ is (p,h)-convex for every $p>0$.

For more information regarding the functions (p,h)-convex and (p,h)-log-convex and their various inequalities, we refer the reader to [4, 7, 14] and the references therein.

Several recent investigations have focused on possible improvements of inequalities related to the functions (p,h)-convex and (p,h)-log-convex. For example, in 2023, Ighachane and Bouchangour [7] established the following inequalities for (p,h)-convex, which generalizes an important result due to Sababheh [17].

THEOREM 1. ([7]) *If f is a positive (p,h)-convex function for a non-negative supermultiplicative and super-additive function h, then we have*

$$h^\lambda\left(\frac{\alpha}{\beta}\right)\leqslant\frac{(h(1-\alpha)f(a)+h(\alpha)f(b))^\lambda-f^\lambda\left[((1-\alpha)a^p+\alpha b^p)^{\frac{1}{p}}\right]}{(h(1-\beta)f(a)+h(\beta)f(b))^\lambda-f^\lambda\left[((1-\beta)a^p+\beta b^p)^{\frac{1}{p}}\right]}\leqslant h^\lambda\left(\frac{1-\alpha}{1-\beta}\right), \tag{4}$$

where $\lambda\geqslant 1$, $p\in\mathbb{R}\backslash\{0\}$ and $0\leqslant\alpha\leqslant\beta\leqslant 1$.

The main objective of this paper is to provide a unified treatment of (p,h)-convex and (p,h)-log-convex functions. More precisely, we will present a general improvement of Theorem 1.

This paper is organized as follows. After the forgoing section, we state and prove our main results concerning the refinement of Theorem 1 using the weak sub-majorization in the second section. Then by carefully selecting suitable (p,h)-log-convex functions we refine and reverse certain Hölder-type inequalities for τ-measurable operators.

2. Some new inequalities for (p,h)-convex functions via weak sub-majorization

In this section, we give an improved version of Theorem 1. We begin with recalling the theory of weak sub-majorization. Throughout this section, we denote by $x^* = (x_1^*,\dots,x_n^*)$ the vector obtained from the vector $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ by rearranging the components of it in decreasing order. Then, for two vectors $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ in $\mathbb{R}^n$, x is said to be weakly sub-majorized by y, written $x\prec_w y$, if

$$\sum_{i=1}^{k} x_i^* \leqslant \sum_{i=1}^{k} y_i^* \quad \text{for all } k=1,\dots,n.$$

The following lemma states an important feature of the theory of weak sub-majorization that will be used in proofs of our results.

LEMMA 1. [13, pp. 13] *Let $x=(x_i)_{i=1}^n, y=(y_i)_{i=1}^n\in\mathbb{R}^n$ and let ϕ be a continuous increasing convex function defined on a real interval containing the components of x and y. If $x\prec_w y$ then*

$$\sum_{i=1}^{n}\phi(x_i)\leqslant\sum_{i=1}^{n}\phi(y_i).$$

In order to accomplish our results, we need the following auxiliary lemmas.

LEMMA 2. *Let h be a non-negative super-multiplicative and super-additive function on J, $f:[a,b]\to[0,+\infty)$ be a (p,h)-convex function and let $0<\alpha<1$. Then we have*

$$f(a)\nabla^{h(\alpha)}f(b)\geqslant f(a\sharp_{p,\alpha}b)+h(2r)\left(f(a)\nabla^{h\left(\frac{1}{2}\right)}f(b)-f\left(a\sharp_{p,\frac{1}{2}}b\right)\right), \tag{5}$$

with $r=\min\{\alpha,1-\alpha\}$.

Proof. To prove (5), we discuss two cases:

(1) *First case*: If we assume that $0<\alpha\leqslant\dfrac{1}{2}$, then $r=\alpha$. So, by letting $\lambda=1$ and $\beta=\dfrac{1}{2}$ in the first inequality of Theorem 1, we get the desired inequality.

(2) *Second case*: If we assume that $\frac{1}{2} \leqslant \alpha < 1$, then $0 < 1-\alpha \leqslant \frac{1}{2}$ and $r = 1-\alpha$. So the desired inequality is obtained by changing a, b and α by b, a and $1-\alpha$ in the first case, respectively.

This completes the proof. □

REMARK 1. It is easy to see that under assumptions of Lemma 2, we can shows a better inequality of the following form

$$f(a)\nabla^{h(\alpha)} f(b) \geqslant f(a\sharp_{p,\alpha} b) + 2h(r)\left(f(a)\nabla f(b) - f\left(a\sharp_{p,\frac{1}{2}} b\right)\right), \tag{6}$$

where $r = \min\{\alpha, 1-\alpha\}$. To see that (6) is better than (5), due to the super-multiplicative and super-addtive of h with its range in $[0,\infty)$, we can show that h is increasing on J and if $\alpha \in \left[0, \frac{1}{2}\right]$. Then via Jensen inequality, we have

$$\begin{aligned} f(a)\nabla^{h(\alpha)} f(b) - f(a\sharp_{p,\alpha} b) &\geqslant 2h(\alpha)\left(f(a)\nabla f(b) - f\left(a\sharp_{p,\frac{1}{2}} b\right)\right) \\ &\geqslant h(\alpha)[f(a)+f(b)] - h(2\alpha) f\left(a\sharp_{p,\frac{1}{2}} b\right) \\ &\geqslant h(2\alpha)\left[f(a)\nabla^{h\left(\frac{1}{2}\right)} f(b) - f\left(a\sharp_{p,\frac{1}{2}} b\right)\right]. \end{aligned}$$

A familiar argument holds also true for $\alpha \in \left[\frac{1}{2}, 1\right]$. Thus, thanks to (6), we can make the better results.

LEMMA 3. *Let h be a non-negative super-multiplicative and super-additive function on J, $f : [a,b] \to [0,+\infty)$ be a (p,h)-convex function and let $0 < \alpha \leqslant \beta < 1$. Then we have*

$$\begin{aligned} f(a)\nabla^{h(\alpha)} f(b) \geqslant f(a\sharp_{p,\alpha} b) &+ h\left(\frac{\alpha}{\beta}\right)\left(f(a)\nabla^{h(\beta)} f(b) - f(a\sharp_{p,\beta} b)\right) \\ &+ h(2r_0)\left(f(a)\nabla^{h\left(\frac{1}{2}\right)} f(a\sharp_{p,\beta} b) - f\left(a\sharp_{p,\frac{\beta}{2}} b\right)\right), \end{aligned} \tag{7}$$

where $r_0 = \min\left\{\frac{\alpha}{\beta}, 1-\frac{\alpha}{\beta}\right\}$.

Proof. Since h is super-multiplicative and super-additive, we have

$$\begin{aligned} & f(a)\nabla^{h(\alpha)} f(b) - h\left(\frac{\alpha}{\beta}\right)\left(f(a)\nabla^{h(\beta)} f(b) - f(a\sharp_{p,\beta} b)\right) \\ = & \left(h(1-\alpha) - h(1-\beta) h\left(\frac{\alpha}{\beta}\right)\right) f(a) + \left(h(\alpha) - h\left(\frac{\alpha}{\beta}\right) h(\beta)\right) f(b) \\ & + h\left(\frac{\alpha}{\beta}\right) f(a\sharp_{p,\beta} b) \end{aligned}$$

$$\geqslant h\left(1-\frac{\alpha}{\beta}\right)f(a)+h\left(\frac{\alpha}{\beta}\right)f(a\natural_{p,\beta}b)$$
$$\geqslant f\left[\left(\left(1-\frac{\alpha}{\beta}\right)a^p+\frac{\alpha}{\beta}(a\natural_{p,\beta}b)^p\right)^{\frac{1}{p}}\right]$$
$$+h(2r_0)\left(f(a)\nabla^{h\left(\frac{1}{2}\right)}f(a\natural_{p,\beta}b)-f\left(a\natural_{p,\frac{\beta}{2}}b\right)\right) \text{ [by applying (5)]}$$
$$=f(a\natural_{p,\alpha}b)+h(2r_0)\left(f(a)\nabla^{h\left(\frac{1}{2}\right)}f(a\natural_{p,\beta}b)-f\left(a\natural_{p,\frac{\beta}{2}}b\right)\right). \quad \square$$

REMARK 2. Notice that Lemma 3 presents one refinement term of the first inequality in Theorem 1, for $\lambda = 1$.

LEMMA 4. *Let h be a non-negative super-multiplicative and super-additive function on J, f be a positive (p,h)-convex function on $[a,b]$, $0<\alpha\leqslant\beta<1$ and $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)\in\mathbb{R}^3$ be two vectors with components*

$$x_1=f(a\natural_{p,\alpha}b),\quad x_2=h\left(\frac{\alpha}{\beta}\right)\left(f(a)\nabla^{h(\beta)}f(b)\right),$$
$$x_3=h(2r_0)\left(f(a)\nabla^{h\left(\frac{1}{2}\right)}f(a\natural_{p,\beta}b)\right),$$

and

$$y_1=f(a)\nabla^{h(\alpha)}f(b),\quad y_2=h\left(\frac{\alpha}{\beta}\right)f(a\natural_{p,\beta}b),\quad y_3=h(2r_0)f\left(a\natural_{p,\frac{\beta}{2}}b\right),$$

where $r_0=\min\left\{\frac{\alpha}{\beta},1-\frac{\alpha}{\beta}\right\}$. Then, we have $x\prec_w y$, namely, the vectors x^* and y^* have components satisfying that

$$x_1^*\leqslant y_1^*, \tag{8}$$
$$x_1^*+x_2^*\leqslant y_1^*+y_2^*, \tag{9}$$
$$x_1^*+x_2^*+x_3^*\leqslant y_1^*+y_2^*+y_3^*. \tag{10}$$

Proof. The inequality (10) comes directly from Lemma 3 and we have

$$x_1+x_2+x_3\leqslant y_1+y_2+y_3. \tag{11}$$

Let us now show the inequality (8). First, notice that y_1^* equals y_1. Indeed, on one hand we have

$$y_1-x_2=\left(h(1-\alpha)-h(1-\beta)h\left(\frac{\alpha}{\beta}\right)\right)f(a)+\left(h(\alpha)-h\left(\frac{\alpha}{\beta}\right)h(\beta)\right)f(b)$$
$$\geqslant h\left(1-\frac{\alpha}{\beta}\right)f(a)\geqslant 0. \tag{12}$$

This shows that $y_1 \geqslant x_2$. Since $x_2 \geqslant y_2$, then we get that $y_1 \geqslant y_2$. On the other hand, it is easy to verify that

$$r_0 = \begin{cases} \dfrac{\alpha}{\beta} & \text{if } \alpha \in \left(0, \frac{\beta}{2}\right], \\ 1 - \dfrac{\alpha}{\beta} & \text{if } \alpha \in \left[\frac{\beta}{2}, \beta\right]. \end{cases} \tag{13}$$

From (13), it follows that $1-\alpha-r_0 \in [0,1]$, in addition, if $\alpha \in \left[\frac{\beta}{2}, \beta\right]$ then $0 \leqslant \left(\frac{2\alpha}{\beta}-1\right)(1-\beta) \leqslant 1$ and $0 \leqslant 2\alpha-\beta \leqslant 1$. Also, if $\alpha \in \left(0, \frac{\beta}{2}\right]$ then $0 < 1-\frac{2\alpha}{\beta} \leqslant 1$. From this we get that

$$\begin{aligned} y_1 - x_3 &= \left(h(1-\alpha) - h(2r_0)h\left(\frac{1}{2}\right)\right) f(a) + h(\alpha) f(b) - h(2r_0)h\left(\frac{1}{2}\right) f(a\sharp_{p,\beta} b) \\ &\geqslant h(1-\alpha-r_0) f(a) + h(\alpha) f(b) - h(r_0) f(a\sharp_{p,\beta} b) \\ &\geqslant h(1-\alpha-r_0) f(a) + h(\alpha) f(b) - h(r_0)\Big[h(1-\beta) f(a) + h(\beta) f(b)\Big] \\ &= \Big[h(1-\alpha-r_0) - h(r_0)h(1-\beta)\Big] f(a) + \Big[h(\alpha) - h(r_0)h(\beta)\Big] f(b) \\ &\geqslant \Big[h(1-\alpha-r_0) - h(r_0 - r_0\beta)\Big] f(a) + \Big[h(\alpha) - h(r_0\beta)\Big] f(b) \\ &\geqslant h\big(1-\alpha-2r_0+r_0\beta\big) f(a) + \Big[h(\alpha) - h(r_0\beta)\Big] f(b), \end{aligned}$$

so,

$$\begin{aligned} y_1 - x_3 &\geqslant \left[h\Big(\frac{2\alpha}{\beta} - 1 + \beta - 2\alpha\Big) f(a) + h(2\alpha-\beta) f(b)\right] \chi_{\left[\frac{\beta}{2},\beta\right]}(\alpha) \\ &\quad + \left[h\Big(1-\frac{2\alpha}{\beta}\Big) f(a)\right] \chi_{\left(0,\frac{\beta}{2}\right]}(\alpha) \\ &\geqslant \left[h\left(\Big(\frac{2\alpha}{\beta}-1\Big)\Big(1-\beta\Big)\right) f(a) + h(2\alpha-\beta) f(b)\right] \chi_{\left[\frac{\beta}{2},\beta\right]}(\alpha) \\ &\quad + \left[h\Big(1-\frac{2\alpha}{\beta}\Big) f(a)\right] \chi_{\left(0,\frac{\beta}{2}\right]}(\alpha) \\ &\geqslant 0, \end{aligned}$$

where χ_A stands for the characteristic function of an interval A. Hence, $y_1 \geqslant x_3$. Consequently, $y_1 \geqslant y_3$, because $x_3 \geqslant y_3$. Moreover, by the previous notes, we have $x_i \leqslant y_1$ for every $i = 1,2,3$. In particular, we obtain the inequality (8).

It remains to prove the inequality (9). To do this, it is sufficient to show the following inequalities.

$$x_1 + x_2 \leqslant y_1 + y_2, \tag{14}$$

$$x_1 + x_3 \leqslant y_1 + y_3, \tag{15}$$

$$x_2 + x_3 \leqslant y_1 + y_2. \tag{16}$$

Observe that the inequality (14) follows immediately from the first inequality in Theorem 1 for $\lambda = 1$. On the other side, it follows from (11) that

$$x_1 + x_3 \leqslant y_1 + y_3 - (x_2 - y_2) \leqslant y_1 + y_3. \tag{17}$$

Hence, the inequality (15) is obtained by using (17) together with the fact that $y_2 \leqslant x_2$. Let us now treat our last inequality (16). Since $= \min\{\frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta}\}$, we have

$$\begin{aligned}
y_1 + y_2 - (x_2 + x_3) &= f(a)\nabla^{h(\alpha)} f(b) + h\left(\frac{\alpha}{\beta}\right) f(a\sharp_{p,\beta} b) \\
&\quad - \left(h\left(\frac{\alpha}{\beta}\right) f(a)\nabla^{h(\beta)} f(b) + h(2r_0)\left(f(a)\nabla^{h\left(\frac{1}{2}\right)} f(a\sharp_{p,\beta} b)\right)\right) \\
&= \left(h(1-\alpha) - h\left(\frac{\alpha}{\beta}\right) h(1-\beta) - h(2r_0) h\left(\frac{1}{2}\right)\right) f(a) \\
&\quad + \left(h(\alpha) - h\left(\frac{\alpha}{\beta}\right) h(\beta)\right) f(b) \\
&\quad + \left(h\left(\frac{\alpha}{\beta}\right) - h(2r_0) h\left(\frac{1}{2}\right)\right) f(a\sharp_{p,\beta} b) \\
&\geqslant h\left(1 - \frac{\alpha}{\beta} - r_0\right) f(a) + h\left(\frac{\alpha}{\beta} - r_0\right) f(a\sharp_{p,\beta} b) \\
&\geqslant 0.
\end{aligned}$$

This complete the proof. □

In the following, we state our first main results. Our arguments are influenced by the ones given in [6]. The following results as mentioned before generalize the results given by Ighachane and Bouchangour in [7].

THEOREM 2. *Let h be a non-negative super-multiplicative and super-additive function on J, f be a (p,h)-convex function on $[a,b]$ and ϕ be a strictly increasing convex function defined on $\mathbb{R}^+$. Then for $0 < \alpha \leqslant \beta < 1$, we have*

$$\begin{aligned}
&\phi\left(f(a)\nabla^{h(\alpha)} f(b)\right) \\
&\geqslant \phi \circ f(a\sharp_{p,\alpha} b) + \phi\left(h\left(\frac{\alpha}{\beta}\right)\left(f(a)\nabla^{h(\beta)} f(b)\right)\right) - \phi\left(h\left(\frac{\alpha}{\beta}\right) f(a\sharp_{p,\beta} b)\right) \\
&\quad + \phi\left(h(2r_0)\left(f(a)\nabla^{h\left(\frac{1}{2}\right)} f(a\sharp_{p,\beta} b)\right)\right) - \phi\left(h(2r_0) f\left(a\sharp_{p,\frac{\beta}{2}} b\right)\right),
\end{aligned} \tag{18}$$

where $r_0 = \min\left\{\dfrac{\alpha}{\beta}, 1 - \dfrac{\alpha}{\beta}\right\}$.

Proof. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ two vectors in $\mathbb{R}^3$ with the same components as in Lemma 4. Since $x \prec_w y$, it follows from Lemma 1 that

$$\phi(x_1) + \phi(x_2) + \phi(x_3) \leqslant \phi(y_1) + \phi(y_2) + \phi(y_3),$$

or equivalently,

$$\phi(y_1) \geqslant \phi(x_1) + (\phi(x_2) - \phi(y_2)) + (\phi(x_3) - \phi(y_3)).$$

This complete the proof. □

The following theorem is the reversed version of the previous one. We prove it using the previous theorem and some specific changes of variables. Furthermore, we point out that we can prove this result by adopting the same ideas as in Theorem 2.

THEOREM 3. *Let h be a non-negative super-multiplicative and super-additive function on J, f be a (p,h)-convex function on $[a,b]$ and ϕ be a strictly increasing convex function defined on $\mathbb{R}^+$. Then for $0 < \alpha \leqslant \beta < 1$, we have*

$$\begin{aligned} &\phi\left(f(a)\nabla^{h(\beta)} f(b)\right) \\ \geqslant\ & \phi \circ f(a\sharp_{p,\beta} b) + \phi\left(h\left(\frac{1-\beta}{1-\alpha}\right)(f(a)\nabla^{h(\alpha)} f(b))\right) - \phi\left(h\left(\frac{1-\beta}{1-\alpha}\right) f(a\sharp_{p,\alpha} b)\right) \\ & -\phi\left(h(2R_0)(f(b)\nabla^{h\left(\frac{1}{2}\right)} f(a\sharp_{p,\alpha} b))\right) - \phi\left(h(2R_0) f\left(a\sharp_{p,\frac{1+\alpha}{2}} b\right)\right), \end{aligned} \tag{19}$$

where $R_0 = \min\left\{\dfrac{1-\beta}{1-\alpha}, 1 - \dfrac{1-\beta}{1-\alpha}\right\}$.

Proof. On one hand, if the function f is (p,h)-convex on the interval J, then its the same for the function g defined on the same interval by $g(x) = f\left([a^p + b^p - x^p]^{\frac{1}{p}}\right)$. On the other hand, notice that if $0 \leqslant \alpha \leqslant \beta \leqslant 1$ then $0 \leqslant 1-\beta \leqslant 1-\alpha \leqslant 1$. Hence, by changing α, β and f by $1-\beta, 1-\alpha$ and g, respectively, in Theorem 2. We get the desired results. □

It's worth noting that Theorems 2 and 3 present respectively the general version and the general reversed version of Lemma 3. In the following, by choosing some appropriate convex functions, we derive some very nice and interesting refinements for the correspondent inequalities for (p,h)-convex and (p,h)-log-convex functions, which improve the main results of [7].

Replacing f by $\log f$ in (18) and (19), we obtain the following inequalities for (p,h)-log-convex functions.

COROLLARY 1. *Let h be a non-negative super-multiplicative and super-additive function on J, f be a (p,h)-log-convex function on $[a,b]$ and ϕ be a strictly increasing convex function defined on $\mathbb{R}^+$. Then for $0 < \alpha \leqslant \beta < 1$, we have*

$$\begin{aligned} &\phi\left(\log\left(f^{h(1-\alpha)}(a) f^{h(\alpha)}(b)\right)\right) \geqslant \phi \circ \log f(a\sharp_{p,\alpha} b) \\ &+\phi\left(\log\left(\left(f^{h(1-\beta)}(a) f^{h(\beta)}(b)\right)^{h\left(\frac{\alpha}{\beta}\right)}\right)\right) - \phi\left(\log\left(f^{h\left(\frac{\alpha}{\beta}\right)}(a\sharp_{p,\beta} b)\right)\right) \\ &+\phi\left(\log\left(f^{h\left(\frac{1}{2}\right)}(a) f^{h\left(\frac{1}{2}\right)}(a\sharp_{p,\beta} b)\right)^{h(2r_0)}\right) - \phi\left(\log\left(f^{h(2r_0)}(a\sharp_{p,\frac{\beta}{2}} b)\right)\right), \end{aligned} \tag{20}$$

where $r_0 = \min\left\{\frac{\alpha}{\beta}, 1-\frac{\alpha}{\beta}\right\}$, *and*

$$\begin{aligned}
&\phi\left(\log\left(f^{h(1-\beta)}(a)f^{h(\beta)}(b)\right)\right) \geqslant \phi\circ\log f(a\sharp_{p,\beta}b) \\
&+\phi\left(\log\left(\left(f^{h(1-\alpha)}(a)f^{h(\alpha)}(b)\right)^{h\left(\frac{1-\beta}{1-\alpha}\right)}\right)\right) - \phi\left(\log\left(f^{h\left(\frac{1-\beta}{1-\alpha}\right)}(\alpha)\right)\right) \\
&+\phi\left(\log\left(\left(f^{h(\frac{1}{2})}(b)f^{h(\frac{1}{2})}(a\sharp_{p,\alpha}b)\right)^{h(2R_0)}\right)\right) \\
&-\phi\left(\log\left(f^{h(2R_0)}\left(a\sharp_{p,\frac{1+\alpha}{2}}b\right)\right)\right),
\end{aligned} \tag{21}$$

where $R_0 = \min\left\{\frac{1-\beta}{1-\alpha}, 1-\frac{1-\beta}{1-\alpha}\right\}$.

The following corollary comes directly by choosing $\phi(x) = x^\lambda$ $(\lambda \geqslant 1)$ in Theorems 2 and 3.

COROLLARY 2. *Let* h *be a non-negative super-multiplicative and super-additive function on* J, f *be a* (p,h)*-convex function on* $[a,b]$, $0 < \alpha \leqslant \beta < 1$ *and* $\lambda > 1$. *Then we have*

$$\begin{aligned}
&\left(f(a)\nabla^{h(\alpha)}f(b)\right)^\lambda \\
&\geqslant f^\lambda(a\sharp_{p,\alpha}b) + h^\lambda\left(\frac{\alpha}{\beta}\right)\left(\left(f(a)\nabla^{h(\beta)}f(b)\right)^\lambda - f^\lambda(a\sharp_{p,\beta}b)\right) \\
&\quad +h^\lambda(2r_0)\left(\left(f(a)\nabla^{h(\frac{1}{2})}f(a\sharp_{p,\beta}b)\right)^\lambda - f^\lambda\left(a\sharp_{p,\frac{\beta}{2}}b\right)\right),
\end{aligned} \tag{22}$$

where $r_0 = \min\left\{\frac{\alpha}{\beta}, 1-\frac{\alpha}{\beta}\right\}$, *and*

$$\begin{aligned}
&\left(f(a)\nabla^{h(\beta)}f(b)\right)^\lambda \\
&\geqslant f^\lambda(a\sharp_{p,\beta}b) + h^\lambda\left(\frac{1-\beta}{1-\alpha}\right)\left(\left(f(a)\nabla^{h(\alpha)}f(b)\right)^\lambda - f^\lambda(a\sharp_{p,\alpha}b)\right) \\
&\quad +h^\lambda(2R_0)\left(\left(f(b)\nabla^{h(\frac{1}{2})}f(a\sharp_{p,\alpha}b)\right)^\lambda - f^\lambda\left(a\sharp_{p,\frac{1+\alpha}{2}}b\right)\right),
\end{aligned} \tag{23}$$

where $R_0 = \min\left\{\frac{1-\beta}{1-\alpha}, 1-\frac{1-\beta}{1-\alpha}\right\}$.

By selecting $\phi(x) = \exp(\lambda x)$ (with $\lambda > 0$) in Corollary 1, we get the following new and important refinement and reversed for (p,h)-log-convex functions.

COROLLARY 3. *Let h be a non-negative super-multiplicative and super-additive function on J, f be a (p,h)-log-convex function on $[a,b]$. Then for $\lambda > 0$ and $0 < \alpha \leqslant \beta < 1$, we have*

$$\begin{aligned}&\left(f^{h(1-\alpha)}(a)f^{h(\alpha)}(b)\right)^{\lambda}\\ &\geqslant f^{\lambda}(a\sharp_{p,\alpha}b)+\left(\left(f^{h(1-\beta)}(a)f^{h(\beta)}(b)\right)^{\lambda h\left(\frac{\alpha}{\beta}\right)}-f^{\lambda h\left(\frac{\alpha}{\beta}\right)}(a\sharp_{p,\beta}b)\right)\\ &\quad+\left(f^{h(\frac{1}{2})}(a)f^{h(\frac{1}{2})}(a\sharp_{p,\beta}b)\right)^{\lambda h(2r_0)}-f^{\lambda h(2r_0)}\left(a\sharp_{p,\frac{\beta}{2}}b\right),\end{aligned} \tag{24}$$

where $r_0=\min\left\{\dfrac{\alpha}{\beta},1-\dfrac{\alpha}{\beta}\right\}$, *and*

$$\begin{aligned}&\left(f^{h(1-\beta)}(a)f^{h(\beta)}(b)\right)^{\lambda}\\ &\geqslant f^{\lambda}(a\sharp_{p,\beta}b)+\left((f^{h(1-\alpha)}(a)f^{h(\alpha)}(b))^{\lambda h\left(\frac{1-\beta}{1-\alpha}\right)}-f^{\lambda h\left(\frac{1-\beta}{1-\alpha}\right)}(a\sharp_{p,\alpha}b)\right)\\ &\quad+\left(f^{h(\frac{1}{2})}(b)f^{h(\frac{1}{2})}(a\sharp_{p,\alpha}b)\right)^{\lambda h(2R_0)}-f^{\lambda h(2R_0)}\left(a\sharp_{p,\frac{1+\alpha}{2}}b\right),\end{aligned} \tag{25}$$

where $R_0=\min\left\{\dfrac{1-\beta}{1-\alpha},1-\dfrac{1-\beta}{1-\alpha}\right\}$.

REMARK 3. Before proceeding to further results, we explain a little about the relation among the Corollary 2 and Theorem 1. Notice that the first inequality in Theorem 1 can be written as follows

$$h^{\lambda}\left(\frac{\alpha}{\beta}\right)\left[\left(f(a)\nabla^{h(\beta)}f(b)\right)^{\lambda}-f^{\lambda}(a\sharp_{p,\beta}b)\right]\leqslant\left(f(a)\nabla^{h(\alpha)}f(b)\right)^{\lambda}-f^{\lambda}(a\sharp_{p,\alpha}b), \tag{26}$$

with $0\leqslant\alpha<\beta\leqslant 1$ and $\lambda\geqslant 1$. While the second inequality in the same theorem can be stated in the following way

$$h^{\lambda}\left(\frac{1-\beta}{1-\alpha}\right)\left[\left(f(a)\nabla^{h(\alpha)}f(b)\right)^{\lambda}-f^{\lambda}(a\sharp_{p,\alpha}b)\right]\leqslant\left(f(a)\nabla^{h(\beta)}f(b)\right)^{\lambda}-f^{\lambda}(a\sharp_{p,\beta}b), \tag{27}$$

where $0\leqslant\alpha<\beta\leqslant 1$ and $\lambda\geqslant 1$. Consequently, the first inequality in Corollary 2 present one refining term of (26), while the second inequality in Corollary 2 present one refining term of (27). Therefore, Corollary 2 gives a considerable refinement of Theorem 1. Since Theorem 1 was a generalization of the results in [1, 6, 7, 17], it follows that our results in this section provide better new estimates than the results in these references. This is the main significance of our results.

REMARK 4. If we take $h(x)=x$ and $p=1$ in this work then we obtain the main results of [8].

Since the main result of [8] is a generalization of the results in [9, 11, 12]. It follows that our main results in this section provide better new estimates than the results in these references.

3. New inequalities for τ-measurable operators via log-convexity of norms

Let $\mathscr{H}$ be a seprable Hilbert space and $B(\mathscr{H})$ the algebra of all bounded linear operator on $\mathscr{H}$. Throughout this section, we denote by $\mathscr{M} \subset B(\mathscr{H})$ a finite von Neumann algebra on $\mathscr{H}$ and by $\mathscr{M}^+$ the set of all operators $A \in \mathscr{M}$ such that $A \geqslant 0$. Recall that a trace on the von Neumann algebra $\mathscr{M}$, denoted by τ, is an additive, positively homogeneous and unitarily invariant map from $\mathscr{M}^+$ to $[0,+\infty)$. The unitary invariant of τ is defined as follows $\tau(T) = \tau(U^*TU)$ for all $T \in \mathscr{M}^+$ and unitary $U \in \mathscr{M}$.

We say that an operator $A : \mathscr{D}(A) \to \mathscr{H}$ is τ-measurable if A affiliated with $\mathscr{M}$ (that is $AU = UA$ for all unitary $U \in \mathscr{M}$) and there exists $\delta > 0$ such that $\tau\left(e^{|A|}(\delta,\infty)\right)$ $< \infty$. For $0 < p < +\infty$, $L_p(\mathscr{M},\tau)$ is defined as the set of all τ-measurable operators A affiliated with $\mathscr{M}$ such that

$$\|A\|_p = \tau(|A|^p)^{\frac{1}{p}} < +\infty.$$

$L_p(\mathscr{M},\tau)$ is a Banach space under $\|.\|_p$ for $1 \leqslant p < +\infty$, see [15] for more information. From now on, $\mathscr{E}$ denotes a symmetric Banach space on $(0,\infty)$ and $\mu(A)$ is the decreasing rearrangement function of A (cf. [5]) defined by $t \mapsto \inf\{\delta > 0 \ : \ \tau\left(e^{|A|}(\delta,\infty)\right) \leqslant t\}$.

Next, we consider the non-commutative symmetric Banach space $(\mathscr{E}(\mathscr{M}), \|.\|_{\mathscr{E}(\mathscr{M})})$ (cf. [19]), defined by

$$\mathscr{E}(\mathscr{M}) := \{A \in L_0(\mathscr{M}) \ : \ \mu(A) \in \mathscr{E}\} \text{ and } \|A\|_{\mathscr{E}(\mathscr{M})} = \|\mu(A)\|_{\mathscr{E}},$$

As known $(L_p(\mathscr{M}), \|.\|_p)$, $0 < p < \infty$ becomes a special case of the previous construction and the same for $L_\infty(\mathscr{M}) = \mathscr{M}$. Moreover, for $0 < r < \infty$, define

$$\mathscr{E}(\mathscr{M})^{(r)} := \{A \in L_0(\mathscr{M}) \ : \ |A|^r \in \mathscr{E}\} \text{ and } \|A\|_{\mathscr{E}(\mathscr{M})^{(r)}} = \||A|^r\|_{\mathscr{E}(\mathscr{M})}^{\frac{1}{r}}.$$

It is well-known that if $\mathscr{E}$ is a symetric (quasi) Banach space, then it is the same for $\mathscr{E}(\mathscr{M})^{(r)}$ (cf. [3, Proposition 3.1]). Recall that a norm $\|.\|$ on $\mathscr{M}$ is symmetric if $\|UAV\| = \|A\|$ for all $A \in \mathscr{M}$ and all unitary $U,V \in \mathscr{M}$.

In this section, we refine and reverse certain Hölder-type inequalities for τ-measurable operators by carefully selecting suitable log-convex functions.

The famous Hölder's inequality for τ-measurable operators is stated as follows:

$$\|A^{1-t}XB^t\|_{\mathscr{E}(\mathscr{M})^{(r)}} \leqslant \|AX\|_{\mathscr{E}(\mathscr{M})^{(r)}}^{1-t} \|XB\|_{\mathscr{E}(\mathscr{M})^{(r)}}^{t}, \tag{28}$$

for every $r > 0$, $A,B \in \mathscr{M}^+$, $X \in \mathscr{E}(\mathscr{M})^{(r)}$ and $0 < t < 1$.

It has been proven in [16] that for $A,B \in \mathscr{M}^+$ and $X \in \mathscr{E}(\mathscr{M})^{(r)}$, the function

$$f_1(t) = \|A^{1-t}XB^t\|_{\mathscr{E}(\mathscr{M})^{(r)}}$$

is log-convex on $[0,1]$, for any symmetric norm $\|.\|_{\mathscr{E}(\mathscr{M})^{(r)}}$. By applying Corollary 3, with $h(x)=x$ and $p=1$, to the function f_1 we get the following theorem which refines and reverses the corresponding Hölder-type inequality (28) for τ-measurable operators.

THEOREM 4. *Let $r>0$, $A,B\in\mathscr{M}^+$ and $X\in\mathscr{E}(\mathscr{M})^{(r)}$. Then, for $0\leqslant\alpha\leqslant\beta\leqslant 1$, we have*

$$\begin{aligned}&\left(\|AX\|^{1-\beta}_{\mathscr{E}(\mathscr{M})^{(r)}}\|XB\|^{\beta}_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{\frac{\alpha}{\beta}}-\left\|A^{1-\beta}XB^{\beta}\right\|^{\frac{\alpha}{\beta}}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&+\left(\sqrt{\|AX\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{1-\beta}XB^{\beta}\|_{\mathscr{E}(\mathscr{M})^{(r)}}}\right)^{2r_0}-\left\|A^{1-\frac{\beta}{2}}XB^{\frac{\beta}{2}}\right\|^{2r_0}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&\leqslant\|AX\|^{1-\alpha}_{\mathscr{E}(\mathscr{M})^{(r)}}\|XB\|^{\alpha}_{\mathscr{E}(\mathscr{M})^{(r)}}-\left\|A^{1-\alpha}XB^{\alpha}\right\|_{\mathscr{E}(\mathscr{M})^{(r)}},\end{aligned}$$

where $r_0=\min\left\{\frac{\alpha}{\beta},1-\frac{\alpha}{\beta}\right\}$, *and*

$$\begin{aligned}&\left(\|AX\|^{1-\alpha}_{\mathscr{E}(\mathscr{M})^{(r)}}\|XB\|^{\alpha}_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{\frac{1-\beta}{1-\alpha}}-\left\|A^{1-\alpha}XB^{\alpha}\right\|^{\frac{1-\beta}{1-\alpha}}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&+\left(\sqrt{\|XB\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{1-\alpha}XB^{\alpha}\|_{\mathscr{E}(\mathscr{M})^{(r)}}}\right)^{2R_0}-\left\|A^{1-\frac{1+\alpha}{2}}XB^{\frac{1+\alpha}{2}}\right\|^{2R_0}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&\leqslant\|AX\|^{1-\beta}_{\mathscr{E}(\mathscr{M})^{(r)}}\|XB\|^{\beta}_{\mathscr{E}(\mathscr{M})^{(r)}}-\left\|A^{1-\beta}XB^{\beta}\right\|_{\mathscr{E}(\mathscr{M})^{(r)}},\end{aligned}$$

where $R_0=\min\left\{\frac{1-\beta}{1-\alpha},1-\frac{1-\beta}{1-\alpha}\right\}$. *In particular, if $\mathscr{M}$ is a finite von Neumann algebra, then*

$$\begin{aligned}&\left(\tau(A)^{1-\beta}\tau(B)^{\beta}\right)^{\frac{\alpha}{\beta}}-\tau^{\frac{\alpha}{\beta}}\left(A^{1-\beta}B^{\beta}\right)+\left(\sqrt{\tau(A)\tau(A^{1-\beta}B^{\beta})}\right)^{2r_0}\\&-\tau^{2r_0}\left(A^{1-\frac{\beta}{2}}B^{\frac{\beta}{2}}\right)\\&\leqslant\tau(A)^{1-\alpha}\tau(B)^{\alpha}-\tau\left(A^{1-\alpha}B^{\alpha}\right),\end{aligned}$$

where $r_0=\min\left\{\frac{\alpha}{\beta},1-\frac{\alpha}{\beta}\right\}$, *and*

$$\begin{aligned}&\left(\tau(A)^{1-\beta}\tau(B)^{\beta}\right)^{\frac{1-\beta}{1-\alpha}}-\tau^{\frac{1-\beta}{1-\alpha}}\left(A^{1-\beta}B^{\beta}\right)+\left(\sqrt{\tau(B)\tau(A^{1-\alpha}B^{\alpha})}\right)^{2R_0}\\&-\tau^{2R_0}\left(A^{1-\frac{1+\alpha}{2}}B^{\frac{1+\alpha}{2}}\right)\\&\leqslant\tau(A)^{1-\beta}\tau(B)^{\beta}-\tau\left(A^{1-\beta}B^{\beta}\right),\end{aligned}$$

where $R_0=\min\left\{\frac{1-\beta}{1-\alpha},1-\frac{1-\beta}{1-\alpha}\right\}$.

Furthermore, it has been established in [16] that for $A,B\in\mathscr{M}^+$ and $X\in\mathscr{E}(\mathscr{M})^{(r)}$, the function $f_2(t)=\|A^tXB^t\|_{\mathscr{E}(\mathscr{M})^{(r)}}$ is log-convex on $[0,1]$ for any symmetric norm $\|.\|_{\mathscr{E}(\mathscr{M})^{(r)}}$. Applying again Corollary 3, with $h(x)=x$ and $p=1$, to the function f_2, we obtain the following theorem.

THEOREM 5. *Let $r>0$, $A,B\in\mathscr{M}^+$ and $X\in\mathscr{E}(\mathscr{M})^{(r)}$. Then, for $0\leqslant\alpha\leqslant\beta\leqslant 1$, we have*

$$\begin{aligned}&\left(\|X\|^{1-\beta}_{\mathscr{E}(\mathscr{M})^{(r)}}\|AXB\|^{\beta}_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{\frac{\alpha}{\beta}}-\left\|A^{\beta}XB^{\beta}\right\|^{\frac{\alpha}{\beta}}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&+\left(\sqrt{\|X\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{\beta}XB^{\beta}\|_{\mathscr{E}(\mathscr{M})^{(r)}}}\right)^{2r_0}-\left\|A^{\frac{\beta}{2}}XB^{\frac{\beta}{2}}\right\|^{2r_0}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&\leqslant\|X\|^{1-\alpha}_{\mathscr{E}(\mathscr{M})^{(r)}}\|AXB\|^{\alpha}_{\mathscr{E}(\mathscr{M})^{(r)}}-\|A^{\alpha}XB^{\alpha}\|_{\mathscr{E}(\mathscr{M})^{(r)}},\end{aligned}$$

where $r_0=\min\left\{\frac{\alpha}{\beta},1-\frac{\alpha}{\beta}\right\}$, *and*

$$\begin{aligned}&\left(\|X\|^{1-\alpha}_{\mathscr{E}(\mathscr{M})^{(r)}}\|AXB\|^{\alpha}_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{\frac{1-\beta}{1-\alpha}}-\|A^{\alpha}XB^{\alpha}\|^{\frac{1-\beta}{1-\alpha}}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&+\left(\sqrt{\|AXB\|_{E(\mathscr{M})^{(r)}}\|A^{\alpha}XB^{\alpha}\|_{\mathscr{E}(\mathscr{M})^{(r)}}}\right)^{2R_0}-\left\|A^{\frac{1+\alpha}{2}}XB^{\frac{1+\alpha}{2}}\right\|^{2R_0}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&\leqslant\|X\|^{1-\beta}_{\mathscr{E}(\mathscr{M})^{(r)}}\|XB\|^{\beta}_{\mathscr{E}(\mathscr{M})^{(r)}}-\left\|A^{\beta}XB^{\beta}\right\|_{\mathscr{E}(\mathscr{M})^{(r)}},\end{aligned}$$

where $R_0=\min\left\{\frac{1-\beta}{1-\alpha},1-\frac{1-\beta}{1-\alpha}\right\}$.

In particular, if $X=I$, we get

$$\begin{aligned}&\left(\|AB\|^{\beta}_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{\frac{\alpha}{\beta}}-\left\|A^{\beta}B^{\beta}\right\|^{\frac{\alpha}{\beta}}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&+\left(\sqrt{\|A^{\beta}B^{\beta}\|_{\mathscr{E}(\mathscr{M})^{(r)}}}\right)^{2r_0}-\left\|A^{\frac{\beta}{2}}B^{\frac{\beta}{2}}\right\|^{2r_0}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&\leqslant\|AB\|^{\alpha}_{\mathscr{E}(\mathscr{M})^{(r)}}-\|A^{\alpha}B^{\alpha}\|_{\mathscr{E}(\mathscr{M})^{(r)}},\end{aligned}$$

where $r_0=\min\left\{\frac{\alpha}{\beta},1-\frac{\alpha}{\beta}\right\}$, *and*

$$\begin{aligned}&\left(\|AB\|^{\alpha}_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{\frac{1-\beta}{1-\alpha}}-\|A^{\alpha}B^{\alpha}\|^{\frac{1-\beta}{1-\alpha}}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&+\left(\sqrt{\|AB\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{\alpha}B^{\alpha}\|_{\mathscr{E}(\mathscr{M})^{(r)}}}\right)^{2R_0}-\left\|A^{\frac{1+\alpha}{2}}B^{\frac{1+\alpha}{2}}\right\|^{2R_0}_{\mathscr{E}(\mathscr{M})^{(r)}}\\&\leqslant\|AB\|^{\beta}_{\mathscr{E}(\mathscr{M})^{(r)}}-\left\|A^{\beta}B^{\beta}\right\|_{\mathscr{E}(\mathscr{M})^{(r)}},\end{aligned}$$

where $R_0=\min\left\{\frac{1-\beta}{1-\alpha},1-\frac{1-\beta}{1-\alpha}\right\}$.

It has been shown in [16] that for $A,B \in \mathscr{M}^+$ and $X \in \mathscr{E}(\mathscr{M})^{(r)}$, the function

$$f_3(t) = \|A^{1-t}XB^t\|_{\mathscr{E}(\mathscr{M})^{(r)}} \|A^tXB^{1-t}\|_{\mathscr{E}(\mathscr{M})^{(r)}},$$

is log-convex on $[0,1]$ for any symmetric norm $\|.\|_{\mathscr{E}(\mathscr{M})^{(r)}}$. Therefore, applying Corollary 3 with $h(x) = x$ and $p = 1$, we obtain the following theorem.

THEOREM 6. *Let $r > 0$, $A,B \in \mathscr{M}^+$ and $X \in \mathscr{E}(\mathscr{M})^{(r)}$. Then, for $0 \leqslant \alpha \leqslant \beta \leqslant 1$, we have*

$$\begin{aligned}
&\left(\|AX\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|XB\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{\frac{\alpha}{\beta}} - \left(\|A^{1-\beta}XB^{\beta}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{\beta}XB^{1-\beta}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{\frac{\alpha}{\beta}} \\
&+\left(\sqrt{\|AX\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|XB\|_{\mathscr{E}(\mathscr{M})^{(r)}}\left(\|A^{1-\beta}XB^{\beta}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{\beta}XB^{1-\beta}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right)}\right)^{2r_0} \\
&-\left(\|A^{1-\frac{\beta}{2}}XB^{\frac{\beta}{2}}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{\frac{\beta}{2}}XB^{1-\frac{\beta}{2}}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{2r_0} \\
\leqslant &\left(\|AX\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|XB\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right) - \left(\|A^{1-\alpha}XB^{\alpha}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{\alpha}XB^{1-\alpha}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right),
\end{aligned}$$

where $r_0 = \min\left\{\dfrac{\alpha}{\beta}, 1-\dfrac{\alpha}{\beta}\right\}$, *and*

$$\begin{aligned}
&\left(\|AX\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|XB\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{\frac{1-\beta}{1-\alpha}} - \left(\|A^{1-\alpha}XB^{\alpha}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{\alpha}XB^{1-\alpha}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{\frac{1-\beta}{1-\alpha}} \\
&+\left(\sqrt{\|AX\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|XB\|_{\mathscr{E}(\mathscr{M})^{(r)}}\left(\|A^{1-\alpha}XB^{\alpha}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{\alpha}XB^{1-\alpha}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right)}\right)^{2R_0} \\
&-\left(\|A^{1-\frac{1+\alpha}{2}}XB^{\frac{1+\alpha}{2}}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{\frac{1+\alpha}{2}}XB^{1-\frac{1+\alpha}{2}}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right)^{2R_0} \\
\leqslant &\left(\|AX\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|XB\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right) - \left(\|A^{1-\beta}XB^{\beta}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\|A^{\beta}XB^{1-\beta}\|_{\mathscr{E}(\mathscr{M})^{(r)}}\right),
\end{aligned}$$

where $R_0 = \min\left\{\dfrac{1-\beta}{1-\alpha}, 1-\dfrac{1-\beta}{1-\alpha}\right\}$.

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