Operators
and
Matrices
Volume 17, Number 4 (2023), 1023–1036 doi:10.7153/oam-2023-17-67

COMMUTING MAPS ON STRICTLY UPPER TRIANGULAR MATRIX RINGS

SHU-WEN KO AND CHENG-KAI LIU*

(*Communicated by E. Poon*)

Abstract. Let R be either a ring with 1 or a semiprime ring not necessarily with 1 and let $N_n(R)$ be the $n\times n$ strictly upper triangular matrix ring over R, where $n\geqslant 3$ is an integer. We completely characterize additive maps $f:N_n(R)\to N_n(R)$ satisfying $[f(x),x]=0$ for all $x\in N_n(R)$. Our theorem naturally generalizes a recent result obtained by Bounds [3] for strictly upper triangular matrix rings over a field of characteristic 0.

1. Introduction and results

Throughout here, R denotes an associative ring with center $Z(R)$. R is called prime if for any $a,b\in R$, $aRb=0$ implies $a=0$ or $b=0$ and R is called semiprime if for any $a\in R$, $aRa=0$ implies $a=0$. For $a,b\in R$, we let $[a,b]=ab-ba$ be the commutator of a and b. A map $f:R\to R$ is called additive if $f(x+y)=f(x)+f(y)$ for all $x,y\in R$. A map $f:R\to R$ is said to be commuting if $[f(x),x]=0$ for all $x\in R$. The usual goal when treating a commuting map is to describe its form. The study of additive commuting maps was initiated by Divinsky and Posner. In 1955 Divinsky [12] proved that if a simple artinian ring R admits a commuting automorphism σ, then either R is commutative or σ is the identity map. On the other hand, in 1957 Posner [22] proved that if a prime ring R admits a commuting derivation d, then either R is commutative or $d=0$. In 1993 Brešar [4] extended above two results to general additive maps and proved that if R is a prime ring with the extended centroid C and $f:R\to R$ is an additive commuting map, then f must be of the form $f(x)=\lambda x+\mu(x)$ for all $x\in R$, where $\lambda\in C$ and $\mu:R\to C$ is an additive map. This influential result has been extended to semiprime rings, superalgebras, von Neumann algebras, C^*-algebras, Lie algebras and matrix algebras etc. We refer the reader to the book [5] for the development of the theory of commuting maps. Recently, additive commuting maps on subrings or subsets of matrix rings have been widely investigated in the literature (see [7]–[11], [13]–[21], [23]–[28] for instance). In 2000 Beidar, Brešar and Chebotar [1] proved that if $T_n(F)$ is the ring of all $n\times n$ upper triangular matrices over a field F and $f:T_n(F)\to T_n(F)$ is a linear commuting map, where $n\geqslant 2$ is an integer, then f is of

Mathematics subject classification (2020): 15A78, 15A27, 16R60, 16N60.
Keywords and phrases: Commuting map, functional identity, strictly upper triangular matrix ring, semiprime ring.
* Corresponding author.

Zagreb
Paper OaM-17-67

the form $f(x) = \lambda x + \mu(x)$ for all $x \in T_n(F)$, where $\lambda \in F$ and $\mu : T_n(F) \to Z(T_n(F))$ is a linear map. This result was later extended to linear commuting maps on the ring of all upper triangular matrices over a commutative ring with 1 by Cheung in [6] and extended to additive commuting maps on the ring of all upper triangular matrices over an arbitrary ring with 1 by Eremita in [13]. In 2016 Bounds [3] successfully characterized linear commuting maps on the ring of all strictly upper triangular matrices over a field of characteristic 0. As usual, let R be a ring with 1, let $M_n(R)$ be the ring of all $n \times n$ matrices over R and let $\{e_{i,j} \mid 1 \leqslant i, j \leqslant n\}$ be the set of matrix units in $M_n(R)$. Precisely, Bounds proved the following:

THEOREM JB. ([3]) *Let $n \geqslant 4$ be an integer and let $N_n(F)$ be the ring of all $n \times n$ strictly upper triangular matrices over a field F of characteristic 0. Suppose that $f : N_n(F) \to N_n(F)$ is a linear map such that $[f(x), x] = 0$ for all $x \in N_n(F)$. Then there exist $\lambda \in F$ and a linear map $\mu : N_n(F) \to \Omega$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in N_n(F)$, where $\Omega = \{\alpha e_{1,n-1} + \beta e_{1,n} + \gamma e_{2,n} : \alpha, \beta, \gamma \in F\}$.*

The proof of Theorem JB depends heavily on the well-known fact about the centralizers of a nonderogatory matrix over a field F of characteristic 0. Up to now, it is still an open problem whether Theorem JB holds true for any field F of positive characteristic $p \geqslant 2$.

PROBLEM 1. *Let $n \geqslant 4$ be an integer and let F be a field of characteristic $p \geqslant 2$. Assume that $f : N_n(F) \to N_n(F)$ is a linear commuting map. Can we describe the form of f?*

The goal of this paper is to give an affirmative answer to Problem 1. Moreover, we extend Theorem JB to strictly upper triangular matrix rings over an arbitrary ring R with 1. Precisely, we prove the following:

THEOREM 1.1. *Let R be a ring with 1 and with center $Z(R)$. Let $N_n(R)$ be the ring of all $n \times n$ strictly upper triangular matrices over R with center $\mathcal{Z}$, where $n \geqslant 3$ is an integer. Suppose that $f : N_n(R) \to N_n(R)$ is an additive map. Then $[f(x), x] = 0$ for all $x \in N_n(R)$ if and only if there exist $\lambda \in Z(R)$, an additive map $\mu : N_n(R) \to \mathcal{Z}$, and an additive map $\nu : N_n(R) \to \Omega$ such that $f(x) = \lambda x + \mu(x) + \nu(x)$ for all $x \in N_n(R)$, where $\Omega = \{\alpha e_{1,n-1} + \beta e_{2,n} : \alpha, \beta \in R\}$ and ν is defined by some $a \in R$ such that $\nu(x) = e_{1,1} x a e_{2,n-1} + e_{2,n-1} a x e_{n,n}$ for all $x \in N_n(R)$.*

In case R is a semiprime ring not necessarily with 1, we prove the following:

THEOREM 1.2. *Let R be a semiprime ring with the multiplier ring $M(R)$ and with the centroid $C(R)$. Let $N_n(R)$ be the ring of all $n \times n$ strictly upper triangular matrices over R with center $\mathcal{Z}$, where $n \geqslant 3$ is an integer. Suppose that $f : N_n(R) \to N_n(R)$ is an additive map. Then $[f(x), x] = 0$ for all $x \in N_n(R)$ if and only if there exist $\lambda \in C(R)$, an additive map $\mu : N_n(R) \to \mathcal{Z}$, and an additive map $\nu : N_n(R) \to \Omega$ such that $f(x) = \lambda x + \mu(x) + \nu(x)$ for all $x \in N_n(R)$, where $\Omega = \{\alpha e_{1,n-1} + \beta e_{2,n} : \alpha, \beta \in R\}$ and ν is defined by some $a \in M(R)$ such that $\nu(x) = e_{1,1} x a e_{2,n-1} + e_{2,n-1} a x e_{n,n}$ for all $x \in N_n(R)$.*

2. Preliminaries

The following lemma is essential to our proof.

LEMMA 2.1. *Let R be either a ring with 1 or a semiprime ring and let $Z(R)$ be the center of R. Suppose that $g : R \to R$ is an additive map. If $xg(x) = 0$ for all $x \in R$, then $g = 0$. Similarly, if $g(x)x = 0$ for all $x \in R$, then $g = 0$.*

Proof. By assumption,

$$xg(x) = 0 \tag{2.1}$$

for all $x \in R$. Assume first that $1 \in R$. Setting $x = 1$ in (2.1), we have $g(1) = 0$. Replacing x with $x+1$ in (2.1) and using $g(1) = 0 = xg(x)$, we obtain $g(x) = 0$ for all $x \in R$, as desired. Assume now that R is semiprime. Replacing x with $x+y$ in (2.1) and using $xg(x) = yg(y) = 0$, we obtain

$$xg(y) + yg(x) = 0 \tag{2.2}$$

for all $x,y \in R$. Replacing y with zy in (2.2), we obtain $xg(zy) + zyg(x) = 0$ for all $x,y,z \in R$. Multiplying (2.2) by z from the left, we obtain $zxg(y) + zyg(x) = 0$ for all $x,y,z \in R$. The difference of the last two equations yields

$$xg(zy) - zxg(y) = 0 \tag{2.3}$$

for all $x,y,z \in R$. Replacing x with wx in (2.3), we obtain $wxg(zy) - zwxg(y) = 0$ for all $x,y,z,w \in R$. Multiplying (2.3) by w from the left, we obtain $wxg(zy) - wzxg(y) = 0$ for all $x,y,z,w \in R$. The difference of the last two equations yields $(zw - wz)xg(y) = 0$ for all $x,y,z,w \in R$. Thus $(g(y)w - wg(y))x(g(y)w - wg(y)) = 0$ for all $x,y,w \in R$. By semiprimeness of R, $g(y)w - wg(y) = 0$ for all $y,w \in R$. This implies that $g(y) \in Z(R)$ for all $y \in R$. Using $g(x) \in Z(R)$ and $0 = xg(x) = g(x)x$ for all $x \in R$, by (2.2), we have $0 = g(x)(xg(y) + yg(x)) = g(x)yg(x)$ for all $x,y \in R$. By semiprimeness of R, $g(x) = 0$ for all $x \in R$. This proves the lemma. $\square$

Throughout the rest of this section, R denotes either a ring with 1 or a semiprime ring not necessarily with 1, $M_n(R)$ denotes the ring of all $n \times n$ matrices over R, $N_n(R)$ denotes the ring of all $n \times n$ strictly upper triangular matrices over R, where $n \geqslant 3$ is an integer and $f : N_n(R) \to N_n(R)$ is an additive map such that $[f(x), x] = 0$ for all $x \in N_n(R)$, that is,

$$f(x)x = xf(x) \tag{2.4}$$

for all $x \in N_n(R)$. Replacing x with $x+y$ in (2.4), we obtain

$$f(x)y - yf(x) = xf(y) - f(y)x \tag{2.5}$$

for all $x,y \in N_n(R)$. As usual, we let $\{e_{i,j} \mid 1 \leqslant i,j \leqslant n\}$ be the set of matrix units in $M_n(R^*)$, where $R^* = R$ if $1 \in R$ and R^* is the ring extension of R adjoint with 1 if $1 \notin R$. Then $N_n(R) = \sum_{i,j=1, i<j}^{n} Re_{i,j}$ and the center $Z(N_n(R))$ of $N_n(R)$ coincides with $Re_{1,n}$. For two distinct integers i,j with $1 \leqslant i < j \leqslant n$, we write

$$f(\alpha e_{i,j}) = \sum_{s,t=1, s<t}^{n} c_{s,t}^{i,j}(\alpha) e_{s,t}$$

for all $\alpha \in R$, where each $c_{s,t}^{i,j} : R \to R$ is an additive map.

LEMMA 2.2. *Let i, j be distinct integers such that $1 \leqslant i < j \leqslant n$. Then $c_{\ell,i}^{i,j} = 0$ for every integer ℓ with $1 \leqslant \ell < i$ and $c_{j,\ell}^{i,j} = 0$ for every integer ℓ with $j < \ell \leqslant n$.*

Proof. Setting $x = \alpha e_{i,j}$ in (2.4), we have

$$f(\alpha e_{i,j})\alpha e_{i,j} = \alpha e_{i,j} f(\alpha e_{i,j}) \tag{2.6}$$

for all $\alpha \in R$. Let ℓ be an integer such that $1 \leqslant \ell < i$. Multiplying (2.6) by $e_{\ell,\ell}$ from the left and by $e_{j,j}$ from the right, we obtain $e_{\ell,\ell} f(\alpha e_{i,j})\alpha e_{i,j} = 0$. This implies $c_{\ell,i}^{i,j}(\alpha)\alpha = 0$ for all $\alpha \in R$. By Lemma 2.1, $c_{\ell,i}^{i,j} = 0$. Let ℓ be an integer such that $j < \ell \leqslant n$. Multiplying (2.6) by $e_{i,i}$ from the left and by $e_{\ell,\ell}$ from the right, we obtain $0 = \alpha e_{i,j} f(\alpha e_{i,j}) e_{\ell,\ell}$. This implies that $\alpha c_{j,\ell}^{i,j}(\alpha) = 0$ for all $\alpha \in R$. By Lemma 2.1, $c_{j,\ell}^{i,j} = 0$. □

LEMMA 2.3. *Let $n \geqslant 4$ be an integer and let i, j be distinct integers such that $1 \leqslant i < j \leqslant n-2$. Then $c_{\ell,k}^{i,j} = 0$ for every integers ℓ, k with $j < \ell < k \leqslant n$.*

Proof. Let ℓ, k be integers with $j < \ell < k \leqslant n$. Setting $x = \alpha e_{i,j}$ and $y = \beta e_{j,\ell}$ in (2.5), we have

$$f(\alpha e_{i,j})\beta e_{j,\ell} - \beta e_{j,\ell} f(\alpha e_{i,j}) = \alpha e_{i,j} f(\beta e_{j,\ell}) - f(\beta e_{j,\ell})\alpha e_{i,j} \tag{2.7}$$

for all $\alpha, \beta \in R$. Multiplying (2.7) by $e_{j,j}$ from the left and by $e_{k,k}$ from the right, we obtain $-\beta e_{j,\ell} f(\alpha e_{i,j}) e_{k,k} = 0$. This implies that $\beta c_{\ell,k}^{i,j}(\alpha) = 0$ for all $\alpha, \beta \in R$. By Lemma 2.1, $c_{\ell,k}^{i,j} = 0$. □

LEMMA 2.4. *Let i, j be distinct integers such that $1 \leqslant i < j \leqslant n$. Then $c_{i,\ell}^{i,j} = 0$ for every integer ℓ with $i < \ell < j$.*

Proof. Let ℓ be an integer such that $i < \ell < j$. Setting $x = \alpha e_{i,j}$ and $y = \beta e_{\ell,j}$ in (2.5), we have

$$f(\alpha e_{i,j})\beta e_{\ell,j} - \beta e_{\ell,j} f(\alpha e_{i,j}) = \alpha e_{i,j} f(\beta e_{\ell,j}) - f(\beta e_{\ell,j})\alpha e_{i,j} \tag{2.8}$$

for all $\alpha, \beta \in R$. Note that $\alpha e_{i,j} f(\beta e_{\ell,j}) e_{j,j} = e_{i,i} f(\beta e_{\ell,j})\alpha e_{i,j} = 0$ as $f(\beta e_{\ell,j}) \in N_n(R)$. Multiplying (2.8) by $e_{i,i}$ from the left and by $e_{j,j}$ from the right, we obtain $e_{i,i} f(\alpha e_{i,j})\beta e_{\ell,j} = 0$. This implies that $c_{i,\ell}^{i,j}(\alpha)\beta = 0$ for all $\alpha, \beta \in R$. By Lemma 2.1, $c_{i,\ell}^{i,j} = 0$. □

LEMMA 2.5. *Let i, j be distinct integers such that $1 \leqslant i < j \leqslant n$. Then $c_{\ell,k}^{i,j} = 0$ for every integers ℓ, k with $i < \ell < j$ and $\ell < k \leqslant n$.*

Proof. Let ℓ, k be integers such that $i < \ell < j$ and $\ell < k \leqslant n$. Setting $x = \alpha e_{i,j}$ and $y = \beta e_{i,\ell}$ in (2.5), we have

$$f(\alpha e_{i,j})\beta e_{i,\ell} - \beta e_{i,\ell} f(\alpha e_{i,j}) = \alpha e_{i,j} f(\beta e_{i,\ell}) - f(\beta e_{i,\ell})\alpha e_{i,j} \tag{2.9}$$

for all $\alpha,\beta \in R$. Note that $e_{ii}f(\beta e_{i,\ell})\alpha e_{i,j} = 0$ as $f(\beta e_{i,\ell}) \in N_n(R)$. Multiplying (2.9) by $e_{i,i}$ from the left and by $e_{k,k}$ from the right, we obtain

$$-\beta e_{i,\ell} f(\alpha e_{i,j}) e_{kk} = \alpha e_{i,j} f(\beta e_{i,\ell}) e_{k,k} \tag{2.10}$$

for all $\alpha,\beta \in R$. Assume first that $k \leqslant j$. Then $\alpha e_{i,j} f(\beta e_{i,\ell}) e_{k,k} = 0$ as $f(\beta e_{i,\ell}) \in N_n(R)$. With this, by (2.10), we obtain $\beta e_{i,\ell} f(\alpha e_{i,j}) e_{k,k} = 0$. This implies that $\beta c^{i,j}_{\ell,k}(\alpha) = 0$ for all $\alpha,\beta \in R$. By Lemma 2.1, $c^{i,j}_{\ell,k} = 0$, as desired. Assume now that $k > j$. In this case, $i < \ell < j < k \leqslant n$. By (2.10), $-\beta c^{i,j}_{\ell,k}(\alpha) = \alpha c^{i,\ell}_{j,k}(\beta)$ for all $\alpha,\beta \in R$. In view of Lemma 2.3, $c^{i,\ell}_{j,k} = 0$. So we have $\beta c^{i,j}_{\ell,k}(\alpha) = 0$ for all $\alpha,\beta \in R$. By Lemma 2.1, $c^{i,j}_{\ell,k} = 0$, as desired. □

LEMMA 2.6. *Let $n \geqslant 4$ be an integer and let i, j be distinct integers such that $3 \leqslant i < j \leqslant n$. Then $c^{i,j}_{\ell,k} = 0$ for every integers ℓ, k with $1 \leqslant \ell < k < i$.*

Proof. Let ℓ, k be integers such that $1 \leqslant \ell < k < i$. Clearly, $1 \leqslant \ell < k < i < j \leqslant n$. Setting $x = \alpha e_{i,j}$ and $y = \beta e_{k,i}$ in (2.5), we have

$$f(\alpha e_{i,j})\beta e_{k,i} - \beta e_{k,i} f(\alpha e_{i,j}) = \alpha e_{i,j} f(\beta e_{k,i}) - f(\beta e_{k,i})\alpha e_{i,j} \tag{2.11}$$

for all $\alpha,\beta \in R$. Multiplying (2.11) by $e_{\ell,\ell}$ from the left and by $e_{i,i}$ from the right, we obtain $e_{\ell,\ell} f(\alpha e_{i,j})\beta e_{k,i} = 0$. This implies that $c^{i,j}_{\ell,k}(\alpha)\beta = 0$ for all $\alpha,\beta \in R$. By Lemma 2.1, $c^{i,j}_{\ell,k} = 0$. □

LEMMA 2.7. *Let $n \geqslant 4$ be an integer and let i, j be distinct integers such that $2 \leqslant i < j \leqslant n$. Then $c^{i,j}_{\ell,k} = 0$ for every integers ℓ, k with $1 \leqslant \ell < i < k < j$.*

Proof. Let ℓ, k be integers such that $1 \leqslant \ell < i < k < j$. Setting $x = \alpha e_{i,j}$ and $y = \beta e_{k,j}$ in (2.5), we have

$$f(\alpha e_{i,j})\beta e_{k,j} - \beta e_{k,j} f(\alpha e_{i,j}) = \alpha e_{i,j} f(\beta e_{k,j}) - f(\beta e_{k,j})\alpha e_{i,j} \tag{2.12}$$

for all $\alpha,\beta \in R$. Multiplying (2.12) by $e_{\ell,\ell}$ from the left and by $e_{j,j}$ from the right, we obtain $e_{\ell,\ell} f(\alpha e_{i,j})\beta e_{k,j} = -e_{\ell,\ell} f(\beta e_{k,j})\alpha e_{i,j}$. This implies that $c^{i,j}_{\ell,k}(\alpha)\beta = -c^{k,j}_{\ell,i}(\beta)\alpha$ for all $\alpha,\beta \in R$. By Lemma 2.6, $c^{k,j}_{\ell,i} = 0$ as $\ell < i < k < j$. Thus we have $c^{i,j}_{\ell,k}(\alpha)\beta = 0$ for all $\alpha,\beta \in R$. By Lemma 2.1, $c^{i,j}_{\ell,k} = 0$. □

LEMMA 2.8. *Let $n \geqslant 4$ be an integer and let i, j be distinct integers such that $2 \leqslant i < j \leqslant n-1$. Then $c^{i,j}_{\ell,k} = 0$ for every integers ℓ, k with $1 \leqslant \ell < i < j \leqslant k \leqslant n-1$.*

Proof. Let ℓ, k be integers such that $1 \leqslant \ell < i < j \leqslant k \leqslant n-1$. Setting $x = \alpha e_{i,j}$ and $y = \beta e_{k,n}$ in (2.5), we have

$$f(\alpha e_{i,j})\beta e_{k,n} - \beta e_{k,n} f(\alpha e_{i,j}) = \alpha e_{i,j} f(\beta e_{k,n}) - f(\beta e_{k,n})\alpha e_{i,j} \tag{2.13}$$

for all $\alpha,\beta \in R$. Multiplying (2.13) by $e_{\ell,\ell}$ from the left and by $e_{n,n}$ from the right, we obtain $e_{\ell,\ell} f(\alpha e_{i,j})\beta e_{k,n} = 0$. This implies that $c^{i,j}_{\ell,k}(\alpha)\beta = 0$ for all $\alpha,\beta \in R$. By Lemma 2.1, $c^{i,j}_{\ell,k} = 0$. □

LEMMA 2.9. *Let $n \geqslant 4$ be an integer and let i,j be distinct integers such that $2 \leqslant i < j \leqslant n-1$. Then $c_{i,k}^{i,j} = 0$ for every integer k with $j < k \leqslant n$.*

Proof. Let k be an integer such that $j < k \leqslant n$. Clearly, $2 \leqslant i < j < k \leqslant n$. Setting $x = \alpha e_{i,j}$ and $y = \beta e_{1,i}$ in (2.5), we have

$$f(\alpha e_{i,j})\beta e_{1,i} - \beta e_{1,i} f(\alpha e_{i,j}) = \alpha e_{i,j} f(\beta e_{1,i}) - f(\beta e_{1,i})\alpha e_{i,j} \tag{2.14}$$

for all $\alpha, \beta \in R$. Multiplying (2.14) by $e_{1,1}$ from the left and by $e_{k,k}$ from the right, we obtain $-\beta e_{1,i} f(\alpha e_{i,j}) e_{k,k} = 0$. This implies that $\beta c_{i,k}^{i,j}(\alpha) = 0$ for all $\alpha, \beta \in R$. By Lemma 2.1, $c_{i,k}^{i,j} = 0$. $\square$

LEMMA 2.10. *Let $n \geqslant 5$ be an integer and let i,j be distinct integers such that $3 \leqslant i < j \leqslant n-1$. Then $c_{\ell,n}^{i,j} = 0$ for every integer ℓ with $2 \leqslant \ell < i$.*

Proof. Let ℓ be an integer such that $2 \leqslant \ell < i$. Clearly, $2 \leqslant \ell < i < j \leqslant n-1$. Setting $x = \alpha e_{i,j}$ and $y = \beta e_{1,\ell}$ in (2.5), we have

$$f(\alpha e_{i,j})\beta e_{1,\ell} - \beta e_{1,\ell} f(\alpha e_{i,j}) = \alpha e_{i,j} f(\beta e_{1,\ell}) - f(\beta e_{1,\ell})\alpha e_{i,j} \tag{2.15}$$

for all $\alpha, \beta \in R$. Multiplying (2.15) by $e_{1,1}$ from the left and by $e_{n,n}$ from the right, we obtain $-\beta e_{1,\ell} f(\alpha e_{i,j}) e_{n,n} = 0$. This implies that $\beta c_{\ell,n}^{i,j}(\alpha) = 0$ for all $\alpha, \beta \in R$. By Lemma 2.1, $c_{\ell,n}^{i,j} = 0$. $\square$

3. Proof of Theorems 1.1 and 1.2

LEMMA 3.1. *Let R be either a ring with 1 or a semiprime ring. Suppose that $f : N_n(R) \to N_n(R)$ is an additive map such that $[f(x), x] = 0$ for all $x \in N_n(R)$, where $n \geqslant 4$ is an integer. Then for every distinct integers i,j with $2 \leqslant i < j \leqslant n-1$, there exist additive maps $c_{i,j}^{i,j}, c_{1,n}^{i,j} : R \to R$ such that $f(\alpha e_{i,j}) = c_{i,j}^{i,j}(\alpha) e_{i,j} + c_{1,n}^{i,j}(\alpha) e_{1,n}$ for all $\alpha \in R$.*

Proof. Let i,j be two distinct integers such that $2 \leqslant i < j \leqslant n-1$. Write $f(\alpha e_{i,j}) = \sum_{s,t=1, s<t}^{n} c_{s,t}^{i,j}(\alpha) e_{s,t}$ for all $\alpha \in R$, where each $c_{s,t}^{i,j} : R \to R$ is an additive map. By Lemmas 2.2, 2.4 and 2.9, $c_{s,i}^{i,j} = c_{i,t}^{i,j} = 0$ for all integers s,t with $1 \leqslant s < i < t \leqslant n$ and $t \neq j$. Next by Lemmas 2.2, 2.5 and 2.8, $c_{s,j}^{i,j} = c_{j,t}^{i,j} = 0$ for all integers s,t with $1 \leqslant s < j < t \leqslant n$ and $s \neq i$. Finally, by Lemmas 2.3, 2.5, 2.6, 2.7, 2.8 and 2.10, $c_{s,t}^{i,j} = 0$ for all integers s,t with $1 \leqslant s < t \leqslant n$, $s,t \notin \{i,j\}$ and $(s,t) \neq (1,n)$. With these, we obtain $f(\alpha e_{i,j}) = c_{i,j}^{i,j}(\alpha) e_{i,j} + c_{1,n}^{i,j}(\alpha) e_{1,n}$ for all $\alpha \in R$, as desired. $\square$

LEMMA 3.2. *Let R be either a ring with 1 or a semiprime ring. Suppose that $f : N_n(R) \to N_n(R)$ is an additive map such that $[f(x), x] = 0$ for all $x \in N_n(R)$, where $n \geqslant 4$ is an integer. Then the following conditions hold:*

(1) *There exist additive maps $c_{1,n}^{1,n}, c_{2,n}^{2,n}, c_{1,n}^{2,n} : R \to R$ such that $f(\alpha e_{1,n}) = c_{1,n}^{1,n}(\alpha) e_{1,n}$ and $f(\alpha e_{2,n}) = c_{2,n}^{2,n}(\alpha) e_{2,n} + c_{1,n}^{2,n}(\alpha) e_{1,n}$ for all $\alpha \in R$;*

(2) *There exist additive maps* $c_{n-1,n}^{n-1,n}, c_{2,n}^{n-1,n}, c_{1,n}^{n-1,n}: R \to R$ *such that* $f(\alpha e_{n-1,n}) = c_{n-1,n}^{n-1,n}(\alpha)e_{n-1,n} + c_{2,n}^{n-1,n}(\alpha)e_{2,n} + c_{1,n}^{n-1,n}(\alpha)e_{1,n}$ *for all* $\alpha \in R$;

(3) *If* $n \geqslant 5$, *then for every integer* i *with* $3 \leqslant i \leqslant n-2$, *there exist additive maps* $c_{i,n}^{i,n}, c_{1,n}^{i,n}: R \to R$ *such that* $f(\alpha e_{i,n}) = c_{i,n}^{i,n}(\alpha)e_{i,n} + c_{1,n}^{i,n}(\alpha)e_{1,n}$ *for all* $\alpha \in R$.

Proof. (1) Applying Lemmas 2.4 and 2.5 to $f(\alpha e_{1,n})$ and applying Lemmas 2.2, 2.4, 2.5 and 2.7 to $f(\alpha e_{2,n})$, we are done.

(2) By Lemmas 2.2 and 2.6, $f(\alpha e_{n-1,n}) = \sum_{\ell=1}^{n-1} c_{\ell,n}^{n-1,n}(\alpha)e_{\ell,n}$, where each $c_{\ell,n}^{n-1,n}: R \to R$ is an additive map. If $n=4$, then $n-1=3$ and then

$$\begin{aligned} f(\alpha e_{n-1,n}) &= \sum_{\ell=1}^{n-1} c_{\ell,n}^{n-1,n}(\alpha)e_{\ell,n} = \sum_{\ell=1}^{3} c_{\ell,n}^{n-1,n}(\alpha)e_{\ell,n} \\ &= c_{n-1,n}^{n-1,n}(\alpha)e_{n-1,n} + c_{2,n}^{n-1,n}(\alpha)e_{2,n} + c_{1,n}^{n-1,n}(\alpha)e_{1,n} \end{aligned}$$

for all $\alpha \in R$, as desired. So we may assume $n \geqslant 5$. Let ℓ be an integer such that $3 \leqslant \ell \leqslant n-2$. By Lemma 3.1, $f(\beta e_{2,\ell}) = c_{2,\ell}^{2,\ell}(\beta)e_{2,\ell} + c_{1,n}^{2,\ell}(\beta)e_{1,n}$ for all $\beta \in R$, where $c_{2,\ell}^{2,\ell}, c_{1,n}^{2,\ell}: R \to R$ are additive maps. Setting $x = \alpha e_{n-1,n}$ and $y = \beta e_{2,\ell}$ in (2.5), we have

$$f(\alpha e_{n-1,n})\beta e_{2,\ell} - \beta e_{2,\ell} f(\alpha e_{n-1,n}) = \alpha e_{n-1,n} f(\beta e_{2,\ell}) - f(\beta e_{2,\ell})\alpha e_{n-1,n} \tag{3.1}$$

for all $\alpha, \beta \in R$. Note that $\alpha e_{n-1,n} f(\beta e_{2,\ell})e_{n,n} = 0$ as $f(\beta e_{2,\ell}) \in N_n(R)$. Multiplying (3.1) by $e_{2,2}$ from the left and by $e_{n,n}$ from the right, we get $-\beta e_{2,\ell} f(\alpha e_{n-1,n})e_{n,n} = -e_{2,2} f(\beta e_{2,\ell})\alpha e_{n-1,n}$. Since $\ell \leqslant n-2 < n-1$ and $f(\beta e_{2,\ell}) = c_{2,\ell}^{2,\ell}(\beta)e_{2,\ell} + c_{1,n}^{2,\ell}(\beta)e_{1,n}$, we get $e_{2,2} f(\beta e_{2,\ell})\alpha e_{n-1,n} = 0$. Thus $\beta e_{2,\ell} f(\alpha e_{n-1,n})e_{n,n} = 0$. This implies that $\beta c_{\ell,n}^{n-1,n}(\alpha) = 0$ for all $\alpha, \beta \in R$. By Lemma 2.1, $c_{\ell,n}^{n-1,n} = 0$. Hence $c_{\ell,n}^{n-1,n} = 0$ for every integer ℓ with $3 \leqslant \ell \leqslant n-2$. This proves the result.

(3) Let i be an integer such that $3 \leqslant i \leqslant n-2$. By Lemmas 2.2, 2.4, 2.5, 2.6 and 2.7, $f(\alpha e_{i,n}) = \sum_{\ell=1}^{i} c_{\ell,n}^{i,n}(\alpha)e_{\ell,n}$, where each $c_{\ell,n}^{i,n}: R \to R$ is an additive map. Let ℓ be an integer such that $2 \leqslant \ell < i < n-1$. By Lemma 3.1, $f(\alpha e_{i,n-1}) = c_{i,n-1}^{i,n-1}(\alpha)e_{i,n-1} + c_{1,n}^{i,n-1}(\alpha)e_{1,n}$ for all $\alpha \in R$, where $c_{i,n-1}^{i,n-1}, c_{1,n}^{i,n-1}: R \to R$ are additive maps. Write $f(\beta e_{1,\ell}) = \sum_{s,t=1,s<t}^{n} c_{s,t}^{1,\ell}(\beta)e_{s,t}$ for all $\beta \in R$, where each $c_{s,t}^{1,\ell}: R \to R$ is an additive map. Setting $x = \beta e_{1,\ell}$ and $y = \alpha e_{i,n-1}$ in (2.5), we have

$$f(\beta e_{1,\ell})\alpha e_{i,n-1} - \alpha e_{i,n-1} f(\beta e_{1,\ell}) = \beta e_{1,\ell} f(\alpha e_{i,n-1}) - f(\alpha e_{i,n-1})\beta e_{1,\ell} \tag{3.2}$$

for all $\alpha, \beta \in R$. Multiplying (3.2) by $e_{1,1}$ from the left and by $e_{n-1,n-1}$ from the right, we obtain $e_{1,1} f(\beta e_{1,\ell})\alpha e_{i,n-1} = \beta e_{1,\ell} f(\alpha e_{i,n-1})e_{n-1,n-1}$. Since $2 \leqslant \ell < i$ and $f(\alpha e_{i,n-1}) = c_{i,n-1}^{i,n-1}(\alpha)e_{i,n-1} + c_{1,n}^{i,n-1}(\alpha)e_{1,n}$, we have $e_{1,\ell} f(\alpha e_{i,n-1})e_{n-1,n-1} = 0$. Thus $e_{1,1} f(\beta e_{1,\ell})\alpha e_{i,n-1} = 0$. This implies that $c_{1,i}^{1,\ell}(\beta)\alpha = 0$ for all $\alpha, \beta \in R$. By Lemma 2.1, $c_{1,i}^{1,\ell} = 0$.

Next setting $x = \alpha e_{i,n}$ and $y = \beta e_{1,\ell}$ in (2.5), we have

$$f(\alpha e_{i,n})\beta e_{1,\ell} - \beta e_{1,\ell} f(\alpha e_{i,n}) = \alpha e_{i,n} f(\beta e_{1,\ell}) - f(\beta e_{1,\ell})\alpha e_{i,n} \tag{3.3}$$

for all $\alpha,\beta \in R$. Multiplying (3.3) by $e_{1,1}$ from the left and by $e_{n,n}$ from the right, we obtain $-\beta e_{1,\ell} f(\alpha e_{i,n})e_{n,n} = -e_{1,1} f(\beta e_{1,\ell})\alpha e_{i,n}$. Note that $e_{1,1} f(\beta e_{1,\ell})\alpha e_{i,n} = c_{1,i}^{1,\ell}(\beta)\alpha e_{1,n} = 0$ as $c_{1,i}^{1,\ell} = 0$. Thus $\beta e_{1,\ell} f(\alpha e_{i,n})e_{n,n} = 0$. This implies that $\beta c_{\ell,n}^{i,n}(\alpha) = 0$ for all $\alpha,\beta \in R$. By Lemma 2.1, $c_{\ell,n}^{i,n} = 0$. Hence $c_{\ell,n}^{i,n} = 0$ for every integer ℓ with $2 \leqslant \ell < i < n-1$. This proves the result. □

LEMMA 3.3. *Let R be either a ring with 1 or a semiprime ring. Suppose that $f : N_n(R) \to N_n(R)$ is an additive map such that $[f(x),x] = 0$ for all $x \in N_n(R)$, where $n \geqslant 4$ is an integer. Then the following conditions hold*:

(1) *There exist additive maps $c_{1,n-1}^{1,n-1}, c_{1,n}^{1,n-1} : R \to R$ such that $f(\alpha e_{1,n-1}) = c_{1,n-1}^{1,n-1}(\alpha)e_{1,n-1} + c_{1,n}^{1,n-1}(\alpha)e_{1,n}$ for all $\alpha \in R$;*

(2) *There exist additive maps $c_{1,2}^{1,2}, c_{1,n-1}^{1,2}, c_{1,n}^{1,2} : R \to R$ such that $f(\alpha e_{1,2}) = c_{1,2}^{1,2}(\alpha)e_{1,2} + c_{1,n-1}^{1,2}(\alpha)e_{1,n-1} + c_{1,n}^{1,2}(\alpha)e_{1,n}$ for all $\alpha \in R$;*

(3) *If $n \geqslant 5$, then for every integer j with $3 \leqslant j \leqslant n-2$, there exist additive maps $c_{1,j}^{1,j}, c_{1,n}^{1,j} : R \to R$ such that $f(\alpha e_{1,j}) = c_{1,j}^{1,j}(\alpha)e_{1,j} + c_{1,n}^{1,j}(\alpha)e_{1,n}$ for all $\alpha \in R$.*

Proof. (1) By Lemmas 2.2, 2.4 and 2.5, we are done.

(2) By Lemmas 2.2 and 2.3, $f(\alpha e_{1,2}) = \sum_{k=2}^{n} c_{1,k}^{1,2}(\alpha)e_{1,k}$ for all $\alpha \in R$, where each $c_{1,k}^{1,2} : R \to R$ is an additive map. If $n = 4$, then $n-1 = 3$ and $f(\alpha e_{1,2}) = \sum_{k=2}^{n} c_{1,k}^{1,2}(\alpha)e_{1,k} = \sum_{k=2}^{4} c_{1,k}^{1,2}(\alpha)e_{1,k} = c_{1,2}^{1,2}(\alpha)e_{1,2} + c_{1,n-1}^{1,2}(\alpha)e_{1,n-1} + c_{1,n}^{1,2}(\alpha)e_{1,n}$ for all $\alpha \in R$, as desired. So we may assume $n \geqslant 5$. Let k be an integer such that $3 \leqslant k \leqslant n-2$. By Lemma 3.1, $f(\beta e_{k,n-1}) = c_{k,n-1}^{k,n-1}(\beta)e_{k,n-1} + c_{1,n}^{k,n-1}(\beta)e_{1,n}$ for all $\beta \in R$, where $c_{k,n-1}^{k,n-1}, c_{1,n}^{k,n-1} : R \to R$ are additive maps. Setting $x = \alpha e_{1,2}$ and $y = \beta e_{k,n-1}$ in (2.5), we have

$$f(\alpha e_{1,2})\beta e_{k,n-1} - \beta e_{k,n-1} f(\alpha e_{1,2}) = \alpha e_{1,2} f(\beta e_{k,n-1}) - f(\beta e_{k,n-1})\alpha e_{1,2} \tag{3.4}$$

for all $\alpha,\beta \in R$. Clearly, $n-1 > 2$. Multiplying (3.4) by $e_{1,1}$ from the left and by $e_{n-1,n-1}$ from the right, we obtain $e_{1,1} f(\alpha e_{1,2})\beta e_{k,n-1} = \alpha e_{1,2} f(\beta e_{k,n-1})e_{n-1,n-1}$. Recall that $3 \leqslant k$ and $f(\beta e_{k,n-1}) = c_{k,n-1}^{k,n-1}(\beta)e_{k,n-1} + c_{1,n}^{k,n-1}(\beta)e_{1,n}$. Thus we have $e_{1,2} f(\beta e_{k,n-1})e_{n-1,n-1} = 0$. So $e_{1,1} f(\alpha e_{1,2})\beta e_{k,n-1} = 0$. This implies that $c_{1,k}^{1,2}(\alpha)\beta = 0$ for all $\alpha,\beta \in R$. By Lemma 2.1, $c_{1,k}^{1,2} = 0$. Hence $c_{1,k}^{1,2} = 0$ for every integer k with $3 \leqslant k \leqslant n-2$. This proves the result.

(3) Let j be an integer such that $3 \leqslant j \leqslant n-2$. By Lemmas 2.2, 2.3, 2.4 and 2.5, $f(\alpha e_{1,j}) = \sum_{k=j}^{n} c_{1,k}^{1,j}(\alpha)e_{1,k}$, where each $c_{1,k}^{1,j} : R \to R$ is an additive map. Let k be an integer such that $3 \leqslant j < k \leqslant n-1$. Setting $x = \alpha e_{1,j}$ and $y = \beta e_{k,n}$ in (2.5), we have

$$f(\alpha e_{1,j})\beta e_{k,n} - \beta e_{k,n} f(\alpha e_{1,j}) = \alpha e_{1,j} f(\beta e_{k,n}) - f(\beta e_{k,n})\alpha e_{1,j} \tag{3.5}$$

for all $\alpha,\beta \in R$. Multiplying (3.5) by $e_{1,1}$ from the left and by $e_{n,n}$ from the right, we obtain

$$e_{1,1} f(\alpha e_{1,j})\beta e_{k,n} = \alpha e_{1,j} f(\beta e_{k,n})e_{n,n} \tag{3.6}$$

for all $\alpha,\beta \in R$. By Lemma 3.2, $f(\beta e_{k,n}) = c^{k,n}_{k,n}(\beta)e_{k,n} + c^{k,n}_{1,n}(\beta)e_{1,n}$ for all $\beta \in R$ if $k \neq n-1$ and $f(\beta e_{k,n}) = c^{k,n}_{k,n}(\beta)e_{k,n} + c^{k,n}_{2,n}(\beta)e_{2,n} + c^{k,n}_{1,n}(\beta)e_{1,n}$ for all $\beta \in R$ if $k = n-1$, where $c^{k,n}_{k,n}, c^{k,n}_{2,n}, c^{k,n}_{1,n} : R \to R$ are additive maps. Thus $e_{1,j}f(\beta e_{k,n})e_{n,n} = 0$ as $3 \leqslant j < k$. So by (3.6), $e_{1,1}f(\alpha e_{1,j})\beta e_{k,n} = 0$. This implies that $c^{1,j}_{1,k}(\alpha)\beta = 0$ for all $\alpha,\beta \in R$. By Lemma 2.1, $c^{1,j}_{1,k} = 0$. Hence $c^{1,j}_{1,k} = 0$ for every integer k with $j < k \leqslant n-1$. This proves the result. $\square$

LEMMA 3.4. *Let R be a ring with 1 and with center $Z(R)$.*

(1) *Let $g : R \to R$ and $h : R \to R$ be additive maps such that $g(x)y = xh(y)$ for all $x,y \in R$. Then there exists $a \in R$ such that $g(x) = xa$ and $h(x) = ax$ for all $x \in R$.*

(2) *Let $g : R \to R$ be an additive map such that $g(x)y = xg(y)$ for all $x,y \in R$. Then there exists $\lambda \in Z(R)$ such that $g(x) = \lambda x$ for all $x \in R$.*

Proof. (1) Clearly, $g(x) = xh(1)$ and $g(1)y = h(y)$ for all $x,y \in R$. Thus $(xh(1))y = g(x)y = xh(y) = x(g(1)y)$ for all $x,y \in R$. So $h(1) = g(1)$, as desired. (2) By (1), there exists $a \in R$ such that $g(x) = ax = xa$ for all $x \in R$. Clearly, $a \in Z(R)$, as desired. $\square$

Let R be a semiprime ring. An ideal I of R is called essential if $I \cap J \neq 0$ for every nonzero ideal J of R. The symmetric Martindale ring of quotients of R, denoted by $Q_s(R)$, is also a semiprime ring and can be characterized as a ring satisfying the following four axioms [2, Proposition 2.2.3]:

(Q1) R is a subring of $Q_s(R)$.
(Q2) For any $a \in Q_s(R)$, there exists an essential ideal I of R such that $aI \cup Ia \subseteq R$.
(Q3) If $a \in Q_s(R)$ and I is an essential ideal of R, then $aI = 0$ if and only if $a = 0$.
(Q4) Given an essential ideal I of R, a left R-module homomorphism $g : I \to R$ and a right R-module homomorphism $h : I \to R$ such that $g(x)y = xh(y)$ for all $x,y \in I$, there exists $a \in Q_s(R)$ such that $g(x) = xa$ and $h(x) = ax$ for all $x \in I$.

We denote by $M(R)$ the multiplier ring of R, that is,

$$M(R) = \{a \in Q_s(R) \mid aR + Ra \subseteq R\}$$

and by $C(R)$ the centroid of R, that is, $C(R) = Z(Q_s(R)) \cap M(R)$. We refer the reader to the book [2] for the basic terminology and notation.

LEMMA 3.5. *Let R be a semiprime ring with the multiplier ring $M(R)$ and with the centroid $C(R)$*

(1) *Let $g : R \to R$ and $h : R \to R$ be additive maps such that $g(x)y = xh(y)$ for all $x,y \in R$. Then there exists $a \in M(R)$ such that $g(x) = xa$ and $h(x) = ax$ for all $x \in R$.*

(2) *Let $g : R \to R$ be an additive map such that $g(x)y = xg(y)$ for all $x,y \in R$. Then there exists $\lambda \in C(R)$ such that $g(x) = \lambda x$ for all $x \in R$.*

Proof. (1) Clearly, $g(zx)y = zxh(y)$ and $z(g(x)y) = z(xh(y))$ for all $x,y,z \in R$. The difference of the last two equations yields $g(zx)y = zg(x)y$ for all $x,y,z \in R$. Thus $(g(zx) - zg(x))R = 0$ for all $x,z \in R$. By semiprimeness of R, $g(zx) = zg(x)$ for all $x,z \in R$. This implies that g is a left R-module homomorphism. By symmetry, h is a right R-module homomorphism. By axiom (Q4), there exists $a \in Q_s(R)$ such that $g(x) = xa$ and $h(x) = ax$ for all $x \in R$. Clearly, $aR = h(R) \subseteq R$ and $Ra = g(R) \subseteq R$. Hence $a \in M(R)$. (2) By (1), there exists $\lambda \in M(R)$ such that $g(x) = \lambda x = x\lambda$ for all $x \in R$. By [2, Remark 2.3.1], $\lambda \in Z(Q_s(R))$. Hence $\lambda \in C(R)$. $\square$

LEMMA 3.6. *Let R be a ring with 1 and with center $Z(R)$ (resp. a semiprime ring with the multiplier ring $M(R)$ and with the centroid $C(R)$). Suppose that $f : N_n(R) \to N_n(R)$ is an additive map such that $[f(x), x] = 0$ for all $x \in N_n(R)$, where $n \geqslant 4$ is an integer. Then there exist $\lambda \in Z(R)$ (resp. $\lambda \in C(R)$), $a \in R$ (resp. $a \in M(R)$) and additive maps $c_{1,n}^{1,2}, c_{1,n}^{n-1,n}, c_{1,n}^{i,i+1} : R \to R$ such that $f(\alpha e_{1,2}) = (\lambda\alpha)e_{1,2} + (\alpha a)e_{1,n-1} + c_{1,n}^{1,2}(\alpha)e_{1,n}$, $f(\alpha e_{n-1,n}) = (\lambda\alpha)e_{n-1,n} + (a\alpha)e_{2,n} + c_{1,n}^{n-1,n}(\alpha)e_{1,n}$ and $f(\alpha e_{i,i+1}) = (\lambda\alpha)e_{i,i+1} + c_{1,n}^{i,i+1}(\alpha)e_{1,n}$ for all $\alpha \in R$ and $2 \leqslant i \leqslant n-2$.*

Proof. By Lemmas 3.2 (2) and 3.3 (2), there exist additive maps $c_{1,2}^{1,2}, c_{1,n-1}^{1,2}, c_{1,n}^{1,2} : R \to R$ such that $f(\alpha e_{1,2}) = c_{1,2}^{1,2}(\alpha)e_{1,2} + c_{1,n-1}^{1,2}(\alpha)e_{1,n-1} + c_{1,n}^{1,2}(\alpha)e_{1,n}$ and there exist additive maps $c_{n-1,n}^{n-1,n}, c_{2,n}^{n-1,n}, c_{1,n}^{n-1,n} : R \to R$ such that $f(\alpha e_{n-1,n}) = c_{n-1,n}^{n-1,n}(\alpha)e_{n-1,n} + c_{2,n}^{n-1,n}(\alpha)e_{2,n} + c_{1,n}^{n-1,n}(\alpha)e_{1,n}$ for all $\alpha \in R$. And by Lemma 3.1, for every integer i with $2 \leqslant i \leqslant n-2$, there exist additive maps $c_{i,i+1}^{i,i+1}, c_{1,n}^{i,i+1} : R \to R$ such that $f(\alpha e_{i,i+1}) = c_{i,i+1}^{i,i+1}(\alpha)e_{i,i+1} + c_{1,n}^{i,i+1}(\alpha)e_{1,n}$ for all $\alpha \in R$.

Let i be an integer such that $1 \leqslant i \leqslant n-2$. Setting $x = \alpha e_{i,i+1}$ and $y = \beta e_{i+1,i+2}$ in (2.5), we have

$$f(\alpha e_{i,i+1})\beta e_{i+1,i+2} - \beta e_{i+1,i+2} f(\alpha e_{i,i+1}) = \alpha e_{i,i+1} f(\beta e_{i+1,i+2}) - f(\beta e_{i+1,i+2})\alpha e_{i,i+1} \tag{3.7}$$

for all $\alpha, \beta \in R$. Multiplying (3.7) by $e_{i,i}$ from the left and by $e_{i+2,i+2}$ from the right, we obtain $e_{i,i} f(\alpha e_{i,i+1})\beta e_{i+1,i+2} = \alpha e_{i,i+1} f(\beta e_{i+1,i+2})e_{i+2,i+2}$. This implies

$$c_{i,i+1}^{i,i+1}(\alpha)\beta = \alpha c_{i+1,i+2}^{i+1,i+2}(\beta) \tag{3.8}$$

for all $\alpha, \beta \in R$ and $i = 1, \ldots, n-2$. Since $n \geqslant 4$, by (3.8) we have $c_{1,2}^{1,2}(\alpha)\beta = \alpha c_{2,3}^{2,3}(\beta)$ and $c_{2,3}^{2,3}(\alpha)\beta = \alpha c_{3,4}^{3,4}(\beta)$ for all $\alpha, \beta \in R$. By Lemma 3.4 (1) (resp. Lemma 3.5 (1)), there exist $u, v \in R$ (resp. $u, v \in M(R)$) such that $c_{2,3}^{2,3}(\alpha) = u\alpha$ and $c_{2,3}^{2,3}(\alpha) = \alpha v$ for all $\alpha \in R$. With these, we have $c_{2,3}^{2,3}(\alpha) = u\alpha = \alpha v$ and then $c_{2,3}^{2,3}(\alpha)\beta = (u\alpha)\beta = u(\alpha\beta) = (\alpha\beta)v = \alpha(\beta v) = \alpha c_{2,3}^{2,3}(\beta)$ for all $\alpha, \beta \in R$. By Lemma 3.4 (2) (resp. Lemma 3.5 (2)), there exists $\lambda \in Z(R)$ (resp. $\lambda \in C(R)$) such that $c_{2,3}^{2,3}(\alpha) = \lambda\alpha$ for all $\alpha \in R$. Then $c_{1,2}^{1,2}(\alpha)\beta = \alpha c_{2,3}^{2,3}(\beta) = \alpha(\lambda\beta) = (\lambda\alpha)\beta$ for all $\alpha, \beta \in R$. Thus $(c_{1,2}^{1,2}(\alpha) - \lambda\alpha)\beta = 0$ for all $\alpha, \beta \in R$. By Lemma 2.1, $c_{1,2}^{1,2}(\alpha) = \lambda\alpha = c_{2,3}^{2,3}(\alpha)$ for all $\alpha \in R$. Similarly, we have $c_{2,3}^{2,3}(\alpha) = \lambda\alpha = c_{3,4}^{3,4}(\alpha)$ for all $\alpha \in R$. Now using (3.8) repeatedly, we obtain $c_{i,i+1}^{i,i+1}(\alpha) = \lambda\alpha$ for all $\alpha \in R$ and $i = 1, \ldots, n-1$.

Setting $x = \alpha e_{1,2}$ and $y = \beta e_{n-1,n}$ in (2.5), we have

$$f(\alpha e_{1,2})\beta e_{n-1,n} - \beta e_{n-1,n} f(\alpha e_{1,2}) = \alpha e_{1,2} f(\beta e_{n-1,n}) - f(\beta e_{n-1,n})\alpha e_{1,2} \tag{3.9}$$

for all $\alpha,\beta \in R$. Multiplying (3.9) by $e_{1,1}$ from the left and by $e_{n,n}$ from the right, we obtain $e_{1,1}f(\alpha e_{1,2})\beta e_{n-1,n} = \alpha e_{1,2}f(\beta e_{n-1,n})e_{n,n}$. This implies $c^{1,2}_{1,n-1}(\alpha)\beta = \alpha c^{n-1,n}_{2,n}(\beta)$ for all $\alpha,\beta \in R$. Thus by Lemma 3.4 (1) (resp. Lemma 3.5 (1)), there exists $a \in R$ (resp. $a \in M(R)$) such that $c^{1,2}_{1,n-1}(\alpha) = \alpha a$ and $c^{n-1,n}_{2,n}(\beta) = a\beta$ for all $\alpha,\beta \in R$. This proves the lemma. $\square$

LEMMA 3.7. *Let R be either a ring with 1 or a semiprime ring. Suppose that $f : N_n(R) \to N_n(R)$ is an additive map such that $[f(x),x] = 0$ for all $x \in N_n(R)$, where $n \geqslant 4$ is an integer. Let λ and a be the elements described in Lemma 3.6. Then for every integer i with $2 \leqslant i \leqslant n-2$, there exists an additive map $c^{i,n}_{1,n} : R \to R$ such that $f(\alpha e_{i,n}) = (\lambda\alpha)e_{i,n} + c^{i,n}_{1,n}(\alpha)e_{1,n}$ for all $\alpha \in R$.*

Proof. By Lemma 3.2 (1) and (3), for every integer i with $2 \leqslant i \leqslant n-2$, there exist additive maps $c^{i,n}_{i,n}, c^{i,n}_{1,n} : R \to R$ such that $f(\alpha e_{i,n}) = c^{i,n}_{i,n}(\alpha)e_{i,n} + c^{i,n}_{1,n}(\alpha)e_{1,n}$ for all $\alpha \in R$. Let i be an integer such that $2 \leqslant i \leqslant n-2$. By Lemma 3.6, $f(\beta e_{i-1,i}) = (\lambda\beta)e_{1,2} + (\beta a)e_{1,n-1} + c^{1,2}_{1,n}(\beta)e_{1,n}$ for all $\beta \in R$ if $i = 2$ and $f(\beta e_{i-1,i}) = (\lambda\beta)e_{i-1,i} + c^{i-1,i}_{1,n}(\beta)e_{1,n}$ for all $\beta \in R$ if $3 \leqslant i \leqslant n-2$. In particular, $e_{i-1,i-1}f(\beta e_{i-1,i})e_{i,n} = \lambda\beta e_{i-1,n}$ as $i < n-1$. Setting $x = \alpha e_{i,n}$ and $y = \beta e_{i-1,i}$ in (2.5), we have

$$f(\alpha e_{i,n})\beta e_{i-1,i} - \beta e_{i-1,i}f(\alpha e_{i,n}) = \alpha e_{i,n}f(\beta e_{i-1,i}) - f(\beta e_{i-1,i})\alpha e_{i,n} \tag{3.10}$$

for all $\alpha,\beta \in R$. Multiplying (3.10) by $e_{i-1,i-1}$ from the left and by $e_{n,n}$ from the right, we see that $-\beta e_{i-1,i}f(\alpha e_{i,n})e_{n,n} = -e_{i-1,i-1}f(\beta e_{i-1,i})\alpha e_{i,n}$. This implies that $\beta c^{i,n}_{i,n}(\alpha) = \lambda\beta\alpha$ for all $\alpha,\beta \in R$. Thus $\beta(c^{i,n}_{i,n}(\alpha) - \lambda\alpha) = 0$ for all $\alpha,\beta \in R$. By Lemma 2.1, $c^{i,n}_{i,n}(\alpha) = \lambda\alpha$ for all $\alpha \in R$, proving the lemma. $\square$

LEMMA 3.8. *Let R be either a ring with 1 or a semiprime ring. Suppose that $f : N_n(R) \to N_n(R)$ is an additive map such that $[f(x),x] = 0$ for all $x \in N_n(R)$, where $n \geqslant 4$ is an integer. Let λ and a be the elements described in Lemma 3.6. Then for every distinct integers i,j with $1 \leqslant i < j \leqslant n-1$ and $(i,j) \neq (1,2)$, there exists an additive map $c^{i,j}_{1,n} : R \to R$ such that $f(\alpha e_{i,j}) = (\lambda\alpha)e_{i,j} + c^{i,j}_{1,n}(\alpha)e_{1,n}$ for all $\alpha \in R$.*

Proof. By Lemma 3.1 and Lemma 3.3 (1) and (3), for every distinct integers i,j with $1 \leqslant i < j \leqslant n-1$ and $(i,j) \neq (1,2)$, there exist additive maps $c^{i,j}_{i,j}, c^{i,j}_{1,n} : R \to R$ such that $f(\alpha e_{i,j}) = c^{i,j}_{i,j}(\alpha)e_{i,j} + c^{i,j}_{1,n}(\alpha)e_{1,n}$ for all $\alpha \in R$. Let i,j be distinct integers such that $1 \leqslant i < j \leqslant n-1$ and $(i,j) \neq (1,2)$. By Lemmas 3.6 and 3.7, $f(\beta e_{j,n}) = (\lambda\beta)e_{j,n} + c^{j,n}_{1,n}(\beta)e_{1,n}$ for all $\beta \in R$ if $2 \leqslant j \leqslant n-2$ and $f(\beta e_{j,n}) = (\lambda\beta)e_{n-1,n} + (a\beta)e_{2,n} + c^{n-1,n}_{1,n}(\beta)e_{1,n}$ for all $\beta \in R$ if $j = n-1$. In particular, we have $e_{i,j}f(\beta e_{j,n})e_{n,n} = \lambda\beta e_{i,n}$ as $j \geqslant 2$ and $n-1 \geqslant 3$.

Setting $x = \alpha e_{i,j}$ and $y = \beta e_{j,n}$ in (2.5), we have

$$f(\alpha e_{i,j})\beta e_{j,n} - \beta e_{j,n}f(\alpha e_{i,j}) = \alpha e_{i,j}f(\beta e_{j,n}) - f(\beta e_{j,n})\alpha e_{i,j} \tag{3.11}$$

for all $\alpha,\beta \in R$. Multiplying (3.11) by $e_{i,i}$ from the left and by $e_{n,n}$ from the right, we obtain $e_{i,i}f(\alpha e_{i,j})\beta e_{j,n} = \alpha e_{i,j}f(\beta e_{j,n})e_{n,n}$. This implies that $c^{i,j}_{i,j}(\alpha)\beta = \alpha\lambda\beta$ for all

$\alpha, \beta \in R$. Thus $(c_{i,j}^{i,j}(\alpha) - \lambda\alpha)\beta = 0$ for all $\alpha, \beta \in R$. By Lemma 2.1, $c_{i,j}^{i,j}(\alpha) = \lambda\alpha$ for all $\alpha \in R$, proving the lemma. □

LEMMA 3.9. *Let R be a ring with 1 (resp. a semiprime ring with the multiplier ring $M(R)$) and let $N_3(R)$ be the ring of all 3×3 strictly upper triangular matrices over R with center $\mathcal{Z}$. Suppose that $f : N_3(R) \to N_3(R)$ is an additive map. Then $[f(x), x] = 0$ for all $x \in N_3(R)$ if and only if there exist an additive map $\mu : N_3(R) \to \mathcal{Z}$ and an additive map $\nu : N_3(R) \to \Omega$ such that $f(x) = \mu(x) + \nu(x)$ for all $x \in N_3(R)$, where $\Omega = \{\alpha e_{1,2} + \beta e_{2,3} : \alpha, \beta \in R\}$ and ν is defined by some $a \in R$ (resp. $a \in M(R)$) such that $\nu(x) = e_{1,1}xae_{2,2} + e_{2,2}axe_{3,3}$ for all $x \in N_3(R)$.*

Proof. The implication "$\Leftarrow$" is trivial. For the implication "$\Rightarrow$": For two distinct integers i, j with $1 \leqslant i < j \leqslant 3$ and write $f(\alpha e_{ij}) = \sum_{s,t=1,s<t}^{3} c_{st}^{ij}(\alpha)e_{st}$ for all $\alpha \in R$, where each $c_{st}^{ij} : R \to R$ is an additive map. By Lemma 2.2, $c_{2,3}^{1,2} = 0$ and $c_{1,2}^{2,3} = 0$. Thus $f(\alpha e_{1,2}) = c_{1,2}^{1,2}(\alpha)e_{1,2} + c_{1,3}^{1,2}(\alpha)e_{1,3}$ and $f(\beta e_{2,3}) = c_{2,3}^{2,3}(\beta)e_{2,3} + c_{1,3}^{2,3}(\beta)e_{1,3}$ for all $\alpha, \beta \in R$. Setting $x = \alpha e_{1,2}$ and $y = \beta e_{2,3}$ in (2.5), we obtain

$$
\begin{aligned}
0 &= f(\alpha e_{1,2})\beta e_{2,3} - \beta e_{2,3} f(\alpha e_{1,2}) - \alpha e_{1,2} f(\beta e_{2,3}) + f(\beta e_{2,3})\alpha e_{1,2} \\
&= (c_{1,2}^{1,2}(\alpha)e_{1,2} + c_{1,3}^{1,2}(\alpha)e_{1,3})\beta e_{2,3} - \beta e_{2,3}(c_{1,2}^{1,2}(\alpha)e_{1,2} + c_{1,3}^{1,2}(\alpha)e_{1,3}) \\
&\quad - \alpha e_{1,2}(c_{2,3}^{2,3}(\beta)e_{2,3} + c_{1,3}^{2,3}(\beta)e_{1,3}) + (c_{2,3}^{2,3}(\beta)e_{2,3} + c_{1,3}^{2,3}(\beta)e_{1,3})\alpha e_{1,2} \\
&= (c_{1,2}^{1,2}(\alpha)\beta - \alpha c_{2,3}^{2,3}(\beta))e_{1,3}
\end{aligned}
$$

for all $\alpha, \beta \in R$. Thus $c_{1,2}^{1,2}(\alpha)\beta - \alpha c_{2,3}^{2,3}(\beta) = 0$ for all $\alpha, \beta \in R$. By Lemma 3.4 (resp. Lemma 3.5), there exists $a \in R$ (resp. $a \in M(R)$) such that $c_{1,2}^{1,2}(\alpha) = \alpha a$ and $c_{2,3}^{2,3}(\alpha) = a\alpha$ for all $\alpha \in R$. Recall that $\mathcal{Z} = Re_{13}$. Thus $f(\alpha e_{1,2}) - (\alpha a)e_{1,2} = c_{1,3}^{1,2}(\alpha)e_{1,3} \in \mathcal{Z}$ and $f(\alpha e_{2,3}) - (a\alpha)e_{2,3} = c_{1,3}^{2,3}(\alpha)e_{1,3} \in \mathcal{Z}$ for all $\alpha \in R$. By Lemmas 2.4 and 2.5, $c_{1,2}^{1,3} = 0$ and $c_{2,3}^{1,3} = 0$. So $f(\alpha e_{1,3}) = c_{1,3}^{1,3}(\alpha)e_{1,3} \in \mathcal{Z}$ for all $\alpha \in R$. Let $\nu : N_3(R) \to \Omega$ be the additive map defined by $\nu(x) = e_{1,1}xae_{2,2} + e_{2,2}axe_{3,3}$ for all $x \in N_3(R)$, where $\Omega = \{\alpha e_{1,2} + \beta e_{2,3} : \alpha, \beta \in R\}$. Then $f(x) - \nu(x) \in \mathcal{Z}$ for all $x \in N_3(R)$. Hence $f(x) = \mu(x) + \nu(x)$ for all $x \in N_3(R)$, where $\mu : N_3(R) \to \mathcal{Z}$ is the additive map defined by $\mu(x) = f(x) - \nu(x)$ for all $x \in N_3(R)$. This proves the lemma. □

We are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. The implication "$\Leftarrow$" is trivial. For the implication "$\Rightarrow$": By Lemma 3.9, we may assume $n \geqslant 4$. Let λ and a be the elements described in Lemma 3.6 and let $\Omega = \{\alpha e_{1,n-1} + \beta e_{2,n} : \alpha, \beta \in R\}$. Let $\nu : N_n(R) \to \Omega$ be the additive map defined by $\nu(x) = e_{1,1}xae_{2,n-1} + e_{2,n-1}axe_{n,n}$ for all $x \in N_n(R)$. Clearly, $\nu(\alpha e_{i,j}) = 0$ for all $\alpha \in R$ and distinct integers i, j with $1 \leqslant i < j \leqslant n$ and $(i, j) \notin \{(1,2), (n-1,n)\}$. By Lemmas 3.6, 3.7 and 3.8, $f(\alpha e_{i,j}) - \lambda(\alpha e_{i,j}) - \nu(\alpha e_{i,j}) \in Re_{1,n}$ for all $\alpha \in R$ and distinct integers i, j with $1 \leqslant i < j \leqslant n$ and $(i, j) \neq (1, n)$. Moreover, in view of Lemma 3.2 (1), $f(\alpha e_{1,n}) \in Re_{1,n}$ and hence $f(\alpha e_{1,n}) - \lambda(\alpha e_{1,n}) - \nu(\alpha e_{1,n}) \in Re_{1,n}$ for all $\alpha \in R$. Recall that $\mathcal{Z} = Re_{1,n}$. So

$f(x)-\lambda x-\nu(x)\in\mathcal{Z}$ for $x\in N_n(R)$. Let $\mu: N_n(R)\to\mathcal{Z}$ be the additive map defined by $\mu(x)=f(x)-\lambda x-\nu(x)$ for $x\in N_n(R)$. Consequently, $f(x)=\lambda x+\mu(x)+\nu(x)$ for all $x\in N_n(R)$. This proves the theorems. $\square$

Acknowledgements. The authors are thankful to the referee for the very thorough reading of the paper and valuable suggestions.

REFERENCES

[1] K. I. BEIDAR, M. BREŠAR AND M. A. CHEBOTAR, *Functional identities on upper triangular matrix algebras*, J. Math. Sci. (New York) **102** (2000), 4557–4565.
[2] K. I. BEIDAR, W. S. MARTINDALE 3RD AND A. V. MIKHALEV, *Rings with Generalized Identities*, Marcel Dekker, Inc., New York–Basel–Hong Kong, 1996.
[3] J. BOUNDS, *Commuting maps over the ring of strictly upper triangular matrices*, Linear Algebra Appl. **507** (2016), 132–136.-
[4] M. BREŠAR, *Centralizing mappings and derivations in prime rings*, J. Algebra **156** (1993), 385–394.
[5] M. BREŠAR, M. A. CHEBOTAR AND W. S. MARTINDALE III, *Functional Identities*, Frontiers in Mathematics. Basel, Birkhauser Verlag, 2007.
[6] W. S. CHEUNG, *Commuting maps of triangular algebras*, J. London Math. Soc. **63** (2001), 117–127.
[7] W. L. CHOOI, K. H. KWA AND L. Y. TAN, *Commuting maps on rank k triangular matrices*, Linear Multilinear Algebra **68** (2020), 1021–1030.
[8] W. L. CHOOI, K. H. KWA AND L. Y. TAN, *Commuting maps on invertible triangular matrices over* $\mathbb{F}_2$, Linear Algebra Appl. **583** (2019), 77–101.
[9] W. L. CHOOI, M. H. A. MUTALIB AND L. Y. TAN, *Commuting maps on rank one triangular matrices*, Linear Algebra Appl. **626** (2021), 34–55.
[10] W. L. CHOOI AND Y. N. TAN, *A note on commuting additive maps on rank k symmetric matrices*, Electron. J. Linear Algebra **37** (2021), 734–746.
[11] P.-H. CHOU AND C.-K. LIU, *Power commuting additive maps on rank-k linear transformations*, Linear Multilinear Algebra **69** (2021), 403–427.
[12] N. DIVINSKY, *On commuting automorphisms of rings*, Trans. Roy. Soc. Canada. Sect. III **49** (1955), 19–22.
[13] D. EREMITA, *Functional identities of degree 2 in triangular rings*, Linear Algebra Appl. **438** (2013), 584–597.
[14] W. FRANCA, *Commuting maps on some subsets of matrices that are not closed under addition*, Linear Algebra Appl. **437** (2012), 388–391.
[15] W. FRANCA, *Commuting maps on rank-k matrices*, Linear Algebra Appl. **438** (2013), 2813–2815.
[16] W. FRANCA, *Weakly commuting maps on the set of rank-1 matrices*, Linear Multilinear Algebra **65** (2017), 479–495.
[17] W. FRANCA AND N. LOUZA, *Commuting maps on rank-1 matrices over noncommutative division rings*, Comm. Algebra **45** (2017), 4696–4706.
[18] W. FRANCA AND N. LOUZA, *Generalized commuting maps on the set of singular matrices*, Electron. J. Linear Algebra **35** (2019), 533–542.
[19] C.-K. LIU, *Centralizing maps on invertible or singular matrices over division rings*, Linear Algebra Appl. **440** (2014), 318–324.
[20] C.-K. LIU AND J.-J. YANG, *Power commuting additive maps on invertible or singular matrices*, Linear Algebra Appl. **530** (2017), 127–149.
[21] Y. LI AND F. WEI, *Semi-centralizing maps of generalized matrix algebras*, Linear Algebra Appl. **436** (2012), 1122–1153.
[22] E. C. POSNER, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.
[23] R. SLOWIK AND D. A. H. AHMED, *M-commuting maps on triangular and strictly triangular infinite matrices*, Electron. J. Linear Algebra **37** (2021), 247–255.
[24] Y. WANG, *Functional identities of degree 2 in arbitrary triangular rings*, Linear Algebra Appl. **479** (2015), 171–184.

[25] Y. WANG, *On functional identities of degree 2 and centralizing maps in triangular rings*, Oper. Matrices **10** (2016), 485–499.

[26] Y. WANG, *Functional identities in upper triangular matrix rings revisited*, Linear Multilinear Algebra **67** (2019), 348–359.

[27] Z.-K. XIAO AND F. WEI, *Commuting mappings of generalized matrix algebras*, Linear Algebra Appl. **433** (2010), 2178–2197.

[28] X. XU AND X. YI, *Commuting maps on rank-k matrices*, Electron. J. Linear Algebra **27** (2014), 735–741.

(Received March 16, 2023)

Shu-Wen Ko
Department of Mathematics
National Changhua University of Education
Changhua 500, Taiwan

Cheng-Kai Liu
Department of Mathematics
National Changhua University of Education
Changhua 500, Taiwan
e-mail: ckliu@cc.ncue.edu.tw