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# SOME INEQUALITIES RELATED TO NUMERICAL RADIUS AND DISTANCE FROM SCALAR OPERATORS IN HILBERT SPACES

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*Abstract.* In this paper, we characterize bounded linear operators $A,B$ on a complex Hilbert space such that $\inf_{\lambda\in\mathbb{C}}\|A+B-\lambda I\| = \inf_{\lambda\in\mathbb{C}}\|A-\lambda I\| + \inf_{\lambda\in\mathbb{C}}\|B-\lambda I\|$, where $I$ is the identity operator. We also establish some inequalities satisfied by the distance from scalar operators for products of two complex Hilbert space operators.

## 1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the induced norm. Let $\mathscr{B}(\mathscr{H})$ denote the collection of all bounded linear operators acting on $\mathscr{H}$, the operator norm is also denoted by $\|\cdot\|$. For $A\in\mathscr{B}(\mathscr{H})$, the *numerical range* of $A$ is given by

$$W(A)=\Big\{\langle Ax,x\rangle : x\in\mathscr{H} \text{ and } \|x\|=1\Big\}.$$

It is known that $W(A)$ is a nonempty (when $\mathscr{H}\neq\{0\}$) bounded convex subset (not necessarily closed) of the complex plane. To measure the location and relative size of $W(A)$, one frequently used quantity; *numerical radius* of $A$. It is denoted and given by

$$w(A)=\sup\Big\{|\lambda| : \lambda\in W(A)\Big\}.$$

It is well-known that $w(\cdot)$ is a norm on $\mathscr{B}(\mathscr{H})$ and

$$\frac{1}{2}\|A\|\leqslant w(A)\leqslant\|A\| \tag{1}$$

for all $A\in\mathscr{B}(\mathscr{H})$, that is $w(\cdot)$ defines an equivalent norm to $\|\cdot\|$ on $\mathscr{B}(\mathscr{H})$. Also, it is a basic fact that the norm $w(\cdot)$ is self-adjoint (i.e., $w(A^*)=w(A)$ for all $A\in\mathscr{B}(\mathscr{H})$)

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where $A^*$ is the adjoint of $A$). For more material about the numerical radius and other information on the basic theory of numerical range, we refer the reader to [10].

In [16], Stampfli introduced the center of mass of $A \in \mathscr{B}(\mathscr{H})$ as the (unique) value $\lambda \in \mathbb{C}$ at which the minimum of $\|A - \lambda I\|$ (where $I$ is the identity operator on $\mathscr{H}$) is attained, we denote $d(A) := \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$.

In 1963 Björck and Thomée [5] proved, that if $A$ is a normal operator acting on a Hilbert space, then the radius $R_A$ of the smallest circular disc containing the spectrum of $A$, $\sigma(A)$, is equal to $d(A)$. Later, in 1980 Garske [8] generalized Björck and Thomée's result for an arbitrary operator $A \in \mathscr{B}(\mathscr{H})$ and showed that in the general case we have only the inequality $R_A \leqslant d(A)$. In [3], Ando has proved that

LEMMA 1. ([3]) *Let $A \in \mathscr{B}(\mathscr{H})$. Then,*

*(i)* $d(A) = \sup\{|\langle Ax, y\rangle| : x, y \in \mathscr{H},\ \|x\| = \|y\| = 1$ *and* $\langle x, y\rangle = 0\}$;

*(ii)* $d(A) = \sup\{\|(I-P)AP\| : P \in \mathscr{B}(\mathscr{H})$ *is an orthogonal projection of rank one*$\}$

($P \in \mathscr{B}(\mathscr{H})$ *is called an orthogonal projection if* $P^2 = P = P^*$).

Many authors have obtained several refinements and reverses for the inequalities in (1) see e.g., [7, 9, 14, 15]. It has been shown in [7, 12], that if $A \in \mathscr{B}(\mathscr{H})$, then

$$\|A\|^2 \leqslant w^2(A) + d^2(A),$$

it is shown in [2] that if $A, B \in \mathscr{B}(\mathscr{H})$, then

$$w(AB) \leqslant w(A)\big(d(B) + \|B\|\big).$$

In [11, 13], the authors proved that

$$w(AB) \leqslant w(A)w(B) + d(A)d(B).$$

The numerical radius equality $\big(w(A+B) = w(A) + w(B)\big)$, has been discussed in [1], the authors proved that

THEOREM 1. ([1]) *Let $A, B \in \mathscr{B}(\mathscr{H})$, the following conditions are equivalent:*

*(i)* $w(A+B) = w(A) + w(B)$;

*(ii) there exists a sequence of unit vectors* $\{x_n\}$ *in* $\mathscr{H}$ *such that*

$$\lim_{n\to+\infty} \langle A^* x_n, x_n\rangle \langle Bx_n, x_n\rangle = w(A)w(B).$$

The proof of Theorem 1 is similar to that of norm equality $\big(\|A+B\| = \|A\| + \|B\|\big)$ which says that $\|A+B\| = \|A\| + \|B\|$ if and only if there exists a sequence $\{x_n\}$ of unit vectors in $\mathscr{H}$ such that $\lim_{n\to\infty} \langle A^*Bx_n, x_n\rangle = \|A\|\|B\|$, see [4].

The purpose of this paper is to characterize operators $A, B \in \mathscr{B}(\mathscr{H})$ such that $d(A+B) = d(A) + d(B)$. We also establish some inequalities satisfied by $d(\cdot)$ for products of two complex Hilbert space operators.

## 2. Main results

Our first main result in this section reads as follows.

THEOREM 2. *Let* $A, B \in \mathscr{B}(\mathscr{H})$ *with* $A$ *is normaloid* $\big($*i.e.,* $\|A\| = w(A)\big)$ *and*

$$w(A+B) = w(A) + w(B).$$

*Then,*

$$w(A)\,w(B) \leqslant \min\{w(AB), w(BA)\}.$$

*Proof.* If $dim(\mathscr{H}) = 1$, then the inequality holds without any hypothesis.
Using Theorem 1, there exists a sequence of unit vectors $\{x_n\}$ in $\mathscr{H}$ such that

$$\lim_{n\to+\infty} \langle B^* x_n, x_n\rangle \langle A x_n, x_n\rangle = w(A)\,w(B).$$

Let $\{y_n\} \subset \mathscr{H}$ be a sequence of unit vectors with $\langle x_n, y_n\rangle = 0$ and $Ax_n = \langle Ax_n, x_n\rangle x_n + \langle Ax_n, y_n\rangle y_n$, for all $n \in \mathbb{N}$, it follows that

$$\langle BAx_n, x_n\rangle = \langle Ax_n, x_n\rangle \langle Bx_n, x_n\rangle + \langle Ax_n, y_n\rangle \langle By_n, x_n\rangle, \tag{2}$$

and

$$\|Ax_n\|^2 = |\langle Ax_n, x_n\rangle|^2 + |\langle Ax_n, y_n\rangle|^2. \tag{3}$$

Since $|\langle Ax_n, x_n\rangle \langle Bx_n, x_n\rangle| \leqslant w(A)w(B)$ and

$$\lim_{n\to+\infty} |\langle Ax_n, x_n\rangle \langle Bx_n, x_n\rangle| = \lim_{n\to+\infty} |\langle B^* x_n, x_n\rangle \langle Ax_n, x_n\rangle| = w(A)\,w(B),$$

$\lim\limits_{n\to+\infty} |\langle Bx_n, x_n\rangle| = w(B)$ and $\lim\limits_{n\to+\infty} |\langle Ax_n, x_n\rangle| = w(A) = \|A\|$ (because $A$ is normaloid).
From (3), we have

$$\|Ax_n\|^2 = |\langle Ax_n, x_n\rangle|^2 + |\langle Ax_n, y_n\rangle|^2 \geqslant |\langle Ax_n, x_n\rangle|^2$$

this implies that $\lim\limits_{n\to+\infty} \langle Ax_n, y_n\rangle = 0$. Now, from (2) we derive that

$$w(A)\,w(B) \leqslant w(BA). \tag{4}$$

Now, we apply the same argument to $A^*$ and $B^*$, we get

$$w(A^*)\,w(B^*) \leqslant w(B^*A^*) = w(AB).$$

So,

$$w(A)\,w(B) \leqslant w(AB). \tag{5}$$

From the inequalities (4) and (5), we get

$$w(A)\,w(B) \leqslant \min\{w(AB), w(BA)\},$$

and this completes the proof. $\square$

REMARK 1. It is worth mentioning that in the absence of normaloidity, the result of Theorem 2 need not be true. To see this, consider

$$A=\begin{bmatrix}0&1\\0&0\end{bmatrix},$$

then $w(A)=\frac{1}{2}$ and $w\left(A^2\right)=0$.

COROLLARY 1. *Under the conditions of Theorem 2, if also $B$ is normaloid, then*

$$w(AB)=w(BA)=w(A)\,w(B).$$

In the following theorem, we give a necessary and sufficient condition for the equality $d(A+B)=d(A)+d(B)$ where $A,B\in\mathscr{B}(\mathscr{H})$.

THEOREM 3. *Let $A,B\in\mathscr{B}(\mathscr{H})$ with $dim(\mathscr{H})\geqslant 2$. The following conditions are equivalent:*

*(i)* $d(A+B)=d(A)+d(B)$;

*(ii) there exist two sequences of unit vectors $\{x_n\},\{y_n\}\subset\mathscr{H}$ with $\langle x_n,y_n\rangle=0$, for all $n\in\mathbb{N}$ such that*

$$\lim_{n\to+\infty}\langle A^*y_n,x_n\rangle\langle Bx_n,y_n\rangle=d(A)\,d(B).$$

*Proof.* (i)$\Longrightarrow$(ii) Assume that $d(A+B)=d(A)+d(B)$. By Lemma 1, there exist two sequences of unit vectors $\{x_n\},\{y_n\}\subset\mathscr{H}$ with $\langle x_n,y_n\rangle=0$, for all $n\in\mathbb{N}$ such that

$$\begin{aligned}
d^2(A+B) &= \lim_{n\to+\infty}|\langle (A+B)x_n,y_n\rangle|^2\\
&= \lim_{n\to+\infty}|\langle Ax_n,y_n\rangle+\langle Bx_n,y_n\rangle|^2\\
&= \lim_{n\to+\infty}\Big(|\langle Ax_n,y_n\rangle|^2+|\langle Bx_n,y_n\rangle|^2+2Re\left(\langle A^*y_n,x_n\rangle\langle Bx_n,y_n\rangle\right)\Big)\\
&\leqslant \lim_{n\to+\infty}\Big(|\langle Ax_n,y_n\rangle|^2+|\langle Bx_n,y_n\rangle|^2+2\,|\langle A^*y_n,x_n\rangle\langle Bx_n,y_n\rangle|\Big)\\
&= \lim_{n\to+\infty}\Big(|\langle Ax_n,y_n\rangle|+|\langle Bx_n,y_n\rangle|\Big)^2\\
&\leqslant \big(d(A)+d(B)\big)^2=d^2(A+B).
\end{aligned}$$

This implies that

$$\lim_{n\to+\infty}\langle A^*y_n,x_n\rangle\langle Bx_n,y_n\rangle=d(A)\,d(B).$$

(ii)$\Longrightarrow$(i) Suppose that there exist two sequences of unit vectors $\{x_n\},\{y_n\}\subset\mathscr{H}$ and $\langle x_n,y_n\rangle=0$, for all $n\in\mathbb{N}$ such that

$$\lim_{n\to+\infty}\langle A^*y_n,x_n\rangle\langle Bx_n,y_n\rangle=d(A)\,d(B).$$

Since $|\langle Ax_n, y_n\rangle| \leqslant d(A)$ and $|\langle Bx_n, y_n\rangle| \leqslant d(B)$,

$$\lim_{n\to+\infty} |\langle A^* y_n, x_n\rangle| = d(A) \text{ and } \lim_{n\to+\infty} |\langle Bx_n, y_n\rangle| = d(B).$$

Therefore,

$$\begin{aligned}\lim_{n\to+\infty} |\langle (A+B)x_n, y_n\rangle|^2 &= d^2(A) + d^2(B) + 2d(A)d(B) \\ &= \big(d(A) + d(B)\big)^2.\end{aligned}$$

This implies that

$$d(A+B) \geqslant d(A) + d(B). \tag{6}$$

Further, we have

$$\begin{aligned} d(A+B) &= \sup\{|\langle (A+B)x, y\rangle| : \|x\| = \|y\| = 1 \text{ and } \langle x, y\rangle = 0\} \\ &\leqslant d(A) + d(B). \end{aligned} \tag{7}$$

By (6) and (7), we conclude that

$$d(A+B) = d(A) + d(B). \quad \square$$

The following lemma will be useful in the proof of the next result.

LEMMA 2. *Let $A \in \mathscr{B}(\mathscr{H})$ with $dim(\mathscr{H}) \geqslant 3$ and $d(A) = \|A\|$. Let $\{x_n\}, \{y_n\} \subset \mathscr{H}$ be two sequences of unit vectors such that $\lim_{n\to+\infty} |\langle Ax_n, y_n\rangle| = d(A)$ and $\langle x_n, y_n\rangle = 0$ for all $n \in \mathbb{N}$. Let $\{z_n\} \subset \mathscr{H}$ be a sequence of unit vectors with $\langle x_n, z_n\rangle = \langle y_n, z_n\rangle = 0$ and $Ax_n = \alpha_n x_n + \beta_n y_n + \gamma_n z_n$, $(\alpha_n, \beta_n, \gamma_n \in \mathbb{C})$ for all $n \in \mathbb{N}$. Then*

$$\lim_{n\to+\infty} \langle Ax_n, x_n\rangle = \lim_{n\to+\infty} \langle Ax_n, z_n\rangle = 0.$$

*Proof.* The following identity holds true:

$$\|Ax_n\|^2 = |\langle Ax_n, x_n\rangle|^2 + |\langle Ax_n, y_n\rangle|^2 + |\langle Ax_n, z_n\rangle|^2.$$

Since

$$\lim_{n\to+\infty} |\langle Ax_n, y_n\rangle| = d(A),$$

we obtain

$$\lim_{n\to+\infty} |\langle Ax_n, x_n\rangle|^2 + |\langle Ax_n, z_n\rangle|^2 = 0$$

thus,

$$\lim_{n\to+\infty} \langle Ax_n, x_n\rangle = \lim_{n\to+\infty} \langle Ax_n, z_n\rangle = 0. \quad \square$$

An application of Theorem 3 is the following result.

THEOREM 4. *Let $A,B \in \mathscr{B}(\mathscr{H})$ with $B$ is self-adjoint. If $d(A) = \|A\|$ and $d(A+B) = d(A) + d(B)$, then $d(A)d(B) \in \overline{W(BA)}$, $d(A)d(B) \in \overline{W\left((AB)^*\right)}$ and so*

$$d(A)d(B) \leqslant \min\{w(AB), w(BA)\}.$$

*Proof.* If $dim(\mathscr{H}) = 1$, then the results hold.

Assume that $dim(\mathscr{H}) \geqslant 3$. In view of Theorem 3, there exist two sequences of unit vectors $\{x_n\}, \{y_n\} \subset \mathscr{H}$ with $\langle x_n, y_n\rangle = 0$, for all $n \in \mathbb{N}$ such that

$$\lim_{n\to+\infty} \langle B^* y_n, x_n\rangle \langle Ax_n, y_n\rangle = d(A)d(B).$$

Let $\{z_n\} \subset \mathscr{H}$ be a sequence of unit vectors with $\langle x_n, z_n\rangle = \langle y_n, z_n\rangle = 0$ and $Ax_n = \alpha_n x_n + \beta_n y_n + \gamma_n z_n$, for all $n \in \mathbb{N}$, it follows that

$$\langle BAx_n, x_n\rangle = \langle Ax_n, x_n\rangle \langle Bx_n, x_n\rangle + \langle By_n, x_n\rangle \langle Ax_n, y_n\rangle + \langle Bz_n, x_n\rangle \langle Ax_n, z_n\rangle.$$

By Lemma 2, we obtain

$$\lim_{n\to+\infty} \langle Ax_n, x_n\rangle = \lim_{n\to+\infty} \langle Ax_n, z_n\rangle = 0.$$

So,

$$\begin{aligned}\lim_{n\to+\infty} \langle BAx_n, x_n\rangle &= \lim_{n\to+\infty} \langle By_n, x_n\rangle \langle Ax_n, y_n\rangle \\ &= \lim_{n\to+\infty} \langle B^* y_n, x_n\rangle \langle Ax_n, y_n\rangle \text{ (since } B^* = B) \\ &= d(A)d(B).\end{aligned}$$

This implies that $d(A)d(B) \in \overline{W(BA)}$. Hence,

$$d(A)d(B) \leqslant w(BA). \tag{8}$$

On the other hand, we have $d(A^*) = d(A) = \|A\| = \|A^*\|$ and

$$d\left((A+B)^*\right) = d(A+B) = d(A) + d(B) = d(A^*) + d(B^*).$$

Hence,

$$d(A^*)d(B^*) \in \overline{W(B^*A^*)}$$

thus,

$$d(A)d(B) \in \overline{W\left((AB)^*\right)}.$$

Consequently, we obtain

$$d(A)d(B) \leqslant w(AB). \tag{9}$$

So, it follows from (8) and (9) that

$$d(A)d(B) \leqslant \min\{w(AB), w(BA)\}.$$

Hence, the desired results are proved.

If $dim(\mathscr{H}) = 2$, the proof is similar to that in the case $dim(\mathscr{H}) \geqslant 3$; take $z_n = 0$. □

Our next result is stated as follows.

PROPOSITION 1. *Let* $A,B\in\mathscr{B}(\mathscr{H})$. *Then*

$$d(AB)\leqslant w(A)\,d(B)+\|B\|\,d(A).$$

*Proof.* If $dim(\mathscr{H})=1$ then the inequality holds.

Let $x,y\in\mathscr{H}$ such that $\|x\|=\|y\|=1$ and $\langle x,y\rangle=0$. Put $Bx=\alpha y+\beta y^{\perp}$ with $\alpha=\langle Bx,y\rangle$, $\beta=\left\langle Bx,y^{\perp}\right\rangle$ and $y^{\perp}$ is orthogonal to $y$. Now, it can be observed that

$$\begin{aligned}\langle ABx,y\rangle &= \alpha\langle Ay,y\rangle+\beta\left\langle Ay^{\perp},y\right\rangle\\ &= \langle Bx,y\rangle\langle Ay,y\rangle+\left\langle Bx,y^{\perp}\right\rangle\left\langle Ay^{\perp},y\right\rangle.\end{aligned}$$

This implies that

$$|\langle ABx,y\rangle|\leqslant|\langle Bx,y\rangle|\,|\langle Ay,y\rangle|+\left|\left\langle Bx,y^{\perp}\right\rangle\right|\left|\left\langle Ay^{\perp},y\right\rangle\right|.$$

Thus, by taking the supremum in the above inequality over all $x,y\in\mathscr{H}$ with $\|x\|=\|y\|=1$ and $\langle x,y\rangle=0$, we get

$$d(AB)\leqslant w(A)\,d(B)+\|B\|\,d(A).$$

Hence, the proof of the proposition is complete. □

Now, we are in a position to prove the following result.

PROPOSITION 2. *Let* $A,B\in\mathscr{B}(\mathscr{H})$. *If* $\dim\mathscr{H}=2$, *then*

$$d(AB)\leqslant w(A)\,d(B)+w(B)\,d(A).$$

*Proof.* Let $x,y\in\mathscr{H}$ with $\|x\|=\|y\|=1$ and $\langle x,y\rangle=0$. Put $Bx=\alpha x+\beta y$ with $\alpha=\langle Bx,x\rangle$ and $\beta=\langle Bx,y\rangle$. It follows that

$$\begin{aligned}|\langle ABx,y\rangle| &= |\alpha\langle Ax,y\rangle+\beta\langle Ay,y\rangle|\\ &= |\langle Bx,x\rangle\langle Ax,y\rangle+\langle Bx,y\rangle\langle Ay,y\rangle|\\ &\leqslant |\langle Bx,x\rangle\langle Ax,y\rangle|+|\langle Bx,y\rangle\langle Ay,y\rangle|.\end{aligned}$$

Taking the supremum in the above inequality over all $x,y\in\mathscr{H}$ with $\|x\|=\|y\|=1$ and $\langle x,y\rangle=0$, we get

$$d(AB)\leqslant w(B)\,d(A)+w(A)\,d(B).$$

This completes the proof. □

Next we need the following inequality, known as Buzano's inequality [6].

LEMMA 3. ([6]) *Let* $x,y,z\in\mathscr{H}$, *then*

$$|\langle x,z\rangle\langle z,y\rangle|\leqslant\frac{\|z\|^2}{2}\left(\|x\|\,\|y\|+|\langle x,y\rangle|\right).$$

The following lemma is useful in the proof of the next results.

LEMMA 4. *Let $A \in \mathscr{B}(\mathscr{H})$ be a positive operator and let $x, y \in \mathscr{H}$. Then*

$$|\langle Ax, y\rangle| \leqslant \frac{\|A\|}{2}\Big(\|x\|\,\|y\| + |\langle x, y\rangle|\Big).$$

*Proof.* By using Lemma 3, we obtain

$$|\langle Ax, y\rangle\,\langle Ax, x\rangle| \leqslant \frac{\|Ax\|^2}{2}\Big(\|x\|\,\|y\| + |\langle x, y\rangle|\Big).$$

If $\langle Ax, x\rangle \neq 0$, then

$$|\langle Ax, y\rangle| \leqslant \frac{\|Ax\|^2}{2\,\langle Ax, x\rangle}\left(\|x\|\,\|y\| + |\langle x, y\rangle|\right).$$

Therefore,

$$\begin{aligned}
\frac{\|Ax\|^2}{\langle Ax, x\rangle} = \frac{\|Ax\|^2}{\left\|A^{\frac{1}{2}}x\right\|^2} &\leqslant \frac{\left\|A^{\frac{1}{2}}\right\|^2 \left\|A^{\frac{1}{2}}x\right\|^2}{\left\|A^{\frac{1}{2}}x\right\|^2} \\
&= \left\|A^{\frac{1}{2}}\right\|^2 \\
&= \|A\|.
\end{aligned}$$

If $\langle Ax, x\rangle = 0$, then $A^{\frac{1}{2}}x = 0$ and so $Ax = 0$. Therefore, we get the desired inequality. This completes the proof. $\square$

Next, we introduce the following theorem.

THEOREM 5. *Let $T, S, A \in \mathscr{B}(\mathscr{H})$ be such that $A$ is positive. Then,*

$$d\,(SAT) \leqslant \frac{\|A\|}{2}\Big(\|T\|\,\|S\| + d\,(ST)\Big).$$

*Proof.* If $dim(\mathscr{H}) = 1$, then the inequality holds.

Let $x, y \in \mathscr{H}$ such that $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = 0$. By using Lemma 4, it can be observed that

$$\begin{aligned}
|\langle SATx, y\rangle| &= |\langle ATx, S^*y\rangle| \\
&\leqslant \frac{\|A\|}{2}\left(|\langle Tx, S^*y\rangle| + \|Tx\|\,\|S^*y\|\right) \\
&\leqslant \frac{\|A\|}{2}\left(|\langle STx, y\rangle| + \|T\|\,\|S\|\right) \\
&\leqslant \frac{\|A\|}{2}\Big(\|T\|\,\|S\| + d\,(ST)\Big). \quad \square
\end{aligned}$$

As an application of Theorem 5, we derive the following inequality.

COROLLARY 2. *Let $T,S,A \in \mathscr{B}(\mathscr{H})$ be such that $A$ is positive and let $r \geqslant 1$. Then,*

$$d^r(SAT) \leqslant \frac{\|A\|^r}{2}\Big(\|T\|^r\|S\|^r + d^r(ST)\Big).$$

*Proof.* Since $t \mapsto t^r$, $r \geqslant 1$ is a convex increasing function on $[0,+\infty)$,

$$\begin{aligned} d^r(SAT) &\leqslant \|A\|^r\left(\frac{\|T\|\|S\| + d(ST)}{2}\right)^r \\ &\leqslant \frac{\|A\|^r}{2}\Big(\|T\|^r\|S\|^r + d^r(ST)\Big). \quad \square \end{aligned}$$

Our final result is the following

THEOREM 6. *Let $T \in \mathscr{B}(\mathscr{H})$ with the polar decomposition $T = U|T|$. Then,*

$$d(T) \leqslant \frac{1}{2}\left(\|T\| + \|T\|^{\frac{1}{2}} d\big(U|T|^{\frac{1}{2}}\big)\right).$$

*Proof.* For any $x,y \in \mathscr{H}$, we have

$$\begin{aligned} |\langle Tx,y\rangle| &= |\langle U|T|x,y\rangle| \\ &= \left|\left\langle |T|^{\frac{1}{2}}x, |T|^{\frac{1}{2}}U^*y\right\rangle\right|. \end{aligned}$$

Using Lemma 4, we get

$$\begin{aligned} \left|\left\langle |T|^{\frac{1}{2}}x, |T|^{\frac{1}{2}}U^*y\right\rangle\right| &\leqslant \frac{\left\||T|^{\frac{1}{2}}\right\|}{2}\left(\left|\left\langle x, |T|^{\frac{1}{2}}U^*y\right\rangle\right| + \|x\|\left\||T|^{\frac{1}{2}}U^*y\right\|\right) \\ &\leqslant \frac{\|T\|^{\frac{1}{2}}}{2}\left(\left|\left\langle U|T|^{\frac{1}{2}}x, y\right\rangle\right| + \|x\|\|y\|\|T\|^{\frac{1}{2}}\|U^*\|\right). \end{aligned}$$

Since $\|U\| = \|U^*\| = 1$ and taking the supremum over all unit vectors $x,y \in \mathscr{H}$ such that $\langle x,y\rangle = 0$, we obtain

$$d(T) \leqslant \frac{1}{2}\left(\|T\| + \|T\|^{\frac{1}{2}} d\left(U|T|^{\frac{1}{2}}\right)\right),$$

as desired. $\square$

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