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SOME PROPERTIES OF EXACT PHASE RETRIEVABLE SUBSPACES

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Abstract. Sometimes a frame is not exact phase-retrievable, but it can be used to perform exact phase retrieval for some subsets of the Hilbert space. Hence the main purpose of this paper is to discuss some properties of the exact phase-retrievable subspaces. Firstly, according to the operator theory, the necessary and sufficient condition for a subspace to become an exact phase-retrievable subspace can be obtained. With the help of this property, the necessary and sufficient condition for the existence of the k-dimensional exact phase-retrievable subspace is discussed. Furthermore, we find that the subspace G of an exact phase-retrievable subspace M may not be an exact phase-retrievable subspace, although there is a frame that has the exact PR-redundancy property of G. Finally, the properties about the union of two exact phase-retrievable frames and the union of two exact PR subspaces are obtained.

1. Introduction

As a new research direction, the frame theory involves functional analysis [1], operator theory [2], signal processing [3], and many other disciplines. As we all know, a frame as a generalization of a basis for a Hilbert space was first introduced by Duffin and Schaeffer to deal with some problems concerning the nonharmonic Fourier series in 1952 [5]. It not only inherits some properties of the basis but also has excellent characteristics that the basis does not have, such as redundancy [3]. Hence the frame gradually replaced the basis and has been applied in many fields such as signal and image processing [6], quantization [7], capacity of transmission channels [9], coding theory [10], data transmission technology [20], and so on.

DEFINITION 1.1. [11] Let H be a Hilbert space. A collection of vectors $F = \{f_i\}_{i=1}^k \subset H$ is a frame for a Hilbert space H if there exists constants $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{i=1}^{k} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2,$$

for any $f \in H$. The constants A and B are lower and upper frame bounds, respectively. Moreover, if $A = B$, then the frame is a tight frame for H. Especially, if $A = B = 1$, the frame is a Parseval frame.

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In recent years, frames have been used to discuss the phase retrieval problem and play an important role in physics and engineering such as X-ray crystallography, image signal processing, deep learning, machine learning, and many more [13, 12, 8]. Balan et al are one of the pioneers who began to study the phase retrieval problem by using linear measures for frames (see [17]). For the case of complex Hilbert space, Conca et al improved the conclusions of Balan in [18]. They proved that $4n-4$ general intensity measures in complex Hilbert space are sufficient for phase retrieval.

Before introducing the definition of phase retrievable, we provide an equivalence relationship "$\sim$" between two vectors $x, y \in H$ as follows.

DEFINITION 1.2. [14] Let H be a Hilbert space. The vectors $x \in H$ and $y \in H$ are similar if and only if there is a complex constant z of unit magnitude and $|z| = 1$, such that $y = zx$. In this case, we say $x \sim y$. Moreover, let $\hat{H} = H/\sim$ be the quotient space.

In order to better introduce the phase-retrievable frame, we consider the following nonlinear operator,

$$\alpha : \hat{H} \to R^m, (\alpha(\hat{x}))_k = |\langle x, f_k\rangle|, 1 \leqslant k \leqslant m.$$

Furthermore, we obtain the following concept.

DEFINITION 1.3. [15] A frame $\{f_k\}_{k=1}^m$ for a Hilbert space H is called phase-retrievable if the map α which mentioned above is injective.

In addition, Han Deguang and others pointed out that even if a frame is not phase-retrievable, it is still possible that it can be used to perform phase retrieval for some subsets of the Hilbert space. Moreover, he studied the maximal phase-retrievable subspaces for a given frame in [19]. And he introduced a special phase-retrievable frame, which is the exact phase-retrievable frame. Inspired by their research, we want to know even if a frame is not exact phase-retrievable, can it be used to perform exact phase retrieval for some subsets of the Hilbert space? And we also care about some properties of exact phase-retrievable frames. Hence in this paper, we mainly discuss the exact phase-retrievable frame and the exact phase-retrievable subspaces. We not only focus on finding the existence of the k-dimensional exact phase-retrievable subspace but are also interested in discussing some similarities and differences between phase-retrievable frames and exact phase-retrievable frames.

Then we introducing the concept of an exact phase-retrievable frame. Before that, we need to give a property of a phase-retrievable frame.

Let $S^{1,1}$ denote the following space of symmetric operators over a Hilbert space H [21],

$$S^{1,1}(H) = \{T \in Sym(H), rank(T) \leqslant 2, S_p(T) = \{\lambda_{\max}, 0, \lambda_{\min}\}, \lambda_{\max} \geqslant 0 \geqslant \lambda_{\min}\},$$

where Sym(H) denotes the set of symmetric operators (matrices) on the Hilbert space H, $S_p(T)$ denotes the spectrum (set of eigenvalues) of T, and $\lambda_{\max}$, $\lambda_{\min}$ denote the largest, and smallest eigenvalue of the corresponding operator, respectively.

Then according to the definition of phase-retrievable frame, we have the following equivalent description of the phase-retrievable frame.

PROPOSITION 1.4. [16] *A frame $\{\Lambda_k\}_{k=1}^m$ for a Hilbert space H is phase-retrievable if and only if*

$$ker(\Theta_{L(F)}) \cap S^{1,1} = 0,$$

where $\Theta_{L(F)}$ is the analysis operator of $L(F) = L(f_i)$, and $L(f_i) = f_i \bigotimes f_i$. Furthermore, for each $f, g \in H$, let $(f \bigotimes g)x = \langle x, g\rangle f$ for every $x \in H$, and $\langle A, B\rangle = tr(AB^*)$.

Then we can get the definition of the exact phase-retrievable frame. Let $\{f_k\}_{k=1}^m$ be a frame for H. For each subset Λ of $\{1,\dots,m\}$, let $F_\Lambda = \{f_i\}_{i\in\Lambda}$ and use $|\Lambda|$ to denote the cardinality of Λ.

DEFINITION 1.5. [4] Given a frame $\{f_k\}_{k=1}^m$ for H. Let k be the smallest integer such that there exists a subset Λ of $\{1,\dots,m\}$ with the property that $|\Lambda| = k$ and

$$ker(\Theta_{L(F_\Lambda)}) \cap S^{1,1} = ker(\Theta_{L(F)}) \cap S^{1,1}.$$

Then we call m/k the phase-retrievable of frame $\{f_k\}_{k=1}^m$. Moreover, we call a frame $\{f_k\}_{k=1}^m$ for H have the exact phase-retrievable property if its phase-retrievable is 1. In this case, we call $\{f_k\}_{k=1}^m$ have the exact PR-redundancy property. A phase-retrievable frame with the exact PR-redundancy property will be called an exact phase-retrievable frame or an exact PR frame.

However, it is difficult to directly determine whether a frame has the exact PR-redundancy property based on its definition. Hence we introduce a property of the exact phase-retrievable frame. In the discussion of this paper, we often use this property to determine whether a frame has the exact PR-redundancy property.

PROPOSITION 1.6. [19] *Given a frame $\{f_k\}_{k=1}^m$ for H. From the above Definition1.5 we have the following:*

(i) F has the exact PR-redundancy property if and only if for any proper subset Λ of $\{1,\dots,m\}$, there exist two vectors $x, y \in H$ such that

$$|\langle x, f_j\rangle| = |\langle y, f_j\rangle|$$

for every $j \in \Lambda$, but $|\langle x, f_j\rangle| \neq |\langle y, f_j\rangle|$ for some $i \in \Lambda^c$.

(ii) If a frame $\{f_k\}_{k=1}^m$ is phase-retrievable, then it is an exact phase-retrievable frame if and only if F_Λ is no longer phase-retrievable for any proper subset Λ of $\{1,\dots,m\}$.

In this paper, if a frame is not the exact phase-retrievable for H, then its exact phase-retrievable subspace $G \subset H$ is studied. on the one hand, we explore the existence of the k-dimensional exact phase-retrievable subspace. On the other hand, we explore some characterizations of the exact phase-retrievable subspace. Such as the necessary and sufficient condition to make a subspace to be an exact phase-retrievable subspace, the union or the product of two exact phase-retrievable subspaces, and so on.

For the convenience of the following description, we introduce these symbols which appear in the following text.

H	The Hilbert space
$B(X,Y)$	The set of bounded linear operators from X to Y
$B(H)$	The set of bounded linear operators from H to H
Θ_F	The analysis operator of F
$tr(T)$	The sum of all eigenvalues of the matrix T
$ker(T)$ or ker_T	The kernel of T
T^*	The adjoint operator of T
$\|\|f\|\|$	The norm of f
$\|\|T\|\|$	The operator norm of operator T

2. The exact PR subspace

Sometimes a frame $F=\{f_i\}_{i=1}^n$ may not be exact phase-retrievable for H. But after being mapped by the orthogonal projection operator, it can be an exact phase-retrievable frame for a subspace of H. Hence in this section, we mainly introduce some properties of the exact PR subspace. Firstly similar to the phase-retrievable subspaces, we give the definition of the exact phase-retrievable subspaces.

DEFINITION 2.1. Let $F=\{f_i\}_{i=1}^n$ be a frame for H and M be a subspace of H. We say that M is an exact phase-retrievable subspace with respect to F if $\{P_M f_i\}_{i=1}^n$ is an exact phase-retrievable frame for M, where P_M is the orthogonal projection from H onto M. In this paper, we call M an exact F-PR subspace.

Then we discuss the necessary and sufficient condition for the existence of the k-dimensional exact F-PR subspace. Before that, let's introduce a nature of the exact F-PR subspace.

PROPOSITION 2.2. *Let H be a Hilbert space, $\{f_i\}_{i=1}^m$ be a frame for H, and T be an invertible operator. Then subspace $M\subset H$ is an exact F-PR subspace if and only if $T^{-1}M$ is an exact T^* F-PR subspace.*

Proof. First of all, we prove that M is a F-PR subspace if and only if $T^{-1}M$ is a T^* F-PR subspace.

According to Definition 2.1, we just need to show that $\{Pf_i\}_{i=1}^m$ is a phase-retrievable frame for M if and only if $\{P'T^*f_i\}_{i=1}^m$ is a phase-retrievable frame for $T^{-1}M$, where P and P' are the orthogonal projection onto M and $T^{-1}M$ respectively.

(1) On the one hand, let $\{Pf_i\}_{i=1}^m$ be a phase-retrievable frame for M. For any $u,v\in T^{-1}M$, if $|\langle u,P'T^*f_i\rangle|=|\langle v,P'T^*f_i\rangle|$, then there are two vectors $x,y\in M$ such that $u=T^{-1}x, v=T^{-1}y$, and

$$|\langle T^{-1}x,P'T^*f_i\rangle|=|\langle T^{-1}y,P'T^*f_i\rangle| \tag{1}$$

for any $i = 1, 2\cdots, m$.

The equation (1) is equivalent to the following equation,

$$|\langle T^{-1}x, T^{*}f_i\rangle| = |\langle T^{-1}y, T^{*}f_i\rangle|$$

for any $i = 1, 2\cdots, m$, since P' is the orthogonal projection onto T^{-1}M.

After simple deformation, we have

$$|\langle TT^{-1}x, f_i\rangle| = |\langle TT^{-1}y, f_i\rangle|$$

for any $i = 1, 2\cdots, m$.

Thus

$$|\langle x, Pf_i\rangle| = |\langle y, Pf_i\rangle|$$

for any $i = 1, 2\cdots, m$, since P is the orthogonal projection onto M.

So we can obtain that $\widehat{x} = \widehat{y}$, since $\{Pf_i\}_{i=1}^{m}$ is a phase-retrievable frame for M.

Moreover,

$$\widehat{u} = \widehat{T^{-1}x} = \widehat{T^{-1}y} = \widehat{v}$$

can be obtained.

Hence $\{P'T^{*}f_i\}_{i=1}^{m}$ is a phase-retrievable frame for $T^{-1}M$.

(2) On the other hand, let $\{P'T^{*}f_i\}_{i=1}^{m}$ be a phase-retrievable frame for $T^{-1}M$. For any $x, y \in M$, if

$$|\langle x, Pf_i\rangle| = |\langle y, Pf_i\rangle|$$

for any $i = 1, 2\cdots, m$, then

$$|\langle TT^{-1}x, f_i\rangle| = |\langle TT^{-1}y, f_i\rangle|$$

for any $i = 1, 2\cdots, m$.

Thus

$$|\langle T^{-1}x, P'T^{*}f_i\rangle| = |\langle T^{-1}y, P'T^{*}f_i\rangle|$$

for any $i = 1, 2\cdots, m$, and $T^{-1}x, T^{-1}y \in T^{-1}M$.

Hence we can get that $\widehat{T^{-1}x} = \widehat{T^{-1}y}$, since $\{P'T^{*}f_i\}_{i=1}^{m}$ is a phase-retrievable frame for $T^{-1}M$.

That is to say $\widehat{x} = \widehat{y}$. So $\{Pf_i\}_{i=1}^{m}$ is a phase-retrievable frame for M. Thus we've proved that M is a F-PR subspace if and only if T^{-1}M is a T^{*}F-PR subspace. Then we prove that M is an exact F-PR subspace if and only if T^{-1}M is an exact T^{*}F-PR subspace.

In fact, if $\{Pf_i\}_{i=1}^{m}$ is an exact phase-retrievable frame for M, then for any $\Lambda \subset \{1, 2, \cdots, m\}$, there are two vectors $x, y \in M$ such that

$$|\langle x, Pf_j\rangle| = |\langle y, Pf_j\rangle|$$

for every $j \in \Lambda$, but $|\langle x, Pf_j\rangle| \neq |\langle y, Pf_j\rangle|$ for some $i \in \Lambda^{c}$.

It is not difficult to find by proof (1) that there are two vectors $T^{-1}x, T^{-1}y \in M$ such that

$$|\langle T^{-1}x, P'T^{*}f_j\rangle| = |\langle T^{-1}y, P'T^{*}f_j\rangle|$$

for every $j \in \Lambda$, but $|\langle T^{-1}x, P'T^* f_j\rangle| \neq |\langle T^{-1}y, P'T^* f_j\rangle|$ for some $i \in \Lambda^c$.

Hence $\{P'T^* f_j\}_{i=1}^m$ has the exact PR-redundancy property. Based on the conclusion of (1), we know $\{P'T^* f_j\}_{i=1}^m$ is an exact phase-retrievable frame for $T^{-1}M$.

By the same reason, we can get that if $\{P'T^* f_j\}_{i=1}^m$ is an exact phase-retrievable frame for $T^{-1}M$, then $\{Pf_i\}_{i=1}^m$ is an exact phase-retrievable frame for M. That is to say, M is an exact F-PR subspace if and only if T^{-1}M is an exact T^*F-PR subspace. $\square$

From the following Lemma 2.3, we can obtain the necessary and sufficient condition for the existence of the k-dimensional exact phase-retrievable subspace.

LEMMA 2.3. *Let H be an n-dimensional real Hilbert space. Then for every integer N with $2n-1 \leqslant N \leqslant n(n+1)/2$, there exists an exact phase-retrievable frame of length N.*

Theorem 2.4 shows that if M is an exact F-PR subspace, then there is a certain relationship between the dimension of the subspace M and the dimension of the real Hilbert space H.

THEOREM 2.4. *Let $F = \{f_i\}_{i=1}^n$ be a basis for a real Hilbert space H. Then there exists a k-dimensional exact F-PR subspace if and only if $[(\sqrt{1+8n}-1)/2] \leqslant k \leqslant [(n+1)/2]$.*

Proof. (1) On the one hand, if M is a k-dimensional exact F-PR subspace, then according to Lemma 2.3 we can obtain that $2k-1 \leqslant n \leqslant k(k+1)/2$. After simple calculation, it can be concluded that $[(\sqrt{1+8n}-1)/2] \leqslant k \leqslant [(n+1)/2]$.

(2) On the other hand, according to Proposition 2.2, we know that M is an exact F-PR subspace if and only if T^{-1}M is an exact T^*F-PR subspace.

Hence we just need to prove that for any k-dimensional subspace M with $[(\sqrt{1+8n}-1)/2] \leqslant k \leqslant [(n+1)/2]$, there is a basis $U = \{u_i\}_{i=1}^n$ such that M is an exact U-PR subspace.

Since $[(\sqrt{1+8n}-1)/2] \leqslant k \leqslant [(n+1)/2]$, then we have $2k-1 \leqslant n \leqslant k(k+1)/2$. By Lemma 2.3, there is an exact phase-retrievable frame $\varphi = \{\varphi_i\}_{i=1}^n$ of M.

Then we let $\{\varphi_i\}_{i=1}^k$ be an orthonormal basis for M. And it can be extended into an orthonormal basis $\{e_i\}_{i=1}^n$ for H, where $e_i = \varphi_i$ for any $i = 1,2,\dots k$.

Now we construct a basis $U = \{u_i\}_{i=1}^n$ for H, where

$$u_i = e_i = \varphi_i,$$

for all $i = 1,2,\dots k$, and

$$u_i = e_i + \varphi_i,$$

for all $i = k, k+1, \dots n$.

Obviously, U is a basis for H. Let P_M be the orthogonal projection onto M. According to the properties of orthogonal projection, we have

$$\{P_M u_i\} = \{\varphi_1, \varphi_2, \dots, \varphi_k, \varphi_{k+1}, \dots, \varphi_n\}.$$

Hence $\{P_M u_i\}_{i=1}^n = \varphi = \{\varphi_i\}_{i=1}^n$ is an exact phase-retrievable frame of M. That is to say M is an exact U-PR subspace. $\square$

The remaining content of this section introduces some properties of the exact F-PR subspace. Firstly, we discuss the relationship between the exact F-PR subspace M and the subspace $G \supset M$.

PROPOSITION 2.5. *Let $F = \{f_i\}_{i=1}^n$ be a frame for a real Hilbert space H, M and G be the subspaces of H, and $M \subset G \subset H$. Let P and P' be the orthogonal projection onto M and G respectively. If $\{Pf_i\}_{i=1}^n$ have the exact PR-redundancy property of M, then $\{P'f_i\}_{i=1}^n$ have the exact PR-redundancy property of G.*

Proof. Since $\{Pf_i\}_{i=1}^n$ have the exact PR-redundancy property of M, then for any $\Lambda \subset \{1,2,\cdots,n\}$, there are two vectors $x,y \in M$ such that

$$|\langle x, f_j\rangle| = |\langle x, Pf_j\rangle| = |\langle y, Pf_j\rangle| = |\langle y, f_j\rangle|$$

for every $j \in \Lambda$, but $|\langle x, Pf_j\rangle| \neq |\langle y, Pf_j\rangle|$ for some $j \in \Lambda^c$.

According to $M \subset G \subset H$, we can obtain that $x,y \in G$, and

$$|\langle x, f_j\rangle| = |\langle x, P'f_j\rangle|, \ \ |\langle y, f_j\rangle| = |\langle y, P'f_j\rangle|.$$

Thus for any $\Lambda \subset \{1,2,\cdots,n\}$, there are two vectors $x,y \in G$ such that

$$|\langle x, P'f_j\rangle| = |\langle y, P'f_j\rangle|$$

for every $j \in \Lambda$, but $|\langle x, P'f_j\rangle| \neq |\langle y, P'f_j\rangle|$ for some $j \in \Lambda^c$.

Hence $\{P'f_i\}_{i=1}^n$ have the exact PR-redundancy property of G. $\square$

It is worth noting that, we can not obtain that G is also an exact F-PR subspace, when M is an exact F-PR subspace. That's because if M is a F-PR subspace, then G may not be a F-PR subspace, although there is a frame $\{P'f_i\}_{i=1}^n$ have the exact PR-redundancy property of G. The following Example 2.6 shows that.

EXAMPLE 2.6. Let $F = \{e_1, e_2, e_1+e_2+e_3\}$ be a frame of $\mathbb{R}^3$, $A = span\{e_1,e_2\}$ be a subspace of $\mathbb{R}^3$, $B = span\{e_1,e_2,e_3\}$.

Then $A \subset B$, and

$$PF = \{e_1,e_2,e_1+e_2\}, P'F = \{e_1,e_2,e_1+e_2+e_3\},$$

where P and P' are the orthogonal projection onto A and B respectively.

It's easy to know $PF = \{e_1,e_2,e_1+e_2\}$ is an exact phase-retrievable frame of A, and $P'F = \{e_1,e_2,e_1+e_2+e_3\}$ have the exact PR-redundancy property of B by simple calculation.

Let $a = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \in B$, $b = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \in B$, and we can obtain that

$$\begin{aligned} |\langle a, P'F_1\rangle| = 1 = |\langle b, P'F_1\rangle| \\ |\langle a, P'F_2\rangle| = 1 = |\langle b, P'F_2\rangle| \\ |\langle a, P'F_3\rangle| = 1 = |\langle b, P'F_3\rangle|. \end{aligned}$$

However, it is easy to check that $\widehat{a} \neq \widehat{b}$. Hence $P'F = \{e_1, e_2, e_1 + e_2 + e_3\}$ is not a phase-retrievable frame of B even though it have the exact PR-redundancy property. Thus $P'F = \{e_1, e_2, e_1 + e_2 + e_3\}$ is not an exact phase-retrievable frame of B.

Furthermore, Corollary 2.7 as a corollary to Proposition 2.5, shows that if M is an exact F-PR subspace, then its subspace G may not be an exact F-P'R subspace.

COROLLARY 2.7. *Let $F = \{f_i\}_{i=1}^n$ be a frame for a complex Hilbert space H, M and G be the subspaces of H, and $G \subset M \subset H$. Let P and P' be the orthogonal projection onto M and G respectively. If M is an exact F-PR subspace, then G is also a F-P'R subspace. However, G may not be an exact F-P'R subspace.*

The proof of Corollary 2.7 is similar to Proposition 2.5, so we won't repeat it here.

In frame theory, whether the sum of two frames is still a frame has always been a hot topic of research. And we know that the sum of two frames may not necessarily be a frame. Hence we want to know the properties of the union of two exact phase-retrievable frames and the union of two exact PR subspaces. First of all, we discuss the union of two exact PR subspaces.

THEOREM 2.8. *Suppose that M and N are the subspaces of H, P and Q are the orthogonal projections onto M and N respectively. If $\{Pf_i\}_{i=1}^m$ and $\{Qf_i\}_{i=1}^m$ are the exact phase-retrievable frames of M and N respectively, and $M \perp N$, then we can obtain $\{g_i\}_{i \in I} = \{Pf_i\}_{i=1}^m \cup \{Qf_i\}_{i=1}^m$ is an exact phase-retrievable frame of $M \cup N$.*

Proof. (1) First of all, we show that $\{Pf_i\}_{i=1}^m \cup \{Qf_i\}_{i=1}^m$ is a phase-retrievable frame of $M \cup N$.

For any $x, y \in M \cup N$, there are three cases, $x, y \in M$, $x, y \in N$, and $x \in M, y \in N$.

(i) $x, y \in M$.

In this case, if $|\langle x, g_i \rangle| = |\langle y, g_i \rangle|$ for any $i \in I$, then

$$|\langle x, Pf_i \rangle| = |\langle y, Pf_i \rangle|$$

for every $i = 1, 2 \cdots, m$.

Hence $\widehat{x} = \widehat{y}$, since $\{Pf_i\}_{i=1}^m$ is the phase-retrievable frame of M. Moreover we can obtain that $\{g_i\}_{i \in I} = \{Pf_i\}_{i=1}^m \cup \{Qf_i\}_{i=1}^m$ is a phase-retrievable frame of $M \cup N$.

(ii) $x, y \in N$.

The proof for this case is similar to (i), so we won't repeat it here.

(iii) $x \in M, y \in N$.

In this case, if $|\langle x, g_i \rangle| = |\langle y, g_i \rangle|$ for any $i \in I$, then

$$|\langle x, Pf_i \rangle| = |\langle y, Pf_i \rangle| = 0$$

for every $i = 1, 2 \cdots, m$, since $M \perp N$.

And

$$0 = |\langle x, Qf_i \rangle| = |\langle y, Qf_i \rangle|$$

for every $i = 1,2\cdots,m$, since $M \perp N$.

Hence we can obtain that $x = y = 0$. Moreover, according to Definition1.3, we know that $\{g_i\}_{i\in I} = \{Pf_i\}_{i=1}^m \cup \{Qf_i\}_{i=1}^m$ is a phase-retrievable frame of $M \cup N$.

(2) Now we show that $\{g_i\}_{i\in I} = \{Pf_i\}_{i=1}^m \cup \{Qf_i\}_{i=1}^m$ is an exact phase-retrievable frame of $M \cup N$.

For any $\Lambda \subset I$, we also have three cases, they are $\{g_i\}_{i\in\Lambda} \subset \{Pf_i\}_{i=1}^m$, $\{g_i\}_{i\in\Lambda} \subset \{Qf_i\}_{i=1}^m$, and $\{g_i\}_{i\in\Lambda_1} \subset \{Pf_i\}_{i=1}^m, \{g_i\}_{i\in\Lambda_2} \subset \{Qf_i\}_{i=1}^m$, where $\Lambda = \Lambda_1 \bigoplus \Lambda_1$.

(i) The first case $\{g_i\}_{i\in\Lambda} \subset \{Pf_i\}_{i=1}^m$.

Since $\{Pf_i\}_{i=1}^m$ is the exact phase-retrievable frame of M, then according to Proposition 1.6 there are two vectors $x,y \in M$ such that

$$|\langle x, Pf_i\rangle| = |\langle y, Pf_i\rangle|$$

for every $i \in \Lambda$, but $|\langle x, Pf_i\rangle| \neq |\langle y, Pf_i\rangle|$ for some $i \in \Lambda^c$.

Thus $\{g_i\}_{i\in I} = \{Pf_i\}_{i=1}^m \cup \{Qf_i\}_{i=1}^m$ is an exact phase-retrievable frame of $M \cup N$.

(ii) The second case $\{g_i\}_{i\in\Lambda} \subset \{Qf_i\}_{i=1}^m$.

The proof for this case is similar to the first case, hence we won't repeat it here.

(iii) The third case $\{g_i\}_{i\in\Lambda_1} \subset \{Pf_i\}_{i=1}^m$ and $\{g_i\}_{i\in\Lambda_2} \subset \{Qf_i\}_{i=1}^m$.

According to $\{g_i\}_{i\in\Lambda_1} \subset \{Pf_i\}_{i=1}^m$, then there are two vectors $x,y \in M$ such that

$$|\langle x, Pf_i\rangle| = |\langle y, Pf_i\rangle|$$

for every $i \in \Lambda_1$, but $|\langle x, Pf_i\rangle| \neq |\langle y, Pf_i\rangle|$ for some $i \in \Lambda_1^c$. And

$$|\langle x, Qf_i\rangle| = |\langle y, Qf_i\rangle| = 0$$

for every $i \in \Lambda_2$.

That is to say there are two vectors $x,y \in M \cup N$ such that

$$|\langle x, g_i\rangle| = |\langle y, g_i\rangle|$$

for every $i \in \Lambda$, but $|\langle x, g_i\rangle| \neq |\langle y, g_i\rangle|$ for some $i \in \Lambda^c$.

Thus $\{g_i\}_{i\in I} = \{Pf_i\}_{i=1}^m \cup \{Qf_i\}_{i=1}^m$ is an exact phase-retrievable frame of $M \cup N$. $\square$

REMARK. Note that the union of two exact phase retrievable frames may not be exact, although it may become phase retrievable. For example, suppose that M is the subspace of H, and P is the orthogonal projection onto M. If $\{Pf_i\}_{i=1}^m$ and $\{Pg_i\}_{i=1}^m$ are the exact phase-retrievable frames of M respectively, then $\{u_i\}_{i\in I} = \{Pf_i\}_{i=1}^m \cup \{Pg_i\}_{i=1}^m$ may not be an exact phase-retrievable frame of M. Moreover, according to Proposition 2.9, we can get that $\{(Pf_i, Pg_i)\}_{i=1}^m$ is a frame of (M,M) with the exact PR-redundancy property.

PROPOSITION 2.9. *Suppose that M is the subspace of H, and P is the orthogonal projection onto M. If $\{Pf_i\}_{i=1}^m$ and $\{Pg_i\}_{i=1}^m$ are the frames of M with the exact PR-redundancy property respectively, then $\{(Pf_i, Pg_i)\}_{i=1}^m$ is a frame of (M,M) with the exact PR-redundancy property.*

Proof. Since $\{Pf_i\}_{i=1}^m$ is a frame of M with the exact PR-redundancy property, then for any subset $\Lambda \subset \{1,2,\cdots,m\}$, there are two vectors $x,y \in M$, such that

$$|\langle x, Pf_j\rangle| = |\langle y, Pf_j\rangle|$$

for every $j \in \Lambda$, but $|\langle x, Pf_j\rangle| \neq |\langle y, Pf_j\rangle|$ for some $i \in \Lambda^c$.

Hence we can obtain that, there are two vectors $x' = (x,0),\ y' = (y,0) \in (M,M)$, such that

$$\begin{aligned}|\langle x', (Pf_j, Pg_j)\rangle| &= |\langle x, Pf_j\rangle| \\ &= |\langle y, Pf_j\rangle| \\ &= |\langle y', (Pf_j, Pg_j)\rangle|\end{aligned}$$

for every $j \in \Lambda$. And $|\langle x', (Pf_j, Pg_j)\rangle| \neq |\langle y', (Pf_j, Pg_j)\rangle|$ for some $i \in \Lambda^c$.

That is to say $\{(Pf_i, Pg_i)\}_{i=1}^m$ is a frame of (M,M) with the exact PR-redundancy property. $\square$

The following Example 2.10 shows that for two phase-retrievable frames F and G, even though $\{(Pf_i, Pg_i)\}_{i=1}^m$ is a frame of (M,M) with the exact PR-redundancy property, it may not be an exact phase-retrievable frame, since $\{(Pf_i, Pg_i)\}_{i=1}^m$ is not a phase-retrievable frame.

EXAMPLE 2.10. Let $F = \{e_1, e_2, 2e_1 - e_2\}$, $G = \{e_1, e_2, e_1 + e_2\}$. Obviously F and G are phase-retrievable frames of $\mathbb{R}^2$. But $\{(F,G)\}_{i=1}^6\}$ is not a phase-retrievable frames of $\begin{bmatrix}\mathbb{R}^2\\ \mathbb{R}^2\end{bmatrix}$.

Proof. By simple calculation we can get

$$\begin{aligned}U &= \{(F,G)\}_{i=1}^6\} \\ &= \left\{ \begin{bmatrix}e_1\\ e_2\end{bmatrix}, \begin{bmatrix}e_1\\ e_1\end{bmatrix}, \begin{bmatrix}e_1\\ e_1+e_2\end{bmatrix}, \begin{bmatrix}e_2\\ e_1\end{bmatrix}, \begin{bmatrix}e_2\\ e_2\end{bmatrix}, \right. \\ &\qquad \left. \begin{bmatrix}e_2\\ e_1+e_2\end{bmatrix}, \begin{bmatrix}2e_1-e_2\\ e_2\end{bmatrix}, \begin{bmatrix}2e_1-e_2\\ e_1\end{bmatrix}, \begin{bmatrix}2e_1-e_2\\ e_1+e_2\end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\1\end{bmatrix} \right\}.\end{aligned}$$

Then let $x = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, we have

$$\begin{aligned} &|\langle x,u_1\rangle| = 1 = |\langle y,u_1\rangle|,\ |\langle x,u_2\rangle| = 1 = |\langle y,u_2\rangle| \\ &|\langle x,u_3\rangle| = 1 = |\langle y,u_3\rangle|,\ |\langle x,u_4\rangle| = 1 = |\langle y,u_4\rangle| \\ &|\langle x,u_5\rangle| = 1 = |\langle y,u_5\rangle|,\ |\langle x,u_6\rangle| = 1 = |\langle y,u_6\rangle| \\ &|\langle x,u_7\rangle| = 1 = |\langle y,u_7\rangle|,\ |\langle x,u_8\rangle| = 1 = |\langle y,u_8\rangle| \\ &|\langle x,u_9\rangle| = 1 = |\langle y,u_9\rangle|. \end{aligned}$$

That is to say, there are two vectors $x, y \in \begin{bmatrix} \mathbb{R}^2 \\ \mathbb{R}^2 \end{bmatrix}$ and $\widehat{x} \neq \widehat{y}$ such that $|\langle x,u_i\rangle\rangle| = |\langle y,u_i\rangle|$, for all $i = 1,2,\cdots,9$. Hence $\{(F,G)\}_{i=1}^{6}\}$ is not a phase-retrievable frames of $\begin{bmatrix} \mathbb{R}^2 \\ \mathbb{R}^2 \end{bmatrix}$. $\square$

Statements and declarations

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