Operators
and
Matrices
Volume 17, Number 4 (2023), 939–952
doi:10.7153/oam-2023-17-62

NONADDITIVE COMMUTING MAPPINGS ON TRIANGULAR n–MATRIX RINGS

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(*Communicated by M. Omladič*)

Abstract. Let $\mathcal{A}$ be any ring. A nonadditive mapping $\varphi : \mathcal{A} \to \mathcal{A}$ is said to be commuting if $[\varphi(a), b] = [a, \varphi(b)]$ for all $a, b \in \mathcal{A}$. In this paper, we mainly describe the general form of nonadditive commuting mappings on triangular n-matrix rings. The result is then applied to triangular rings.

1. Introduction and preliminaries

Let $\mathcal{A}$ be an associative ring (or an algebra over a field $\mathbb{F}$) and $\mathcal{Z}(\mathcal{A})$ be the center of $\mathcal{A}$. Recall that an additive (a linear) mapping $\varphi : \mathcal{A} \to \mathcal{A}$ is called commuting if $[\varphi(a), a] = 0$ for all $a \in \mathcal{A}$. Clearly, in the case of $\mathcal{A}$ 2-torsion free, the additive (linear) mapping φ is commuting if and only if $[\varphi(a), b] = [a, \varphi(b)]$ for all $a, b \in \mathcal{A}$. A commuting mapping φ of $\mathcal{A}$ is called proper if it is of the form $\varphi(a) = za + \tau(a)$ for all $a \in \mathcal{A}$, where $z \in \mathcal{Z}(\mathcal{A})$ and τ is an additive (a linear) mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$. There is a well-written survey paper [2], in which the author presented the development of the theory of commuting mappings and their applications in detail. Brešar [2] showed that both commuting mappings on simple unital algebras and commuting mappings on prime algebras are proper. Cheung in [4] discussed commuting mappings on triangular algebras and determined the class of triangular algebras for which every commuting linear mapping is proper. Xiao and Wei [11] considered the sufficient and necessary conditions for commuting mappings of the generalized matrix algebras to be proper. For other related results on additive or linear commuting mappings, see [1, 5, 6, 8, 10] and the references therein.

In the case of nonadditive mapping, we say that a nonadditive mapping φ from a ring $\mathcal{A}$ into itself is commuting if $[\varphi(a), b] = [a, \varphi(b)]$ for all $a, b \in \mathcal{A}$. In [9], Qi and Feng gave a characterization of nonadditive commuting mappings on a class of ring. More precisely, suppose that $\mathcal{A}$ is a unital ring with a nontrivial idempotent e_1 and $\mathcal{A}$ satisfies the following conditions:

$$\begin{cases} e_1ae_1 \cdot e_1\mathcal{A}e_2 = \{0\} = e_2\mathcal{A}e_1 \cdot e_1ae_1 \Rightarrow e_1ae_1 = 0, \\ e_1\mathcal{A}e_2 \cdot e_2ae_2 = \{0\} = e_2ae_2 \cdot e_2\mathcal{A}e_1 \Rightarrow e_2ae_2 = 0, \end{cases} \tag{$*$}$$

Mathematics subject classification (2020): Primary 16W10; Secondary 47B47.
Keywords and phrases: Triangular n-matrix rings, triangular rings, commuting mappings.
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Paper OaM-17-62

for all $a \in \mathcal{A}$, where $e_2 = I - e_1$. Then every nonadditive commuting mapping of $\mathcal{A}$ can be represented the sum of a proper form and two special central-valued maps.

Recently, Ferreira [7] defined a class of ring called triangular n-matrix ring as follows.

DEFINITION 1.1. ([7]) Let $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n$ be unital rings and $\mathcal{M}_{ij}$ be $(\mathcal{R}_i, \mathcal{R}_j)$-bimodules with $\mathcal{M}_{ii} = \mathcal{R}_i$ for all $1 \leqslant i \leqslant j \leqslant n$. Let $\varphi_{ijk} : \mathcal{M}_{ij} \otimes_{\mathcal{R}_j} \mathcal{M}_{jk} \to \mathcal{M}_{ik}$ be $(\mathcal{R}_i, \mathcal{R}_k)$-bimodules homomorphisms with $\varphi_{iij} : \mathcal{R}_i \otimes_{\mathcal{R}_i} \mathcal{M}_{ij} \to \mathcal{M}_{ij}$ and $\varphi_{ijj} : \mathcal{M}_{ij} \otimes_{\mathcal{R}_j} \mathcal{R}_j \to \mathcal{M}_{ij}$ the canonical multiplication maps for all $1 \leqslant i \leqslant j \leqslant k \leqslant n$. Write $ab = \varphi_{ijk}(a \otimes b)$ for all $a \in \mathcal{M}_{ij}$ and $b \in \mathcal{M}_{jk}$. Assume that $\mathcal{M}_{ij}$ is faithful as a left $\mathcal{R}_i$-module and faithful as a right $\mathcal{R}_j$-module for all $1 \leqslant i < j \leqslant n$. Moreover, suppose that $a(bc) = (ab)c$ for all $a \in \mathcal{M}_{ik}$, $b \in \mathcal{M}_{kl}$ and $c \in \mathcal{M}_{lj}$ with $1 \leqslant i \leqslant k \leqslant l \leqslant j \leqslant n$. The set

$$\begin{aligned}
\mathcal{T} &= \mathcal{T}_n(\mathcal{R}_i; \mathcal{M}_{ij}) \\
&= \left\{ \begin{pmatrix} r_{11} & m_{12} & \cdots & m_{1(n-1)} & m_{1n} \\ 0 & r_{22} & \cdots & m_{2(n-1)} & m_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{(n-1)(n-1)} & m_{(n-1)n} \\ 0 & 0 & \cdots & 0 & r_{nn} \end{pmatrix} : r_{ii} \in \mathcal{R}_i, m_{ij} \in \mathcal{M}_{ij}, 1 \leqslant i < j \leqslant n \right\}.
\end{aligned}$$

under the usual matrix operations is called triangular n-matrix ring.

Note that triangular n-matrix rings do not satisfy the condition $(*)$ in [9]. So it is nature to ask what is the structure of nonadditive commuting mappings on triangular n-matrix rings. The purpose of the present paper is to characterize the general form of nonadditive commuting mappings on triangular n-matrix rings.

In the rest part of this paper, we shall use the following result.

PROPOSITION 1.2. ([3, Lemma 2.1]) *Let $\mathcal{T} = \mathcal{T}_n(\mathcal{R}_i; \mathcal{M}_{ij})$ be a triangular n-matrix ring. The center of $\mathcal{T}$ is*

$$\mathcal{Z}(\mathcal{T}) = \left\{ \bigoplus_{i=1}^{n} r_{ii} \,\middle|\, r_{ii} m_{ij} = m_{ij} r_{jj} \ \textit{for all} \ m_{ij} \in \mathcal{M}_{ij}, \ i < j \right\}.$$

Moreover, $\mathcal{Z}(\mathcal{T})_{ii} \cong \pi_{\mathcal{R}_i}(\mathcal{Z}(\mathcal{T})) \subseteq \mathcal{Z}(\mathcal{R}_i)$, and there exists a unique ring isomorphism τ_i^j from $\pi_{\mathcal{R}_i}(\mathcal{Z}(\mathcal{T}))$ to $\pi_{\mathcal{R}_j}(\mathcal{Z}(\mathcal{T}))$ $i \neq j$ such that $r_{ii} m_{ij} = m_{ij} \tau_i^j(r_{ii})$ for all $m_{ij} \in \mathcal{M}_{ij}$.

Here, $\overset{n}{\underset{i=1}{\oplus}} r_{ii}$ denotes the element

$$\begin{pmatrix} r_{11} & 0 & \cdots & 0 \\ & r_{22} & \cdots & 0 \\ & & \ddots & \vdots \\ & & & r_{nn} \end{pmatrix},$$

and $\pi_{\mathcal{R}_i} : \mathcal{T} \to \mathcal{R}_i$ $(1 \leqslant i \leqslant n)$ is the natural projection defined by $\pi_{\mathcal{R}_i}(m_{ij}) = r_{ii}$.

Fix any $i \in \{1, 2, \ldots, n\}$. Let E_i stand for the nontrivial idempotent in $\mathcal{T}$ with (i,i) position 1 and other positions 0. Write $P_i = E_1 + E_2 + \cdots + E_i$ and $Q_i = I - P_i$. Denote by $\mathcal{A}_i = P_i \mathcal{T} P_i$, $\mathcal{B}_i = Q_i \mathcal{T} Q_i$ and $\mathcal{M}_i = P_i \mathcal{T} Q_i$. Hence, $\mathcal{T} = \mathcal{A}_i + \mathcal{M}_i + \mathcal{B}_i$. Furthermore, for any $A_i \in \mathcal{A}_i$, $M_i \in \mathcal{M}_i$ and $B_i \in \mathcal{B}_i$, we identify

$$A_i \cong \begin{pmatrix} r_{11} & m_{12} & \cdots & m_{1i} \\ 0 & r_{22} & \cdots & m_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{ii} \end{pmatrix}, \quad M_i \cong \begin{pmatrix} m_{1,i+1} & m_{1,i+2} & \cdots & m_{1n} \\ m_{2,i+1} & m_{2,i+2} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{i,i+1} & m_{i,i+2} & \cdots & m_{in} \end{pmatrix}$$

and

$$B_i \cong \begin{pmatrix} r_{i+1,i+1} & m_{i+1,i+2} & \cdots & m_{i+1,n} \\ 0 & r_{i+2,i+2} & \cdots & m_{i+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{pmatrix}.$$

We define two natural projections $\pi_{\mathcal{A}_i} : \mathcal{T} \to \mathcal{A}_i$ and $\pi_{\mathcal{B}_i} : \mathcal{T} \to \mathcal{B}_i$ by

$$\pi_{\mathcal{A}_i}(A_i + M_i + B_i) = A_i$$

and

$$\pi_{\mathcal{B}_i}(A_i + M_i + B_i) = B_i.$$

Then $\pi_{\mathcal{A}_i}(\mathcal{Z}(\mathcal{T})) \subseteq \mathcal{Z}(\mathcal{A}_i)$ and $\pi_{\mathcal{B}_i}(\mathcal{Z}(\mathcal{T})) \subseteq \mathcal{Z}(\mathcal{B}_i)$.

PROPOSITION 1.3. ([3, Lemma 2.2]) *Let $\mathcal{T} = \mathcal{T}_n(\mathcal{R}_i; \mathcal{M}_{ij})$ be a triangular n-matrix ring. Then there exists a unique ring isomorphism $\pi : \pi_{\mathcal{A}_i}(\mathcal{Z}(\mathcal{T})) \to \pi_{\mathcal{B}_i}(\mathcal{Z}(\mathcal{T}))$ such that $A_i M_i = M_i \pi(A_i)$ for all $M_i \in \mathcal{M}_i$ and $A_i \in \pi_{\mathcal{A}_i}(\mathcal{Z}(\mathcal{T}))$, and moreover, $A_i + \pi(A_i) \in \mathcal{Z}(\mathcal{T})$.*

2. Result and proof

In this section, we mainly discuss the general structure of nonadditive commuting mappings on triangular n-matrix rings. The main result is the following.

THEOREM 2.1. *Let $\mathcal{T}$ be a 2-torsion free triangular n-matrix ring. Assume that $\mathcal{Z}(P_{[n/2]} \mathcal{T} P_{[n/2]}) = \mathcal{Z}(\mathcal{T}) P_{[n/2]}$ and $\mathcal{Z}(Q_{[n/2]} \mathcal{T} Q_{[n/2]}) = \mathcal{Z}(\mathcal{T}) Q_{[n/2]}$. Then a nonadditive mapping $\varphi : \mathcal{T} \to \mathcal{T}$ is commuting if and only if it has the form*

$$\varphi(T) = ZT + f(T) + h(P_{[n/2]} T Q_{[n/2]}) Q_{[n/2]}$$

for all $T \in \mathcal{T}$, where $Z \in \mathcal{Z}(\mathcal{T})$, $f : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ is a map and $h : \mathcal{M}_{[n/2]} \to \mathcal{Z}(\mathcal{T})$ is an additive mapping satisfying $h(M_{[n/2]}) M_{[n/2]} = 0$ for all $M_{[n/2]} \in \mathcal{M}_{[n/2]}$. Here, $[k]$ is the integer part of k.

Proof. For the "if" part, assume that $\varphi(T)=ZT+f(T)+h(P_{[n/2]}TQ_{[n/2]})Q_{[n/2]}$ for all $T\in\mathcal{T}$, where $Z\in\mathcal{Z}(\mathcal{T})$, $f:\mathcal{T}\to\mathcal{Z}(\mathcal{T})$ is a map and $h:\mathcal{M}_{[n/2]}\to\mathcal{Z}(\mathcal{T})$ is an additive mapping satisfying $h(M_{[n/2]})M_{[n/2]}=0$ for all $M_{[n/2]}\in\mathcal{M}_{[n/2]}$. Let $T=A_{[n/2]}+M_{[n/2]}+B_{[n/2]}$ and $S=A'_{[n/2]}+M'_{[n/2]}+B'_{[n/2]}$. Since h is additive and $h(M_{[n/2]})M_{[n/2]}=0$ for all $M_{[n/2]}$ in $\mathcal{M}_{[n/2]}$, it is easy to check that

$$h(M_{[n/2]})M'_{[n/2]}+h(M'_{[n/2]})M_{[n/2]}=0$$

holds for all $M_{[n/2]}$, $M'_{[n/2]}\in\mathcal{M}_{[n/2]}$. Furthermore, we have

$$\begin{aligned}[\varphi(T),S]&=[ZT+f(T)+h(M_{[n/2]})Q_{[n/2]},S]\\&=Z[T,S]+[h(M_{[n/2]})Q_{[n/2]},M'_{[n/2]}]\\&=Z[T,S]-M'_{[n/2]}h(M_{[n/2]})Q_{[n/2]}\\&=Z[T,S]-h(M_{[n/2]})M'_{[n/2]}\end{aligned}$$

and

$$\begin{aligned}[T,\varphi(S)]&=[T,ZS+f(S)+h(M'_{[n/2]})Q_{[n/2]}]\\&=Z[T,S]+[M_{[n/2]},h(M'_{[n/2]})Q_{[n/2]}]\\&=Z[T,S]+M_{[n/2]}h(M'_{[n/2]})Q_{[n/2]}\\&=Z[T,S]+h(M'_{[n/2]})M_{[n/2]}.\end{aligned}$$

Combining the above three equalities, we obtain that

$$[\varphi(T),S]=[T,\varphi(S)]$$

for all T, $S\in\mathcal{T}$, as desired.

For the "only if" part, we shall organize the proof in a series of claims.

CLAIM 1. For any T and $S\in\mathcal{T}$, the following statements hold:
(i) $[\varphi(T),T]=0$;
(ii) the map $(T,S)\to[\varphi(T),S]$ is double additive.

Since $[\varphi(T),T]=[T,\varphi(T)]$ and $\mathcal{T}$ is 2-torsion free, we have

$$[\varphi(T),T]=0$$

for all $T\in\mathcal{T}$.

Clearly, we only need to show that the map is additive with respect to the first component. Indeed, for any T,W and S in $\mathcal{T}$, we have

$$\begin{aligned}[\varphi(T+W),S]&=[T+W,\varphi(S)]\\&=[T,\varphi(S)]+[W,\varphi(S)]\\&=[\varphi(T),S]+[\varphi(W),S]\\&=[\varphi(T)+\varphi(W),S].\end{aligned}$$

Hence the map is additive with respect to the first component.

CLAIM 2. The following statements hold:
(i) $\varphi(P_i) \in \mathcal{A}_i + \mathcal{B}_i$, $i \in \{1,2,\ldots,n-1\}$;
(ii) $\varphi(Q_i) \in \mathcal{A}_i + \mathcal{B}_i$, $i \in \{1,2,\ldots,n-1\}$.

By Claim 1 (i), we have $[\varphi(P_i),P_i] = 0$, which implies that $P_i\varphi(P_i)Q_i = 0$. Then $\varphi(P_i) \in \mathcal{A}_i + \mathcal{B}_i$.

Similarly, one can obtain $\varphi(Q_i) \in \mathcal{A}_i + \mathcal{B}_i$.

CLAIM 3. For any $A_i \in \mathcal{A}_i$, $B_i \in \mathcal{B}_i$, $i \in \{1,2,\ldots,n-1\}$, the following statements hold:
(i) $P_i\varphi(A_i)Q_i = P_i\varphi(B_i)Q_i = 0$;
(ii) $[P_i\varphi(A_i)P_i, A_i] = [Q_i\varphi(B_i)Q_i, B_i] = 0$;
(iii) $P_i\varphi(B_i)P_i \in \mathcal{Z}(\mathcal{A}_i)$;
(iv) $Q_i\varphi(A_i)Q_i \in \mathcal{Z}(\mathcal{B}_i)$.

Since $[\varphi(A_i),P_i] = [A_i,\varphi(P_i)]$ and $\varphi(P_i) \in \mathcal{A}_i + \mathcal{B}_i$, we have

$$-P_i\varphi(A_i)Q_i = A_iP_i\varphi(P_i)P_i - P_i\varphi(P_i)P_iA_i.$$

Multiplying Q_i from the right side of the above equation, we arrive at $P_i\varphi(A_i)Q_i = 0$. Similarly, we have $P_i\varphi(B_i)Q_i = 0$.

By Claim 1 (i), we have $[\varphi(A_i),A_i] = 0$. This together with the fact $\varphi(A_i) \in \mathcal{A}_i + \mathcal{B}_i$ implies that $[P_i\varphi(A_i)P_i, A_i] = 0$. Similarly, we get $[Q_i\varphi(B_i)Q_i, B_i] = 0$.

By Claim 1 (i)–(ii) and Claim 3 (ii), we have

$$\begin{aligned}
0 &= [\varphi(A_i+B_i), A_i+B_i] \\
&= [\varphi(A_i)+\varphi(B_i), A_i+B_i] \\
&= [\varphi(A_i), B_i] + [\varphi(B_i), A_i] \\
&= [Q_i\varphi(A_i)Q_i, B_i] + [P_i\varphi(B_i)P_i, A_i],
\end{aligned}$$

which implies $Q_i\varphi(A_i)Q_i \in \mathcal{Z}(\mathcal{B}_i)$ and $P_i\varphi(B_i)P_i \in \mathcal{Z}(\mathcal{A}_i)$.

In particular, it follows from Claim 3 (iii)–(iv) that $Q_{[n/2]}\varphi(P_{[n/2]})Q_{[n/2]} \in \mathcal{Z}(\mathcal{B}_{[n/2]})$ and $P_{[n/2]}\varphi(Q_{[n/2]})P_{[n/2]} \in \mathcal{Z}(\mathcal{A}_{[n/2]})$. By the assumption of theorem, we see that there exists some $Z(P_{[n/2]})$, $Z(Q_{[n/2]}) \in \mathcal{Z}(\mathcal{T})$ such that

$$Q_{[n/2]}\varphi(P_{[n/2]})Q_{[n/2]} = Z(P_{[n/2]})Q_{[n/2]}$$

and

$$P_{[n/2]}\varphi(Q_{[n/2]})P_{[n/2]} = Z(Q_{[n/2]})P_{[n/2]}.$$

CLAIM 4. The following statements hold:
(i) $P_i\varphi(P_i)P_i \in \mathcal{Z}(\mathcal{A}_i)$, $i \in \{1,2,\ldots,n-1\}$;
(ii) $Q_i\varphi(Q_i)Q_i \in \mathcal{Z}(\mathcal{B}_i)$, $i \in \{1,2,\ldots,n-1\}$.

For any $A_i \in \mathcal{A}_i$, by Claim 2 (i) and Claim 3 (i), we have

$$\begin{aligned}[P_i\varphi(P_i)P_i, A_i] &= [\varphi(P_i), A_i] \\ &= [P_i, \varphi(A_i)] \\ &= P_i\varphi(A_i)Q_i \\ &= 0,\end{aligned}$$

which means $P_i\varphi(P_i)P_i \in \mathcal{Z}(\mathcal{A}_i)$.

Similarly, we can get $Q_i\varphi(Q_i)Q_i \in \mathcal{Z}(\mathcal{B}_i)$, as desired.

In particular, we see that $P_{[n/2]}\varphi(P_{[n/2]})P_{[n/2]} \in \mathcal{Z}(\mathcal{A}_{[n/2]})$ and $Q_{[n/2]}\varphi(Q_{[n/2]})Q_{[n/2]} \in Z(\mathcal{B}_{[n/2]})$. By the assumption of theorem, there exists some $Z'(P_{[n/2]}) \in \mathcal{Z}(\mathcal{T})$ and $Z'(Q_{[n/2]}) \in \mathcal{Z}(\mathcal{T})$ such that

$$P_{[n/2]}\varphi(P_{[n/2]})P_{[n/2]} = Z'(P_{[n/2]})P_{[n/2]}$$

and

$$Q_{[n/2]}\varphi(Q_{[n/2]})Q_{[n/2]} = Z'(Q_{[n/2]})Q_{[n/2]}.$$

CLAIM 5. For any $M_{[n/2]} \in \mathcal{M}_{[n/2]}$, the following statements hold:
(i) $P_{[n/2]}\varphi(M_{[n/2]})Q_{[n/2]} = (Z'(P_{[n/2]}) - Z(P_{[n/2]}))M_{[n/2]}$;
(ii) $P_{[n/2]}\varphi(M_{[n/2]})Q_{[n/2]} = M_{[n/2]}(Z'(Q_{[n/2]}) - Z(Q_{[n/2]}))$;
(iii) $P_{[n/2]}\varphi(M_{[n/2]})P_{[n/2]} = Z(M_{[n/2]})P_{[n/2]}$ for some $Z(M_{[n/2]}) \in \mathcal{Z}(\mathcal{T})$;
(iv) $Q_{[n/2]}\varphi(M_{[n/2]})Q_{[n/2]} = Z'(M_{[n/2]})Q_{[n/2]}$ for some $Z'(M_{[n/2]}) \in \mathcal{Z}(\mathcal{T})$.

For any $M_{[n/2]} \in \mathcal{M}_{[n/2]}$, we have $[\varphi(M_{[n/2]}), P_{[n/2]}] = [M_{[n/2]}, \varphi(P_{[n/2]})]$. This together with Claim 3 (iv) and Claim 4 (i) leads to

$$\begin{aligned}-P_{[n/2]}\varphi(M_{[n/2]})Q_{[n/2]} &= M_{[n/2]}\varphi(P_{[n/2]})Q_{[n/2]} - P_{[n/2]}\varphi(P_{[n/2]})M_{[n/2]} \\ &= M_{[n/2]}Z(P_{[n/2]}) - Z'(P_{[n/2]})M_{[n/2]} \\ &= (Z(P_{[n/2]}) - Z'(P_{[n/2]}))M_{[n/2]}.\end{aligned}$$

Similarly, one can check that (ii) is true.

For any $A_{[n/2]} \in \mathcal{A}_{[n/2]}$, by Claim 1 (i)–(ii), we have

$$\begin{aligned}0 &= [\varphi(A_{[n/2]} + M_{[n/2]}), A_{[n/2]} + M_{[n/2]}] \\ &= [\varphi(A_{[n/2]}) + \varphi(M_{[n/2]}), A_{[n/2]} + M_{[n/2]}] \\ &= [\varphi(A_{[n/2]}), M_{[n/2]}] + [\varphi(M_{[n/2]}), A_{[n/2]}] \\ &= P_{[n/2]}\varphi(A_{[n/2]})P_{[n/2]}M_{[n/2]} - M_{[n/2]}Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]} \\ &\quad + [P_{[n/2]}\varphi(M_{[n/2]})P_{[n/2]}, A_{[n/2]}] - A_{[n/2]}P_{[n/2]}\varphi(M_{[n/2]})Q_{[n/2]}.\end{aligned}$$

This leads to $[P_{[n/2]}\varphi(M_{[n/2]})P_{[n/2]}, A_{[n/2]}] = 0$. Thus

$$P_{[n/2]}\varphi(M_{[n/2]})P_{[n/2]} \in \mathcal{Z}(\mathcal{A}_{[n/2]}).$$

By the assumption of theorem, there exists some $Z(M_{[n/2]}) \in \mathcal{Z}(\mathcal{T})$ such that

$$P_{[n/2]}\varphi(M_{[n/2]})P_{[n/2]} = Z(M_{[n/2]})P_{[n/2]}.$$

So (iii) is true.

Similarly, one can get (iv).

CLAIM 6. Let $Z = (Z'(P_{[n/2]}) - Z(P_{[n/2]}))P_{[n/2]} + (Z'(Q_{[n/2]}) - Z(Q_{[n/2]}))Q_{[n/2]}$. We claim that $Z \in \mathcal{Z}(\mathcal{T})$.

In fact, for any $T \in \mathcal{T}$, we have

$$[\varphi(I), T] = [I, \varphi(T)] = 0,$$

which means $\varphi(I) \in \mathcal{Z}(\mathcal{T})$, then $\varphi(P_{[n/2]}) + \varphi(Q_{[n/2]}) \in \mathcal{Z}(\mathcal{T})$. It follows that

$$\begin{aligned}
Z &= (Z'(P_{[n/2]}) - Z(P_{[n/2]}))P_{[n/2]} + (Z'(Q_{[n/2]}) - Z(Q_{[n/2]}))Q_{[n/2]} \\
&= P_{[n/2]}\varphi(P_{[n/2]})P_{[n/2]} - Z(P_{[n/2]})P_{[n/2]} \\
&\quad + Q_{[n/2]}\varphi(Q_{[n/2]})Q_{[n/2]} - Z(Q_{[n/2]})Q_{[n/2]} \\
&= \varphi(P_{[n/2]}) - Z(P_{[n/2]})Q_{[n/2]} - Z(P_{[n/2]})P_{[n/2]} \\
&\quad + \varphi(Q_{[n/2]}) - Z(Q_{[n/2]})P_{[n/2]} - Z(Q_{[n/2]})Q_{[n/2]} \\
&= \varphi(P_{[n/2]}) + \varphi(Q_{[n/2]}) - Z(P_{[n/2]}) - Z(Q_{[n/2]}) \in \mathcal{Z}(\mathcal{T}),
\end{aligned}$$

as desired.

In the sequel, Z is the central element in Claim 6.

Now, for any $T = A_{[n/2]} + M_{[n/2]} + B_{[n/2]} \in \mathcal{T}$, we define two mappings $\phi : \mathcal{T} \to \mathcal{T}$ and $\gamma : \mathcal{T} \to \mathcal{T}$ by

$$\begin{aligned}
\phi(T) = P_{[n/2]}\varphi(A_{[n/2]})P_{[n/2]} - \pi^{-1}(Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]}) \\
+ Q_{[n/2]}\varphi(B_{[n/2]})Q_{[n/2]} - \pi(P_{[n/2]}\varphi(B_{[n/2]})P_{[n/2]}) + \varphi(M_{[n/2]})
\end{aligned}$$

and

$$\gamma(T) = \varphi(T) - \phi(T)$$

for all $T \in \mathcal{T}$. Thus, by Claim 1 (iii) and Proposition 1.3, we have

$$\begin{aligned}
\gamma(T) = Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]} + \pi^{-1}(Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]}) \\
+ P_{[n/2]}\varphi(B_{[n/2]})P_{[n/2]} + \pi(P_{[n/2]}\varphi(B_{[n/2]})P_{[n/2]}) \\
+ \varphi(T) - \varphi(A_{[n/2]}) - \varphi(B_{[n/2]}) - \varphi(M_{[n/2]}) \in \mathcal{Z}(\mathcal{T})
\end{aligned}$$

for all $T = A_{[n/2]} + M_{[n/2]} + B_{[n/2]} \in \mathcal{T}$ So ϕ is a commuting mapping on $\mathcal{T}$. Moreover, we see that $\phi(A_{[n/2]}) \in \mathcal{A}_{[n/2]}$, $\phi(B_{[n/2]}) \in \mathcal{B}_{[n/2]}$.

CLAIM 7. ϕ is additive on $\mathcal{A}_{[n/2]}$ and $\mathcal{B}_{[n/2]}$.

For any $A_{[n/2]}$, $A'_{[n/2]} \in \mathcal{A}_{[n/2]}$ and $M_{[n/2]} \in \mathcal{M}_{[n/2]}$, we have

$$[\phi(A_{[n/2]}+A'_{[n/2]})-\phi(A_{[n/2]})-\phi(A'_{[n/2]}),M_{[n/2]}]=0.$$

Since $\mathcal{M}_{ij}$ is a faithful left $\mathcal{R}_i$-module, the above equation implies

$$E_i(\phi(A_{[n/2]}+A'_{[n/2]})-\phi(A_{[n/2]})-\phi(A'_{[n/2]}))E_i=0 \tag{2.1}$$

for all $i=1,2,\dots,[n/2]$. Moreover, let $i\in\{1,2,\dots,[n/2]-1\}$. On the one hand, we have

$$\begin{aligned}[\phi(Q_i),A_{[n/2]}+A'_{[n/2]}]&=[Q_i,\phi(A_{[n/2]}+A'_{[n/2]})]\\&=-P_i\phi(A_{[n/2]}+A'_{[n/2]})Q_i.\end{aligned}$$

On the other hand,

$$\begin{aligned}[\phi(Q_i),A_{[n/2]}+A'_{[n/2]}]&=[\phi(Q_i),A_{[n/2]}]+[\phi(Q_i),A'_{[n/2]}]\\&=[Q_i,\phi(A_{[n/2]})]+[Q_i,\phi(A'_{[n/2]})]\\&=-P_i(\phi(A_{[n/2]})+\phi(A'_{[n/2]}))Q_i.\end{aligned}$$

Comparing these two equations, we get

$$P_i(\phi(A_{[n/2]}+A'_{[n/2]})-\phi(A_{[n/2]})-\phi(A'_{[n/2]}))Q_i=0 \tag{2.2}$$

for all $i=1,2,\dots,[n/2]-1$. So Eqs. (2.5)–(2.6) together imply that

$$\phi(A_{[n/2]}+A'_{[n/2]})=\phi(A_{[n/2]})+\phi(A'_{[n/2]}),$$

which means ϕ is additive on $\mathcal{A}_{[n/2]}$.

Similarly, one can verify that ϕ is additive on $\mathcal{B}_{[n/2]}$.

CLAIM 8. The following statements hold:
(i) $\phi(A_{[n/2]})=ZA_{[n/2]}$ for all $A_{[n/2]}\in\mathcal{A}_{[n/2]}$;
(ii) $\phi(B_{[n/2]})=ZB_{[n/2]}$ for all $B_{[n/2]}\in\mathcal{B}_{[n/2]}$.

We only need to check (i), and the proof of (ii) is similar. For any $A_{[n/2]}\in\mathcal{A}_{[n/2]}$, $M_{[n/2]}\in\mathcal{M}_{[n/2]}$, since

$$[\phi(A_{[n/2]}),M_{[n/2]}]=[A_{[n/2]},\phi(M_{[n/2]})],$$

we have

$$\begin{aligned}\phi(A_{[n/2]})M_{[n/2]}&=[A_{[n/2]},\phi(M_{[n/2]})]\\&=[A_{[n/2]},P_{[n/2]}\phi(M_{[n/2]})Q_{[n/2]}]\\&=A_{[n/2]}P_{[n/2]}\phi(M_{[n/2]})Q_{[n/2]}\\&=A_{[n/2]}ZM_{[n/2]},\end{aligned}$$

which implies
$$(\phi(A_{[n/2]})-ZA_{[n/2]})M_{[n/2]}=0.$$
Note that $\mathcal{M}_{ij}$ is a faithful left $\mathcal{R}_i$-module. It follows that
$$E_i(\phi(A_{[n/2]})-ZA_{[n/2]})E_i=0,\ \ i=1,2,\dots,[n/2]. \tag{2.3}$$
Writing $A_{[n/2]}=(a_{kl})_{[n/2]\times[n/2]}$, we get
$$A_{[n/2]}=\sum_{1\leqslant k\leqslant l\leqslant [n/2]} A_{kl},$$
where A_{kl} is the matrix with (k,l) position a_{kl} and other positions 0. Since ϕ is additive on $\mathcal{A}_{[n/2]}$, one only needs to check that
$$\phi(A_{kl})=ZA_{kl} \text{ forall } 1\leqslant k\leqslant l\leqslant [n/2].$$
Now, we divide the proof into the following two steps.

Step 1. $\phi(A_{kk})=ZA_{kk}$ for any $A_{kk}\in\mathcal{A}_{[n/2]}$, $1\leqslant k\leqslant [n/2]$.

In fact, replacing $A_{[n/2]}$ with A_{kk}, $k\neq i$ in Eq. (2.7), we get
$$E_i\phi(A_{kk})E_i=E_iZA_{kk}E_i=0,$$
that is,
$$E_i\phi(A_{kk})E_i=0\ ,\ 1\leqslant i\neq k\leqslant [n/2]. \tag{2.4}$$
For any $i\in\{1,2,\dots,[n/2]-1\}$, by Claim 3 (iv) and Claim 4 (i), we have
$$[P_i,\phi(A_{kk})]=[\phi(P_i),A_{kk}]=0,$$
which implies
$$P_i\phi(A_{kk})Q_i=0\ ,\ i=1,2,\dots,[n/2]-1. \tag{2.5}$$
Eqs. (2.8)–(2.9) together imply that
$$\phi(A_{kk})=ZA_{kk},\ \ 1\leqslant k\leqslant [n/2].$$

Step 2. $\phi(A_{kl})=ZA_{kl}$ for any $A_{kl}\in\mathcal{A}_{[n/2]}$, $1\leqslant k<l\leqslant [n/2]$.

In fact, replacing $A_{[n/2]}$ with A_{kl}, $k<l$ in Eq. (2.7), we have
$$E_i\phi(A_{kl})E_i=E_iZA_{kl}E_i=0,$$
that is,
$$E_i\phi(A_{kl})E_i=0,\ \ i=1,2,\dots,[n/2]. \tag{2.6}$$
For any $i\in\{1,2,\dots,[n/2]\}$, by the fact $[\phi(A_{kl}),A_{ii}]=[A_{kl},\phi(A_{ii})]$ and Step 1, we have
$$\phi(A_{kl})A_{ii}-A_{ii}\phi(A_{kl})=[A_{kl},ZA_{ii}]. \tag{2.7}$$
If $i\neq k,l$, by taking $A_{ii}=E_i$ in Eq. (2.11), then we have
$$\phi(A_{kl})E_i-E_i\phi(A_{kl})=0;$$

If $i=k$, by taking $A_{ii}=E_k$ in Eq. (2.11), then we have

$$\phi(A_{kl})E_k - E_k\phi(A_{kl}) = -ZA_{kl};$$

If $i=l$, by taking $A_{ii}=E_l$ in Eq. (2.11), then we have

$$\phi(A_{kl})E_l - E_l\phi(A_{kl}) = ZA_{kl}.$$

Let $1 \leqslant i < j \leqslant [n/2]$. Comparing the above three equations, we get

$$E_i\phi(A_{kl})E_j = \begin{cases} ZA_{kl}, & \text{if } (i,j)=(k,l) \\ 0, & \text{if } (i,j)\neq(k,l) \end{cases}. \tag{2.8}$$

Eq. (2.10) and Eq. (2.12) together imply that

$$\phi(A_{kl}) = ZA_{kl}\ ,\ 1 \leqslant k < l \leqslant [n/2].$$

Now, by Step 1 and Step 2, we can infer that

$$\phi(A_{[n/2]}) = ZA_{[n/2]}.$$

The proof of the claim is completed.

CLAIM 9. $\varphi(T) = ZT + f(T) + h(P_{[n/2]}TQ_{[n/2]})Q_{[n/2]}$ for all $T \in \mathcal{T}$, where $Z \in \mathcal{Z}(\mathcal{T}), f : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ and $h : \mathcal{M}_{[n/2]} \to \mathcal{Z}(\mathcal{T})$ are two maps.

For any $T \in \mathcal{T}$, by Claim 3 (i), (iii), (iv), Claim 5 (i), (iii), (iv) and Claim 8 (i)–(ii), we have

$$\varphi(A_{[n/2]}) = ZA_{[n/2]} + P_{[n/2]}\gamma(A_{[n/2]})P_{[n/2]} + Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]}, \tag{2.9}$$

$$\varphi(M_{[n/2]}) = ZM_{[n/2]} + Z(M_{[n/2]})P_{[n/2]} + Z'(M_{[n/2]})Q_{[n/2]}, \tag{2.10}$$

and

$$\varphi(B_{[n/2]}) = ZB_{[n/2]} + P_{[n/2]}\varphi(B_{[n/2]})P_{[n/2]} + Q_{[n/2]}\gamma(B_{[n/2]})Q_{[n/2]}. \tag{2.11}$$

Moreover, we see that

$$\begin{aligned}\gamma(A_{[n/2]}) &= \varphi(A_{[n/2]}) - \phi(A_{[n/2]}) \\ &= \varphi(A_{[n/2]}) - P_{[n/2]}\varphi(A_{[n/2]})P_{[n/2]} + \pi^{-1}(Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]}) \\ &= Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]} + \pi^{-1}(Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]}),\end{aligned}$$

which implies

$$P_{[n/2]}\gamma(A_{[n/2]})P_{[n/2]} = \pi^{-1}(Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]}). \tag{2.12}$$

Similarly, we have

$$Q_{[n/2]}\gamma(B_{[n/2]})Q_{[n/2]} = \pi(P_{[n/2]}\varphi(B_{[n/2]})P_{[n/2]}). \tag{2.13}$$

Combining Eqs. (2.13)–(2.17), we have

$$\begin{aligned}\varphi(T) &= \varphi(A_{[n/2]}) + \varphi(M_{[n/2]}) + \varphi(B_{[n/2]}) + Z_\varphi \\ &= ZT + Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]} + \pi^{-1}(Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]}) \\ &\quad + P_{[n/2]}\varphi(B_{[n/2]})P_{[n/2]} + \pi(P_{[n/2]}\varphi(B_{[n/2]})P_{[n/2]}) \\ &\quad + Z(M_{[n/2]})P_{[n/2]} + Z'(M_{[n/2]})Q_{[n/2]} + Z_\varphi,\end{aligned}$$

where $Z_\varphi \in \mathcal{Z}(\mathcal{T})$. Let

$$\begin{aligned}f(T) &= Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]} + \pi^{-1}(Q_{[n/2]}\varphi(A_{[n/2]})Q_{[n/2]}) \\ &\quad + P_{[n/2]}\varphi(B_{[n/2]})P_{[n/2]} + \pi(P_{[n/2]}\varphi(B_{[n/2]})P_{[n/2]}) \\ &\quad + Z(M_{[n/2]}) + Z_\varphi\end{aligned}$$

and

$$h(M_{[n/2]}) = Z'(M_{[n/2]}) - Z(M_{[n/2]}).$$

Then

$$\varphi(T) = ZT + f(T) + h(M_{[n/2]})Q_{[n/2]}$$

for all $T \in \mathcal{T}$. It is clear that f is a map from $\mathcal{T}$ into $\mathcal{Z}(\mathcal{T})$ and h is a map from $\mathcal{M}_{[n/2]}$ into $\mathcal{Z}(\mathcal{T})$.

CLAIM 10. *h is additive and $h(M_{[n/2]})M_{[n/2]} = 0$ for all $M_{[n/2]} \in \mathcal{M}_{[n/2]}$.*

For any $M_{[n/2]}, M'_{[n/2]} \in \mathcal{M}_{[n/2]}$, we have

$$[\varphi(M_{[n/2]}), M'_{[n/2]}] = [M_{[n/2]}, \varphi(M'_{[n/2]})]$$

and

$$\begin{aligned}\varphi(M_{[n/2]}) &= Z(M_{[n/2]})P_{[n/2]} + ZM_{[n/2]} + Z'(M_{[n/2]})Q_{[n/2]} \\ &= Z(M_{[n/2]}) + ZM_{[n/2]} + (Z'(M_{[n/2]}) - Z(M_{[n/2]}))Q_{[n/2]} \\ &= Z(M_{[n/2]}) + ZM_{[n/2]} + h(M_{[n/2]})Q_{[n/2]}.\end{aligned}$$

Combining the above two equation, we obtain that

$$\begin{aligned}&[Z(M_{[n/2]}) + ZM_{[n/2]} + h(M_{[n/2]})Q_{[n/2]}, M'_{[n/2]}] \\ &= [M_{[n/2]}, Z(M'_{[n/2]}) + ZM'_{[n/2]} + h(M'_{[n/2]})Q_{[n/2]}],\end{aligned}$$

that is,

$$[h(M_{[n/2]})Q_{[n/2]}, M'_{[n/2]}] = [M_{[n/2]}, h(M'_{[n/2]})Q_{[n/2]}],$$

and hence

$$h(M_{[n/2]})M'_{[n/2]} + h(M'_{[n/2]})M_{[n/2]} = 0 \tag{2.14}$$

for all $M_{[n/2]}, M'_{[n/2]} \in \mathcal{M}_{[n/2]}$. In particular, we have

$$h(M_{[n/2]})M_{[n/2]} = 0$$

for all $M_{[n/2]} \in \mathcal{M}_{[n/2]}$.

Furthermore, for any $M_{[n/2]}$, $M'_{[n/2]}$, $M^*_{[n/2]} \in \mathcal{M}_{[n/2]}$, by Eq. (2.18), we have

$$[h(M_{[n/2]} + M^*_{[n/2]}) - h(M_{[n/2]}) - h(M^*_{[n/2]})]M'_{[n/2]} = 0.$$

So

$$[h(M_{[n/2]} + M^*_{[n/2]}) - h(M_{[n/2]}) - h(M^*_{[n/2]})]P_{[n/2]} = 0. \tag{2.15}$$

Similarly, we can obtain

$$[h(M_{[n/2]} + M^*_{[n/2]}) - h(M_{[n/2]}) - h(M^*_{[n/2]})]Q_{[n/2]} = 0. \tag{2.16}$$

Therefore, by Eqs. (2.19)–(2.20), we get

$$h(M_{[n/2]} + M^*_{[n/2]}) = h(M_{[n/2]}) + h(M^*_{[n/2]}).$$

Now, by Claim 9–10, we complete the proof of the theorem. □

If the center of triangular n-matrix ring $\mathcal{T} = \mathcal{T}_n(\mathcal{R}_i; \mathcal{M}_{ij})$ satisfies the following condition:

$$\mathcal{Z}(\mathcal{T}) = \{A + B :\ A \in \mathcal{Z}(P_{[n/2]}\mathcal{T}P_{[n/2]}),\ B \in \mathcal{Z}(Q_{[n/2]}\mathcal{T}Q_{[n/2]}),\\ AM_0 = M_0B \text{ forsome } M_0 \in \mathcal{M}_{[n/2]}\},$$

then we have the following result.

THEOREM 2.2. *Let $\mathcal{T}$ be a 2-torsion free triangular n-matrix ring. Suppose that*
(i) $Z(P_{[n/2]}\mathcal{T}P_{[n/2]}) = \mathcal{Z}(\mathcal{T})P_{[n/2]}$,
(ii) $Z(Q_{[n/2]}\mathcal{T}Q_{[n/2]}) = \mathcal{Z}(\mathcal{T})Q_{[n/2]}$,
(iii) $\mathcal{Z}(\mathcal{T}) = \{A + B : A \in \mathcal{Z}(P_{[n/2]}\mathcal{T}P_{[n/2]}),\ B \in \mathcal{Z}(Q_{[n/2]}\mathcal{T}Q_{[n/2]}),\ AM_0 = M_0B$ for some $M_0 \in \mathcal{M}_{[n/2]}\}$.

Then a nonadditive map $\varphi : \mathcal{T} \to \mathcal{T}$ is commuting if and only if $\varphi(T) = ZT + f(T)$ holds for all $T \in \mathcal{T}$, where $Z \in \mathcal{Z}(\mathcal{T})$, $f : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ is a map.

Proof. The "if" part is obvious. We only need to prove "only if" part.

Using Theorem 2.1, we have

$$\varphi(T) = ZT + f(T) + h(P_{[n/2]}TQ_{[n/2]})Q_{[n/2]}$$

for all $T \in \mathcal{T}$, where $Z \in \mathcal{Z}(\mathcal{T})$, $f : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ is a map, $h : \mathcal{M}_{[n/2]} \to \mathcal{Z}(\mathcal{T})$ is a additive map satisfying $h(M_{[n/2]})M_{[n/2]} = 0$ for all $M_{[n/2]} \in \mathcal{M}_{[n/2]}$.

Now, we check that $h = 0$.

Indeed, using assumption (iii) of the theorem and the fact $h(M_0)M_0 = 0$, we have $h(M_0)P_{[n/2]} \in \mathcal{Z}(\mathcal{T})$, and hence

$$h(M_0)P_{[n/2]} = 0. \tag{2.17}$$

For any $M \in \mathcal{M}_{[n/2]}$, by Eq. (2.21), we have

$$0 = h(M+M_0)(M+M_0) = h(M)M_0.$$

By the assumption (iii) again, we get

$$h(M)P_{[n/2]} = 0$$

for all $M \in \mathcal{M}_{[n/2]}$.

Similarly, we can obtain $h(M)Q_{[n/2]} = 0$ for all $M \in \mathcal{M}_{[n/2]}$. Hence $h = 0$. □

As an application of Theorem 2.1 and Theorem 2.2, we consider the triangular ring case.

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital rings, and let $\mathcal{M}$ be a unital $(\mathcal{A},\mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module; that is, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $A\mathcal{M} = \mathcal{M}B = \{0\}$ imply $A = 0$ and $B = 0$. The set

$$\mathcal{U} = Tri(\mathcal{A},\mathcal{M},\mathcal{B}) = \left\{ \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} : A \in \mathcal{A}, M \in \mathcal{M}, B \in \mathcal{B} \right\}$$

under the usual matrix addition and formal matrix multiplication is called a triangular ring. Clearly, $\mathcal{U}$ is unital with unit $I = \begin{pmatrix} I_{\mathcal{A}} & 0 \\ 0 & I_{\mathcal{B}} \end{pmatrix}$. We denote the non-trivial idempotent $P = \begin{pmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = I - P$.

It is obvious that triangular rings are triangular 2-matrix rings. So we have the following corollaries.

COROLLARY 2.3. *Let $\mathcal{U} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular ring. Assume that $\mathcal{Z}(\mathcal{U})P = \mathcal{Z}(P\mathcal{U}P)$ and $\mathcal{Z}(\mathcal{U})Q = \mathcal{Z}(Q\mathcal{U}Q)$. Then a nonadditive map $\varphi : \mathcal{U} \to \mathcal{U}$ is commuting if and only if $\varphi(X) = ZX + f(X) + h(PXQ)$ for all $X \in \mathcal{U}$, where $Z \in \mathcal{Z}(\mathcal{U})$, $f : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a map and $h : \mathcal{M} \to \mathcal{Z}(\mathcal{U})$ is an additive mapping satisfying $h(M)M = 0$ for all $M \in \mathcal{M}$.*

COROLLARY 2.4. *Let $\mathcal{U} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular ring. Assume that $\mathcal{Z}(\mathcal{U})P = \mathcal{Z}(P\mathcal{U}P)$, $\mathcal{Z}(\mathcal{U})Q = \mathcal{Z}(Q\mathcal{U}Q)$ and there exists some element $M_0 \in \mathcal{M}$ such that $\mathcal{Z}(\mathcal{U}) = \{A + B : A \in \mathcal{Z}(\mathcal{A}),\ B \in \mathcal{Z}(\mathcal{B}),\ AM_0 = M_0B\}$. Then a nonadditive map $\varphi : \mathcal{U} \to \mathcal{U}$ is commuting if and only if $\varphi(X) = ZX + f(X)$ for all $X \in \mathcal{U}$, where $Z \in \mathcal{Z}(\mathcal{U})$ and $f : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a map.*

Acknowledgement. This work is supported by the National Natural Science Foundation of China (No. 12071134) and Natural Science Foundation of Shaanxi Province (No. 2021JM-119).

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(Received November 21, 2022)

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