Operators and Matrices
Volume 17, Number 4 (2023), 875–900 doi:10.7153/oam-2023-17-59

SURFACE LOCALIZATION IN IMPURITY BAND WITH ARBITRARY SINGULAR DISORDER AND LONG–RANGE POTENTIALS

VICTOR CHULAEVSKY

(*Communicated by G. Teschl*)

Abstract. We consider a variety of Anderson-type random Hamiltonians in disordered media with a special layer of nonzero co-dimension ("surface" models), and prove spectral and strong dynamical localization in such models, with exponential decay of eigenfunctions and sub-exponential decay of eigenfunction correlators. The main novelty is that we allow arbitrarily singular disorder, and assume that the media-particle interactions feature a power-law decay at infinity.

1. Introduction

1.1. Motivation

In the mathematical theory of disordered media, a vast majority of works consider the models where disorder, usually understood as *randomness* (or some form of almost periodicity), is more or mess evenly distributed over the configuration space where quantum particles or waves, acoustic or electromagnetic (cf., e.g., [17, 18, 19]), are localized, contrary to the situation where the fundamental properties of the ambient media sample evolve periodically, as in perfect crystals. There are by now several excellent monographs where the principal technical tools, analytic and probabilistic, are presented in full detail; cf., e.g., [1, 28, 32].

The so-called surface localization models have also been explored over the last two decades, but to a lesser extent. Perhaps, the terminology used in this area deserves a few comments. In applications to physical models, there are indeed situations where a geometrical boundary of a large sample has properties different from those of the sample's bulk. Such a boundary may be two-dimensional (in 3D samples) or one-dimensional (in thin, quasi-two-dimensional films). However, one can also artificially design samples where a thin inner layer of codimension 1 (in 3D or 2D samples) or 2 (a quasi-one-dimensional channel in 3D samples) carries atoms different from those constituting the bulk, or is otherwise made special. In certain aspects, such inner layers share some common properties with the genuine boundary layers. The questions of physical motivation for such models have been addressed, e.g., in the paper [23].

From the spectral point of view, the disorder may be created by the local potentials induced by constitutive atoms (or ions), often referred to as "scatterer" or "bump"

Mathematics subject classification (2020): 35Q40, 47B80, 60H25.
Keywords and phrases: Anderson localization, surface localization, Bernoulli disorder.

Paper OaM-17-59

potentials, and/or by structural properties (substitutions/displacements). One can also consider the "strips" or "wires" with ragged border; cf. e.g., the paper [24] on ragged waveguides, and the references therein.

Until now, a majority of mathematical works consider the models of disordered systems where the local scatterer potentials are compactly supported. In such a case, it suffices to make use only of the sites close to the surface to obtain satisfactory Wegner-type estimates (in the framework of multi-scale analysis) or bounds on fractional moments of resolvents (in the framework of the Aizenman–Molchanov method). The latter, when applicable, provides the strongest possible (viz. exponential) decay estimates. Jakšić and Molchanov used this approach long ago in [22] and proved spectral localization in the discrete case with the help of the Simon–Wolff criterium [30].

The work [27] addressed a continuous model, and strong dynamical localization was proved with the help of multi-scala analysis. The variant of the MSA used in [27], pre-dating the bootstrap MSA introduced in [21], resulted in power-law decay of eigenfunction correlators. The reader can find in [27] a general discussion of surface models along with a number of useful references. In their work, only the surface layer carries randomness, while in the rest of the space, one has a non-random constant or, more generally, periodic non-negative potential.

Apart from random surface Hamiltonians, almost periodic (viz. quasi-periodic) surface models have also been considered in the past. In our paper, we do not address this class of surface models.

Much less is known about the disordered systems with infinite-range scatterer potentials. The present paper is a follow-up of the author's works [7, 6, 8] where long-range models have been studied. As in [7], we treat the case where the technical difficulties due to the infinite range of the scatterer potentials are combined with those due to an arbitrarily singular nature of the random amplitudes modulating these potentials, thus creating the disorder in the system at hand. However, unlike [7], we address several types of *surface* localization models.

One of the messages conveyed by the present work is that, on the technical level, the difference between the "surface" Anderson Hamiltonians and their more classical counterparts with "bulk" disorder is less pronounced in the models with infinite-range media-particle interactions than in the case of short-range interactions.

1.2. The models

We consider two classes of models: in $\mathbb{R}^d$ and in $\mathbb{Z}^d$, with $d \geqslant 2$: at least one dimension required for the "surface", and the latter has a nonzero co-dimension.

In $\mathbb{R}^d$, we consider the random Schrödinger operators of the form

$$H(\omega, \vartheta) = -h\Delta + V(x; \omega; \vartheta), \;\; h > 0, \tag{1.1}$$

acting in the Hilbert space $L^2(\mathbb{R}^d)$. Quite often, one puts a factor $g > 0$ (coupling constant) in front of the potential, but up to a change of the energy scale, the operator $-h\Delta + V$ is equivalent to $-\Delta + gV$ with $g = h^{-1}$. In the latter case, g measures the strength of disorder, so $h \gg 1$ corresponds to a weak disorder.

To specify the structure of the random potential $x \mapsto V(x;\omega;\vartheta)$, introduce the following conventions and notations. We assume that a periodic lattice with d linearly independent generators is chosen in $\mathbb{R}^d$; for definiteness, we choose the integer lattice $\mathbb{Z}^d \subset \mathbb{R}^d$, but our results can be easily adapted to more general lattices. The sites of the lattice $\mathbb{Z}^d$ carry local potentials (often called scatterer, or bump, potentials).

It is to be stressed from the beginning that all local potentials have infinite range, and we shall specify below some conditions on the rate of their decay at infinity.

For clarity, we denote the "surface" layer by $\mathcal{Z}^{d_0}$ instead of $\mathbb{Z}^{d_0}$. Specifically, $\mathcal{Z}^{d_0} = \left\{ \mathrm{b} \in \mathbb{Z}^d : \mathrm{b} = (\mathrm{a}, 0) \in \mathbb{Z}^{d_0} \times \mathbb{Z}^{d-d_0} \right\}$, this allows us to write $\mathbb{Z}^d \setminus \mathcal{Z}^{d_0}$ instead of a more ambiguous expression $\mathbb{Z}^d \setminus \mathbb{Z}^{d_0}$.

Section 2, building on the results of [7], is devoted to the Wegner-type estimates, i.e., to the eigenvalue concentration analysis. The estimates obtained here can be easily adapted to various models considered in this paper.

In Sections 3–5, we study a base model (**model 1**) in $\mathbb{R}^d$; its discrete analog (**model 2**) is considered in Section 6. Further extensions are discussed in Section 7.

The base model: sparse surface impurities in non-ergodic bulk

In all models considered in the present paper, we work with two types of scatterer potentials, $\mathfrak{u}^+$ and $\mathfrak{u}^-$. The potential V induced by any sample without "impurities" is non-negative, while an impurity at $\mathrm{b} \in \mathbb{Z}^{d_0}$ (implemented by a site potential $\mathfrak{u}^-(\cdot - \mathrm{b})$) produces a *negative potential well* in a ball $\mathrm{B}_{\mathrm{r}_0}(\mathrm{b})$ of some radius r_0. We assume that this negative well cannot be destroyed by any configuration of atom types and random scatterers with centers outside some neighborhood of the site b. For this and some other reasons, all scatterer amplitudes are uniformly bounded.

Further, we assume in the base model that some periodic sub-lattice $\hat{\mathcal{Z}}^{d_0} \subset \mathcal{Z}^{d_0}$ carries only positive random potentials $\omega_{\mathrm{b}}\, \mathfrak{u}^+(\cdot - \mathrm{b})$, so the impurities can appear only in the complement $\mathcal{Z}^{d_0} \setminus \hat{\mathcal{Z}}^{d_0}$. Such an assumption is quite realistic from the physical point of view: if a crystal is composed of $n \geqslant 2$ types of atoms, then it may happen that the impurity atoms could replace only a specific kind of atoms, and not all. Technically, this assumption simplifies application of crucial Wegner-type estimates established in our prior work [7], but it can be significantly relaxed; see Section 7.

The disorder in our models is represented by two independent random fields:

- $\omega : \Omega \times \mathbb{Z}^d \to [0, \bar{s}]$; in other terms, a family of random variables (often abbreviated to r.v.) $\left\{ \omega_{\mathrm{b}},\, \mathrm{b} \in \mathbb{Z}^d \right\}$ with values in an interval $[0, \bar{s}]$, defined on a probability space $(\Omega, \mathfrak{F}_\Omega, \mathbb{P}_\Omega)$; see the hypothesis $(\mathsf{V1}(1))$ below. Here $\Omega = [0, \bar{s}]^{\mathbb{Z}^d}$, and $\mathfrak{F}_\Omega$ is the cylindric σ-algebra induced by the Lebesgue σ-algebra on $[0, \bar{s}]$.

- $\vartheta : \Theta \times \mathbb{Z}^{d_0} \to \{0, 1\}$, i.e., a family of r.v. $\left\{ \vartheta_{\mathrm{b}},\, \mathrm{b} \in \mathbb{Z}^{d_0} \right\}$ with values 0 and 1, on some probability space $(\Theta, \mathfrak{F}_\Theta, \mathbb{P}_\Theta)$; see the hypothesis $(\mathsf{V3}(1))$ below.

Clearly, one can define both random fields on a common, product probability space $\left([0, \bar{s}]^{\mathbb{Z}^d} \times \Theta, \mathfrak{F}_\Omega \times \mathfrak{F}_\Theta, \mathbb{P}_\Omega \times \mathbb{P}_\Theta\right)$, so we keep the same notation $\mathbb{P}$ for the probability of various events relative to ω and/or ϑ.

In order to make more transparent comparison of several models with specific sets *of assumptions, we use a double numeration of hypotheses, where the second, smaller number in parentheses indicates the model number. For example, in Section 6, we introduce the hypotheses* (V1(2))–(V4(2)).

Now we turn to the formal description of the base model. The random potential has the form $V(x,\omega,\vartheta) = V^{(\mathrm{B})}(x) + V^{(\mathrm{S})}(x;\omega;\vartheta)$ where

$$V^{(\mathrm{B})}(x) = \sum_{\mathrm{b}\in\mathbb{Z}^d\setminus\mathcal{Z}^{d_0}} \mathrm{s}_{\mathrm{b}}\, \mathfrak{u}^{+}(|x-\mathrm{b}|) \geqslant 0, \quad \mathrm{s}_{\mathrm{b}} \in [0,\overline{s}], \tag{1.2}$$

with some fixed, non-random amplitudes s_{b}, and

$$\begin{aligned} V^{(\mathrm{S})}(x;\omega;\vartheta) = &\sum_{\mathrm{b}\in\hat{\mathcal{Z}}^{d_0}} \omega_{\mathrm{b}} \mathfrak{u}^{+}(|x-\mathrm{b}|) \\ &+ \sum_{\mathcal{Z}^{d_0}\setminus\hat{\mathcal{Z}}^{d_0}\ni\mathrm{b}:\,\vartheta_{\mathrm{b}}=0} \mathfrak{u}^{+}(|x-\mathrm{b}|) + \sum_{\mathcal{Z}^{d_0}\setminus\hat{\mathcal{Z}}^{d_0}\ni\mathrm{b}:\,\vartheta_{\mathrm{b}}=1} \mathfrak{u}^{-}(|x-\mathrm{b}|). \end{aligned} \tag{1.3}$$

Apart from $\mathrm{s}_{\mathrm{b}} \in [0,\overline{s}]$, no additional assumption on s_{b} is made, so the function $\mathrm{b} \mapsto \mathrm{s}_{\mathrm{b}}$ on $\mathbb{Z}^d$ may be constant, or periodic. However, in general, with arbitrarily chosen s_{b}, one cannot rely on the so-called Lifshits tails phenomenon (cf. [25]) providing the initial length scale estimate (serving as the base of the scale induction; cf. Section 4) for arbitrarily large $h > 0$ in $H = -h\Delta + V$, i.e., for an arbitrarily weak disorder.

Until Section 6, our main hypotheses are as follows.

(V1(1)) *The random variables* $\{\omega_{\mathrm{b}},\, \mathrm{b}\in\mathbb{Z}^{d_0}\}$ *are IID (independent and identically distributed). The support of their common probability measure* μ *contains at least two points, and one has* $0 \in \operatorname{supp}\mu \subset [0,\overline{s}]$.

(V2(1)) *For* $r \geqslant 1/2$ *and some* $A = 3d + 2\gamma$, $\gamma > 0$,

$$-\mathfrak{u}^{-}(r) = |\mathfrak{u}^{-}(r)| \leqslant \mathrm{C}_1 r^{-A}, \quad \mathrm{C}_1 \in (0,+\infty); \tag{1.4}$$

$$\mathfrak{u}^{+}(r) = r^{-A}. \tag{1.5}$$

For $r \in [0,1/2]$, $\mathfrak{u}^{+}(r) = \tilde{c}_{+} \geqslant 0$ and $\mathfrak{u}^{-}(r) = \tilde{c}_{-} < 0$.

(V3(1)) *The random field* ϑ, *independent of* ω, *satisfies the following conditions:*

$$\mathbb{P}\Big\{\operatorname{card}\big\{\mathrm{b}\in\mathbb{Z}^{d_0} : \vartheta_{\mathrm{b}} = 1\big\} = +\infty\Big\} = 1; \tag{1.6}$$

$$\rho := \sup_{\mathrm{b}\in\mathbb{Z}^{d_0}} \mathbb{P}\{\vartheta_{\mathrm{b}} = 1\} \in (0,1). \tag{1.7}$$

(V4(1)) *There exist* $E_* < 0$, $\mathscr{E} > E_*$, $r_0 \in (0,1)$, *and* $\phi \in L^2(\mathbb{R}^d)$, *with* $\|\phi\|_2 = 1$ *and* $\operatorname{supp}\phi \subset \mathrm{B}_{r_0}(0)$, *such that, denoting* $\phi_{\mathrm{a}} = \phi(\cdot - \mathrm{a})$, *one has*

$$\sup_{\mathrm{a}\in\mathbb{Z}^{d_0}\setminus\hat{\mathcal{Z}}^{d_0}} \Big(\phi_{\mathrm{a}}, \Big[-h\Delta + \mathfrak{u}^{-}(|x-\mathrm{a}|) + \sum_{\mathrm{b}\in\mathbb{Z}^d\setminus\{0\}} \overline{s}\mathfrak{u}^{+}(|x-\mathrm{b}|)\Big]\phi_{\mathrm{a}}\Big) < E_*, \tag{1.8}$$

$$\inf_{\mathrm{a}\in\mathbb{Z}^d} \inf_{x\in\mathrm{B}_1(\mathrm{a})} \sum_{\mathrm{b}\in\mathcal{Z}^{d_0}\setminus\{\mathrm{a}\}} \mathfrak{u}^{-}(|x-\mathrm{b}|) \geqslant E_* + \mathscr{E}. \tag{1.9}$$

Comments. • The values assigned to $\mathfrak{u}^{\pm}(r)$ for $0 \leqslant r \leqslant 1/2$ are of little importance; they only affect implicitly (1.8)–(1.9) but do not play any role in the proofs.

• Clearly, the condition (1.6) is rather weak: we assume neither ergodicity nor even translation invariance of the random field ϑ. Also, (1.7) is essentially the definition of an important parameter ρ to be used in the hypotheses of Theorem 1, except that it says that ϑ_{b} is not a.s. (almost surely) equal to 1. In the terminology of random point fields, (1.7) is an upper bound on the first correlation function of the field ϑ. A reader not interested in a great generality can replace (V3(1)) with a simpler condition:

(V3'(1)) *The random field ϑ on $\mathbb{Z}^{d_0}$ has IID values, independent of ω, with*

$$\rho := \mathbb{P}\{\vartheta_{\mathrm{b}} = 1\} = 1 - \mathbb{P}\{\vartheta_{\mathrm{b}} = 0\} \in (0,1). \tag{1.10}$$

• The condition (1.8) is deterministic, so the variational inequality (1.8) holds true almost surely, if the parameters s_{b} are replaced with ω_{b}. The assumption (1.8) guarantees the a.s. existence of nontrivial essential spectrum of $H(\omega,\vartheta)$ in $(-\infty, E_*)$, owing to (1.7)–(1.8). Since $\|V(\cdot,\omega,\vartheta)\|_\infty < +\infty$, $H(\omega,\vartheta)$ is deterministically semi-bounded and has a nontrivial spectrum in some bounded interval $I_* \subset (-\infty, E_*)$.

On the other hand, (1.9) guarantees a gap of size at least $\mathscr{E} > 0$ between $(-\infty, E_*)$ and the spectrum of a restriction $H_{\mathrm{B}}(\omega,\vartheta)$ of $H(\omega,\vartheta)$ to any cube B with no impurities inside it, hence only with positive potentials $x \mapsto \mathrm{s}_{\mathrm{b}}\mathfrak{u}^{+}(|x-\mathrm{b}|)$, $\mathrm{b} \in \mathrm{B}$, even if all the sites $\mathrm{b} \in \mathcal{Z}^{d_0} \setminus \mathrm{B}$ carry negative potentials $\mathfrak{u}^{-}(|x-\mathrm{b}|)$, and even if $\mathrm{s}_{\mathrm{b}} = 0$ for all $\mathrm{b} \notin \mathcal{Z}^{d_0}$. If we had considered a model where $\mathrm{s}_{\mathrm{b}} \geqslant c_+ > 0$, (1.9) should have been replaced with

$$\inf_{\mathrm{a}\in\mathbb{Z}^d} \inf_{x\in \mathrm{B}_1(\mathrm{a})} \Big(c_+ \mathfrak{u}^{+}(|x-\mathrm{a}|) + \sum_{\mathrm{b}\in\mathcal{Z}^{d_0}\setminus\{\mathrm{a}\}} \mathfrak{u}^{-}(|x-\mathrm{b}|) \Big) \geqslant E_* + \mathscr{E}. \tag{1.11}$$

In the setting of **model 1**, $\mathrm{s}_{\mathrm{b}} \in [0,\overline{s}]$, so we must allow also for $\mathrm{s}_{\mathrm{b}} = 0$, thus rely exclusively on a nonzero decay of $|\mathfrak{u}^{-}(|x|)|$ between $\mathrm{B}_{1/2}(0)$, where $\mathfrak{u}^{-}(|x|)$ takes a constant negative value, and $\mathbb{R}^d \setminus \mathrm{B}_1(0)$. This decay must result in a gap of size $\mathscr{E} > 0$ in the cumulative potential induced by all negative scatterers. In turn, this makes the “bulk” $\mathbb{R}^d \setminus \cup_{\mathrm{b}\in\mathcal{Z}^{d_0}} \mathrm{B}_1(\mathrm{b})$ a forbidden zone for particles with energy below E_*.

• In the second sum in the RHS of (1.3), one could also have put random amplitudes ω_{b} in front of $\mathfrak{u}^{+}(|x-\mathrm{b}|)$ with $\mathrm{b} \in \mathcal{Z}^{d_0} \setminus \hat{\mathcal{Z}}^{d_0}$ such that $\vartheta_{\mathrm{b}} = 0$. This would have no impact on the Wegner-type estimates, for we would have included these terms in the “background” operator H_0 and condition on the σ-algebra generated by the respective ω_{b}. In essence, more randomness carried by $\omega_{\mathrm{b}}\mathfrak{u}^{+}(|x-\mathrm{b}|)$ such that $\vartheta_{\mathrm{b}} = 0$ would be only welcome, but its contribution would not be crucial to the final results.

• Our techniques and results can be adapted to the models where random amplitudes modulate both types of scatterer potentials, $\mathfrak{u}^{+}$ and $\mathfrak{u}^{-}$, or only $\mathfrak{u}^{-}$. However, in the latter case, $\mathfrak{u}^{-}$ must have a specific analytic form; a mere upper bound on its decay would be insufficient. Indeed, a compactly supported function fulfills *any* decay condition, but the eigenvalue concentration estimates in Theorems 2 and 3 require both upper and lower decay bounds.

1.3. Main results

THEOREM 1. *Under the assumptions* (V1(1))–(V4(1)), *there exist* $\hat{\rho} > 0$ *and an interval* $I_* \subset (-\infty, 0)$ *such that the following holds with* $\rho \in (0, \hat{\rho}]$ *(cf.* (1.7)*).*

(A) *With probability one, the spectrum of* $H(\omega, \vartheta)$ *in the interval* I_* *is nontrivial and pure point, and all its eigenfunctions* ψ *of* $H(\omega, \vartheta)$ *with eigenvalues* $E_\psi \in I_*$ *decay exponentially at infinity: for some* $m > 0$ *one has*

$$\forall x \in \mathbb{Z}^d \;\; |\psi(x, \omega, \vartheta)| \leqslant C_\psi(\omega, \vartheta) \mathrm{e}^{-m|x|} . \tag{1.12}$$

(B) *Let* $\chi_x = \mathbf{1}_{\mathrm{B}_1(x)}$, $x \in \mathbb{R}^d$. *For any* $\zeta \in (0, 1/3)$ *and some* $C_\zeta \in (0, +\infty)$, *one has:*

$$\mathbb{E}\Big[\sup_{\phi \in \mathcal{B}(I_*)} \big\| \chi_x \, \phi_{I_*}(H(\omega, \vartheta)) \, \chi_y \big\| \Big] \leqslant C_\zeta \mathrm{e}^{-|x-y|^\zeta} , \tag{1.13}$$

where $\mathcal{B}(I_*)$ *is the set of Borel functions* ϕ *with* $\operatorname{supp} \phi_{I_*} \subset I_*$ *and* $\|\phi_{I_*}\|_\infty \leqslant 1$.

REMARK 1. The condition $\rho \leqslant \hat{\rho}$ is used only in Lemma 2. Since ϑ and ω are independent, we shall pick any ϑ required for Lemma 2, and work mainly with the random operator $\omega \mapsto H(\omega, \vartheta)$, ϑ being sometimes dropped from notation.

2. Wegner-type estimates

Introduce some useful notations. Given a point $u = (u_1, \dots, u_d) \in \mathbb{R}^d$ and $L \in \mathbb{R}_+$, we define an open cube centered at u of radius L by

$$\mathrm{B}_L(u) = \left\{ x \in \mathbb{R}^d : |x - u|_\infty < L \right\}, \quad |x|_\infty = \max_{1 \leqslant i \leqslant d} |x_i|. \tag{2.1}$$

We also define the "boundary" of a cube $\mathrm{B}_L(u)$ by $\partial \mathrm{B}_L(u) := \mathrm{B}_L(u) \setminus \mathrm{B}_{L-2}(u)$.

The spectrum of an operator H is denoted by $\Sigma(H)$, and $\Sigma^I(H)$ stands for the restriction $\Sigma(H) \cap I$, $I \subset \mathbb{R}$.

The two main results of this section, Theorems 2 and 3, provide two kinds of Wegner-type estimates required for the multi-scale analysis in its various forms. Theorem 2 is crucial to the fixed-energy scale induction (see Section 4); it suffices for the proof of decay bounds on the Green functions in cubes of growing sizes L_k, $k \geqslant 0$. The latter rule out the almost sure a.c. (absolutely continuous) spectrum of the Hamiltonian $H(\omega, \vartheta)$ in $L^2(\mathbb{R}^d)$ (cf. [26]). Theorem 3 is required to derive spectral and strong dynamical localization from the decay of the Green functions; see Section 5.

These theorems are minor adaptations of similar results proved in [7], so their proofs are omitted (see the Figures 1–2). However, it seems instructive to give here the proof of a simpler result, Lemma 1, also proved in [7] in a lattice context, for it presents the main tools of the harmonic analysis of infinite convolutions of singular measures. The latter relies only on the random scatterer potentials $\omega_\mathrm{b} \mathfrak{u}^+(|\cdot - \mathrm{b}|)$, while the rest of $V(\cdot, \omega, \vartheta)$ including $\vartheta_\mathrm{b} \mathfrak{u}^-(|\cdot - \mathrm{b}|)$ is irrelevant to this analysis. As was said, the probability bounds relative to the random field ϑ are used only in the initial length scale estimates (Lemma 3). For this reason, we fix a random sample ϑ as required in Lemma 3, and then work with it in the rest of the proof, often dropping it from notation.

THEOREM 2. (Cf. [7, Thm. 2.5]) *Fix some* $A > 3d$, $\tau > 8d$, $\kappa > 0$, *and a cube* $\mathrm{B} = \mathrm{B}_L(u)$, *and let* $\bar{\mathrm{B}} := \mathrm{B}_{L^{(1+\kappa)\tau}}(u)$. *Then for some* $\kappa' > \kappa + \frac{1}{3A}$, *one has:*

$$\mathbb{P}\left\{ \omega_{\bar{\mathrm{B}}} \,\middle|\, \exists\, \omega^{\perp}_{\bar{\mathrm{B}}} : \ \mathrm{dist}\left(\Sigma(H_{\mathrm{B}}(\omega_{\bar{\mathrm{B}}} + \omega^{\perp}_{\bar{\mathrm{B}}}), E\right) \leqslant L^{-(1+\kappa)A\tau} \right\} \leqslant L^{-A\kappa'\tau + d}. \tag{2.2}$$

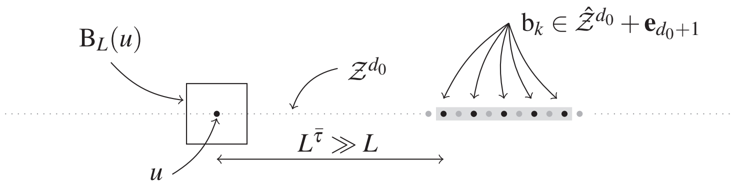

Figure 1: *Example for Theorem 2. Here* $d_0 = 1$, $d = 2$, $\mathbf{e}_{d_0+1} = \mathbf{e}_2 = (0,1)$, *so* $\hat{\mathcal{Z}}^{d_0} + \mathbf{e}_2 \subset \mathbb{Z}^d \setminus \mathcal{Z}^{d_0}$. *The sites* $\mathrm{b}_k \in \mathcal{Z}^{d_0} + \mathbf{e}_{d_0+1}$ *carry the random potentials* $\omega_{\mathrm{b}_k} \mathrm{u}^+_0(\cdot - \mathrm{b}_k)$. *In the proof, one makes use of a collection of random potentials supported by the elements of* $\mathcal{B}_L = \{\mathrm{b}_k, 1 \leqslant k \leqslant n\}$ *such that* $n \asymp \mathrm{diam}\,\mathcal{B}_L \asymp L^{\varrho}$ *with* $0 < \varrho < 1$. *The light gray dots represent the sites* $\mathrm{b} \in \mathcal{Z}^{d_0} \setminus \hat{\mathcal{Z}}^{d_0}$ *which may carry impurities with nonrandom potentials* u^-_1.

REMARK 2. In this paper, we set

$$\kappa := 1.$$

In [7, Thm. 2.5], there is no special "surface" layer, and one has a bound applicable to our model, with any fixed sample of the random field ϑ:

$$\mathbb{P}\left\{ \omega_{\bar{\mathrm{B}}} \,\middle|\, \exists\, \omega^{\perp}_{\bar{\mathrm{B}}} : \ \mathrm{dist}\left(\Sigma(H_{\mathrm{B}}(\omega_{\bar{\mathrm{B}}} + \omega^{\perp}_{\bar{\mathrm{B}}}), E\right) \leqslant \varepsilon_L \right\} \leqslant L^{-A(\kappa + \frac{\varrho}{2A})\tau + d}.$$

Here an auxiliary value ϱ can be chosen in $(0, 1 - \tau^{-1})$. We assume $\tau > 8d$, so one can take $\frac{2}{3} < \varrho < \frac{7}{8}$, getting $\kappa' = \kappa + \frac{1}{3A} + \upsilon$, $\upsilon > \frac{1}{10A}$. Thus (2.2) takes the form used in the rest of the paper:

$$\mathbb{P}\left\{ \omega_{\bar{\mathrm{B}}} \,\middle|\, \exists\, \omega^{\perp}_{\bar{\mathrm{B}}} : \ \mathrm{dist}\left(\Sigma(H_{\mathrm{B}}(\omega_{\bar{\mathrm{B}}} + \omega^{\perp}_{\bar{\mathrm{B}}}, \vartheta), E\right) \leqslant L^{-2A\tau} \right\} \leqslant L^{-\frac{\tau}{10A}} L^{-(A+\frac{1}{3})\tau + d}, \tag{2.3}$$

for L large enough and with fixed ϑ. Consequently, the above RHS is of order of $o\left(L^{-(A+\frac{1}{3})\tau + d}\right)$, hence we are free to replace the first factor $L^{-\frac{\tau}{10A}}$ with any positive constant, if required. ▷

The next Theorem will not be used until Section 5.1, in the context different from Section 4 where the scale induction is carried out; see Remark 3 and the discussion in Section 5.1, where the main difference between Theorems 2 and 2 is explained.

THEOREM 3. (Cf. [7, Thm. 2.7]) *Let be given two cubes* $\mathrm{B}' = \mathrm{B}_L(u')$, $\mathrm{B}'' = \mathrm{B}_L(u'')$ *with* $|u'-u''| = L^\sigma$, $1 < \sigma < \tau/2$, $\tau > 8d$, *and denote* $H' = H_{\mathrm{B}'}$, $H'' = H_{\mathrm{B}''}$. *Let* $A^* := A + 1 + \sigma\tau^{-1}\left(1 - \frac{1}{3A}\right)$. *Then for any* $\kappa > 0$ *and* $\varepsilon \geqslant L^{-A^*(\kappa+1)\tau}$,

$$\mathbb{P}\left\{ \operatorname{dist}\left[\Sigma(H'), \Sigma(H'')\right] \leqslant \varepsilon \right\} \leqslant L^{A^*\tau + 2d}\varepsilon. \tag{2.4}$$

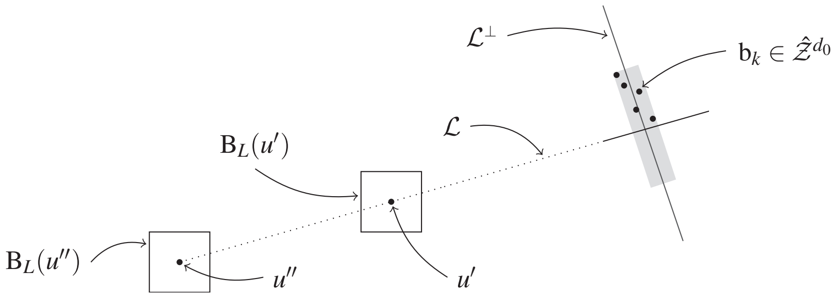

Figure 2: *Example for Theorem 3. Here* $d_0 = 2$, *and the drawing is made in projection onto* $\mathbb{R}^{d_0} \supset \mathcal{Z}^{d_0}$. *The sites* $\mathrm{b} \in \hat{\mathcal{Z}}^{d_0} \subset \mathcal{Z}^{d_0}$ *carry the random potentials* $\omega_{\mathrm{b}} \mathfrak{u}_0^+(\cdot - \mathrm{b})$. *By hypotheses of Theorem 3, here* $L \ll |u'-u''| \asymp L^\sigma \ll |\hat{x} - u'| \asymp L^\tau$. *In the proof, one makes use of a collection of random potentials supported by the elements of* $\mathcal{B}_L = \{\mathrm{b}_k, 1 \leqslant k \leqslant n\}$ *such that* $n \asymp \operatorname{diam} \mathcal{B}_L \asymp L^\varrho$ *with* $0 < \varrho < 1$. $\mathcal{L}$ *is the affine line passing though* u' *and* u'', *and* $\mathcal{L}^\perp$ *is an affine hyperplane in* $\mathbb{R}^{d_0}$ *orthogonal to* $\mathcal{L}$.

REMARK 3. $\varepsilon \leqslant L^{-A^*(\kappa+1)\tau}$ implies that $L^{A^*+2d} \leqslant \varepsilon^{-c}$, with

$$c < \frac{A^*\tau + 2d}{A^*\tau(\kappa+1)} < \frac{1 + \frac{2d}{A^*\tau}}{\kappa+1} < \frac{1 + \frac{1}{4A}}{\kappa+1}, \tag{2.5}$$

(we used $A^* > A$ and $\tau > 8d$), so with $\kappa = 1$ and $A > 3d \geqslant 3$,

$$\mathbb{P}\left\{ \operatorname{dist}\left[\Sigma(H'), \Sigma(H'')\right] \leqslant \varepsilon \right\} \leqslant \varepsilon^{\mathfrak{b}}, \quad \mathfrak{b} > \frac{1}{3}. \tag{2.6}$$

Unlike Theorem 2 where one needs strong non-resonance (SNR) property of the Hamiltonians in the balls of size L_k, $k \geqslant 0$, Section 5.1 operates with NR (non-resonance) condition and not its stronger counterpart, SNR. For this reason, the value of τ can be chosen arbitrarily large, resulting in admissible values of $\varepsilon = L^{-A^*(\kappa+1)\tau} = L^{-2A^*\tau}$ (if one sets $\kappa = 1$) with arbitrarily large $\tau > 1$, hence arbitrarily small $\varepsilon > 0$. *Concluding, we shall use in Section 5.1 a particular case (2.6) of (2.4).* $\triangleright$

REMARK 4. Later, we will assume that $\tau > \max\left[8d, \frac{2d+2}{A-3d}\right]$.

The next statement will be used in Section 6 where we shall prove a variant of the initial length scale bound for a lattice model (cf. Lemma 10).

LEMMA 1. *Let be given a family of IID random variables with zero mean,*

$$\mathrm{X}_{n,k}(\omega),\ n\in\mathbb{N}^*,\ \ 1\leqslant k\leqslant K_n,\ \ K_n\asymp n^{d-1},\ \ d\geqslant 1.$$

Assume that their common characteristic function $\varphi_{\mathrm{X}}(t)=\mathbb{E}\left[\mathrm{e}^{\mathrm{i}t\mathrm{X}_{\bullet,\bullet}}\right]$ *obeys*

$$\forall s\in[-s_0,s_0]\quad \ln\left|\varphi_{\mathrm{X}}(s)\right|^{-1}\geqslant C_\varphi s^2,\quad s_0\in(0,+\infty). \tag{2.7}$$

Let $S(\omega)=\sum_{n\geqslant 1}\sum_{k=1}^{K_n}\mathfrak{a}_{n,k}\mathrm{X}_{n,k}(\omega)$, $\mathfrak{a}_{n,k}\asymp n^{-A}$ *and* $S_{M,N}(\omega)=\sum_{n=M}^{N}\sum_{k=1}^{K_n}\mathfrak{a}_{n,k}\mathrm{X}_{n,k}(\omega)$, $M\leqslant N$. *Then the following holds true.*

(A) *There exist some* $C,c\in(0,+\infty)$ *such that* $\forall t\in\mathbb{R}\ \left|\mathbb{E}\left[\mathrm{e}^{\mathrm{i}tS(\omega)}\right]\right|\leqslant C\mathrm{e}^{-c|t|^{d/A}}$. *Therefore,* $S(\omega)$ *admits a probability density* $\mathrm{p}_S\in\mathcal{C}^\infty(\mathbb{R})$.

(B) *For* $N\geqslant(1+c')M\geqslant 1$ *with* $c'>0$, *and* $|t|M^{-A}\leqslant s_0$,

$$\ln\left|\mathbb{E}\left[\mathrm{e}^{\mathrm{i}tS_{M,N}(\omega)}\right]\right|^{-1}\gtrsim M^{-2A+d}t^2. \tag{2.8}$$

(C) *For any* $\varepsilon\geqslant N^{-A}$, $\sup_{a\in\mathbb{R}}\mathbb{P}\left\{S_{M,N}(\omega)\in[a,a+\varepsilon]\right\}\lesssim M^{A-\frac{d}{2}}\varepsilon$.

Here and below, we use the notations "$\gtrsim$", "$\lesssim$", and "$\asymp$" for comparison of functions of some parameter. For example, $f(t)\gtrsim g(t)$ means that $f(t)\geqslant Cg(t)$ for some $C\in(0,+\infty)$, "$\lesssim$" is its counterpart for upper bounds, and $f(t)\asymp g(t)\iff\left(f(t)\lesssim g(t)\lesssim f(t)\right)$.

In applications of Lemma 1 to **model 2** (cf. Section 6), we operate with $\mathrm{X}_{n,k}=\omega_{\mathrm{b}_{n,k}}$, and the sequences $(\mathrm{b}_{n,k}),\mathfrak{a}_{n,k}$ are defined as follows. First, decompose the family of the scatterers' sites $\left\{\omega_{\mathrm{b}},0\neq\mathrm{b}\in\mathbb{Z}^d\right\}$ into a union of sub-families $\mathscr{A}_n:=\left\{\omega_{\mathrm{b}}:|\mathrm{b}|\in[n,n+1)\right\}$, $n\in\mathbb{N}^*$. Next, for each $n\in\mathbb{N}$, order $\mathscr{A}_n$ in some way and denote its elements by $\mathrm{b}_{n,k}$, $1\leqslant k\leqslant K_n\asymp n^{d-1}$. Further, let $\mathfrak{a}_{n,k}=\mathfrak{u}^+(|\mathrm{b}_{n,k}-0|)$, so that for all $\mathrm{b}_{n,k}$ with fixed $n\geqslant 1$, one has $\mathfrak{a}_{n,k}\sim\mathfrak{u}^+(n)=n^{-A}$.

The condition (2.7) is valid for all probability measures with finite moment of order 3. We assume (V1(1)), so the r.v. $\omega_{\bullet,\bullet}$ have finite moments of all orders.

Proof. **(A)** Let $\varphi_S(t):=\mathbb{E}\left[\exp\left(\mathrm{i}t\sum_{n\geqslant 1}\sum_{k=1}^{K_n}\mathfrak{a}_{n,k}\mathrm{X}_{n,k}\right)\right]$. Let $N_t=C_\varphi|t|^{1/A}$ (cf. (2.7)), so $n^{-A}|t|\leqslant N_t^{-A}|t|\in[0,s_0]$ for all $n\geqslant N_t$. By independence of $\mathrm{X}_{\bullet,\bullet}$, we have:

$$\begin{aligned}\ln\left|\varphi_S(t)\right|^{-1}&\geqslant\sum_{n\geqslant 1}\min_k K_n\ln\left|\varphi_\omega\left(\mathfrak{a}_{n,k}t\right)\right|^{-1}\\&\gtrsim\Bigg(\sum_{n\leqslant N_t}+\sum_{n>N_t}\Bigg)n^{d-1}\min_k\ln\left|\varphi_\omega\left(\mathfrak{a}_{n,k}t\right)\right|^{-1}=:\mathcal{S}_1(t)+\mathcal{S}_2(t)\geqslant\mathcal{S}_2(t),\end{aligned} \tag{2.9}$$

since $|\varphi_{\mathrm{X}}(t)|\leqslant 1$ for $t\in\mathbb{R}$, thus $\ln|\varphi_{\mathrm{X}}(t)|^{-1}\geqslant 0$ and $\mathcal{S}_{1,2}(t)\geqslant 0$. We focus on $\mathcal{S}_2(t)$, for this suffices to prove assertion (A): with $\mathfrak{a}_{n,k}\sim n^{-A}$ as $n\to+\infty$, we obtain

$$\mathcal{S}_2(t)\gtrsim t^2\sum_{n>N_t}n^{-2A+d-1}\gtrsim t^2\int_{C_\varphi|t|^{1/A}}^{+\infty}s^{-2A+d-1}\,ds\gtrsim t^2|t|^{-\frac{2A-d}{A}}=|t|^{d/A}. \tag{2.10}$$

Note that, as was shown by Wintner [33], a simple equidistribution lemma by Pólya and Szegö [29, Section II.4.1, Problem 155] implies a similar lower bound $\mathcal{S}_1(t) \gtrsim |t|^{d/A}$.
(B) The proof is similar:

$$\sum_{n=M}^{N} \ln \left| \varphi_X(\mathfrak{a}_{n,k} t) \right|^{-1} \gtrsim t^2 \int_{M}^{(1+c')M} s^{-2A+d-1}\, ds \gtrsim t^2 M^{-2A+d}. \tag{2.11}$$

(C) We need to assess the integrals of the probability measure of $S(\omega)$ over intervals $I_\varepsilon \subset \mathbb{R}$ of length $\mathrm{O}(\varepsilon)$. It suffices to consider the case where $I_\varepsilon = [-\varepsilon, \varepsilon]$ to have less cumbersome formulae. Further, since the main estimate will be achieved in the Fourier representation, it is customary to work with a smoothed indicator function instead of $\mathbf{1}_{I_\varepsilon}$. A convenient choice of smoothing function was found in the theory of asymptotic expansions for the sums of independent r.v. (cf. [2, 11, 15, 16]). Specifically, given a probability measure $\tilde{\mu}$ on $\mathbb{R}$ with characteristic function $\varphi_{\tilde{\mu}}$, one can upper-bound $\tilde{\mu}(I_\varepsilon)$ by assessing $\varphi_{\tilde{\mu}}(t)$ only in a finite interval (cf. [2]):

$$\tilde{\mu}(I_\varepsilon) \leqslant 2\varepsilon \int_{|t| \leqslant \varepsilon^{-1}} \left| \varphi_{\tilde{\mu}}(t) \right| dt. \tag{2.12}$$

Recall that we have assumed the lower bound (2.7). Define a mapping

$$n \mapsto T_n = \sup\left\{ t > 0 : \mathfrak{a}_n t \sim n^{-A} t \leqslant s_0 \right\} = C(s_0) n^A, \tag{2.13}$$

and let $\mathcal{X}_n := \left\{ x \in \mathbb{Z}^d : |x| \in [n, n+1) \right\}, n \in \mathbb{N}$, then

$$\forall t \in [-T_n, T_n] \qquad \sum_{x \in \mathcal{X}_n} \ln \left| \varphi_X(\mathfrak{a}_n t) \right|^{-1} \gtrsim t^2 n^{-2A+d-1}. \tag{2.14}$$

- For $|t| \leqslant T_M$ and $N \geqslant (1+c)M$ with $c > 0$, we have by **(B)**

$$\ln \left| \varphi_{S_{M,N}}(t) \right|^{-1} \gtrsim t^2 M^{-2A+d}. \tag{2.15}$$

- For $T_M \leqslant |t| \leqslant T_N$, hence for $|t| \in \left[T_M, \varepsilon^{-1}\right]$, we have, setting $N_t := C_\varphi |t|^{1/A}$ and assuming $(1+c)N_t \leqslant N$ with some $c > 0$,

$$\ln \left| \varphi_{S_{M,N}}(t) \right|^{-1} \geqslant \sum_{n=N_t}^{N} K_n \ln \left| \varphi_X(\mathfrak{a}_{n,k} t) \right|^{-1} \gtrsim t^2 \sum_{n=N_t}^{N} n^{d-1} \mathfrak{a}_n^2 t^2 \gtrsim t^2 N_t^{-2A+d} \asymp |t|^{\frac{d}{A}}. \tag{2.16}$$

Applying (2.12), we get

$$\mu_{S_{M,N}}(I_\varepsilon) \leqslant 2\varepsilon \int_{-T_M}^{T_M} \left| \varphi_S(t) \right| dt + 2\varepsilon \int_{T_M \leqslant |t| \leqslant \varepsilon^{-1}} \left| \varphi_S(t) \right| dt =: J_- + J_+, \tag{2.17}$$

where, by (2.15),

$$J_- \leqslant 2\varepsilon \int_{-T_M}^{T_M} \left| \varphi_S(t) \right| dt \lesssim \varepsilon \int_{\mathbb{R}} \mathrm{e}^{-t^2 \,\mathrm{Const}\, M^{-2A+d}}\, dt \lesssim M^{A-\frac{d}{2}}\, \varepsilon, \tag{2.18}$$

while for J_+ we have a much smaller upper bound:

$$J_+ \lesssim 2\varepsilon \int_{T_M}^{\varepsilon^{-1}} \mathrm{e}^{-c|t|^{d/A}}\, dt \lesssim \mathrm{e}^{-T_M^{d/A}}\, \varepsilon = \mathrm{o}\left(M^{A-\frac{d}{2}}\,\varepsilon\right), \quad \text{as } M \to \infty. \tag{2.19}$$

This completes the proof: $\mu_{S_{M,N}}(I_\varepsilon) \lesssim M^{A-\frac{d}{2}}\,\varepsilon$, for $\varepsilon \geqslant N^{-A}$. $\square$

3. Initial length scale estimates

3.1. Basic notation and definitions

Given $\tau > 1$ and $L_0 \in \mathbb{N}^*$, set $\alpha = 3\tau$ and define recursively a sequence L_k by

$$L_k := \left\lceil L_{k-1}^{\alpha} \right\rceil, \ \ k \geqslant 1. \tag{3.1}$$

As is rather customary, in a number of calculations involving the length scales, we make a slightly abusive substitution for $L_k = \left\lceil L_{k-1}^{\alpha} \right\rceil$ and use instead $L_{k-1}^{\alpha} \sim \left\lceil L_{k-1}^{\alpha} \right\rceil$ (in the limit $L_0 \to +\infty$). Further, let $m_0 > 0$, and set for $k \geqslant 0$

$$m_k := m_0 \prod_{j=0}^{k} (1-\eta_j), \quad \eta_k := 2L_k^{-c}, \ \ c > 0, \tag{3.2}$$

with $c > 0$ to be specified in the proof of Corollary 1. Clearly, with L_0 large enough we have a convergent product $\prod_{j\geqslant 0}(1-\eta_j) \geqslant \frac{1}{2}$, so $m_k \geqslant m_0/2$.

Given $A > 3d$ and $\tau > 1$, we set $\gamma := \frac{A-3d}{2}$, so that $A = 3d + 2\gamma$, and define two positive sequences:

$$\delta_k := \mathrm{e}^{-m_k L_k}, \quad \varepsilon_k := L_k^{-2A\tau}. \tag{3.3}$$

DEFINITION 1. Let be given a cube $\mathrm{B} = \mathrm{B}_{L_k}(u)$, $k \geqslant 0$. A sample ω is called

(1) (E,δ)-NS (non-singular) in B iff $E \notin \Sigma(H_{\mathrm{B}}(\omega))$ and

$$\left\| \mathbf{1}_{\partial \mathrm{B}_{L_k}} G_{\mathrm{B}_{L_k}}(E,\omega)\, \mathbf{1}_{\mathrm{B}_{L_k/3}} \right\| \leqslant (3L_k)^{-d}\,\delta; \tag{3.4}$$

(2) (E,ε)-NR (non-resonant) in B iff

$$\operatorname{dist}\left[\Sigma(H_{\mathrm{B}_{L_k}}), E\right] \geqslant \varepsilon; \tag{3.5}$$

(3) (E,ε)-CNR (completely non-resonant) in $\mathrm{B}_{L_{k+1}}$ iff it is (E,ε)-NR for in all balls $\mathrm{B}_L(u)$ with $L_k \leqslant L \leqslant L_{k+1}$.

DEFINITION 2. Given $\tau > 1$ and a cube $\mathrm{B} = \mathrm{B}_{L_k}(u)$, $k \geqslant 0$, denote $\bar{\mathrm{B}} = \mathrm{B}_{L_k^{2\tau}}(u)$ and consider the decomposition $\omega = \omega_{\bar{\mathrm{B}}} + \omega_{\bar{\mathrm{B}}}^{\perp}$. Let $\mathscr{P}$ be one of the properties (E,δ)-NS, (E,ε)-NR, or (E,ε)-CNR relative to B. We will say that a sub-sample $\omega_{\bar{\mathrm{B}}}$ has a strong (or stable) property $\mathscr{P}$ iff for any complementary sub-sample $\omega_{\bar{\mathrm{B}}}^{\perp}$ the full sample $\omega = (\omega_{\bar{\mathrm{B}}}, \omega_{\bar{\mathrm{B}}}^{\perp})$ has property $\mathscr{P}$ in B. Respectively, the three aforementioned notations are replaced with (E,δ)-SNS, (E,ε)-SNR, and (E,ε)-SCNR.

Note that an event of the form $\{ \omega_{\bar{\mathrm{B}}}$ has strong property $\mathscr{P} \}$ is measurable with respect to the σ-algebra $\mathfrak{F}_{\bar{\mathrm{B}}}$ generated by $\{\omega_{\mathrm{b}}, \mathrm{a} \in \bar{\mathrm{B}}\}$, hence any family of such events relative to disjoint cubes $\mathrm{B}_L(u_i)$, $i \in [\![1,M]\!]$ is independent.

DEFINITION 3. • A cube $\mathrm{B}_{L_{k+1}}(u)$ is called (E,δ_k,τ,K)-good iff it contains no collection of K (or more) pairwise $L_k^{2\tau}$-distant cubes $\{\mathrm{B}_{L_k}(x_i), 1 \leqslant i \leqslant K\}$, neither of which is (E,δ_k)-NS.

• The cube $\mathrm{B}_{L_{k+1}}(u)$ is called (E,δ_k,τ,K)-strongly-good ((E,δ_k,τ,K)-S-good) iff it contains no collection of K (or more) pairwise L_k^{τ}-distant cubes $\{\mathrm{B}_{L_k}(x_i), 1 \leqslant i \leqslant K\}$, neither of which is (E,δ_k)-SNS.

3.2. Initial length scale (ILS) estimate

Recall that by (V4(1)), with probability 1, the operator $H(\omega,\vartheta)$ has nontrivial spectrum in $(-\infty,E_*)$, hence in some interval $I_* \subset (-\infty,E_*)$ of length $|I_*| > 0$.

LEMMA 2. (ILS for the model with sparse impurities) *Let $\tau > 1$. There exist $C,c,L_* > 0$ and an interval $I_* \subset (-\infty,E_*)$ such that, for any $L_0 \geqslant L_*$, $\mathfrak{s} > 0$, and $\rho \in \big(0,\rho^*(L_0,s)\big)$ with $\rho^*(L_0,s)$ small enough, one has, with $\delta_0 = \mathrm{e}^{-c\mathscr{E}L_0}$:*

$$\sup_{E\in I_*} \mathbb{P}\left\{ \mathrm{B}_{L_0}(x) \text{ is not } (E,\delta_0\,,\tau)\text{-SNS} \right\} \leqslant L_0^{-\mathfrak{s}}\,. \tag{3.6}$$

Proof. Fix $L_0 \in \mathbb{N}$, and consider the operator $H_{\mathrm{B}_{L_0}(u)}(\omega)$. If there are no impurity atoms in an augmented cube $\mathrm{B}_{L_0+r_0}(u)$), i.e., $\forall\, \mathrm{b} \in \mathrm{B}_{L_0+r_0}$ $\theta_{\mathrm{b}} = 0$, then

$$\inf_{x\in \mathrm{B}_{L_0}(u)} V(x,\omega,\vartheta) \geqslant E_* + \mathscr{E},$$

thus

$$\forall E \leqslant E_* \quad \operatorname{dist}\left[\Sigma\big(H_{\mathrm{B}}(\omega,\vartheta)\big),E\right] \geqslant \mathscr{E}.$$

By the Combes–Thomas estimate [10], there exist some $C,c \in (0,+\infty)$ such that

$$\forall E \leqslant E_* \quad \left\| \mathbf{1}_{\partial \mathrm{B}}\, G_{\mathrm{B}(L_0\,,u)}(E)\, \mathbf{1}_{\mathrm{B}(\frac{1}{3}L_0\,,u)} \right\| \leqslant C\mathscr{E}^{-1}\mathrm{e}^{-2c\mathscr{E}L_k}$$

With an appropriate choice of L_*, this implies the (E,δ_0)-non-singularity of $\mathrm{B}_{L_0}(u)$, where $\delta_0 = \mathrm{e}^{-m_0L_0}$ and, e.g., $m_0 = c\mathscr{E} > 0$.

Finally, notice that if $\mathrm{B}_{L_0}(u) \cap \mathcal{Z}^{d_0} = \varnothing$, then $\mathrm{B}_{L_0}(u)$ contains no impurity; otherwise, one has

$$\mathbb{P}\left\{ \exists\, \mathrm{b} \in \mathrm{B}_{L_0+r_0}(u):\ \theta_{\mathrm{b}} \neq 0 \right\} \leqslant |\mathrm{B}_{L_0+r_0}(u)|\rho,$$

and once L_0 is fixed, the RHS is bounded by $L_0^{-\mathfrak{s}}$ for $0 < \rho \leqslant |\mathrm{B}_{L_0+r_0}(u)|^{-1}L_0^{-\mathfrak{s}}$. $\square$

Next, we consider a model briefly discussed in Section 7 (cf. **model 4**), where the impurities carrying negative potentials fill the entire periodic sub-lattice $\mathcal{Z}^{d_0}$, but the amplitudes of the positive potentials supported by the sites $\mathrm{b} \in \mathbb{Z}^d \setminus \mathcal{Z}^{d_0}$ are IID random variables, so the random potential has the form $V(x;\boldsymbol{\omega}) = V^{(\mathrm{B})}(x;\omega) + V^{(\mathrm{S})}(x)$ with non-random, periodic surface potential and random positive bulk potential:

$$V^{(\mathrm{S})}(x) = \sum_{\mathrm{b}\in\mathbb{Z}^{d_0}} \mathfrak{u}^{-}(|x-\mathrm{b}|) \tag{3.7}$$

$$V^{(\mathrm{B})}(x;\omega) = \sum_{\mathrm{b}\in\mathbb{Z}^{d}\setminus\mathcal{Z}^{d_0}} \omega_{\mathrm{b}}\, \mathfrak{u}^{+}(|x-\mathrm{b}|) \geqslant 0. \tag{3.8}$$

We define the background operator $H_0 = -\Delta + V^{(\mathrm{S})}$ and let $E_*^0 = \inf\Sigma(H_0)$.

LEMMA 3. (ILS for periodic impurities in random bulk) *Consider the Hamiltonian* $-h\Delta + V(x;\boldsymbol{\omega})$*, assume the hypotheses* (V1(4))–(V4(4)) *(cf. Section 7), and fix* $\tau > 1$*. Then for any* $q\in(0,1)$*, there exist* $L_*, c > 0$ *such that for any* $L_0 \geqslant L_*$ *one has, with* $\mathrm{B} = \mathrm{B}_{L_0}(u)$ *and* $\delta_0 = \mathrm{e}^{-m_0 L_0}$*,* $m_0 = cL_0^{-q}$*:*

$$\sup_{E\in\left[E_*^0, E_*^0 + L_0^{-q}\right]} \ \sup_{u\in\mathbb{R}^d} \ \mathbb{P}\left\{ \mathrm{B}_{L_0}(u) \textit{ is not } (E,\delta_0,\tau)\textit{-SNS} \right\} \leqslant L_0^{-\mathfrak{s}}. \tag{3.9}$$

Proof. As in Lemma 2, we can apply the Combes–Thomas estimate, so it suffices to prove that, for any $q\in(0,1)$ and with L_0 large enough,

$$\mathbb{P}\left\{ \omega_{\bar{\mathrm{B}}} : \inf_{\omega_{\bar{\mathrm{B}}}^{\perp}} \inf\Sigma\left(H(\omega_{\bar{\mathrm{B}}} + \omega_{\bar{\mathrm{B}}}^{\perp}, \vartheta) \right) \leqslant E_* + 2L_0^{-q} \right\} \leqslant L_0^{-\mathfrak{s}}. \tag{3.10}$$

Since $\omega_{\mathrm{b}} \geqslant 0$ for all $\mathrm{b}\in\mathbb{Z}^d$ by the hypothesis (V1(1)), we have:

$$\forall x\in\mathrm{B} \qquad \inf_{\omega_{\bar{\mathrm{B}}}^{\perp}} V^{(\mathrm{B})}(x,\omega_{\bar{\mathrm{B}}} + \omega_{\bar{\mathrm{B}}}^{\perp}) = V^{(\mathrm{B})}(x,\omega_{\bar{\mathrm{B}}} + \emptyset_{\bar{\mathrm{B}}}^{\perp}),$$

so the claim would follow from the estimate where $\omega_{\bar{\mathrm{B}}}^{\perp}$ in the LHS of (3.10) is replaced with the empty configuration $\emptyset_{\bar{\mathrm{B}}}^{\perp}$. *(As a result, the actual value of* $\tau > 1$ *is irrelevant.)*

Fix any $h > 0$ in $h\Delta$, $q\in(0,1)$, and $L_0 > 1$ required below to be large enough. Pick some $q'\in(0,1)$ to be fixed later, and let $l_0 = \lceil L_0^{q'} \rceil \sim L_0^{q'}$. Partition B into a union of adjacent cubes $\mathrm{B}_{(i)} := \mathrm{B}_{l_0}(u_i)$, $i = 1,\dots,M := \lfloor L_0^d / l_0^d \rfloor$, and let $\Lambda_i = \mathrm{B}_{(i)} \cap \mathbb{Z}^d$. Owing to the Dirichlet–Neumann bracketing, it suffices to work with the Neumann boundary conditions, so in the rest of the proof, we consider the operators $\Delta_{\mathrm{B}}^{\mathrm{N}}$, $H_{\mathrm{B}}^{\mathrm{N}}(\omega) = -h\Delta_{\mathrm{B}}^{\mathrm{N}} + V(\cdot,\omega)$ (here and below, the superscript "N" stands for "Neumann") and their counterparts $\Delta_i^{\mathrm{N}} = \Delta_{\mathrm{B}_{(i)}}^{\mathrm{N}}$, $H_{\mathrm{B}_{(i)}}^{\mathrm{N}}(\omega)$ in smaller cubes $\mathrm{B}_{(i)}$. Since $\Delta_{\mathrm{B}}^{\mathrm{N}} \geqslant \oplus_{i=1}^{M} \Delta_{\mathrm{B}_{(i)}}^{\mathrm{N}}$, we also have $H_{\mathrm{B}}^{\mathrm{N}} \geqslant \oplus_{i=1}^{M} H_{\mathrm{B}_{(i)}}^{\mathrm{N}}$, for any potential $V : \mathrm{B}\to\mathbb{R}$. Therefore,

$$\inf\Sigma\left(H_{\mathrm{B}}^{\mathrm{N}}\right) \geqslant \min_{1\leqslant i\leqslant M} \inf\Sigma\left(H_{\mathrm{B}_{(i)}}^{\mathrm{N}}\right). \tag{3.11}$$

Further, we have $\inf \Sigma\left(H^{\mathrm{N}}_{\mathrm{B}_{(i)}}(\omega)\right) \geqslant E_* + \inf_{x\in \mathrm{B}_{(i)}} V^{(\mathrm{B})}(x,\omega)$. Since the random field $V^{(\mathrm{B})}(\cdot,\omega)$ is translation invariant, (3.11) implies a probabilistic bound: for any $\varepsilon > 0$,

$$\mathbb{P}\left\{ \inf \Sigma\left(H^{\mathrm{N}}_{\mathrm{B}}(\omega)\right) \leqslant E_* + \varepsilon \right\} \leqslant L_0^{d(1-q')}\, \mathbb{P}\Big\{ \inf_{x\in \mathrm{B}_{l_0}(0)} V(x,\omega) \leqslant \varepsilon \Big\}. \tag{3.12}$$

By (V1(1)), the support of the probability measure μ of the IID random variables $\omega_\bullet$ is not a single point, so there exist $s > 0$ and $\nu' > 0$ such that

$$\mathbb{P}\Big\{ \max_{\mathrm{b}\in\Lambda_i} \omega_{\mathrm{b}} < s \Big\} \leqslant \left(\mathrm{e}^{-\nu'}\right)^{l_0^d} = \mathrm{e}^{-\nu' L_0^{q'd}}\,. \tag{3.13}$$

If $\max_{\mathrm{b}\in\Lambda_i} \omega_{\mathrm{b}} \geqslant h$, then $\inf_{x\in \mathrm{B}_{(i)}} gV(x,\omega_{\mathrm{B}}) \geqslant gsl_0^{-A} = gsL_0^{-q'A}$, thus (3.13) implies

$$\mathbb{P}\Big\{ \omega_{\mathrm{B}} : \inf_{x\in\mathrm{B}} gV(x,\omega_{\mathrm{B}}) < gsL_0^{-q'A} \Big\} \leqslant M\mathrm{e}^{-\nu' L_0^{q'd}} \ll L_0^d\, \mathrm{e}^{-\nu' L_0^{q'd}}\,.$$

Therefore, with $\nu = \nu'/2$, $q' = (1+c')q/A$, $c' \ll 1$, $c = q'd$, and L_0 large enough:

$$\mathbb{P}\Big\{ \omega_{\mathrm{B}} : \inf_{x\in \mathrm{B}_{(i)}} gV(x,\omega_{\mathrm{B}}) < L_0^{-q} \Big\} \leqslant \mathrm{e}^{-\nu L_0^c} \leqslant L_0^{-\mathfrak{s}}\,. \tag{3.14}$$

This proves (3.10), and (3.9) follows with the help of the Combes–Thomas bound. $\square$

4. Fixed-energy multi-scale analysis of model 1

The following statement is a standard result of the multi-scale analysis originating in [12, 31], [13, Lemma 4.2], and streamlined in [21, Section 5]. The assumptions and the main statement are quite flexible and can be adapted various models in continuous spaces and on graphs, including periodic lattices; see, e.g., [5, Lemma 2], [9, Lemma 3.1]. Both in continuous and discrete models, the main analytic tool is an analog of Lieb–Simon inequality (also known as geometric resolvent inequality); cf. [21, 32].

LEMMA 4. (Conditions for non-singularity) *Let $u \in \mathbb{R}^d$, and suppose that*
(i) $\mathrm{B}_{L_{k+1}}(u)$ *is* (E, ε_{k+1})*-NR with* $\varepsilon_{k+1} \geqslant \delta_k^{1-c}$, *for some* $\varepsilon_{k+1}, \delta_k, c \in (0,1)$*;*
(ii) $\mathrm{B}_{L_{k+1}}(u)$ *is* (E, δ_k, K)*-good, with* $K \geqslant 0$ *such that*

$$N := \lfloor L_{k+1}/L_k \rfloor - 10K \lfloor L_k^{2\tau} \rfloor \geqslant 1\,. \tag{4.1}$$

Then $\mathrm{B}_{L_{k+1}}(u)$ *is* (E, δ_k^{N+c})*-NS.*

In our case $\delta_k = \mathrm{e}^{-m_k L_k}$ and $\varepsilon_{k+1} = \mathrm{e}^{-\mathrm{O}(\ln L_{k+1})}$, hence the condition $\varepsilon_{k+1} \geqslant \delta_k^{1-c}$ in hypothesis **(ii)** is fulfilled for any $c \in (0,1)$ and large L_0.

COROLLARY 1. (Conditions for strong non-singularity) *Let be given a cube* $\mathrm{B} = \mathrm{B}_{L_{k+1}}(u)$, $k \geqslant 0$, *and suppose that*
(i) B *is* $(E, \varepsilon_{k+1}, \tau)$-SNR*;*
(ii) B *is* (E, δ_k, τ, K)-S-good*, with* $K \geqslant 0$ *such that* (4.1) *holds.*
Then B *is* (E, δ_{k+1}, τ)-SNS.

Proof. Denote $\bar{\mathrm{B}} = \mathrm{B}_{L_{k+1}^{2\tau}}(u)$. One has to show that, with a fixed sample $\omega_{\bar{\mathrm{B}}}^{\perp}$ satisfying the hypotheses **(i)**–**(ii)**, the cube B is (E, δ_{k+1})-NS for the sample $(\omega_{\bar{\mathrm{B}}}, \omega_{\bar{\mathrm{B}}}^{\perp})$ regardless of the complementary sample $\omega_{\bar{\mathrm{B}}}^{\perp}$.

First, notice that the condition **(i)** is already stable with respect to $\omega_{\bar{\mathrm{B}}}^{\perp}$.

Next, by **(ii)** there are at most $K-1$ cubes $\mathrm{B}_{L_k}(x_i)$ which are pairwise $L_k^{2\tau}$-distant and such that any ball $\mathrm{B}_{L_k}(x)$ with $x \notin \cup_{i=1}^{K} \mathrm{B}_{L_k^{2\tau}}(x_i)$ is (E, δ_k, τ)-SNS. The support of $\omega_{\bar{\mathrm{B}}}^{\perp}$ is outside all cubes $\mathrm{B}_{L_k^{2\tau}}(x_i)$, hence such distant samples $\omega_{\bar{\mathrm{B}}}^{\perp}$ cannot affect the strong non-singularity property of the cubes $\mathrm{B}_{L_k}(x_i)$. Applying Lemma 4, we see that the cube B is $(E, \tilde{\delta}_{k+1})$-NS with

$$
\begin{aligned}
-\ln \tilde{\delta}_{k+1} &= N_{k+1} m_k L_k - C' \ln L_{k+1} + \Big(\ln (3L_{k+1})^d - \ln (3L_{k+1})^d\Big) \\
&\geqslant L_k Y_{k+1} m_k \left(1 - \frac{10 K L_k^{2\tau}}{L_k^{\alpha}} - \frac{C'' \ln L_{k+1}}{m_k L_{k+1}}\right) + \ln (3L_{k+1})^d \\
&\geqslant L_{k+1} m_k (1-\eta_k) + \ln (3L_{k+1})^d,
\end{aligned}
\tag{4.2}
$$

where η_k is as in (3.2), with $c = \alpha - 2\tau > 0$. Thus B is (E, δ_{k+1}, τ)-SNS. $\square$

Until the end of this section, we will need to examine only the non-singularity properties of the cubes having non-empty intersection with the sub-lattice $\mathbb{Z}^{d_0}$ (the impurity layer), since we are concerned only with energies $E \in I_*$, and any cube outside this layer has energies above I_*. On any scale, we are free to choose a partition of $\mathbb{R}^d$ into a union of cubes of size L_k, so can we cover first $\mathbb{Z}^{d_0}$ by cubes $\mathrm{B}_{L_k}(\mathrm{b})$ with $\mathrm{b} \in \mathbb{Z}^{d_0}$, and then decompose the rest of the space into L_k-cubes having empty intersection with $\mathbb{Z}^{d_0}$. We will not repeat this fact every time again. Also, it is readily seen that the entire bulk represents a "forbidden zone" for eigenfunctions with eigenvalues $E < E_*$, so the latter decay exponentially, and deterministically, away from $\mathcal{Z}^{d_0}$, and we have to prove their decay only along $\mathcal{Z}^{d_0}$.

LEMMA 5. *Let be given the real numbers $A > 3d$ and $\tau > 1$. Consider a cube $\mathrm{B}_{L_{k+1}}(u)$ and let $\varepsilon_{k+1} = L_{k+1}^{-2A\tau}$. Then*

$$
\mathbb{P}\Big\{ \omega_{\bar{\mathrm{B}}} : \exists\, \omega_{\bar{\mathrm{B}}}^{\perp}\ \mathrm{B}_{L_{k+1}}(u) \text{ is not } (E, \varepsilon_{k+1}, \tau)\text{-SCNR} \Big\} \leqslant \frac{1}{2} L_{k+1}^{-(A+\frac{1}{3})\tau + d + 1}. \tag{4.3}
$$

Proof. By Definition 2, if $\mathrm{B}_{L_{k+1}}(u)$ is not $(E, \varepsilon_{k+1}, \tau)$-SCNR, then for some $R \in [\![L_k, L_{k+1}]\!]$ the cube $\mathrm{B}_R(u)$ is not $(E, \varepsilon_{k+1}, \tau)$-SNR. Even the largest among them, $\mathrm{B}_{L_{k+1}}(u)$, is surrounded by a belt of width L_{k+1}^{τ}, so by Theorem 2 (cf. (2.3)),

$$
\mathbb{P}\Big\{ \omega_{\bar{\mathrm{B}}} : \inf_{\omega_{\bar{\mathrm{B}}}^{\perp}} \ \mathrm{dist}\left[\Sigma\big(H_{\mathrm{B}_R(u)}\big), E\right] \leqslant 2\varepsilon_{k+1} \Big\} = \mathrm{o}\left(L_{k+1}^{-\left(A+\frac{1}{3}\right)\tau + d} \right) \leqslant \frac{1}{2} L_{k+1}^{-\left(A+\frac{1}{3}\right)\tau + d},
$$

which proves (4.3), since $R \in [\![L_k, L_{k+1}]\!]$ takes less than L_{k+1} values. $\square$

LEMMA 6. *Assume that $A = 3d + 2\gamma$, $\gamma > 0$, and let $\tau > \frac{d+1}{\gamma} = \frac{2d+2}{A-3d}$. Set*

$$\alpha = 3\tau, \quad \mathfrak{s} = \left(A + \tfrac{1}{3}\right)\tau - d - 1 > A\tau - d - 1, \tag{4.4}$$

then

$$\mathfrak{s} - \alpha d > \gamma\tau > 0. \tag{4.5}$$

Furthermore, assume that

$$p_k := \sup_{x \in \mathbb{R}^d} \mathbb{P}\left\{ \mathrm{B}_{L_k}(x) \text{ is not } (E, \delta_k, \tau)\text{-SNS} \right\} \leqslant L_k^{-\mathfrak{s}}. \tag{4.6}$$

Fix any integer $M \geqslant 1$ and assume that

$$\frac{1}{2} L_0^{\alpha - 2\tau} \equiv \frac{1}{2} L_0^{\tau} \geqslant K := \left\lceil 2M \cdot 3\mathfrak{s}\gamma^{-1} \right\rceil. \tag{4.7}$$

Then for L_0 large enough

$$\mathbb{P}\left\{ \mathrm{B}_{L_{k+1}}(u) \text{ is not } (E, \delta_{k+1}, K, \tau)\text{-S-good} \right\} \leqslant \frac{1}{2} L_{k+1}^{-M\mathfrak{s}}. \tag{4.8}$$

Proof. (4.5) follows by a simple calculation: with $A = 3d + 2\gamma$ and $\gamma\tau > d + 1$,

$$\mathfrak{s} - \alpha d > A\tau - d - 1 - 3\tau d \geqslant 2 \cdot 1\gamma\tau - (d+1) > \gamma\tau.$$

It follows from Definition 2 that the event $\mathscr{B}_x = \left\{ \omega : \mathrm{B}_{L_k}(x) \text{ is not } (E, \delta_k, \tau)\text{-SNS} \right\}$ is $\mathfrak{F}\left(\mathrm{B}(L_k^{2\tau}, (x)\right)$-measurable, and so if the cubes $\mathrm{B}_{L_k^{2\tau}}(x_i)$, $1 \leqslant i \leqslant K$, are disjoint, then

$$\mathbb{P}\left\{ \cap_{i=1}^{K} \mathscr{B}_{x_i} \right\} = \prod_{i=1}^{K} \mathbb{P}\left\{ \mathscr{B}_{x_i} \right\} \leqslant p_k^K. \tag{4.9}$$

By (4.5), we have $\frac{\mathfrak{s} - \alpha d}{\alpha} > \frac{\gamma\tau}{3\tau} = \frac{\gamma}{3}$, so with $K = \left\lceil 6M\mathfrak{s}\gamma^{-1} \right\rceil$, the random maximal number $\mathcal{S}(\omega)$ of pairwise $L_k^{2\tau}$-distant singular cubes $\mathrm{B}_{L_k}(x_i)$ inside $\mathrm{B}_{L_{k+1}}(u)$ obeys

$$\mathbb{P}\left\{ \mathcal{S}(\omega) \geqslant K \right\} \leqslant C L_{k+1}^{Kd} p_k^K \leqslant C L_{k+1}^{-K\left(\frac{\mathfrak{s}}{\alpha} - d\right)} < \frac{1}{2} L_{k+1}^{-M\mathfrak{s}}. \tag{4.10}$$

This proves the inequality (4.8). $\square$

REMARK 5. Observe that, while (4.7) allows one to operate with any fixed $M \in \mathbb{N}^*$, provided L_0 is large enough, one can take $M = M_k$ by taking $K = K_k$, provided K_k disjoint $L_k^{2\tau}$-cubes can fit into an L_{k+1}-cube, with $L_{k+1} \sim L_k^{\alpha} = L_k^{3\tau}$. This will be used in Section 5.1 to “boost” the probability estimates obtain through the scale induction with a fixed M. It would be pointless to do so in the course of the scale induction due to a weaker, power-law probability bound (4.3). $\triangleright$

LEMMA 7. (Scale induction) *Assume that the bound*

$$\sup_{x\in\mathbb{R}^d}\ \sup_{E\in I_*}\ \mathbb{P}\left\{ \mathrm{B}_{L_j}(x) \text{ is not } (E,\delta_j,\tau)\text{-SNS} \right\} \leqslant L_j^{-\mathfrak{s}}, \quad \mathfrak{s}>0, \tag{4.11}$$

holds for $j=k\geqslant 0$. *Then it also holds for* $j=k+1$.

Proof. By Corollary 1, if the cube $\mathrm{B}_{L_{k+1}}(x)$ is not (E,δ_{k+1},τ)-SNS, then

- either $\mathrm{B}_{L_{k+1}}(x)$ is not $(E,\varepsilon_{k+1},\tau)$-SNR,
- or $\mathrm{B}_{L_{k+1}}(x)$ is not (E,δ_k,K,τ)-S-good.

By Lemmas 5 and 6, the probabilities of both events are bounded by $\frac{1}{2}L_{k+1}^{-\mathfrak{s}}$, so the claim follows. ☐

The base of the scale induction is provided by Lemma 2, and by induction on k, we come to the conclusion of the multi-scale analysis of the base model.

LEMMA 8. *Consider the Hamiltonian of the form* (1.1) *with* $A>3d$. *There exists an interval* $I^*\subset\mathbb{R}_-$ *such that* $H(\omega,\vartheta)$ *has a nontrivial spectrum in* I_*. *In addition, for any* $\mathfrak{s}>0$, *there exists* $L_*\in\mathbb{N}$ *such that, if* $L_0\geqslant L_*$, *then*

$$\forall k\geqslant 0\ \sup_{E\in I_*}\ \sup_{x\in\mathbb{R}^d}\ \mathbb{P}\left\{ \mathrm{B}_{L_k}(x) \text{ is not } (E,\delta_k,\tau)\text{-SNS} \right\} \leqslant L_k^{-\mathfrak{s}}. \tag{4.12}$$

5. Decay of eigenfunctions and of eigenfunction correlators

5.1. Enhancement of the MSA estimates

LEMMA 9. *Under the assumptions and with notations of Lemma 8, one has for some* $\zeta>0$

$$\sup_{x\in\mathbb{Z}}\ \sup_{E\in I_*} \mathbb{P}\left\{ \mathrm{B}_{L_{k+1}}(x) \textit{ is not } (E,\delta_{k+1})\textit{-NS} \right\} \leqslant \mathrm{e}^{-L_k^{\zeta}}. \tag{5.1}$$

Proof. In the proof of Lemma 6, the RHS of (4.11) is obtained as a sum of the probability bounds (4.3) and (4.8). Remark 5 shows that (4.8) can be enhanced by making M scale-dependent. For example, with $M=M_k=c(A,d)\mathfrak{s}^{-1}L_k^{\tau}$ and some $c(A,d)>0$ small enough, the RHS of (4.10) is bounded by

$$L_{k+1}^{-c(A,d)\mathfrak{s}^{-1}L_k^{2\tau}\cdot\mathfrak{s}} \leqslant \mathrm{e}^{-c(A,d)L_{k+1}^{\alpha/3}} \leqslant \mathrm{e}^{-L_{k+1}^{q}}, \quad 0<q<1/3. \tag{5.2}$$

Such an enhancement would be pointless, since (4.3), proved in the framework of the scale induction, provides only a power-law bound on the probability of non-SNR cubes.

The weakness of (4.3) is due to the fact that, in the course of the scale induction, one needs *strong* non-resonance (SNR) condition for the cubes involved. Thus we are allowed to make use of the random amplitudes ω_{b} only in a finite cube $\mathrm{B}_{L_k^{2\tau}}(u)$, whence the lower bound $\varepsilon \geqslant \varepsilon_{L_k} = L_k^{-A2\tau}$. However, Theorem 2, stated for an arbitrary cube size L, evidences that, once the restriction $\mathrm{b} \in \mathrm{B}_{L_k^{A2\tau}}(u)$ on the admissible supports of the scatterers $\omega_{\mathrm{b}}\,\mathfrak{u}^+(\cdot - \mathrm{b})$ is lifted, so is the constraint $\varepsilon \geqslant \varepsilon_{L_k}$, too.

In Lemma 4, the hypothesis **(i)** requires that $\varepsilon_{k+1} \geqslant \delta_k^{1-c}$ for some $c > 0$, so with $\delta_k = \mathrm{e}^{-mL_k}$, we are allowed to assign to ε_{k+1} any value of the form $\mathrm{e}^{-L_{k+1}^q}$.

Turning to the statement of Lemma 9, note that the main event refers to the non-resonance property and not *strong* non-resonance, thus one can apply Lemma 9 making use of ω_{b} with b in arbitrarily large ambient cube of some size L'. Therefore, we can apply Lemma 9 with any $\varepsilon > 0$, provided L' is large enough, so that $\varepsilon_{L'} \leqslant \varepsilon$. The bound (4.3) is no longer the bottleneck, and we can choose $\varepsilon > 0$ as small as required to match the quantity (5.2) (*sub*-exponential in L_{k+1} !), replacing (4.8).

Summarizing, to prove (5.1), we use Lemma 4 instead of Corollary 1, and set $\varepsilon_{k+1} = \mathrm{e}^{-L_{k+1}^q}$, provided $0 < q < 1/3$, $K = K_{k+1} = \frac{1}{2} L_{k+1}^{1-\frac{2\tau}{\alpha}}$, and L_0 is large enough. Then the claim follows in essentially the same way as in the proof of Lemma 7. $\square$

It is worth emphasizing that the enhancement provided by the above Theorem can be achieved on *any* scale L_k only after completion of the scale induction with weaker probability estimates. Indeed, the enhancement on scale L_{k+1} relies upon the MSA bounds on all scales $0 \leqslant k' \leqslant k$: this is only a “bootstrap”, not an independent proof.

5.2. Energy-interval estimates

Until this point, we carried out the fixed-energy multi-scale analysis. The proof of spectral and dynamical localization requires probability estimates for pairs of Hamiltonians $H_{\mathrm{B}'}$, $H_{\mathrm{B}''}$ in distant cubes B' and B'', where the spectral parameter E is no longer fixed but ranges in a certain interval I_*. The reduction of variable-energy MSA estimates to their fixed-energy counterparts is well-understood by now, and various techniques can be used for this purpose. Denote

$$\mathcal{M}_{L,x}(E) := \left\| \mathbf{1}_{\partial \mathrm{B}_{L_k}(x)}\, G_{\mathrm{B}_{L_k}(x)}(E,\omega)\, \mathbf{1}_{\mathrm{B}_{L_k/3}(x)} \right\|.$$

Below we build upon the results of [4] adapting the techniques proposed by Elgart *et al.* [14]. Specifically [4, Theorem 3] allows for various types of probability estimates of the form $\mathbb{P}\{\mathcal{M}_{L,x}(E) > a_L\} \leqslant g(a)$, including the situation where $0 < a_L < \mathrm{e}^{-\frac{1}{3}L^q}$, with $q \in (0,1)$, and $g(a) = \mathrm{e}^{-L^q}$ (cf. [4, Section 5]. This fits our case, so for brevity, we present below an adaptation of the results of [4, Section 5].

PROPOSITION 4. *Assume that the following conditions are fulfilled:*

$$\sup_{E \in I_*} \sup_{x \in \mathbb{Z}} \mathbb{P}\left\{ \mathcal{M}_{L,x}(E) > \mathrm{e}^{-\frac{1}{3}L^q} \right\} \leqslant \mathrm{e}^{-L^q}, \tag{5.3}$$

and for some pair of disjoint cubes $\mathrm{B}_L(x)$ *and* $\mathrm{B}_L(y)$,

$$\mathbb{P}\left\{ \operatorname{dist}\left[\Sigma^{I_*}(H_{\mathrm{B}_L(x)}),\, \Sigma^{I_*}(H_{\mathrm{B}_L(y)}) \leqslant s\right] \right\} \leqslant s^{\mathfrak{b}}, \quad \mathfrak{b} \in (0,1]. \tag{5.4}$$

If L is large enough, then

$$\mathbb{P}\left\{ \exists E \in I_* : \min_{z \in \{x,y\}} \mathcal{M}_{L,z}(E) > \mathrm{e}^{-\frac{1}{3}L^q} \right\} \leqslant 3\mathrm{e}^{-\frac{1}{3}L^q}. \tag{5.5}$$

COROLLARY 2. *For any $q \in (0,1/3)$ and L_0 large enough, one has*

$$\forall k \in \mathbb{N} \quad \mathbb{P}\left\{ \exists E \in I_* : \mathrm{B}_{L_k}(x) \text{ and } \mathrm{B}_{L_k}(y) \text{ are not } (E,\delta_k)\text{-}NS \right\} \leqslant 3\mathrm{e}^{-\frac{1}{3}L^q}. \tag{5.6}$$

With the above estimate at hand, an exponential decay of all generalized eigenfunctions ψ with eigenvalues in I_* (the assertion (A) of Theorem 1) can be established essentially as in [13]; see also [7, Section 7.3, Lemma 7.9].

5.3. Strong dynamical localization

An elegant derivation of strong dynamical localization from the energy-interval MSA estimates was proposed long ago by Germinet and Klein [21] (see also [20]) who operated with the eigenfunction correlators in the entire space. Their argument becomes particularly simple in the situation where one proves first the decay bound for the eigenfunction correlators in finite cubes, where it can be encapsulated in a fairly elementary functional-analytical lemma, as was shown in our prior papers, e.g., [4, 5].

PROPOSITION 5. (Cf. [4, Thm. 7], [5, Thm. 3]) *Suppose that a bound of the form*

$$\mathbb{P}\left\{ \exists E \in I_* : \mathrm{B}_{L_k}(x) \text{ and } \mathrm{B}_{L_k}(y) \text{ are not } (E,\delta_k)\text{-}NS \right\} \leqslant f(L_k) \tag{5.7}$$

holds for some function $f \geqslant 0$ and all $k \geqslant 0$. Then for $|x-y| \in [3L_k, 3L_{k+1}]$, one has, recalling $\chi_x = \mathbf{1}_{\mathrm{B}_1(x)}$,

$$\mathbb{E}\left[\chi_x \phi_I(H(\omega))\chi_y\right] \leqslant C_1|x-y|^d f(|x-y|^{\alpha^{-1}}) + C_2\mathrm{e}^{-m|x-y|}. \tag{5.8}$$

Proof of Theorem 1(B). By Corollary 2, the estimate (5.7) holds in our model with $f(L) = 3\mathrm{e}^{-\frac{1}{3}L^q}$, where $0 < q < 1/3$. It is plain that the second, exponential term in the RHS of (5.8) is much smaller than the first one, which is sub-exponential in L_k. Thus by a straightforward calculation, for any $\zeta \in (0,q)$, we have:

$$\forall x,y \in \mathbb{Z} \quad \mathbb{E}\left[\chi_x \phi_I(H(\omega,\vartheta))\chi_y\right] \leqslant C_\zeta \mathrm{e}^{-|x-y|^\zeta}, \quad C_\zeta \in (0,+\infty). \tag{5.9}$$

Since $0 < q < 1/3$ is arbitrary, so is $\zeta \in (0,q)$. This proves the claim (1.13). □

6. Adaptation to the lattice model

Now we turn to the surface model on a lattice $\mathbb{Z}^{d_0} \equiv \mathbb{Z}^{d_0} \cong \mathbb{Z}^{d_0} \times \left\{0_{d-d_0}\right\}$ embedded into an ambient lattice $\mathbb{Z}^d$. Our proof of localization (in the continuous model) under the assumption of low density of impurity sites carrying negative potentials can be easily extended to the discrete case with minimal modifications.

The definition of the cubes $\mathrm{B}_L(u)$ is similar to (2.1):

$$\mathrm{B}_L(u) = \left\{ x \in \mathbb{Z}^d : |x-u|_\infty \leqslant L \right\}, \quad |x|_\infty = |x_1| + \cdots + |x_d|. \tag{6.1}$$

The definition of a cube's boundary remains unchanged: $\partial \mathrm{B}_L(u) = \mathrm{B}_L(y) \setminus \mathrm{B}_{L-2}(u)$.

As before, we work with a sequence of length scales $(L_k)_{k\geqslant 0}$ defined recursively: $L_{k+1} = \lceil L_k^\alpha \rceil$, with $\alpha = 3\tau,\ \tau > \max\left[8d, \frac{2d+2}{A-3d}\right]$.

As pointed out in Section 4, the main toolbox of the multi-scale analysis easily adapts to various Anderson-type Hamiltonians in Euclidean spaces and on lattices. The principal analytic component requiring adaptation is the Simon-Lieb type inequality, the proof of which is in fact much more elementary for the Hamiltonian on combinatorial graphs, including the lattices $\mathbb{Z}^d$. It has been used in numerous earlier papers. With this inequality at hand, the general induction scheme remains essentially the same in continuous and lattice models. For these reasons, we do not repeat, almost verbatim, the arguments from Section 4.

The enhancements of the probabilistic estimates provided by the fixed-energy scale induction from Section 4, carried out in Section 5.1, also apply to the lattice model, and so do the results of sections 5.2–5.3.

The Hamiltonian $H(\omega,\vartheta)$ is the lattice analog of its counterpart (1.1),

$$H(\omega,\vartheta) = -h\Delta + V(x;\omega;\vartheta) = -h\Delta + V^{(\mathrm{B})}(x) + V^{(\mathrm{S})}(x;\omega;\vartheta), \tag{6.2}$$

acting in the Hilbert space $\ell^2(\mathbb{Z}^d)$, but Δ is now the canonical graph Laplacian on $\mathbb{Z}^d$ endowed with the usual graph structure,

$$(\Delta f)(x) = \sum_{y\in\mathbb{Z}^d:\, |y-x|_\infty = 1} (f(y) - f(x)).$$

The random potential has the form $V(x,\omega,\vartheta) = V^{(\mathrm{B})}(x) + V^{(\mathrm{S})}(x;\omega;\vartheta)$

$$V^{(\mathrm{B})}(x) = \sum_{\in\mathbb{Z}^d\setminus\mathcal{Z}^{d_0}} \mathrm{s}_\mathrm{b}\, \mathfrak{u}^+(|x-\mathrm{b}|) \geqslant 0, \quad \mathrm{s}_\mathrm{b} \in [0,\bar{s}], \tag{6.3}$$

$$V^{(\mathrm{S})}(x;\omega;\vartheta) = \sum_{\mathrm{b}\in\hat{\mathcal{Z}}^{d_0}} \omega_\mathrm{b}\mathfrak{u}^+(|x-\mathrm{b}|) \tag{6.4}$$

$$+ \sum_{\mathcal{Z}^{d_0}\setminus\hat{\mathcal{Z}}^{d_0}\ni \mathrm{b}:\, \vartheta_\mathrm{b}=0} \mathfrak{u}^+(|x-\mathrm{b}|) + \sum_{\mathcal{Z}^{d_0}\setminus\hat{\mathcal{Z}}^{d_0}\ni \mathrm{b}:\, \vartheta_\mathrm{b}=1} \mathfrak{u}^-(|x-\mathrm{b}|). \tag{6.5}$$

Here, again, $\mathrm{s} : \mathbb{Z}^d \setminus \mathcal{Z}^{d_0} \to [0,\bar{s}]$ is a fixed, non-random function.

Now our hypotheses are as follows.

(V1(2)) = (V1(1)) , (V3(2)) = (V3(1))

(V2(2)) *For all* $r \geqslant 0$, $\mathfrak{u}^-(0) < 0$, *and* $\sum_{r\geqslant 1} r^{d-1}|\mathfrak{u}^-(r)| < +\infty$. $\mathfrak{u}^+(0) > 0$, *and for* $r \geqslant 1$ *and some* $A = 3d + 2\gamma$ *with* $\gamma > 0$, $\mathfrak{u}^+(r) = r^{-A}$.

(V4(2)) *Let* $H_0 = -h\Delta + V^{(\mathrm{B})}(x)$. *There exist* $E_* < 0$, $\mathscr{E} > E_*$, *and* $\phi \in L^2(\mathrm{B}_{r_0}(0))$ *with* $\|\phi\|_2 = 1$ *such that*

$$\mathfrak{u}^-(0) + \sum_{\mathrm{b}\in\mathbb{Z}^d\setminus\mathcal{Z}^{d_0}} \mathfrak{u}^+(|0-\mathrm{b}|) < -\|\Delta\| + E_*\,. \tag{6.6}$$

$$\inf_{x\in\mathbb{Z}^d} \sum_{\mathrm{b}\in\mathcal{Z}^{d_0}\setminus\{x\}} \mathfrak{u}^-(|x-\mathrm{b}|) \geqslant E_* + \mathscr{E}\,. \tag{6.7}$$

The proof of the initial length scale estimate, analogous to Lemma 3, requires only minor adaptations, and performing the multi-scale analysis essentially as in Sections 4–5 (again, in the lattice setting), we come to the following

THEOREM 6. *Under the assumptions* (V1(2))–(V4(2)), *there exist* $\hat{\rho} > 0$ *and an interval* $I_* \subset (-\infty, 0)$ *such that the following holds with* $\rho \in (0, \hat{\rho}]$ *(cf.* (1.7)*).*

(A) *With probability one, the spectrum of* $H(\omega,\vartheta)$ *in the interval* I_* *is nontrivial and pure point, and all its eigenfunctions* ψ *of* $H(\omega,\vartheta)$ *with eigenvalues* $E_\psi \in I_*$ *decay exponentially at infinity: for some* $m > 0$ *one has*

$$\forall x \in \mathbb{Z} \ \ |\psi(x;\omega;\vartheta)| \leqslant C_\psi(\omega,\vartheta)\mathrm{e}^{-m|x|}\,. \tag{6.8}$$

(B) *For any* $\zeta \in (0, 1/3)$ *and some* $C_\zeta \in (0, +\infty)$, *one has:*

$$\mathbb{E}\Big[\sup_{\phi\in\mathcal{B}(I_*)} \big\|\mathbf{1}_{\{x\}}\ \phi_{I_*}(H(\omega,\vartheta))\ \mathbf{1}_{\{y\}}\big\|\Big] \leqslant C_\zeta \mathrm{e}^{-|x-y|^\zeta}\,, \tag{6.9}$$

where $\mathcal{B}(I_*)$ *is the set of Borel functions* ϕ *with* $\operatorname{supp}\phi_{I_*} \subset I_*$ *and* $\|\phi_{I_*}\|_\infty \leqslant 1$.

7. Further extensions

The general scheme of the proof of localization for the **models 3–5** remains the same as in Sections 4–5, so we only briefly comment on required modifications.

Model 3. *Strong surface disorder and periodic bulk in the lattice* $\mathbb{Z}^d$. In this model, the entire surface lattice carries the negative potentials $\mathfrak{u}^-$, all modulated by random amplitudes $\omega_\bullet$, while the bulk sites $\mathrm{b} \in \mathbb{Z}^d \setminus \mathcal{Z}^{d_0}$ carry the potentials $\mathfrak{u}^+$ with a constant amplitude, so the bulk potential is periodic. The Hamiltonian has the form $H(\omega, g) = -\Delta + gV(x,\omega)$ with $V(x,\omega) = V^{(\mathrm{B})}(x) + V^{(\mathrm{S})}(x,\omega)$, and we assume the following.

$$V^{(\mathrm{B})}(x) = \sum_{\mathrm{b}\in\mathbb{Z}^d} \mathfrak{u}^+(|x-\mathrm{b}|)\,, \tag{7.1}$$

$$V^{(\mathrm{S})}(x;\omega) = \sum_{\mathrm{b}\in\mathcal{Z}^{d_0}} \omega_\mathrm{b}\,\mathfrak{u}^-(|x-\mathrm{b}|) \tag{7.2}$$

(V1(3)) = (V1(1)) ,

(V2(3)) *For all* $r \geqslant 0$, $\mathfrak{u}^+(0) \geqslant 0$, *and* $\sum_{r\geqslant 1} r^{d-1}|\mathfrak{u}^+(r)| < +\infty$. $\mathfrak{u}^-(0) < 0$, *and for* $r \geqslant 1$ *and some* $C > 0$, $A = 3d + 2\gamma$ *with* $\gamma > 0$, $\mathfrak{u}^-(r) = -Cr^{-A}$.

(V3(3)): none.

(V4(3)) *The potentials* $\mathfrak{u}^-$ *and* $\mathfrak{u}^+$ *fulfill the condition*

$$\mathfrak{u}^-(0) + \sum_{\mathrm{b}\in\mathbb{Z}^d\setminus\mathbb{Z}^{d_0}} \bar{\mathfrak{s}}\mathfrak{u}^+(|0-\mathrm{b}|) < -\|\Delta\| + E_* . \tag{7.3}$$

$$\inf_{x\in\mathbb{Z}^d} \sum_{\mathrm{b}\in\mathcal{Z}^{d_0}\setminus\{x\}} \mathfrak{u}^-(|x-\mathrm{b}|) \geqslant E_* + \mathscr{E} . \tag{7.4}$$

Since only the surface sites carry negative potentials and produce eigenfunctions with negative energies, it suffices to make use of the strong disorder in the surface layer: the bulk represents a “forbidden zone” for such eigenfunctions, so one can use the Combes–Thomas bound to prove ILS estimate in the bulk cubes.

LEMMA 10. (ILS under strong disorder; cf. [7, Lemma 6.5]) *For any* $m > 0$ *and* $\mathfrak{s} > 0$, *there exist* $L_* \in \mathbb{N}$, $\tau > \left[\frac{\mathfrak{s}+d+1}{2(A-d)}, 8d\right]$, *and an interval* $I_* \subset (-\infty, 0)$ *such that for* $L \geqslant L_*$ *and some* $\hat{g}(m,\mathfrak{s},L) > 0$, *for any* $g \geqslant \hat{g}(m,\mathfrak{s},L)$, *one has, with* $\mathrm{B} = \mathrm{B}_L(0)$, $\bar{\mathrm{B}} = \mathrm{B}_{L^{2\tau}}(0)$, *and* $\omega_{\bar{\mathrm{B}}}^{\perp} = \omega_{\mathbb{Z}^d\setminus\bar{\mathrm{B}}}$*:*

$$\sup_{I_*\in\mathbb{R}} \mathbb{P}\left\{ \omega_{\mathrm{B}} : \forall\, \omega_{\bar{\mathrm{B}}}^{\perp}\ \min_{x\in\mathrm{B}} \left|gV(x;\omega_{\bar{\mathrm{B}}} + \omega_{\bar{\mathrm{B}}}^{\perp}) - E\right| \leqslant 2\mathrm{e}^{mL} \right\} \leqslant L^{-\mathfrak{s}} , \tag{7.5}$$

so that by virtue of the min-max principle,

$$\sup_{E\in I_*} \mathbb{P}\left\{ \omega_{\mathrm{B}} : \forall\, \omega_{\bar{\mathrm{B}}}^{\perp}\ \operatorname{dist}\left[\Sigma\big(-\Delta_{\mathrm{B}} + gV\big), E\right] \leqslant \mathrm{e}^{mL} \right\} \leqslant L^{-\mathfrak{s}} . \tag{7.6}$$

Consequently, with $\delta_0 = \mathrm{e}^{mL}$, *one has for* $\forall u \in \mathbb{R}^d$ *and* $E \in I_*$*:*

$$\mathbb{P}\left\{ \mathrm{B}_{L_0}(u) \text{ is not } (E,\delta_0,\tau)\text{-SNS} \right\} \leqslant L^{-\mathfrak{s}} . \tag{7.7}$$

Proof. Decompose $V(x,\omega) = V(x,\omega_{\bar{\mathrm{B}}}) + V(x,\omega_{\bar{\mathrm{B}}}^{\perp})$ and let $W_x(\omega) := V(x,\omega_{\bar{\mathrm{B}}})$, $\zeta_x(\omega_{\bar{\mathrm{B}}}^{\perp}) := V(x,\omega_{\bar{\mathrm{B}}}^{\perp})$, then

$$\forall x \in \mathrm{B} \quad C_1 L^{-2(A-d)\tau} \leqslant \left\|\zeta_x\right\|_\infty \leqslant \eta_L := C_2^{-1} L^{-2(A-d)\tau+1} . \tag{7.8}$$

Next, let $\hat{g} = C_2 L^{2(A-d)\tau-1}\mathrm{e}^{mL} \geqslant \left\|\zeta_x\right\|_\infty^{-1}\mathrm{e}^{mL}$, then for $g \geqslant \hat{g}$ and for all $x \in \mathrm{B}_L(0)$,

$$\begin{aligned}
&\sup_E \mathbb{P}\left\{ \omega_{\bar{\mathrm{B}}} : \inf_{\omega_{\bar{\mathrm{B}}}^{\perp}} |gV(x,\omega_{\bar{\mathrm{B}}} + \omega_{\bar{\mathrm{B}}}^{\perp}) - E| \leqslant \mathrm{e}^{mL} \right\} \\
&= \sup_{E'} \mathbb{P}\left\{ \omega_{\bar{\mathrm{B}}} : \exists \omega_{\bar{\mathrm{B}}}^{\perp}\ |\big(W(x,\omega_{\bar{\mathrm{B}}}) - E'\big) + \zeta_x(\omega_{\bar{\mathrm{B}}}^{\perp})| \leqslant g^{-1}\mathrm{e}^{mL} \right\} \\
&\leqslant \sup_{E'} \mathbb{P}\left\{ |W(x,\omega) - E'| \leqslant \hat{g}^{-1}\mathrm{e}^{mL} + \eta_L \right\}
\end{aligned} \tag{7.9}$$

The goal of the above transformations was to eliminate $\inf_{\omega^{\perp}_{\overline{\mathrm{B}}}}$ from the probability of the event in question. Now that this goal is achieved, we apply again the identity $W(x,\omega) = V(x,\omega) - \zeta_x(\omega)$ and recall that, by Lemma 1, the random variable $V(x,\cdot)$ has a bounded (indeed, $\mathcal{C}^\infty(\mathbb{R})$) density, thus for any $E' \in \mathbb{R}$:

$$\begin{aligned}\mathbb{P}\left\{ |W(x,\omega) - E'| \leqslant \hat{g}^{-1}\mathrm{e}^{mL} + \eta_L \right\} &\leqslant \sup_{E'} \mathbb{P}\left\{ |V(x,\omega) - E'| \leqslant \hat{g}^{-1}\mathrm{e}^{mL} + 2\eta_L \right\} \\ &\leqslant \sup_{E'} \mathbb{P}\left\{ |V(x,\omega) - E'| \leqslant 3\eta_L \right\} \leqslant C''\eta_L \leqslant C'''L^{-2(A-d)\overline{\tau}+1}.\end{aligned} \tag{7.10}$$

It follows that

$$\mathbb{P}\left\{ \omega_{\overline{\mathrm{B}}} : \exists x \in \mathrm{B} \ \inf_{\omega^{\perp}_{\overline{\mathrm{B}}}} |gV(x,\omega_{\overline{\mathrm{B}}} + \omega^{\perp}_{\overline{\mathrm{B}}}) - E| \leqslant \mathrm{e}^{mL} \right\} \leqslant C''|\mathrm{B}|L^{-2(A-d)\tau+1+d}.$$

Now the claim follows from the bound $\tau > \frac{\mathfrak{s}+d+1}{2(A-d)}$, so that $2(A-d)\tau - d - 1 > \mathfrak{s}$. □

THEOREM 7. *Under the assumptions* (V1(3))–(V4(3)), *there exists* $\hat{g} > 0$ *such that the following holds for any* $g \geqslant \hat{g}$ *and some* $E^* = E^*(g) < 0$.

(A) *With probability one, spectrum of* $H(\omega,\vartheta)$ *in the interval* $I_* = (-\infty, E^*]$ *is nontrivial and pure point. Any eigenfunction* $\psi(\cdot,\omega)$ *with eigenvalue* $E \in I_*$ *decays exponentially: for some* $m > 0$,

$$\forall y \in \mathbb{Z} \ |\psi_x(y;\omega)| \leqslant C_\psi(\omega)\mathrm{e}^{-m|y-x|}. \tag{7.11}$$

(B) *There exists some* $\varrho > 0$ *such that, for all* $x \neq y$ *and for any bounded Borel function* ϕ *with* $\operatorname{supp}\phi \subset I_*$ *and* $\|\phi\|_\infty \leqslant 1$, *one has*

$$\mathbb{E}\left[\left\| \mathbf{1}_{\{x\}}\, \phi(H(\omega,\vartheta))\, \mathbf{1}_{\{y\}} \right\| \right] \leqslant \mathrm{e}^{-|x-y|^{\varrho}}. \tag{7.12}$$

Model 4. *Homogeneous random bulk and periodic non-random surface in the configuration space* $\mathbb{R}^d$. This corresponds to impurities in **model 1** filling regularly the entire surface lattice $\mathcal{Z}^{d_0}$ with a constant amplitude of all negative scatterer potentials. Specifically, $V(x;\omega) = V^{(\mathrm{B})}(x,\omega) + V^{(\mathrm{S})}(x)$, where

$$V^{(\mathrm{S})}(x) = \sum_{\mathrm{b}\in\mathcal{Z}^{d_0}} \mathfrak{u}^-(|x-\mathrm{b}|), \quad V^{(\mathrm{B})}(x,\omega) = \sum_{\mathrm{b}\in\mathbb{Z}^d\setminus\mathcal{Z}^{d_0}} \omega_{\mathrm{b}}\mathfrak{u}^+(|x-\mathrm{b}|). \tag{7.13}$$

(V1(4)) *The random variables* $\left\{ \omega_{\mathrm{b}},\, \mathrm{b}\in\mathbb{Z}^d \right\}$ *are IID. The support of their common probability measure* μ *contains at least two points, and* $0 \in \operatorname{supp}\mu \subset [0,\overline{s}]$, $\overline{s} > 0$.

(V2(4)) = (V2(1)) , (V3(4)): none.

$(\mathsf{V4}_{(4)})$ *There exist some* $E_* < 0$ *and* $\phi \in L^2(\mathbb{R}^d)$ *with* $\|\phi\|_2 = 1$ *and* $\operatorname{supp}\phi \subset \mathrm{B}_{r_0}(0)$, $r_0 \in (0,1)$, *such that*

$$\Big(\phi, \big(-h\Delta + \mathfrak{u}^-(|x|) + \sum_{\mathrm{b}\in\mathbb{Z}^d\setminus\{0\}} \bar{s}\,\mathfrak{u}^+(|x-\mathrm{b}|)\big)\,\phi\Big) \leqslant E_*. \tag{7.14}$$

Possible variants:

(i) The potentials $\mathfrak{u}^+(\cdot - \mathrm{b})$ of constant amplitude fill a sublattice $\hat{\mathcal{Z}}^{d_0} \subset \mathcal{Z}^{d_0}$.

(ii) The random potentials $\omega_{\mathrm{b}}\mathfrak{u}^+(\cdot - \mathrm{b})$ appear only in some layer $\mathcal{Z}^{d_0} \times [\![1,R]\!]$, $R \geqslant 1$. Here $[\![1,R]\!] = [1,R] \cap \mathbb{Z}$.

(iii) The potentials $\omega_{\mathrm{b}}\mathfrak{u}^+(\cdot - \mathrm{b})$ appear on an entire periodic sublattice of $\mathbb{Z}^d \setminus \mathcal{Z}^{d_0}$.

Model 5. *Periodic non-random bulk in the configuration space* $\mathbb{R}^d$. The random positive potentials $\omega_{\mathrm{b}}\,\mathfrak{u}^+(|\cdot - \mathrm{b}|)$ are carried by randomly placed sites $\mathrm{b} \in \mathcal{Z}^{d_0}$ where $\vartheta_{\mathrm{b}} = 0$, and the remaining sites $\mathrm{b} \in \mathcal{Z}^{d_0}$ (with $\vartheta_{\mathrm{b}} = 1$) carry non-random potentials $\mathfrak{u}^-(|\cdot - \mathrm{b}|)$. The random potential has the form $V(x;\omega;\vartheta) = V^{(\mathrm{B})}(x) + V^{(\mathrm{S})}(x,\omega,\vartheta)$, where

$$\begin{aligned} V^{(\mathrm{S})}(x;\omega;\vartheta) &= \sum_{\mathrm{b}\in\mathbb{Z}^d\setminus\mathcal{Z}^{d_0}} \mathfrak{u}^+(|x-\mathrm{b}|). \\ V^{(\mathrm{B})}(x) &= \sum_{\mathrm{b}\in\mathcal{Z}^{d_0}} \Big(\big(1-\vartheta_{\mathrm{b}}\big)\omega_{\mathrm{b}}\,\mathfrak{u}^+(|x-\mathrm{b}|) + \vartheta_{\mathrm{b}}\mathfrak{u}^-(|x-\mathrm{b}|)\Big), \end{aligned} \tag{7.15}$$

$(\mathsf{V1}_{(4)}) = (\mathsf{V1}_{(1)})$, $(\mathsf{V2}_{(4)}) = (\mathsf{V2}_{(1)})$, $(\mathsf{V4}_{(4)}) = (\mathsf{V4}_{(1)})$.

$(\mathsf{V3}_{(4)})$ The random field $\vartheta_\bullet$ on $\mathcal{Z}^{d_0}$ is IID with values in $\{0,1\}$ and

$$\mathbb{P}\{\vartheta_{\mathrm{b}} = 1\} = 1 - \mathbb{P}\{\vartheta_{\mathrm{b}} = 0\} \in (0,1). \tag{7.16}$$

Comments on model 5. The eigenvalue concentration analysis in this model requires some additional probabilistic estimates, but these are quite straightforward. As pointed out in Section 2, the proofs of Theorems 2–3, which merely adapt Theorems 2.5 and 2.7 from [7], rely on a possibility to find, for any cube $\mathrm{B}_L(u)$ with $u \in \mathcal{Z}^{d_0}$, a collection $\mathcal{B}_L$ of random potentials $\omega_{\mathrm{b}_k}\mathfrak{u}^+(\cdot - \mathrm{b}_k)$, $1 \leqslant k \leqslant L^\varrho$, where $|\mathrm{b}_k - u| \asymp L^\tau$ and $\varrho \in (0,1)$. This is clearly possible when an entire periodic sublattice $\hat{\mathcal{Z}}^{d_0} \subsetneq \mathcal{Z}^{d_0}$ carries the random potentials $\omega_{\mathrm{b}_k}\mathfrak{u}^+(\cdot - \mathrm{b})$, so in model 5, one has to make sure such a collection can be found. To this end, one can apply the large deviations estimates for the IID random field $\vartheta_\bullet$ with values 0 and 1, both having strictly positive probabilities. In a set of cardinality $n \asymp L^\varrho \gg 1$, the number of sites carrying the value $\vartheta_{\mathrm{b}} = 0$ is asymptotically ρn, with probability at least $1 - \mathrm{e}^{-c'n} \geqslant 1 - \mathrm{e}^{cL^\varrho}$ (cf., e.g., the Chernoff estimate [3]), and such a probability is more than sufficient for the scale induction in Section 4. Thus, from the combinatorial point of view, the situation with model 5 is quite close to that in **model 1**, where the number of sites b with $\vartheta_{\mathrm{b}} = 0$, say, in a cube of cardinality $n \asymp L^\varrho$, is also asymptotically Const n with Const > 0 depending on the period of sublattice $\hat{\mathcal{Z}}^{d_0} \subsetneq \mathcal{Z}^{d_0}$. $\triangleright$

Our final comment is that, as the reader has probably realized by now, one can include a periodic background potential $V^{(0)}$ in all the models listed above and treated in Sections 4–6. Moreover, in the models where the initial length scale bound is inferred from a low density of impurities, as in **model 1**, it suffices to assume that $\|V^{(0)}\|_\infty < \infty$.

Acknowledgements. I thank the reviewer for a thorough reading of the manuscript and making valuable suggestions which contributed to improving the clarity of presentation.

REFERENCES

[1] M. AIZENMAN, S. WARZEL, *Random Operators: Disorder Effects on Quantum Spectra and Dynamics*, Graduate Studies in MathematicsIMS Lecture Notes. Monograph Series, vol. 1668, AMS, USA (2016).

[2] A. C. BERRY, *The accuracy of the Gaussian approximation to the sum of independent random variates*, Trans. Amer. Math. Soc. **49**, 122–136 (1941).

[3] F. CHERNOFF, *A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations*, Ann. Math. Stat. **23** (4), 493–501 (1952).

[4] V. CHULAEVSKY, *From fixed-energy localization analysis to dynamical localization: An elementary path*, J. Stat. Phys. **154**, 1391–1429 (2014).

[5] V. CHULAEVSKY, *Exponential scaling limit for single-particle Anderson models via adaptive feedback scaling*, J. Stat. Phys. **162**, 603–614 (2016).

[6] V. CHULAEVSKY, *Fast decay of eigenfunction correlators in long-range continuous random alloys*, Rand. Oper. Stoch. Equ. **27** (1), 43–51 (2019).

[7] V. CHULAEVSKY, *Universality of high-dimensional disorder and density of states under non-local interactions I. Infinite smoothness and localization*, J. Spec. Theory **10** (2), 515–560 (2020).

[8] V. CHULAEVSKY, *Multi-scale analysis of random alloy models with summable site potentials of infinite range*, Commun. Math. Phys. **381** (2), 557–590 (2021).

[9] V. CHULAEVSKY, Y. SUHOV, *Efficient Anderson localization bounds for large multi-partilce systems*, J. Spec. Theory **7**, 269–320 (2017).

[10] J. M. COMBES, L. THOMAS, *Asymptotic behaviour of eigenfunctions for multi-particle Schrödinger operators*, Commun. Math. Phys. **34**, 251–263 (1973).

[11] H. CRAMÉR, *Random variables and probability distributions*, Cambridge tracts in mathematics and mathematical physics, vol. 36, Cambridge University Press (1937).

[12] H. VON DREIFUS, *On the effect of randomness in ferromagnetic models and Schrödinger operators*, (1987), PhD thesis, New York University.

[13] H. VON DREIFUS, A. KLEIN, *A new proof of localization in the Anderson tight binding model*, Commun. Math. Phys. **124**, 285–299 (1989).

[14] A. ELGART, M. TAUTENHAHN, I. VESELIĆ, *Localization via fractional moments for models on* **Z** *with single-site potentials of finite support*, J. Phys. A **43** (8), 474 021 (2010).

[15] C. ESSEEN, *Fourier analysis of distribution funtion. A mathematical study of the Laplace–Gaussian law*, Acta Math. **77**, 1–125 (1945).

[16] W. FELLER, *An introduction to probability theory and its applications*, vol. 2, John Wiley and Sons, Inc., New York London Sidney (1966).

[17] A. FIGOTIN, A. KLEIN, *Localization of electromagnetic and acoustic waves in random media. Lattice models*, J. Stat. Phys. **76**, 985–1003 (1994).

[18] A. FIGOTIN, A. KLEIN, *Localization of classical waves I: Acoustic waves*, Commun. Math. Phys. **180**, 439–482 (1996).

[19] A. FIGOTIN, A. KLEIN, *Localization of classical waves II: Electromagnetic waves*, Commun. Math. Phys. **184**, 411–441 (1997).

[20] F. GERMINET, S. DE BIÉVRE, *Dynamical localization for discrete and continuous random Schrödinger operators*, Commun. Math. Phys. **194**, 323–341 (1998).
[21] F. GERMINET, A. KLEIN, *Bootstrap multi-scale analysis and localization in random media*, Commun. Math. Phys. **222**, 415–448 (2001).
[22] V. JAKŠIĆ, S. A. MOLCHANOV, *Localization of surface spectra*, Commun. Math. Phys. **208**, 153–172 (1999).
[23] V. JAKŠIĆ, S. A. MOLCHANOV, L. A. PASTUR, *On the propagation of surface waves*, IMA Math. Appl. **96**, 143–154 (1998).
[24] F. KLEESPIES, P. STOLLMANN, *Lifshitz asymptotics and localization for radom quantum waveguides*, Rev. Math. Phys. **12** (10), 1345–1365 (2000).
[25] I. M. LIFSHITZ, *Energy spectrum and quantum states of disordered condensed systems*, Soviet Phys. Uspekhi **7** (4), 617–663 (1965).
[26] F. MARTINELLI, E. SCOPPOLA, *Remark on the absence of absolutely continuous spectrum for d-dimensional Schrödinger operators with random potential for large disorder or low energy*, Commun. Math. Phys. **97**, 465–471 (1985).
[27] A. BOUTET DE MONVEL, P. STOLLMANN, *Dynamical localization for continuum random surface models*, Archiv der Mathematik **80** (1), 87–97 (2003).
[28] L. A. PASTUR, A. L. FIGOTIN, *Spectra of random and almost-periodic operators*, Springer, Berlin (1992).
[29] G. PÓLYA, G. SZEGÖ, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Springer, Berlin (1970).
[30] B. SIMON, T. WOLF, *Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians*, Commun. Pure Appl. Math **39**, 75–90 (1986).
[31] T. SPENCER, *Localization for random and quasi-periodic potentials*, J. Stat. Phys. **51**, 1009–1019 (1988).
[32] P. STOLLMANN, *Caught by disorder*, Progress in Mathematical Physics, vol. 20, Birkhäuser Boston Inc., Boston, MA (2001), Bound states in random media.
[33] A. WINTNER, *On analytic convolutions of Bernoulli distributions*, Amer. J. Math. **56**, 659–663 (1934).

(Received August 2, 2021)

Victor Chulaevsky
Université de Reims
Département de Mathématiques
Moulin de la Housse, B.P. 1039, 51687 Reims Cedex, France
e-mail: victor.tchoulaevski@univ-reims.fr