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SOME INEQUALITIES FOR EIGENVALUES AND POSITIVE LINEAR MAPS

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Abstract. In this paper, we demonstrate some inequalities for positive linear maps on matrices. Moreover, we discuss lower bounds for the spread of nonnegative matrices which provides improvement over the existing bounds.

1. Introduction

Let $\mathbb{M}_n$ denote the algebra of all $n\times n$ complex matrices. Let $A\in\mathbb{M}_n$. We denote $\operatorname{tr}(A)$ and $\sigma(A)$ by the trace of A and the set of all eigenvalues in this paper. Let $A\in\mathbb{M}_n$, $(n\geqslant 3)$, and let $\lambda_1(A),\lambda_2(A),\ldots,\lambda_n(A)$ be the eigenvalues of A. The spread of A denoted $spd(A)$, is defined by $spd(A)=\max_{i,j}\left|\lambda_i(A)-\lambda_j(A)\right|$. This quantity first was introduced by Mirsky [14]. In literature, inequalities for spreads have been studied by several authors; see [1, 12, 14, 15, 17]. A linear map $\Phi:\mathbb{M}_n\longrightarrow\mathbb{M}_k$ is called positive if $\Phi(A)$ is positive semidefinite $(\Phi(A)\geqslant O)$ whenever $A\geqslant O$, and strictly positive if $\Phi(A)>O$ is positive whenever $A>O$. It is called unital if $\Phi(I_n)=I_k$. We use the symbol φ for a linear functional from $\mathbb{M}_n$ into $\mathbb{C}$. Inequalities involving positive linear maps, and spreads have been obtained by many authors in literature; see [2, 3, 4, 5, 6, 10, 11, 18, 19]. For example, Kadison [10] proved that if A is any Hermitian element of $\mathbb{M}_n$, then

$$\Phi(A^2)\geqslant(\Phi(A))^2. \tag{1}$$

An inequality complementary to (1) was obtained by Bhatia and Davis [2]. They proved that if the eigenvalues of a Hermitian matrix A are contained in the interval $[\lambda_{\min}(A),\ \lambda_{\max}(A)]$, then

$$\Phi(A^2)-(\Phi(A))^2\leqslant\frac{1}{4}(spd(A))^2I, \tag{2}$$

for every unital positive linear map. Moreover, they also proved that if $A>0$, then

$$\Phi(A^{-1})\leqslant\frac{(\lambda_{\min}(A)+\lambda_{\max}(A))^2}{4\lambda_{\min}(A)\lambda_{\max}(A)}(\Phi(A))^{-1}. \tag{3}$$

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In [19], Sharma et al. proved that if A is any Hermitian element of $\mathbb{M}_n$, then

$$\varphi(B^4) \leqslant \frac{1}{12}\left(spd\,(A)\right)^4, \tag{4}$$

where φ is any unital positive linear functional and $B = A - \varphi(A)I$.

Wolkowicz and Styan [20] proved that if $A \in \mathbb{M}_n$ has real eigenvalues contained in the interval $[\lambda_{\min}(A),\ \lambda_{\max}(A)]$, then

$$\lambda_{\min}(A) \leqslant \frac{\operatorname{tr}(A)}{n} - \frac{1}{\sqrt{n-1}}\sqrt{\frac{\operatorname{tr}(A^2)}{n} - \left(\frac{\operatorname{tr}(A)}{n}\right)^2} \tag{5}$$

and

$$\lambda_{\max}(A) \geqslant \frac{\operatorname{tr}(A)}{n} + \frac{1}{\sqrt{n-1}}\sqrt{\frac{\operatorname{tr}(A^2)}{n} - \left(\frac{\operatorname{tr}(A)}{n}\right)^2}. \tag{6}$$

The inequalities (5) and (6) were sharpened by Sharma et al. [19]; that is,

$$\lambda_{\min}(A) \leqslant \frac{\operatorname{tr}(A)}{n} - \left(\frac{n^2-3n+3}{n^3(n-1)^3}\right)^{\frac{1}{4}} \frac{\operatorname{tr}(B^2)}{(\operatorname{tr}(B^4))^{\frac{1}{4}}} \tag{7}$$

and

$$\lambda_{\max}(A) \geqslant \frac{\operatorname{tr}(A)}{n} + \left(\frac{n^2-3n+3}{n^3(n-1)^3}\right)^{\frac{1}{4}} \frac{\operatorname{tr}(B^2)}{(\operatorname{tr}(B^4))^{\frac{1}{4}}}, \tag{8}$$

where $B = A - \frac{\operatorname{tr}(A)}{n}I$.

In Section 2 we discuss some inequalities involving eigenvalues and unital positive linear maps; see Theorems 1–6 and Corollaries 1–2. In Section 3 we present lower bounds for the spread of nonnegative matrices; see Theorem 7, Corollary 3, Remark 3 and Theorem 8.

2. Inequalities involving eigenvalues and positive linear maps

We begin with the following proposition of [3].

PROPOSITION 1. *Let P, Q be strictly positive matrices. Then the block matrix* $\begin{bmatrix} P & X \\ X^* & Q \end{bmatrix}$ *is positive if and only if $P \geqslant XQ^{-1}X^*$.*

THEOREM 1. *Let $\Phi : \mathbb{M}_n \longrightarrow \mathbb{M}_k$ be an unital positive linear map. Let A be any normal element of $\mathbb{M}_n$. Then*

$$\Phi\left((A^*A)^r\right) \geqslant \Phi\left(\left(\tfrac{A+A^*}{2}\right)^{2r}\right) \geqslant \left(\Phi\left(\left(\tfrac{A+A^*}{2}\right)^{r}\right)\right)^2, \tag{9}$$

holds for all positive integers r. Furthermore, if $A \in \mathbb{M}_n$ is any nonsingular normal matrix, then

$$\Phi\left(\left(\tfrac{A+A^*}{2}\right)^{2s}\right) \geqslant \Phi\left((A^*A)^s\right), \tag{10}$$

holds for all negative integers s.

Proof. By the spectral theorem of normal matrices, we have

$$A = \sum_{k=1}^n \lambda_k(A)P_k, \qquad A^* = \sum_{k=1}^n \overline{\lambda_k(A)}P_k, \qquad A^*A = \sum_{k=1}^n |\lambda_k(A)|^2 P_k,$$

where $\lambda_k(A)$ are the eigenvalues of A and P_k are the corresponding orthogonal projections with $\sum_{k=1}^n P_k = I$. Therefore, using unital positive linear maps we have

$$\Phi\left((A^*A)^r\right) = \sum_{k=1}^n |\lambda_k(A)|^{2r}\Phi(P_k), \;\; \Phi\left(\left(\tfrac{A+A^*}{2}\right)^r\right) = \sum_{k=1}^n \left(\tfrac{\lambda_k(A)+\overline{\lambda_k(A)}}{2}\right)^r \Phi(P_k),$$

where r is any positive integer, and

$$\sum_{k=1}^n \Phi(P_k) = I_k.$$

For a normal matrix A, the eigenvalues of $\frac{A+A^*}{2}$ are the real parts of the eigenvalues of A. Therefore, for all positive integers r; $|\lambda_k(A)|^{2r} \geqslant (\Re\lambda_k(A))^{2r}$. Now multiplying this inequality by orthogonal projection P_k and summing the resulting inequality for $k = 1, 2, \ldots, n$, and then applying unital positive linear map Φ, we have

$$\sum_{k=1}^n |\lambda_k(A)|^{2r}\Phi(P_k) \geqslant \sum_{k=1}^n \left(\frac{\lambda_k(A)+\overline{\lambda_k(A)}}{2}\right)^{2r} \Phi(P_k),$$

which gives the left-hand side inequality of (9). The right-hand side inequality in (9) follows immediately using (1) because $\frac{A+A^*}{2}$ is Hermitian. Equality holds in the left-hand side of (9) when A is Hermitian.

To prove (10), we will use the same arguments that we used above. We have

$$\sum_{k=1}^n \left(\frac{\lambda_k(A)+\overline{\lambda_k(A)}}{2}\right)^{2s} \Phi(P_k) \geqslant \sum_{k=1}^n |\lambda_k(A)|^{2s}\Phi(P_k),$$

holds for any negative integer s, and hence the inequality (10). □

REMARK 1. If in Theorem 1 we choose $r = 1$ and A is any Hermitian element of $\mathbb{M}_n$, then (9) is equivalent to (1).

THEOREM 2. *Let $\Phi : \mathbb{M}_n \longrightarrow \mathbb{M}_k$ be an unital positive linear map. Let A and B be Hermitian matrices such that $AB = BA$. Then*

$$\Phi\left(\tfrac{A^m+B^m}{2}\right) \geqslant \left(\Phi\left(AB\right)^{\frac{m}{4}}\right)^2, \tag{11}$$

where m is any positive even integer.

Proof. We will prove (11) using the tensor product of matrices. Since matrices A and B are commuting and Hermitian, therefore by the spectral theorem of Hermitian matrices, we have

$$A = \sum_{k=1}^{n} \lambda_k(A) P_k, \qquad B = \sum_{k=1}^{n} \mu_k(A) P_k,$$

where $\lambda_k(A)$ and $\mu_k(B)$ are the eigenvalues of A and B, respectively, and P_k are the corresponding orthogonal projections for both the commuting Hermitian matrices A and B. Also using unital positive linear maps, we have

$$\begin{aligned} &\begin{bmatrix} \Phi\left(\frac{A^m+B^m}{2}\right) & \Phi\left((AB)^{\frac{m}{4}}\right) \\ \Phi\left((AB)^{\frac{m}{4}}\right) & I \end{bmatrix} \\ &= \Sigma_{k=1}^{n} \begin{bmatrix} \frac{(\lambda_k(A))^m+(\mu_k(A))^m}{2} & (\lambda_k(A)\mu_k(A))^{\frac{m}{4}} \\ (\lambda_k(A)\mu_k(A))^{\frac{m}{4}} & 1 \end{bmatrix} \otimes \Phi(P_k). \end{aligned} \tag{12}$$

Now for any even integer m, each summand in (12) is positive and, so is the sum on right-hand side. Finally, apply Proposition 1 to the left-hand side of (12) to obtain (11). $\square$

THEOREM 3. *Let φ be a strictly positive unital linear functional, and let A be a positive definite matrix with $\lambda_{\min}(A) I \leqslant A \leqslant \lambda_{\max}(A) I$. Then*

$$\varphi(A^2)\,(\varphi(A))^{-2} \leqslant \frac{(\lambda_{\min}(A)+\lambda_{\max}(A))^2}{4\lambda_{\min}(A)\,\lambda_{\max}(A)}. \tag{13}$$

Proof. The matrices $A-\lambda_{\min}(A) I$ and $\lambda_{\max}(A) I - A$ are positive and commute with each other. So, $(A-\lambda_{\min}(A) I)(\lambda_{\max}(A) I - A) \geqslant O$. This gives

$$A^2 \leqslant (\lambda_{\min}(A)+\lambda_{\max}(A))A - \lambda_{\min}(A)\,\lambda_{\max}(A)\, I,$$

and hence

$$\varphi(A^2) \leqslant (\lambda_{\min}(A)+\lambda_{\max}(A))\,\varphi(A) - \lambda_{\min}(A)\,\lambda_{\max}(A).$$

Since, $\varphi(A) > 0$, therefore above inequality implies that

$$\varphi(A^2)\,(\varphi(A))^{-2} \leqslant (\lambda_{\min}(A)+\lambda_{\max}(A))\,(\varphi(A))^{-1} - \lambda_{\min}(A)\,\lambda_{\max}(A)\,(\varphi(A))^{-2}. \tag{14}$$

The right-hand side expression of (14) achieves its maximum in the interval $0 < \lambda_{\min}(A) \leqslant \varphi(A) \leqslant \lambda_{\max}(A)$ at $\varphi(A) = \frac{2\lambda_{\min}(A)\lambda_{\max}(A)}{\lambda_{\min}(A)+\lambda_{\max}(A)}$, and hence (13) follows from (14). $\square$

THEOREM 4. *Let φ be a strictly unital positive linear functional defined on $\mathbb{M}_n$, and let $A \in \mathbb{M}_n$ be a Hermitian matrix with $\lambda_{\min}(A) I \leqslant A \leqslant \lambda_{\max}(A) I$. Then*

$$\lambda_{\min}(A) \leqslant \varphi(A) - \sqrt{\varphi(A^2)-(\varphi(A))^2} \qquad \textit{for} \quad \gamma < 0 \tag{15}$$

and

$$\lambda_{\max}(A) \geqslant \varphi(A) + \sqrt{\varphi(A^2) - (\varphi(A))^2} \quad \text{for} \quad \gamma > 0, \tag{16}$$

where $\gamma = \varphi\left((A - \varphi(A)I)^3\right)$.

Proof. We can write (2.20) of the paper [16] in the following equivalent form:

$$a\varphi(P^3) + \frac{\left(a\varphi(P) - \varphi(P^2)\right)^2}{\varphi(P) - a} \leqslant \varphi(P^3) \leqslant b\varphi(P^2) - \frac{\left(b\varphi(P) - \varphi(P^2)\right)^2}{b - \varphi(P)}, \tag{17}$$

where φ is any unital positive linear functional defined on $\mathbb{M}_n$, and $P \in \mathbb{M}_n$ is a Hermitian matrix whose spectrum lies in the interval $[a,b]$; since $\varphi(P^r) = \mu_r'$ for every positive integer r. Now, for a positive definite matrix P, a simple calculation in the left-hand side inequality of (17) leads to the following inequality

$$\varphi(P^3) \geqslant \frac{\left(\varphi(P^2)\right)^2}{\varphi(P)} + \frac{m\left(\varphi(P^2) - (\varphi(P))^2\right)\left(\varphi(P^2) - m\varphi(P)\right)}{\varphi(P)\left(\varphi(P) - m\right)}. \tag{18}$$

The second quantity in the right-hand side of (18) is nonnegative, and hence

$$\varphi(P^3) \geqslant \frac{\left(\varphi(P^2)\right)^2}{\varphi(P)}. \tag{19}$$

Note that $A - xI$ is a positive definite matrix for $x < \lambda_{\min}(A)$. Therefore, we can use (19) for a positive definite matrix $P = A - xI$, we get

$$\frac{\varphi\left((A - xI)^3\right)}{\varphi\left((A - xI)^2\right)} \geqslant \frac{\varphi\left((A - xI)^2\right)}{\varphi(A - xI)}. \tag{20}$$

Since $\lambda_{\min}(A)I \leqslant A \leqslant \lambda_{\max}(A)I$, therefore

$$\lambda_{\min}(A) \leqslant \varphi(A) \leqslant \lambda_{\max}(A). \tag{21}$$

By combining (2) and (21), we obtain that

$$\frac{3\lambda_{\min}(A) - \lambda_{\max}(A)}{2} \leqslant \varphi(A) - \sqrt{\varphi(A^2) - (\varphi(A))^2} \leqslant \lambda_{\max}(A). \tag{22}$$

Also, since $spd(A) = \lambda_{\max}(A) - \lambda_{\min}(A)$ is always nonnegative, therefore we can write

$$\lambda_{\min}(A) \geqslant \frac{3\lambda_{\min}(A) - \lambda_{\max}(A)}{2}. \tag{23}$$

From (22) and (23), we conclude that $\varphi(A) - \sqrt{\varphi(A^2) - (\varphi(A))^2} = x$ (say) lies in the interval $I = I_1 \cup I_2$, where $I_1 = \left[\frac{3\lambda_{\min}(A) - \lambda_{\max}(A)}{2},\ \lambda_{\min}(A)\right)$ and $I_2 = [\lambda_{\min}(A),\ \lambda_{\max}(A)]$.

We need to prove if $\gamma < 0$, then

$$\lambda_{\min}(A) \leqslant x. \tag{24}$$

If $x \in I_2$ then (24) is trivially true without any condition on γ. For $x \in I_1$, we argue contra positively, that is, if $x < \lambda_{\min}(A)$ then $\gamma \geqslant 0$. Let $x \in I_1$, that is,

$$\frac{3\lambda_{\min}(A) - \lambda_{\max}(A)}{2} \leqslant x = \varphi(A) - \sqrt{\varphi(A^2) - (\varphi(A))^2} < \lambda_{\min}(A).$$

By substituting $x = \varphi(A) - \sqrt{\varphi(A^2) - (\varphi(A))^2} < \lambda_{\min}(A)$ in (20), a simple calculation leads to

$$\sqrt{\varphi(A^2) - (\varphi(A))^2}\left[\varphi(A^3) - 3\varphi(A)\varphi(A^2) + 2(\varphi(A))^3\right] \geqslant 0. \tag{25}$$

It follows from (25) that $\gamma = \varphi(A^3) - 3\varphi(A)\varphi(A^2) + 2(\varphi(A))^3 = \varphi\left((A - \varphi(A)I)^3\right) \geqslant 0$; because $\varphi(A^2) - (\varphi(A))^2$ is always nonnegative. This completes the proof of (15). The inequality (16) follows immediately, on applying (15) to the matrix $-A$. $\square$

Now we present improvement in (5) and (6) under the conditions of the following result.

COROLLARY 1. *Let $A \in \mathbb{M}_n$ be a Hermitian matrix with $\lambda_{\min}(A)I \leqslant A \leqslant \lambda_{\max}(A)I$. Let φ be an unital positive linear functional defined on $\mathbb{M}_n$ by $\varphi(A) = \frac{tr(A)}{n}$. Then*

$$\lambda_{\min}(A) \leqslant \frac{tr(A)}{n} - \sqrt{\frac{tr(A^2)}{n} - \left(\frac{tr(A)}{n}\right)^2} \quad \textit{for} \quad \gamma < 0$$

and

$$\lambda_{\max}(A) \geqslant \frac{tr(A)}{n} + \sqrt{\frac{tr(A^2)}{n} - \left(\frac{tr(A)}{n}\right)^2} \quad \textit{for} \quad \gamma > 0,$$

where $\gamma = \frac{1}{n}\left(tr\left(A - \frac{tr(A)}{n}I\right)^3\right)$.

Proof. The proof follows by choosing $\varphi(A) = \dfrac{\mathrm{tr}(A)}{n}$ in Theorem 4. $\square$

THEOREM 5. *Let $A \in \mathbb{M}_n$ and $B \in \mathbb{M}_n$ be commuting positive definite matrices, and let $\lambda_i(A)$ and $\mu_i(B)$ be the eigenvalues of A and B respectively. Let $\Phi : \mathbb{M}_n \longrightarrow \mathbb{M}_k$ be an unital positive linear map. Then*

$$\mu_{\min}(B)\Phi(A) \leqslant \Phi(AB) \leqslant \mu_{\max}(B)\Phi(A) \tag{26}$$

and

$$\lambda_{\min}(A)\Phi(B) \leqslant \Phi(AB) \leqslant \lambda_{\max}(A)\Phi(B). \tag{27}$$

Proof. Since the eigenvalues of positive definite matrices are positive, therefore without loss of generality we assume that

$$\lambda_{\min}(A) \leqslant \lambda_i(A) \leqslant \lambda_{\max}(A) \tag{28}$$

and

$$\mu_{\min}(B) \leqslant \mu_i(B) \leqslant \mu_{\max}(B), \tag{29}$$

for all $i = 1,2,\ldots,n$. From (28) and (29), we find that

$$\mu_{\min}(B)\lambda_i(A) \leqslant \lambda_i(A)\mu_i(B) \leqslant \mu_{\max}(B)\lambda_i(A),$$

and therefore

$$\mu_{\min}(B)\sum_{i=1}^n \lambda_i(A)\Phi(P_i) \leqslant \sum_{i=1}^n \lambda_i(A)\mu_i(B)\Phi(P_i) \leqslant \mu_{\max}(B)\lambda_i(A)\Phi(P_i). \tag{30}$$

The inequality (26) now follows immediately from (30). Likewise, (27) holds on using similar arguments. $\square$

COROLLARY 2. *Let φ be any strictly positive unital linear functional defined on $\mathbb{M}_n$, and let A be a positive definite matrix with $\lambda_{\min}(A)I \leqslant A \leqslant \lambda_{\max}(A)I$. Then for $k \in \mathbb{N}$, we have*

$$(\lambda_{\min}(A))^k \leqslant \frac{\varphi(A^{k+1})}{\varphi(A)} \leqslant (\lambda_{\max}(A))^k. \tag{31}$$

Proof. The inequality (31) follows on choosing $B = A^k$ in (26). $\square$

THEOREM 6. *Let φ be any strictly unital positive linear functional defined on $\mathbb{M}_n$, and Let $A \in \mathbb{M}_n$ be a Hermitian matrix with $\lambda_{\min}(A)I \leqslant A \leqslant \lambda_{\max}(A)I$. Then*

$$(spd(A))^2 \geqslant 3\left(\varphi(A^2) - \varphi(A)^2 + \left(\frac{\gamma}{2}\right)^{\frac{2}{3}}\right), \tag{32}$$

where $\gamma = \varphi\left((A - \varphi(A)I)^3\right)$.

Proof. Let $B = A - \lambda_{\min}(A)I$. Then B is positive semidefinite matrix whose spectrum is lies in $[0, \lambda_{\max}(A) - \lambda_{\min}(A)]$. Applying the Corollary 2 to the matrix B, and choosing $k = 2$, we have

$$\begin{aligned}(spd(A))^2 &\geqslant \frac{\varphi\left((A - \lambda_{\min}(A)I)^3\right)}{\varphi(A - \lambda_{\min}(A)I)} \\ &= \frac{-(\lambda_{\min}(A))^3 + 3(\lambda_{\min}(A))^2\varphi(A) - 3\lambda_{\min}(A)\varphi(A^2) + \varphi(A^3)}{\varphi(A) - \lambda_{\min}(A)}.\end{aligned} \tag{33}$$

A simple calculation in (33) leads to

$$(spd(A))^2 \geqslant y^2 + \mu + \tfrac{\gamma}{y}, \tag{34}$$

where $\mu = 3\left(\varphi(A^2) - (\varphi(A))^2\right)$ and $y = \varphi(A) - \lambda_{\min}(A)$. Now (32) follows from (34), since the function attains its minimum at $y = \left(\frac{\gamma}{2}\right)^{\frac{1}{3}}$ only when $\gamma > 0$. Similarly, for matrix $\lambda_{\max}(A)I - A$ we get the same result when $\gamma < 0$. $\square$

We now present an example which shows the effectiveness of Theorem 3, Theorem 4 and Theorem 6.

EXAMPLE 1. Let

$$A = \begin{bmatrix} 0.3 & 0.02 & 0.015 \\ 0.02 & 0.1 & 0.04 \\ 0.015 & 0.04 & 0.6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{bmatrix}.$$

From (3), we find $\frac{(\lambda_{\min}(A) + \lambda_{\max}(A))^2}{4\lambda_{\min}(A)\lambda_{\max}(A)} \geqslant 1.0402$ while Theorem 3 provides better estimate $\frac{(\lambda_{\min}(A) + \lambda_{\max}(A))^2}{4\lambda_{\min}(A)\lambda_{\max}(A)} \geqslant 1.2$ for the choice of an unital linear functional $\varphi(A) = a_{22}$.

By (7) and (8) we respectively have $\lambda_{\min}(B) \leqslant 3.7414$ and $\lambda_{\max}(B) \geqslant 7.2586$. From Theorem 4, $\lambda_{\min}(B) \leqslant 2.6277$ for $\varphi(B) = \frac{\operatorname{tr}(B)}{n}$; and $\lambda_{\max}(B) \geqslant 7.6955$ for $\varphi(B) = b_{11}$. Likewise using (2) and (4); $spd(B) \geqslant 7.211$ and $spd(B) \geqslant 7.579$ while our Theorem 6 gives better estimate $spd(B) \geqslant 7.6955$ for $\varphi(B) = b_{11}$.

3. Inequalities for nonnegative matrices

We say $M = [m_{ij}] \in \mathbb{M}_{m\times n}$ is nonnegative if all $m_{ij} \geqslant 0$. Denote $\mathbb{M}_n^+$ by the collection of all $n \times n$ nonnegative matrices. Define

$$\rho(M) = \max\{|\lambda(M)| : \lambda(M) \in \sigma(M)\},$$

to be the spectral radius of M. For a nonnegative matrix $\rho(M)$ is an eigenvalue of M called Perron eigenvalue of M and the corresponding nonnegative eigenvector is known as Perron eigenvector. For strictly positive matrix, Perron eigenvalue is simple eigenvalue that is strictly larger than the modulus of any other eigenvalue and corresponding Perron eigenvector has positive components, see [9].

Notice that the spread of a nilpotent matrix is zero; and $spd(cM) = |c|\, spd(M)$ for every complex number c. Therefore, to study the spread of a matrix one can assume, without loss of generality, that $\rho(M) = 1$. In the present context, Drnovšek [7] proved that if $M \in \mathbb{M}_n^+$ with one zero diagonal element and spectral radius $\rho(M) = 1$ then for $n \geqslant 6$,

$$spd(M) > \frac{2}{4 + \sqrt{2(n+3)}}. \tag{35}$$

In 2021, Drnovšek [8] proved that if $M \in \mathbb{M}_n$ then for $n \geqslant 2$,

$$spd\,(M) > \frac{2}{2+\sqrt{2n}}(\rho(M) - m), \tag{36}$$

where $m = \min_{1 \leqslant i \leqslant n} m_{ii}$.

In this section, we present improvement of (35) and (36).

We begin with the following proposition of [7].

PROPOSITION 2. *Let $M \in \mathbb{M}_n^+$ with k zero diagonal elements. Then for all positive integers t*

$$s_1^t \leqslant (n-k)^{t-1} s_t, \tag{37}$$

where $s_t = tr(M^t)$.

We now present a lower bound for the spread of a nonnegative matrix in terms of order of matrix n with k zero diagonal elements.

THEOREM 7. *Let $M \in \mathbb{M}_n^+$ with k zero diagonal elements, and let $\rho(M) = 1$. Then*

$$spd\,(M) \geqslant \frac{2}{(n-1)\left(\frac{2}{n} + \sqrt{\frac{4}{n^2} + \frac{2}{k(n-1)}}\right)}. \tag{38}$$

Proof. Let $\lambda_1(M) = \rho(M), \lambda_2(M), \ldots, \lambda_n(M)$ be the eigenvalues of M. Since $\rho(M) = 1$, and the matrix M with k zero diagonal elements, therefore $spd\,(M) \neq 0$. When $spd\,(M) \geqslant 1$ then result is obvious, thus here we suppose that $spd\,(M) < 1$ hence, $\Re\lambda_i(M) \geqslant 0$. Applying Proposition 2 by taking $t = 2$ in (37), we see that

$$\left(\sum_{i=1}^{n} \lambda_i(M)\right)^2 = s_1^2 \leqslant (n-k)s_2 = (n-k)\sum_{i=1}^{n} \lambda_i^2(M).$$

This inequality can be rewritten as

$$k\sum_{i=1}^{n} \lambda_i^2(M) \leqslant \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left(\lambda_i(M) - \lambda_j(M)\right)^2 \leqslant \frac{n(n-1)}{2}(spd\,(M))^2. \tag{39}$$

Also, one can easily observe that

$$\Re\left(1 - \lambda_i^2(M)\right) \leqslant |1 - \lambda_i^2(M)| = |\rho^2(M) - \lambda_i^2(M)| \leqslant spd\left(M^2\right) \tag{40}$$

and

$$spd\left(M^2\right) = \max_{i,j} |\lambda_i^2(M) - \lambda_j^2(M)| = \max_{i,j}(|\lambda_i(M) - \lambda_j(M)||\lambda_i(M) + \lambda_j(M)|),$$

which gives

$$spd\left(M^2\right) = \max_{i,j} |\lambda_i^2(M) - \lambda_j^2(M)| \leqslant 2\,spd\,(M). \tag{41}$$

Now, we write

$$tr\left(M^2\right) = \sum_{i=1}^{n} \lambda_i^2(M) = \sum_{i=1}^{n} \Re\lambda_i^2(M) = 1 + \sum_{i=2}^{n} \Re\lambda_i^2(M). \tag{42}$$

By combining (40), (41) and (42), a little calculation leads to

$$\sum_{i=1}^{n} \lambda_i^2(M) \geqslant 1 + (n-1)\left(1 - 2\, spd\,(M)\right). \tag{43}$$

By using (39) and (43), we have

$$k\left[1 + (n-1)\left(1 - 2\, spd\,(M)\right)\right] \leqslant \frac{n(n-1)}{2} \left(spd\,(M)\right)^2,$$

which leads to the inequality

$$n\left(n-1\right)\left(spd\,(M)\right)^2 + 4k\left(n-1\right)spd\,(M) - 2nk \geqslant 0. \tag{44}$$

The inequality (38) now follows immediately from (44). □

REMARK 2. By using (39) one can easily observe that for any $M \in \mathbb{M}_n^+$ with k zero diagonal elements,

$$spd\,(M) \geqslant \sqrt{\frac{2k\,\mathrm{tr}\left(M^2\right)}{n(n-1)}}.$$

COROLLARY 3. *Let* $M = [m_{ij}] \in \mathbb{M}_n^+$, $m = \min_{1 \leqslant i \leqslant n} m_{ii}$ *and* $\rho(M) > m$. *Then*

$$spd\,(M) \geqslant \frac{2}{(n-1)\left(\frac{2}{n} + \sqrt{\frac{4}{n^2} + \frac{2}{n-1}}\right)}\left(\rho(M) - m\right). \tag{45}$$

Proof. Let $\lambda_i(M)$ be an eigenvalues of M. Then, one can easily observe that

$$\max_i |\lambda_i(M) - m| \geqslant |\rho(M) - m| = \rho(M) - m,$$

which implies that

$$\rho(M - mI) \geqslant \rho(M) - m. \tag{46}$$

By setting $N = \dfrac{M - mI}{\rho(M - mI)} \in \mathbb{M}_n^+$, one can see that N has at least one zero diagonal element, and $\rho(N) = 1$. Therefore, using Theorem 7 for the matrix N, we have

$$spd\,(M) \geqslant \frac{2}{(n-1)\left(\frac{2}{n} + \sqrt{\frac{4}{n^2} + \frac{2}{n-1}}\right)}\rho(M - mI), \tag{47}$$

since

$$spd\,(N) = \frac{spd\,(M - mI)}{\rho(M - mI)} = \frac{spd\,(M)}{\rho(M - mI)}.$$

The inequality (45) now follows on combining (46) and (47). □

REMARK 3. If we choose at least k diagonal elements of the matrix N are zero in the proof of Corollary 3, then one can easily obtain the following inequality:

$$spd(M) \geqslant \frac{2}{(n-1)\left(\frac{2}{n}+\sqrt{\frac{4}{n^2}+\frac{2}{k(n-1)}}\right)}(\rho(M)-m). \tag{48}$$

It is also notice here that for $k \geqslant 2$ the inequality (48) improves over (36), while for $k=1$, the inequalities (36) and (45) are independent of each other. For $k=1$ and $\rho(M)=1$ the inequality (38) improves over (35).

THEOREM 8. *Let $A=[a_{ij}] \in \mathbb{M}_n^+$ be a strictly positive matrix with Perron eigenvalue $\lambda(A)$ and corresponding Perron eigenvector $x=\{x_i\}>0$, and let $\mu(A)$ be any eigenvalue of A other than $\lambda(A)$. Then*

$$|\mu(A)| \leqslant \lambda(A) - \tfrac{x_m}{x_M}\textstyle\sum_{j=1}^n m_j(A), \tag{49}$$

where $m_j(A)=\min_i\{a_{ij}\}$, $x_m=\min_i\{x_i\}$, $x_M=\max_i\{x_i\}$.

Proof. In [13], Lynn and Timlake have proved that:

$$|\mu(A)| \leqslant \lambda(A) - \textstyle\sum_{j=1}^n \left(\min_i \tfrac{a_{ij}}{x_i}\right) x_j, \tag{50}$$

where $\mu(A)$ is any eigenvalue of $A \in \mathbb{M}_n^+$ other than $\lambda(A)$. Also, one can see that

$$\min_i \tfrac{a_{ij}}{x_i} \geqslant \tfrac{m_j(A)}{x_M}. \tag{51}$$

By combining (50) and (51) we get (49). □

REMARK 4. In particular, for strictly positive (row) stochastic matrix B, that is, all its row sum is unity, we have

$$|\mu(B)| \leqslant 1 - \textstyle\sum_{j=1}^n m_j(B).$$

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