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CBMO ESTIMATES FOR SOME MULTILINEAR OPERATORS ON MIXED HERZ SPACES

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Abstract. In this paper, we develop the CBMO estimates for the multilinear operators of singular integral operator, fractional integral operator, and Hardy-type operators in higher dimensional cases, which include their commutators as special cases, on mixed Herz spaces. Furthermore, some endpoint cases are also obtained, such as Hardy-type and weak-type estimates for multilinear operators. Particularly, it is demonstrated that in some extreme cases, these operators are actually not bounded.

1. Introduction

In the past few decades, mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$, as natural extensions of classical Lebesgue spaces $L^p(\mathbb{R}^n)$, have attracted widespread attention. The theory of mixed-norm function spaces can be traced back to the work of Benedek and Panzone [1], in which authors prove some basic properties and boundedness of Riesz potential operator. Especially, mixed-norm spaces are more suitable for studying partial differential equations problems, which usually involve both space and time variables. In this sense, the topic of function spaces with mixed-norm has received a lot of interest and has seen significant progress in recent years. Nowadays, mixed-norm function spaces, such as mixed-norm Hardy spaces [2], mixed Morrey spaces [3], mixed-norm Besov spaces and Triebel-Lizorkin spaces [4, 5] and mixed Herz spaces [6, 7, 8] are intensively studied in harmonic analysis.

The study of Herz spaces originated from the work of Beurling [9]. In the 1990s, Lu and Yang [10] introduced the Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, which can be seen as a substitute of Lebesgue spaces with power weight. Later, certain special multilinear operators, which recover Coifman commutators in the one order case, are investigated in [11, 12, 13]. Lu and Wu [11] investigated multilinear singular integral commutators on Herz spaces and Lu and Zhao [13] established CBMO estimates for multilinear Hardy operator commutators on Lebesgue space and Herz space. Note that recently Wei [6] introduced mixed Herz spaces $\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$, in which boundedness of some classical operators and commutators are obtained via extrapolation theory on mixed spaces.

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In this paper, we mainly consider the boundedness of various multilinear operators, such as singular integral operators, fractional integral operator and Hardy-type operators in high dimensional cases. As an application, we also obtain the boundedness of Coifman commutators. Furthermore, we also obtain some endpoint estimates for multilinear operators.

Throughout this paper, we use the following notations. The letter $\vec{q}$ will denote n-tuples of the numbers in $(0,\infty]$ $(n \geqslant 1)$, $\vec{q}=(q_1,q_2,\dots,q_n)$. By definition, the inequality $0<\vec{q}<\infty$ means that $0<q_i<\infty$ for all i. For $\vec{q}=(q_1,q_2,\dots,q_n)$, write

$$\frac{1}{\vec{q}}=\left(\frac{1}{q_1},\frac{1}{q_2},\dots,\frac{1}{q_n}\right), \quad \vec{q}'=(q_1',q_2',\dots,q_n') \quad (\vec{q}\in[1,\infty)^n),$$

where $q_i'=q_i/(q_i-1)$ is conjugate exponent of q_i. $|B|$ denotes the volume of the ball B, χ_E is the characteristic function of a set E. $A\sim B$ means that $A\lesssim B$ and $B\lesssim A$, $[a]$ denotes take the integer number for a. Let $B_k=\{x\in\mathbb{R}^n : |x|\leqslant 2^k\}$ and $E_k=B_k\backslash B_{k-1}$ for any $k\in\mathbb{Z}$. Denote $\chi_k=\chi_{E_k}$ for any $k\in\mathbb{Z}$. $m_B(f)$ is the mean value of function f on B.

2. Preliminaries

In this section, we will recall the definition and some properties of the homogeneous mixed Herz space $\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$ and the central bounded mean oscillation space $\mathrm{CBMO}(\mathbb{R}^n)$. The definitions associated to multilinear operators are also given. Let us first recall the mixed Lebesgue spaces.

DEFINITION 2.1. (Mixed Lebesgue spaces) ([1]) Let $\vec{p}=(p_1,p_2,\dots,p_n)\in(0,\infty]^n$. Then the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is defined by the set of all measurable functions f such that

$$\|f\|_{\vec{p}}:=\left(\int_{\mathbb{R}}\cdots\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f(x_1,x_2,\dots,x_n)|^{p_1}\,dx_1\right)^{\frac{p_2}{p_1}}dx_2\right)^{\frac{p_3}{p_2}}\cdots dx_n\right)^{\frac{1}{p_n}}<\infty,$$

If $p_j=\infty$, then we have to make appropriate modifications.

DEFINITION 2.2. ([6]) Let $\alpha\in\mathbb{R}$, $0<p\leqslant\infty$, $0<\vec{q}\leqslant\infty$. The mixed homogeneous Herz space $\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n):=\left\{f\in L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}):\|f\|_{\dot{K}^{\alpha,p}_{\vec{q}}}=\left(\sum_{k\in\mathbb{Z}}2^{k\alpha p}\|f\chi_k\|^p_{\vec{q}}\right)^{1/p}<\infty\right\}.$$

And, if $p=\infty$, we define that $\|f\|_{K^{\alpha,\infty}_{\vec{q}}}=\sup_{k\in\mathbb{Z}}2^{k\alpha}\|f\chi_k\|_{\vec{q}}$.

REMARK 2.1. (i) If $0 < \vec{q} = (q_1, q_2, \ldots, q_n) \leqslant \infty$ and $q_1 = q_2 = \ldots = q_n = q$, then $\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, where $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is the classical Herz space.

(ii) The mixed homogeneous Herz space $\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ is a quasi-Banach space. But, if $\vec{q}, p \geqslant 1$, it is a Banach space. These results can be inferred from definitions of mixed Lebesgue spaces and classical Herz spaces.

The bounded mean oscillation space $\mathrm{BMO}(\mathbb{R}^n)$ is a natural generalize of essentially bounded function space $L^\infty(\mathbb{R}^n)$. $\mathrm{BMO}(\mathbb{R}^n)$ often serves as a substitute for $L^\infty(\mathbb{R}^n)$. For instance, classical singular integrals do not map $L^\infty(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ but map $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$.

A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}} = \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, dx < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

The John-Nirenberg inequality for $\mathrm{BMO}(\mathbb{R}^n)$ tells us that the following norms are equivalent for any $1 \leqslant p < \infty$,

$$\|f\|_{\mathrm{BMO}_p} = \sup_B \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p\, dx \right)^{\frac{1}{p}},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

However, the John-Nirenberg inequality fails for the central bounded mean oscillation space $\mathrm{CBMO}(\mathbb{R}^n)$. For any $1 \leqslant p < \infty$, the definition of $\mathrm{CBMO}(\mathbb{R}^n)$ is defined as follows.

Let $1 \leqslant p < \infty$, a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the central bounded mean oscillation space $\mathrm{CBMO}_p(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{CBMO}_p} = \sup_{r>0} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} \left| f(x) - f_{B(0,r)} \right|^p dx \right)^{\frac{1}{p}} < \infty.$$

In fact, the relationship $\mathrm{CBMO}_{p_2}(\mathbb{R}^n) \subseteq \mathrm{CBMO}_{p_1}(\mathbb{R}^n)$ is true for any $1 \leqslant p_1 < p_2 < \infty$ via Hölder's inequality. In the norm sense, we have $\|f\|_{\mathrm{CBMO}_{p_1}} \leqslant \|f\|_{\mathrm{CBMO}_{p_2}}$ under finite measure case.

Next, we will recall the corresponding mixed-norm version of $\mathrm{BMO}(\mathbb{R}^n)$. Similar to classical cases, by the John-Nirenberg inequality for mixed norm spaces [14], we can easily check that for any $\vec{p} = (p_1, \ldots, p_n) \in [1, \infty)^n$, the mixed-norm bounded mean oscillation spaces $\mathrm{BMO}_{\vec{p}}(\mathbb{R}^n)$ is also equivalent to

$$\|f\|_{\mathrm{BMO}_{\vec{p}}} = \sup_B \frac{\|(f - f_B)\chi_B\|_{\vec{p}}}{\|\chi_B\|_{\vec{p}}}.$$

The definition of the mixed-norm central bounded mean oscillation space $\mathrm{CBMO}_{\vec{p}}(\mathbb{R}^n)$ is as follows.

DEFINITION 2.3. Let $\vec{p}=(p_1,\dots,p_n)\in(1,\infty)^n$. Then the mixed central bounded mean oscillation space $\mathrm{CBMO}_{\vec{p}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{\mathrm{CBMO}_{\vec{p}}}=\sup_{r>0}\frac{\left\|\left(f-f_{B(0,r)}\right)\chi_{B(0,r)}\right\|_{\vec{p}}}{\left\|\chi_{B(0,r)}\right\|_{\vec{p}}}<\infty.$$

REMARK 2.2. Likewise, the space $\mathrm{CBMO}_{\vec{p}}(\mathbb{R}^n)$ does not satisfies the John-Nirenberg inequality, and $\mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)\subseteq\mathrm{CBMO}_{\vec{r}}(\mathbb{R}^n)$ if $1\leqslant\vec{r}<\vec{q}<\infty$. Especially, $\mathrm{CBMO}_{\vec{p}}(\mathbb{R}^n)$ is a Banach space in the sense that two functions differ by a constant are regarded as a function in this space. In fact, $\mathrm{BMO}(\mathbb{R}^n)\subseteq\mathrm{CBMO}_{\vec{p}}(\mathbb{R}^n)$ for all $\vec{p}=(p_1,\dots,p_n)\in(1,\infty)^n$. Similar to $\mathrm{CBMO}(\mathbb{R}^n)$, we have the following equivalent norm of $\mathrm{CBMO}_{\vec{p}}(\mathbb{R}^n)$.

$$\|f\|_{\mathrm{CBMO}_{\vec{p}}}\sim\sup_{r>0}\inf_{c\in\mathbb{C}}\frac{\left\|(f-c)\chi_{B(0,r)}\right\|_{\vec{p}}}{\left\|\chi_{B(0,r)}\right\|_{\vec{p}}}<\infty.$$

LEMMA 2.1. *Suppose that* $f\in\mathrm{CBMO}_{\vec{p}}(\mathbb{R}^n)$, $1\leqslant\vec{p}<\infty$, *and* $r,r'>0$, *then*

$$\frac{\left\|\left(f-f_{B(0,r)}\right)\chi_{B(0,r')}\right\|_{\vec{p}}}{\left\|\chi_{B(0,r')}\right\|_{\vec{p}}}\leqslant C_n\left(1+\left|\log(\frac{r'}{r})\right|\right)\|f\|_{\mathrm{CBMO}_{\vec{p}}}.$$

Proof. By the triangle inequality, we have

$$\begin{aligned}\frac{\left\|\left(f-f_{B(0,r)}\right)\chi_{B(0,r')}\right\|_{\vec{p}}}{\left\|\chi_{B(0,r')}\right\|_{\vec{p}}}&=\frac{\left\|\left(f-f_{B(0,r')}+f_{B(0,r')}-f_{B(0,r)}\right)\chi_{B(0,r')}\right\|_{\vec{p}}}{\left\|\chi_{B(0,r')}\right\|_{\vec{p}}}\\&\leqslant\frac{\left\|\left(f-f_{B(0,r')}\right)\chi_{B(0,r')}\right\|_{\vec{p}}}{\left\|\chi_{B(0,r')}\right\|_{\vec{p}}}+\frac{\left\|\left(f_{B(0,r')}-f_{B(0,r)}\right)\chi_{B(0,r')}\right\|_{\vec{p}}}{\left\|\chi_{B(0,r')}\right\|_{\vec{p}}}.\end{aligned}$$

By computing can easily get,

$$\begin{aligned}\left|f_{B(0,r')}-f_{B(0,r)}\right|=\big|&f_{B(0,r')}-f_{B(0,2r')}+f_{B(0,2r')}-f_{B(0,4r')}+\dots\\&-f_{B(0,r'|\log(r'/r)|)}+f_{B(0,r'|\log(r'/r)|)}-f_{B(0,r)}\big|.\end{aligned}$$

We only estimate the first part in the following since others are similar.

$$\begin{aligned}\left|f_{B(0,r')}-f_{B(0,2r')}\right|&\leqslant\frac{1}{|B(0,r')|}\int_{B(0,r')}\left|f(y)-f_{B(0,2r')}\right|dy\\&\leqslant\frac{1}{|B(0,r')|}\int_{B(0,2r')}\left|f(y)-f_{B(0,2r')}\right|dy\\&\leqslant\frac{1}{|B(0,r')|}\left\|(f-f_{B(0,2r')})\chi_{B(0,2r')}\right\|_{\vec{p}}\left\|\chi_{B(0,2r')}\right\|_{\vec{p}'}.\end{aligned}$$

Taking the mixed-norm of both sides, we get

$$\begin{aligned}\frac{\left\|\left(f_{B(0,r')}-f_{B(0,2r')}\right)\chi_{B(0,r')}\right\|_{\vec{p}}}{\left\|\chi_{B(0,r')}\right\|_{\vec{p}}} &\leqslant \frac{1}{|B(0,r')|}\left\|(f-f_{B(0,2r')})\chi_{B(0,2r')}\right\|_{\vec{p}}\left\|\chi_{B(0,2r')}\right\|_{\vec{p}'}\\ &\leqslant \frac{|B(0,2r')|}{|B(0,r')|}\frac{\left\|\left(f-f_{B(0,2r')}\right)\chi_{B(0,2r')}\right\|_{\vec{p}}}{\left\|\chi_{B(0,2r')}\right\|_{\vec{p}}}.\end{aligned}$$

Combine the above estimate, then

$$\frac{\left\|\left(f-f_{B(0,r)}\right)\chi_{B(0,r')}\right\|_{\vec{p}}}{\left\|\chi_{B(0,r')}\right\|_{\vec{p}}}\leqslant C_n\left(1+\left|\log(\frac{r'}{r})\right|\right)\|f\|_{\mathrm{CBMO}_{\vec{p}}}.$$

This proof is completed. □

In the following part, we will give several operators considered in this paper.

The investigations of singular integral operators originated in Hilbert transform, which was initially defined as a convolution operator with a certain principal value distribution. We define the following singular integral operators via modifying the kernel conditions.

Let T be an L^2 bounded singular integral operator

$$Tf(x)=\text{p.v.}\int_{\mathbb{R}^n}K(x-y)f(y)dy,$$

with the kernel K satisfying the following conditions:

(i) $|K(x)|\leqslant c|x|^{-n}$, if $x\neq 0$;

(ii) $\left|\frac{\partial^\beta}{\partial x^\beta}K(x-y)-\frac{\partial^\beta}{\partial x^\beta}K(x-y')\right|\leqslant C_\beta\frac{|y-y'|}{|x-y|^{n+|\beta|+1}}$, if $|x-y|\geqslant 2|y-y'|$, where $\beta=(\beta_1,\cdots,\beta_n)$ is any multi-index and $|\beta|=\beta_1+\cdots+\beta_n$.

Other than singular integral operators, fractional integral operators are also significant in harmonic analysis For $0<\alpha<n$, the fractional integral operator T_α can be defined by

$$T_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}f(y)dy.$$

The boundedness of the fractional integral operator on $L^p(\mathbb{R}^n)$ can be founded in [15], which is also called the Hardy-Littlewood-Sobolev inequality.

In 1920, Hardy [16] introduced the Hardy operator in one-dimensional case,

$$Hf(x):=\frac{1}{x}\int_0^x f(t)dt\quad x\neq 0.$$

The adjoint operator of H is defined by

$$H^*f(x):=\int_x^\infty\frac{f(t)}{t}dt\quad x\neq 0.$$

Later, Faris [17] defined the n-dimensional Hardy operator and its adjoint operators on $\mathbb{R}^n$,

$$\mathscr{H}f(x) := \frac{1}{|x|^n}\int_{|y|<|x|} f(y)dy, \quad x \in \mathbb{R}^n\backslash\{0\}.$$

And

$$\mathscr{H}^* f(x) := \int_{|y|>|x|} \frac{f(y)}{|y|^n}dy, \quad x \in \mathbb{R}^n\backslash\{0\}.$$

The central bounded mean oscillation space $\mathrm{CBMO}_{\vec{p}}(\mathbb{R}^n)$ plays an important role in the commutator theory of Hardy-type operators. Let us recall the definition of Coifman commutators.

The commutator generated by the function b and the operator T is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x). \tag{1}$$

A well-known result of Coifman, Rochberg and Weiss [18] states that $[b,T]$ is bounded on some $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. In 2007, Fu et al. [19] established the boundedness of Hardy commutators generated by CBMO functions. Recently, Wei [7] used the commutators' boundedness of Hardy operator on mixed Herz space to characterize the central BMO functions $\mathrm{CBMO}_{\vec{p}}(\mathbb{R}^n)$.

We now define the multilinear singular integral, fractional integral and Hardy operator.

Let A be a function on $\mathbb{R}^n$ having derivatives of order one in $\mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)$. For $x, y \in \mathbb{R}^n$, set

$$R(A;x,y) = A(x) - A(y) - \nabla A(y)(x-y).$$

LEMMA 2.2. ([20]) *Let A be a function on $\mathbb{R}^n$ with derivatives of order one in $L^q(\mathbb{R}^n)$ for some $q > n$. Then*

$$|A(x) - A(y)| \leqslant C|x-y| \left(\frac{1}{|Q'(x,y)|}\int_{Q'(x,y)} |\nabla A(z)|^q dz\right)^{1/q},$$

where $Q'(x,y)$ is the cube centered at x with sides parallel to the axes and whose side length is $2\sqrt{n}|x-y|$

REMARK 2.3. Lemma 2.2 can easily be generalized to the mixed-norm case via Hölder's inequality on mixed spaces, that is to say, when $\vec{q} > n$,

$$|A(x) - A(y)| \leqslant C|x-y| \left\|\chi_{Q'(x,y)}\right\|_{\vec{q}}^{-1} \left\|\nabla A\, \chi_{Q'(x,y)}\right\|_{\vec{q}}.$$

We also consider a class of multilinear singular integrals and fractional integral operator, which are defined as follows

$$T^A f(x) = \text{p.v.}\int_{\mathbb{R}^n} \frac{K(x-y)R_m(A;x,y)f(y)}{|x-y|}dy, \tag{2}$$

and

$$T^A_\alpha f(x) = \int_{\mathbb{R}^n} \frac{R(A;x,y)f(y)}{|x-y|^{n-\alpha+1}}dy. \tag{3}$$

Furthermore, define the multilinear Hardy operators $\mathscr{H}_A$ and $\mathscr{H}_A^*$ as

$$\mathscr{H}_A f(x) = \frac{1}{v_n|x|^n}\int_{|y|<|x|} \frac{f(y)}{|x-y|}R(A;x,y)dy, \quad x \in \mathbb{R}^n\backslash\{0\} \tag{4}$$

and

$$\mathscr{H}_A^* f(x) = \frac{1}{v_n}\int_{|y|>|x|} \frac{f(y)}{|y|^n|x-y|}R(A;y,x)dy, \quad x \in \mathbb{R}^n\backslash\{0\}, \tag{5}$$

where v_n denotes the volume of the unit ball in $\mathbb{R}^n$ and $v_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$. The operators $\mathscr{H}_A$ and $\mathscr{H}_A^*$ are adjoint mutually,

$$\int_{\mathbb{R}^n} g(x)\mathscr{H}_A f(x)dx = \int_{\mathbb{R}^n} f(x)\mathscr{H}_A^* g(x)dx.$$

3. Main results and their proofs

3.1. The general case

This part establishes the boundedness results of various multilinear operators on mixed Herz spaces. Furthermore, as a corollary, we gain the boundedness of Coifman commutators.

THEOREM 3.1. *Let T^A be defined as in (2), where A has derivatives of order one in* $\mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)$. *If $0 < p \leqslant \infty,\ 1 < \vec{q}_1, \vec{q}_2 < \infty$, and $\frac{1}{q_{2i}} = \frac{1}{q_i} + \frac{1}{q_{1i}}$, α_1 satisfies $-\sum_{i=1}^n \frac{1}{q_{1i}} < \alpha_1 < n - \sum_{i=1}^n \frac{1}{q_{1i}}$ and $\alpha_2 = \alpha_1 - \sum_{i=1}^n \frac{1}{q_i}$, then*

$$\left\|T^A f\right\|_{\dot{K}^{\alpha_2,p}_{\vec{q}_2}} \leqslant C\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\|_{\dot{K}^{\alpha_1,p}_{\vec{q}_1}}.$$

Proof. We only prove the case $0 < p < \infty$, $p = \infty$ follows after slight modifications.

$$\begin{aligned}
\left\|T^A f\right\|_{\dot{K}^{\alpha_2,p}_{\vec{q}_2}} &\leqslant C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-2}\left\|T^A(f\chi_l)\chi_k\right\|_{\vec{q}_2}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k-1}^{k+1}\left\|T^A(f\chi_l)\chi_k\right\|_{\vec{q}_2}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k+2}^{\infty}\left\|T^A(f\chi_l)\chi_k\right\|_{\vec{q}_2}\right)^p\right\}^{1/p} \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

In the following case, we assume that fixed k,

$$A_k(x) = A(x) - m_{B_k}(\nabla A)y. \tag{6}$$

It is easy to check that $R(A;x,y) = R(A_k;x,y)$.

We first estimate I_1, $x \in E_k$, $y \in E_l$, and $l \leqslant k-2$, then $|x-y| \sim |x| \sim 2^k$, then

$$\begin{aligned}
|R(A_k;x,y)| &\leqslant |A_k(x) - A_k(y)| + \left|\nabla A(y) - m_{B_k}(\nabla A)\right| |x-y| \\
&\lesssim |x-y| \left\{ |B'(x,y)|^{\frac{-1}{n}\sum_{i=1}^{n}\frac{1}{q_i}} \|\nabla A_k \chi_{B'}\|_{\vec{q}} + \left|\nabla A(y) - m_{B_k}(\nabla A)\right| \right\} \\
&\lesssim |x-y| \left(\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} + \left|\nabla A(y) - m_{B_k}(\nabla A)\right| \right).
\end{aligned} \tag{7}$$

Furthermore, by (7) and Hölder's inequality on mixed-norm Lebesgue spaces, we can get

$$\begin{aligned}
\left|T^A(f\chi_l)(x)\right| &\leqslant \int_{E_l} \frac{|R(A_k;x,y)||f(y)|}{|x-y|^{n+1}} dy \\
&\leqslant \int_{E_l} \frac{1}{|x-y|^n} \left(\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} + \left|\nabla A(y) - m_{B_k}(\nabla A)\right| \right) |f(y)| dy \\
&\lesssim 2^{-kn} \left\| \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} + \left|\nabla A(y) - m_{B_k}(\nabla A)\right| \chi_l \right\|_{\vec{q}} \|f\chi_l\|_{\vec{q}_1} 2^{l\left(n - \sum_{i=1}^{n}\frac{1}{q_i} - \sum_{i=1}^{n}\frac{1}{q_{1i}}\right)} \\
&\lesssim 2^{-kn + l\sum_{i=1}^{n}\frac{1}{q_i} + l\left(n - \sum_{i=1}^{n}\frac{1}{q_i} - \sum_{i=1}^{n}\frac{1}{q_{1i}}\right)} \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \|f\chi_l\|_{\vec{q}_1}.
\end{aligned}$$

Then

$$\left\|T^A(f\chi_l)\chi_k\right\|_{\vec{q}_2} \leqslant 2^{(l-k)\left(n - \sum_{i=1}^{n}\frac{1}{q_{2i}}\right)} 2^{l\sum_{i=1}^{n}\frac{1}{q_i}} \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \|f\chi_l\|_{\vec{q}_1}.$$

Taking the mixed Herz (quasi-)norm, one can get

$$\begin{aligned}
I_1 &\leqslant C \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p} \left(\sum_{l=-\infty}^{k-2} \left\|T^A(f\chi_l)\chi_k\right\|_{\vec{q}_2} \right)^p \right\}^{1/p} \\
&\lesssim \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p} \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\left(n - \sum_{i=1}^{n}\frac{1}{q_{2i}}\right)} 2^{l\sum_{i=1}^{n}\frac{1}{q_i}} \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \|f\chi_l\|_{\vec{q}_1} \right)^p \right\}^{1/p}.
\end{aligned}$$

When $0 < p \leqslant 1$,

$$\begin{aligned}
I_1 &\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \left\{ \sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p} \sum_{k=l+2}^{\infty} 2^{(l-k)p\left(n - \sum_{i=1}^{n}\frac{1}{q_{2i}} - \alpha_2\right)} \|f\chi_l\|_{\vec{q}_1}^p \right\}^{1/p} \\
&\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \left\{ \sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p} \|f\chi_l\|_{\vec{q}_1}^p \right\}^{1/p} \lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}}.
\end{aligned}$$

When $1 < p < \infty$,

$$I_1 \lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \left\{ \sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p} \left(\sum_{k=l+2}^{\infty} 2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_{2i}}-\alpha_2\right)p/2} \|f\chi_l\|_{\vec{q}_1}^{p} \right) \right.$$
$$\left. \times \left(\sum_{k=l+2}^{\infty} 2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_{2i}}-\alpha_2\right)p'/2} \|f\chi_l\|_{\vec{q}_1}^{p} \right)^{p/p'} \right\}^{1/p}$$
$$\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}}.$$

For I_2, let $\phi \in C_0^\infty(\mathbb{R}^n)$ and supp $\phi \subset B(0,4)$, $\phi(x) \equiv 1$ when $x \in B(0,2)$. Denote $M = \|\nabla\phi\|_\infty$. Take $y_0 \in E_{k+4}$ and let

$$A_k^\phi(x) = (A_k(x) - A_k(y_0))\,\phi(2^{-k}x). \tag{8}$$

It is easy to show that $R(A;x,y) = R(A_k^\phi;x,y)$. That is to say,

$$T^A(f\chi_l)(x)\chi_k(x) = T^{A_k^\phi}(f\chi_l)(x)\chi_k(x).$$

By $x \in E_k$, $y \in E_l$ and $k-1 \leqslant l \leqslant k+1$, inequality (7), we obtain

$$\begin{aligned}
\left|T^{A_k^\phi}(f\chi_l)(x)\right| &= \int_{E_l} \frac{\left|R(A_k^\phi;x,y)\right|\left|f(y)\right|}{|x-y|^{n+1}}dy \\
&\lesssim \int_{E_l} \frac{|A_k^\phi(x) - A_k^\phi(y) - \nabla A_k^\phi(y)(x-y)||f(y)|}{|x-y|^{n+1}}dy \\
&\lesssim \int_{E_l} \frac{|A_k^\phi(x) - A_k^\phi(y)||f(y)|}{|x-y|^{n+1}}dy + \int_{E_l} \frac{|\nabla A_k^\phi(y)||f(y)|}{|x-y|^{n}}dy \\
&:= II_1 + II_2.
\end{aligned}$$

We first estimate II_2. By $|y_0| - |y| \leqslant |y_0 - y| \leqslant |y + y_0| \sim 2^k$, and $B'(x,y)$ is a Ball centered at x with a radius $|x-y|$. We have

$$\begin{aligned}
\left|\nabla A_k^\phi(y)\right| &= \left|\nabla\left(A_k(y) - A_k(y_0)\right)\phi(2^{-k}y)\right| \\
&\lesssim \left|\nabla A_k(y)\phi(2^{-k}y)\right| + 2^{-k}\left|A_k(y) - A_k(y_0)\right|\left|\nabla\phi(2^{-k}y)\right| \\
&\lesssim M\left(|\nabla A_k(y)| + 2^{-k}|A_k(y) - A_k(y_0)|\right) \\
&\lesssim M\left(|\nabla A_k(y)| + 2^{-k}\frac{1}{|B'(y,y_0)|}\int_{B'} |A_k(y) - A_k(y_0)|\,dz\right) \qquad (9) \\
&\lesssim M\left(|\nabla A_k(y)| + 2^{-k}\frac{1}{|B'(y,y_0)|}\int_{B'} \nabla A_k(z)\,|y - y_0|\,dz\right)
\end{aligned}$$

$$\lesssim M|\nabla A_k(y)| + 2^{-k}|y-y_0|\,|B'(y,y_0)|^{\frac{-1}{n}\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A_k(z)\chi_{B'}\|_{\vec{q}}$$
$$\lesssim M\left(|\nabla A_k(y)| + \|\nabla A_k(z)\|_{\mathrm{CBMO}_{\vec{q}}}\right).$$

By the above estimate and Hölder's inequality on mixed-norm Lebesgue spaces, there holds

$$\begin{aligned}
II_2 &= \int_{E_l}\frac{\left|\nabla A_k^{\phi}(y)\right|\left|f(y)\right|}{|x-y|^n}dy\\
&\lesssim \int_{E_l}\frac{M\left(|\nabla A_k(y)| + \|\nabla A_k(z)\|_{\mathrm{CBMO}_{\vec{q}}}\right)|f(y)|}{|x-y|^n}dy\\
&\lesssim M\left\|\,|\nabla A_k(y)| + \|\nabla A_k(z)\|_{\mathrm{CBMO}_{\vec{q}}}\right\|_{\vec{q}}\|f\chi_l\|_{\vec{q}_1}2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_i}-\sum_{i=1}^{n}\frac{1}{q_{1i}}\right)-kn}\\
&\lesssim M2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_i}-\sum_{i=1}^{n}\frac{1}{q_{1i}}\right)-kn+l\sum_{i=1}^{n}\frac{1}{q_i}}.
\end{aligned}$$

Next, we estimate II_1. Firstly,

$$\begin{aligned}
\left|A_k^{\phi}(x)-A_k^{\phi}(y)\right| &\lesssim |x-y||B'(x,y)|^{\frac{-1}{n}\sum_{i=1}^{n}\frac{1}{q_i}}\left\|\nabla A_k^{\phi}\chi_{B'}\right\|_{\vec{q}}\\
&\lesssim |x-y||B'(x,y)|^{\frac{-1}{n}\sum_{i=1}^{n}\frac{1}{q_i}}\left\|\,|\nabla A_k(y)| + \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\chi_{B'}\right\|_{\vec{q}} \qquad (10)\\
&\lesssim |x-y|\left(|B'(x,y)|^{\frac{-1}{n}\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A_k\chi_{B'}\|_{\vec{q}} + \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\right)\\
&\lesssim |x-y|\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}.
\end{aligned}$$

Consequently, II_1 has the following estimate:

$$\begin{aligned}
II_1 &\lesssim \int_{E_l}\frac{\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}|f(y)|}{|x-y|^n}dy\\
&\lesssim 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_{1i}}\right)-kn}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}\\
&\lesssim 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_{2i}}\right)-kn+l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}.
\end{aligned}$$

Combining the above estimates for II_1 and II_2, we arrive at

$$\left|T^{A_k^{\phi}}(f\chi_l)(x)\right| \lesssim 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_{2i}}\right)-kn+l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}.$$

Furthermore,

$$\begin{aligned}\left\|T^{A_k^\phi}(f\chi_l)\chi_k\right\|_{\vec{q}_2} &\lesssim 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_{2i}}\right)-kn+l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}2^{k\sum_{i=1}^{n}\frac{1}{q_{2i}}}\\&\lesssim 2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_{2i}}\right)}2^{l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}.\end{aligned}$$

Taking the mixed Herz (quasi-)norm on both sides of the above inequalities, it yields

$$\begin{aligned}I_2 &\lesssim \left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=k-1}^{k+1}2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_{2i}}\right)}2^{l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}\right)^p\right\}^{1/p}\\&\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\left\{\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}\left(\sum_{k=l-1}^{l+1}2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_{2i}-\alpha_2}\right)}\|f\chi_l\|_{\vec{q}_1}\right)^p\right\}^{1/p}\\&\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\left\{\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}\|f\chi_l\|_{\vec{q}_1}^p\right\}^{1/p}\\&\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}}.\end{aligned}$$

For I_3, since $x\in E_k$ and $y\in E_l$ when $l\geqslant k+2$, similar to I_1, we have

$$\begin{aligned}\left|T^A(fx_l)\right| &\leqslant \int_{E_l}\frac{|R(A;x,y)||f(y)|}{|x-y|^{n+1}}dy\\&\lesssim \int_{E_l}\frac{\left(\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}+\left|\nabla A(y)-m_{B_k}(\nabla A)\right|\right)|f(y)|}{|x-y|^n}dy\\&\leqslant 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_i}-\sum_{i=1}^{n}\frac{1}{q_{1i}}\right)-ln}\left\|\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}|\nabla A(y)-m_{BK}(\nabla A)|\right\|_{\vec{q}}\|f\chi_l\|_{\vec{q}_1}\\&\lesssim 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_i}-\sum_{i=1}^{n}\frac{1}{q_{1i}}\right)-ln+l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}.\end{aligned}$$

As a result,

$$\begin{aligned}\left\|T^A(f\chi_l)\chi_k\right\|_{\vec{q}_2} &\leqslant 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_i}-\sum_{i=1}^{n}\frac{1}{q_{1i}}\right)-ln+l\sum_{i=1}^{n}\frac{1}{q_i}}2^{k\sum_{i=1}^{n}\frac{1}{q_{2i}}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}\\&\lesssim 2^{(k-l)\sum_{i=1}^{n}\frac{1}{q_{2i}}}2^{l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}.\end{aligned}$$

The above estimates imply that

$$\begin{aligned}I_3 &\lesssim \left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=k+2}^{\infty}\left\|T^A(f\chi_l)\chi_k\right\|_{\vec{q}_2}\right)^p\right\}^{\frac{1}{p}}\\&\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\sum_{l=k+2}^{\infty}\left(2^{(k-l)\sum_{i=1}^{n}\frac{1}{q_{2i}}}2^{l\sum_{i=1}^{n}\frac{1}{q_i}}\|f\chi_l\|_{\vec{q}}\right)^p\right\}^{\frac{1}{p}}.\end{aligned}$$

When $0 < p \leqslant 1$,

$$I_3 \leqslant \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \left\{ \sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p} \sum_{k=-\infty}^{l-2} 2^{(k-l)p\left(\sum_{i=1}^{n} \frac{1}{q_{2i}}+\alpha_2\right)} \|f\chi_l\|_{\vec{q}_1}^{p} \right\}^{\frac{1}{p}}$$
$$\leqslant \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}}.$$

When $1 < p < \infty$,

$$I_3 \lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \left\{ \sum_{l=-\infty}^{\infty} 2^{l\alpha_1 p} \left(\sum_{k=-\infty}^{l-2} 2^{(k-l)\left(\sum_{i=1}^{n} \frac{1}{q_{2i}}+\alpha_2\right)p/2} \|f\chi_l\|_{\vec{q}_1}^{p} \right) \right.$$
$$\left. \times \left(\sum_{k=-\infty}^{l-2} 2^{(k-l)\left(\sum_{i=1}^{n} \frac{1}{q_{2i}}+\alpha_2\right)p'/2} \|f\chi_l\|_{\vec{q}_1}^{p} \right)^{p/p'} \right\}^{1/p}$$
$$\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}}.$$

This proof is completed. □

THEOREM 3.2. *Let* $[b,T]$ *be defined as in* (1)*, where* b *in* $\mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)$*. If* $0 < p \leqslant \infty$*,* $1 < \vec{q}_1, \vec{q}_2 < \infty$*, and* $\frac{1}{q_{2i}} = \frac{1}{q_i} + \frac{1}{q_{1i}}$*,* α_1 *satisfies* $-\sum_{i=1}^{n} \frac{1}{q_{1i}} < \alpha_1 < n - \sum_{i=1}^{n} \frac{1}{q_{1i}}$ *and* $\alpha_2 = \alpha_1 - \sum_{i=1}^{n} \frac{1}{q_i}$*, then*

$$\|[b,T]f\|_{\dot{K}_{\vec{q}_2}^{\alpha_2,p}} \leqslant C\|b\|_{\mathrm{CBMO}_{\vec{q}}} \|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}}.$$

Proof. The commutators $[b,T]$ can be seen as multilinear operators T^A in a certain case that functions A are the continuous functions. Similar to the proof of Theorem 3.1, we can easily get the desired result. Here we omit the details. □

REMARK 3.1. If $q_{1i} = q_1$ and $q_{2i} = q_2$, where $i = 1,2,\cdots,n$, Theorem 3.2 reduces to the corresponding result on classical Herz spaces which was obtained by Grafakos et al. [21].

THEOREM 3.3. *Let* T_α^A *be defined as in* (3)*, where* A *has derivatives of order one in* $\mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)$*. If* $0 < p \leqslant \infty$*,* $1 < \vec{q}_1, \vec{q}_2 < \infty$*, and* $\frac{1}{q_{2i}} = \frac{1}{q_i} + \frac{1}{q_{1i}} - \alpha$*,* α_1 *satisfies* $-\sum_{i=1}^{n} \frac{1}{q_{1i}} < \alpha_1 < n - \alpha - \sum_{i=1}^{n} \frac{1}{q_{1i}}$ *and* $\alpha_2 = \alpha_1 - \sum_{i=1}^{n} \frac{1}{q_i}$*, then*

$$\left\|T_\alpha^A f\right\|_{\dot{K}_{\vec{q}_2}^{\alpha_2,p}} \leqslant C\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}}.$$

Proof. We only prove the case $0 < p < \infty$, $p = \infty$ follows after slight modifications. Write

$$\begin{aligned}
\left\|T_\alpha^A f\right\|_{\dot{K}_{q_2}^{\alpha_2,p}} \leqslant C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-2}\left\|T_\alpha^A(f\chi_l)\chi_k\right\|_{q_2}\right)^p\right\}^{1/p} \\
+C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k-1}^{k+1}\left\|T_\alpha^A(f\chi_l)\chi_k\right\|_{q_2}\right)^p\right\}^{1/p} \\
+C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\left(\sum_{l=k+2}^{\infty}\left\|T_\alpha^A(f\chi_l)\chi_k\right\|_{q_2}\right)^p\right\}^{1/p} \\
:= K_1+K_2+K_3.
\end{aligned}$$

Similar to the proof of Theorem 3.1, for K_1, $x \in E_k$, $y \in E_l$, and $l \leqslant k-2$, one has $|x-y| \sim |x| \sim 2^k$. By using (7) and Hölder's inequality, we get

$$\begin{aligned}
\left|T_\alpha^A(f\chi_l)(x)\right| &\leqslant \int_{E_l}\frac{|R(A;x,y)||f(y)|}{|x-y|^{n-\alpha+1}}dy \\
&\leqslant \int_{E_l}\frac{1}{|x-y|^{n-\alpha}}\left(\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}+\left|\nabla A(y)-m_{B_k}(\nabla A)\right|\right)|f(y)|dy \\
&\lesssim 2^{-k(n-\alpha)}\left\|\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}+\left|\nabla A(y)-m_{B_k}(\nabla A)\right|\chi_l\right\|_{\vec{q}} \\
&\quad\times\|f\chi_l\|_{\vec{q}_1}2^{l\left(n-\sum_{i=1}^n\frac{1}{q_i}-\sum_{i=1}^n\frac{1}{q_{1i}}\right)} \\
&\lesssim 2^{-k(n-\alpha)+l\sum_{i=1}^n\frac{1}{q_i}+l\left(n-\sum_{i=1}^n\frac{1}{q_i}-\sum_{i=1}^n\frac{1}{q_{1i}}\right)}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}.
\end{aligned}$$

As a consequence,

$$\left\|T_\alpha^A(f\chi_l)\chi_k\right\|_{\vec{q}_2} \leqslant 2^{(l-k)\left(n-\sum_{i=1}^n\frac{1}{q_{2i}}\right)}2^{l\sum_{i=1}^n\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}.$$

For K_2, let $\phi \in C_0^\infty(\mathbb{R}^n)$ and supp $\phi \subset B(0,4)$, $\phi(x) \equiv 1$ when $x \in B(0,2)$. Denote, $M = \|\nabla\phi\|_\infty$. Let A_k^ϕ be defined as in (8). Then

$$\begin{aligned}
\left|T_\alpha^A(f\chi_l)(x)\right| &= \int_{E_l}\frac{\left|R(A_k^\phi;x,y)\right|\left|f(y)\right|}{|x-y|^{n-\alpha+1}}dy \\
&\lesssim \int_{E_l}\frac{|A_k^\phi(x)-A_k^\phi(y)-\nabla A_k^\phi(y)(x-y)||f(y)|}{|x-y|^{n-\alpha+1}}dy \\
&\lesssim \int_{E_l}\frac{|A_k^\phi(x)-A_k^\phi(y)||f(y)|}{|x-y|^{n-\alpha+1}}dy+\int_{E_l}\frac{|\nabla A_k^\phi(y)||f(y)|}{|x-y|^{n-\alpha}}dy \\
&:= KI_1+KI_2.
\end{aligned}$$

Similar to the estimates of II_1 and II_2, using the (9) and (10), we have

$$\left|T_\alpha^A(f\chi_l)(x)\right| \lesssim 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_{2i}}\right)-k(n-\alpha)+l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1},$$

which further implies

$$\begin{aligned}\left\|T_\alpha^A(f\chi_l)\chi_k\right\|_{\vec{q}_2} &\lesssim 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_{2i}}\right)-k(n-\alpha)+l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}2^{k\sum_{i=1}^{n}\frac{1}{q_{2i}}}\\ &\lesssim 2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_{2i}}\right)}2^{l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}.\end{aligned}$$

For K_3, similar to the estimate of K_1, the following results can be obtained:

$$\left|T_\alpha^A(fx_l)\right| \lesssim 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_i}-\sum_{i=1}^{n}\frac{1}{q_{1i}}\right)-l(n-\alpha)+l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}.$$

Then

$$\left\|T_\alpha^A(f\chi_l)\chi_k\right\|_{\vec{q}_2} \lesssim 2^{(k-l)\sum_{i=1}^{n}\frac{1}{q_{2i}}}2^{l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}.$$

Taking the corresponding norm for the above three parts, we arrive at

$$\begin{aligned}K_1 &\leqslant C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-2}\left\|T_\alpha^A(f\chi_l)\chi_k\right\|_{\vec{q}_2}\right)^p\right\}^{1/p}\\ &\lesssim \left\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\left(\sum_{l=-\infty}^{k-2}2^{(l-k)\left(n-\alpha-\sum_{i=1}^{n}\frac{1}{q_{2i}}\right)}2^{l\sum_{i=1}^{n}\frac{1}{q_i}}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\chi_l\|_{\vec{q}_1}\right)^p\right\}^{1/p}\\ &\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\left\{\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}\left(\sum_{k=l+2}^{\infty}2^{(l-k)\left(n-\alpha-\sum_{i=1}^{n}\frac{1}{q_{2i}}-\alpha_2\right)}\|f\chi_l\|_{\vec{q}_1}\right)^p\right\}^{1/p}.\end{aligned}$$

When $0<p\leqslant 1$,

$$\begin{aligned}K_1 &\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\left\{\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}\sum_{k=l+2}^{\infty}2^{(l-k)p\left(n-\alpha-\sum_{i=1}^{n}\frac{1}{q_{2i}}-\alpha_2\right)}\|f\chi_l\|_{\vec{q}_1}^p\right\}^{1/p}\\ &\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\left\{\sum_{l=-\infty}^{\infty}2^{l\alpha_1 p}\|f\chi_l\|_{\vec{q}_1}^p\right\}^{1/p} \lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}}.\end{aligned}$$

When $1<p<\infty$,

$$K_1 \lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \left\{ \sum_{l=-\infty}^{\infty} 2^{i\alpha_1 p} \left(\sum_{k=l+2}^{\infty} 2^{(l-k)\left(n-\alpha-\sum_{i=1}^{n}\frac{1}{q_{2i}}-\alpha_2\right)p/2} \|f\chi_l\|_{\vec{q}_1}^{p} \right) \right.$$
$$\left. \times \left(\sum_{k=l+2}^{\infty} 2^{(l-k)\left(n-\alpha-\sum_{i=1}^{n}\frac{1}{q_{2i}}-\alpha_2\right)p'/2} \|f\chi_l\|_{\vec{q}_1}^{p} \right)^{p/p'} \right\}^{1/p}$$
$$\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}}.$$

Using a similar estimate of K_1, one gets for $i=2,3$,

$$K_i \lesssim \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}}.$$

This implies the results of Theorem 3.3 □

THEOREM 3.4. *Let $[b,T_\alpha]$ be defined as in (1), where b in $\mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)$. If $0<p\leqslant\infty$, $1<\vec{q}_1,\vec{q}_2<\infty$, and $\frac{1}{q_{2i}}=\frac{1}{q_i}+\frac{1}{q_{1i}}-\alpha$, α_1 satisfies $-\sum_{i=1}^{n}\frac{1}{q_{1i}}<\alpha_1<n-\alpha-\sum_{i=1}^{n}\frac{1}{q_{1i}}$ and $\alpha_2=\alpha_1-\sum_{i=1}^{n}\frac{1}{q_i}$, then*

$$\|[b,T_\alpha]f\|_{\dot{K}_{\vec{q}_2}^{\alpha_2,p}} \leqslant C\|b\|_{\mathrm{CBMO}_{\vec{q}}} \|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}}.$$

Proof. This proof of Theorem 3.4 is similar to that of Theorem 3.3, and the details are omitted. □

REMARK 3.2. Then multilinear operator T_α^A with rough kernel on Herz spaces was considered by Tang [12], in which the boundedness of multilinear operators and Coifman commutators is obtained.

THEOREM 3.5. *Let $0<p<\infty$, $1<\vec{q}<\infty$ and the function A has derivatives of order one in $\mathrm{CBMO}_{\max\{\vec{s},\vec{u}\}}(\mathbb{R}^n)$. If $\vec{u}\geqslant\vec{q}'$, $\alpha<\sum_{i=1}^{n}\frac{1}{q_i'}$, then*

$$\|\mathscr{H}_A f\|_{\dot{K}_{\vec{q}_2}^{\alpha_2,p}} \leqslant C\|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec{s},\vec{u}\}}} \|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}},$$

and

$$\|\mathscr{H}_A^* f\|_{\dot{K}_{\vec{q}_2}^{\alpha_2,p}} \leqslant C\|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec{s},\vec{u}\}}} \|f\|_{\dot{K}_{\vec{q}_1}^{\alpha_1,p}},$$

where $\max\{\vec{s},\vec{u}\}=(\max\{s_1,u_1\},\max\{s_2,u_2\},\dots,\max\{s_n,u_n\})$.

Proof. We only prove the case $0<p<\infty$, $p=\infty$ just follows after slight modifications. Write

$$\mathscr{H}_A f(x) = \frac{1}{|x|^n}\int_{B(0,|x|)} \frac{f(y)R(A;x,y)}{|x-y|}dy$$
$$\lesssim \frac{1}{|x|^n}\int_{B(0,k)} \frac{f(y)R(A;x,y)}{|x-y|}dy$$

$$\begin{aligned}
&\lesssim \sum_{l=-\infty}^{k} \frac{1}{|x|^n} \int_{E_l} \frac{f(y)R(A;x,y)}{|x-y|} dy \\
&\lesssim \sum_{l=-\infty}^{k-2} \frac{1}{|x|^n} \int_{E_l} \frac{f(y)R(A;x,y)}{|x-y|} dy \\
&\quad + \sum_{l=k-1}^{k} \frac{1}{|x|^n} \int_{E_l} \frac{f(y)R(A;x,y)}{|x-y|} dy \\
&:= J_1 + J_2.
\end{aligned}$$

For J_1, since $y \in E_l$, $x \in E_k$, $l \leqslant k-2$, then $|x-y| \sim |x| \sim 2^k$. Fixed k, let $A_k(y) = A(y) - m_{B_k}(\nabla A)y$. By (7) and Hölder's inequality on mixed-norm Lebesgue spaces, we have

$$\begin{aligned}
|J_1| &\lesssim \sum_{l=-\infty}^{k-2} \frac{1}{|x|^n} \int_{E_l} \frac{|f(y)||R(A;x,y)|}{|x-y|} dy \\
&\lesssim \sum_{l=-\infty}^{k-2} \frac{1}{|x|^n} \int_{E_l} |f(y)| \left(\|\nabla A\|_{\mathrm{CBMO}_{\vec{s}}} + \left|\nabla A(y) - m_{B_k}(\nabla A)\right| \right) dy \\
&\lesssim \sum_{l=-\infty}^{k-2} \frac{1}{|x|^n} \|f_l\|_{\vec{q}}\, 2^{l\sum_{i=1}^{n}\frac{1}{u_i}} \left(\|\nabla A\|_{\mathrm{CBMO}_{\vec{s}}} + \|\nabla A\|_{\mathrm{CBMO}_{\vec{u}}} \right) 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_i}-\sum_{i=1}^{n}\frac{1}{u_i}\right)} \\
&\lesssim \sum_{l=-\infty}^{k-2} \frac{1}{|x|^n} (k-l) \|f_l\|_{\vec{q}}\, 2^{l\sum_{i=1}^{n}\frac{1}{u_i}} \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec{s},\vec{u}\}}} 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_i}-\sum_{i=1}^{n}\frac{1}{u_i}\right)}.
\end{aligned}$$

Taking the mixed-norm for J_1, one can get

$$\begin{aligned}
&\left\| \sum_{l=-\infty}^{k-2} \frac{1}{|x|^n} \int_{E_l} \frac{f(y)R(A;x,y)}{|x-y|} dy \right\|_{\vec{q}} \\
&\lesssim \sum_{l=-\infty}^{k-2} \frac{1}{|x|^n} (k-l) \|f_l\|_{\vec{q}} \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec{s},\vec{u}\}}} 2^{l\left(n-\sum_{i=1}^{n}\frac{1}{q_i}-\sum_{i=1}^{n}\frac{1}{u_i}\right)} 2^{-kn} 2^{k\sum_{i=1}^{n}\frac{1}{q_i}} 2^{l\sum_{i=1}^{n}\frac{1}{u_i}} \\
&\lesssim \sum_{l=-\infty}^{k-2} 2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_i}\right)} \|f_l\|_{\vec{q}} \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec{s},\vec{u}\}}}.
\end{aligned}$$

In the following, we will estimate J_2. By a simple calculation,

$$\begin{aligned}
|J_2| &\leqslant \left| \sum_{l=k-1}^{k} \frac{1}{|x|^n} \int_{E_l} \frac{f(y)R(A;x,y)}{|x-y|} dy \right| \\
&\leqslant \left| \sum_{l=k-1}^{k} \frac{1}{|x|^n} \int_{E_l} \frac{f(y)R\left(A_k^{\phi};x,y\right)}{|x-y|} dy \right| \\
&\leqslant \sum_{l=k-1}^{k} \frac{1}{|x|^n} \int_{E_l} \frac{|f(y)|\left|R\left(A_k^{\phi};x,y\right)\right|}{|x-y|} dy
\end{aligned}$$

$$\begin{aligned}&\lesssim \sum_{l=k-1}^{k} \frac{1}{|x|^n} \int_{E_l} \frac{|f(y)| \left|A_k^{\phi}(x)-A_k^{\phi}(y)\right|}{|x-y|} dy \\ &\quad + \sum_{l=k-1}^{k} \frac{1}{|x|^n} \int_{E_l} |f(y)| \left|\nabla A_k^{\phi}(y)\right| dy \\ &\lesssim J_{21}+J_{22}.\end{aligned}$$

For $x \in E_k$, $y \in E_l$ and $k-1 \leqslant l \leqslant k$, by (9), there holds

$$\left|\nabla A_k^{\phi}(y)\right| \leqslant M\left(|\nabla A_k(y)|+\|\nabla A\|_{\mathrm{CBMO}_{\vec{s}}}\right).$$

As a consequence,

$$\begin{aligned}J_{22} &\lesssim \sum_{l=k-1}^{k} \frac{1}{|x|^n} \int_{E_l} |f(y)| M\left(|\nabla A_k(y)|+\|\nabla A\|_{\mathrm{CBMO}_{\vec{s}}}\right) dy \\ &\lesssim M \sum_{l=k-1}^{k} \frac{1}{|x|^n} \|f_l\|_{\vec{q}} 2^{l \sum_{i=1}^{n} \frac{1}{u_i}} \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec{s},\vec{u}\}}} 2^{l\left(n-\sum_{i=1}^{n} \frac{1}{q_i}-\sum_{i=1}^{n} \frac{1}{u_i}\right)} \\ &\lesssim M \sum_{l=k-1}^{k} 2^{l\left(n-\sum_{i=1}^{n} \frac{1}{q_i}\right)} \frac{1}{|x|^n} \|f_l\|_{\vec{q}} \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec{s},\vec{u}\}}}.\end{aligned}$$

Furthermore, taking the mixed-norm for J_{22}, we obtain

$$\|J_{22}\chi_k\|_{\vec{q}} \lesssim M \sum_{l=k-1}^{k} 2^{(l-k)\left(n-\sum_{i=1}^{n} \frac{1}{q_i}\right)} \|f_l\|_{\vec{q}} \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec{s},\vec{u}\}}}.$$

We also know that via (10),

$$\left|A_k^{\phi}(x)-A_k^{\phi}(y)\right| \leqslant M|k-l||x-y|\|\nabla A\|_{\mathrm{CBMO}_{\vec{s}}}.$$

Thus, by Hölder's inequality, we have

$$\begin{aligned}J_{21} &= \sum_{l=k-1}^{k} \frac{1}{|x|^n} \int_{E_l} \frac{|f(y)| \left|A_k^{\phi}(x)-A_k^{\phi}(y)\right|}{|x-y|} dy \\ &\lesssim \sum_{l=k-1}^{k} \frac{1}{|x|^n} \int_{E_l} |f(y)| M|k-l| \|\nabla A\|_{\mathrm{CBMO}_{\vec{s}}} dy \\ &\lesssim M \sum_{l=k-1}^{k} \frac{1}{|x|^n} \|f_l\|_{\vec{q}} \|\nabla A\|_{\mathrm{CBMO}_{\vec{s}}} 2^{l\left(n-\sum_{i=1}^{n} \frac{1}{q_i}\right)}.\end{aligned}$$

Therefore, taking the corresponding mixed-norm, there holds

$$\begin{aligned}\|J_{21}\|_{\vec{q}} &\lesssim M \sum_{l=k-1}^{k} 2^{l\left(n-\sum_{i=1}^{n} \frac{1}{q_i}\right)} \|f_l\|_{\vec{q}} \|\nabla A\|_{\mathrm{CBMO}_{\vec{s}}} 2^{-kn} 2^{k \sum_{i=1}^{n} \frac{1}{q_i}} \\ &\lesssim M \sum_{l=k-1}^{k} 2^{(l-k)\left(n-\sum_{i=1}^{n} \frac{1}{q_i}\right)} \|f_l\|_{\vec{q}} \|\nabla A\|_{\mathrm{CBMO}_{\vec{s}}}.\end{aligned}$$

The definition of mixed Herz spaces assures

$$
\begin{aligned}
\|\mathscr{H}_A f\|_{\dot K^{\alpha,p}_{\vec q}} &= \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|\mathscr{H}_A f\|^p_{\vec q}\right)^{\frac{1}{p}}\\
&\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec s,\vec u\}}}\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_i}\right)}\|f_l\|_{\vec q}\right)^p\right\}^{1/p}\\
&\quad + M\|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec s,\vec u\}}}\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{l=k-1}^{k} 2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_i}\right)}\|f_l\|_{\vec q}\right)^p\right\}^{1/p}\\
&\quad + M\|\nabla A\|_{\mathrm{CBMO}_{\vec s}}\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{l=k-1}^{k} 2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_i}\right)}\|f_l\|_{\vec q}\right)^p\right\}^{1/p}\\
&\lesssim J_1'+J_2'+J_3'.
\end{aligned}
$$

When $0<p\leqslant 1$,

$$
\begin{aligned}
J_1' &\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec s,\vec u\}}}\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\sum_{l=-\infty}^{k-2} 2^{(l-k)p\left(n-\sum_{i=1}^{n}\frac{1}{q_i}\right)}\|f_l\|^p_{\vec q}\right\}^{1/p}\\
&\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec s,\vec u\}}}\left\{\sum_{l=-\infty}^{\infty} 2^{l\alpha p}\sum_{k=l+2}^{\infty} 2^{(l-k)p\left(\sum_{i=1}^{n}\frac{1}{q_i}-\alpha\right)}\|f_l\|^p_{\vec q}\right\}^{1/p}\\
&\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec s,\vec u\}}}\|f\|_{\dot K^{\alpha,p}_{\vec q}}.
\end{aligned}
$$

For J_2', the following estimate is valid:

$$
\begin{aligned}
J_2' &\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec s,\vec u\}}}\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\sum_{l=k-1}^{k} 2^{(l-k)p\left(n-\sum_{i=1}^{n}\frac{1}{q_i}\right)}\|f_l\|^p_{\vec q}\right\}^{1/p}\\
&\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec s,\vec u\}}}\|f\|_{\dot K^{\alpha,p}_{\vec q}}.
\end{aligned}
$$

Similar to J_1', we have

$$J_3' \lesssim M\|\nabla A\|_{\mathrm{CBMO}_{\max\{\vec s,\vec u\}}}\|f\|_{\dot K^{\alpha,p}_{\vec q}}.$$

When $1<p<\infty$,

$$
\begin{aligned}
J_1' &\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\max(\vec s,\vec u)}}\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_i}\right)}\|f_l\|_{\vec q}\right)^p\right\}^{\frac{1}{p}}\\
&\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\max(\vec s,\vec u)}}\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_i}\right)p/2}\|f_l\|^p_{\vec q}\right)\right.
\end{aligned}
$$

$$\times \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\left(n-\sum\limits_{i=1}^{n}\frac{1}{q_i}\right)p'/2} \right)^{p/p'} \Bigg\}^{1/p}$$

$$\lesssim \|\nabla A\|_{\mathrm{CBMO}_{\max}(\vec{s},\vec{u})} \|f\|_{\dot{K}_{\vec{q}}^{\alpha,p}}.$$

Likewise, for all $i = 2,3$, we obtain

$$J_i' \lesssim \|\nabla A\|_{\mathrm{CBMO}_{\max}(\vec{s},\vec{u})} \|f\|_{\dot{K}_{\vec{q}}^{\alpha,p}}.$$

The proof of Theorem 3.5 is completed. □

3.2. The extreme case

Just as proved in the previous section, Theorem 3.1 is true when α_1 is restricted to $-\sum_{i=1}^{n}\frac{1}{q_{1i}} < \alpha_1 < n - \sum_{i=1}^{n}\frac{1}{q_{1i}}$. But, this is not the case at the endpoints. To establish the endpoint estimates, we resort to mixed Herz-Hardy spaces. In what follows, $\mathscr{S}'(\mathbb{R}^n)$ denotes tempered distribution spaces.

DEFINITION 3.1. ([8]) Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $1 < \vec{q} < \infty$, and $N > N_{\vec{q}} = \left[n\left(1+\frac{1}{q_-}\right)+n+2\right]+1$, where q_- denote $\min\{q_1, q_2, \dots, q_n\}$. The mixed homogeneous Herz-type Hardy space $H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{H\dot{K}_{\vec{q}}^{\alpha,p}} = \|\mathscr{M}_N f\|_{\dot{K}_{\vec{q}}^{\alpha,p}} < \infty \right\},$$

where $\mathscr{M}_N f$ denotes the grand maximal operators of function f.

Corresponding to the classical case, mixed Herz-Hardy spaces also have atom decomposition. Now we will recall the definition of central atoms.

DEFINITION 3.2. ([8]) Let $1 < \vec{q} < \infty$, $n - \sum_{i=1}^{n} 1/q_i \leqslant \alpha < \infty$, and non-negative integer $s \geqslant [\alpha - n + \sum_{i=1}^{n} 1/q_i]$.

A function a on $\mathbb{R}^n$ is said to be a central $(\alpha,\vec{q})$-atom, if it satisfies

(i) $\operatorname{supp} a \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$.

(ii) $\|a\|_{L^{\vec{q}}} \leqslant |B(0,r)|^{-\alpha/n}$.

(ii) $\int_{\mathbb{R}^n} a(x) x^{\beta} dx = 0, |\beta| \leqslant s$.

LEMMA 3.1. ([8]) *Let $1 < \vec{q} < \infty$, $0 < p < \infty$ and $n - \sum_{i=1}^{n} 1/q_i \leqslant \alpha < \infty$. Then $f \in H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ if and only if $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$, in the sense of distribute, where each a_k is a central $(\alpha,\vec{q})$-atom with support contained in B_k and $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$. Moreover,*

$$\|f\|_{H\dot{K}_{\vec{q}}^{\alpha,p}} \approx \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

THEOREM 3.6. *Suppose that* $b \in \mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)$, $1<\vec{q}<\infty$. *If* $0<p<\infty$, $1<\vec{q}_1,\vec{q}_2<\infty$ *and* $1/q_{2i}=1/q_i+1/q_{1i}$, *then the following statements are equivalent:*

(i) $[b,T]$ *maps* $\dot{K}_{\vec{q}_1}^{-\sum_{i=1}^n 1/q_{1i},\infty}(\mathbb{R}^n)$ *continuously into* $\mathrm{CBMO}_{\vec{q}_2}(\mathbb{R}^n)$.

(ii) $[b,T]$ *maps* $H\dot{K}_{\vec{q}_1}^{n-\sum_{i=1}^n 1/q_{1i},p}(\mathbb{R}^n)$ *continuously into* $\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}(\mathbb{R}^n)$.

(iii) b *is a constant. That is,* $[b,T]\equiv 0$.

Proof. From dual theory of Herz and Herz-Hardy spaces [8], $\dot{K}_{\vec{q}_1}^{-\sum_{i=1}^n 1/q_{1i},\infty}(\mathbb{R}^n)$ is the dual space of $\dot{K}_{\vec{q}_1'}^{n-\sum_{i=1}^n 1/q_{1i}',1}(\mathbb{R}^n)$ and $\mathrm{CBMO}_{\vec{q}_2}(\mathbb{R}^n)$ is the dual space of $H\dot{K}_{\vec{q}_2'}^{n-\sum_{i=1}^n 1/q_{2i}',1}(\mathbb{R}^n)$, which imply that (i) can conclude (ii). And, (iii) implies (i) via a simple computation. So, we just need to show (ii) $\Rightarrow$ (iii).

To this end, by Lemma 3.1, we only need to consider the behavior of $[b,T]$ acting on a central $(n-\sum_{i=1}^n 1/q_{1i},\vec{q}_1)$-atom. Let a be such an atom with support B_k. For $u\in B_k$, let

$$
\begin{aligned}
v_1(x) &= \chi_{B_{k+2}}(x)\,[b,T]\,a(x),\\
v_2(x,u) &= \chi_{(B_{k+2})^c}(x)\left[b(x)-m_{B_k}(b)\right]\int_{B_k}(K(x,y)-K(x,u))\,a(y)dy,\\
v_3(x,u) &= \chi_{(B_{k+2})^c}(x)\int_{B_k}(K(x,y)-K(x,u))\left[b(y)-m_{B_k}(b)\right]a(y)dy
\end{aligned}
$$

and

$$
v_4(x,u)=\chi_{(B_{k+2})^c}(x)K(x,u)\int_{B_k}b(y)a(y)dy.
$$

By the vanishing condition of a, it is easy to check that

$$
[b,T]\,a(x)=v_1(x)+v_2(x,u)-v_3(x,u)-v_4(x,u).
$$

It follows from the $\left(L^{\vec{q}},L^{\vec{q}}\right)$ boundedness of the commutator $[b,T]$ that

$$
\begin{aligned}
\|v_1\|_{\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}} &\leqslant \left(\sum_{j=-\infty}^{k+2} 2^{j\left(n-\sum\limits_{i=1}^n \frac{1}{q_{2i}}\right)p}\left\|\chi_j\left([b,T]\,a\right)\right\|_{\vec{q}_2}^p\right)^{1/p}\\
&\leqslant C\left(\sum_{j=-\infty}^{k+2} 2^{j\left(n-\sum\limits_{i=1}^n \frac{1}{q_{2i}}\right)p}\|a\|_{\vec{q}_1}^p\|\chi_k\|_{\vec{q}}^p\right)^{1/p}\\
&\leqslant C.
\end{aligned}
$$

By the size condition of central $(n-\sum_{i=1}^n 1/q_{1i},\vec{q}_1)$-atoms, the kernel difference condition of operator T^A and some standard calculation, we obtain

$$
\left\|v_2\chi_j\right\|_{\vec{q}_2}^p \leqslant \|b\|_{\mathrm{CBMO}_{\vec{q}}}(j-k)2^{j\sum\limits_{i=1}^n \frac{1}{q_i}}2^{(k-j)}2^{-jn}2^{j\sum\limits_{i=1}^n \frac{1}{q_{1i}}}.
$$

Furthermore, the following result can be got,

$$\begin{aligned}\|v_2\|_{\dot{K}_{\vec{q}_2}^{n-\Sigma_{i=1}^n 1/q_{2i},p}} &\leqslant C\left\{\sum_{j=k+3}^{\infty} 2^{j(n-\sum\limits_{i=1}^n \frac{1}{q_{2i}})p}\left\|v_2\chi_j\right\|_{\vec{q}_2}^p\right\}^{1/p}\\ &\leqslant C\left\{\sum_{j=k+3}^{\infty} 2^{p(k-j)}(j-k)^p\|b\|_{\mathrm{CBMO}_{\vec{q}}}^p\right\}^{1/p}\\ &\leqslant C\|b\|_{\mathrm{CBMO}_{\vec{q}}}.\end{aligned}$$

Likewise, for v_3,

$$\|v_3\|_{\dot{K}_{\vec{q}_2}^{n-\Sigma_{i=1}^n 1/q_{2i},p}} \leqslant C\|b\|_{\mathrm{CBMO}_{\vec{q}}}.$$

For v_4, assume that $L(b,a)=\int_{B_k} b(y)a(y)dy$. Then

$$\begin{aligned}\|v_4\|_{\dot{K}_{\vec{q}_2}^{n-\Sigma_{i=1}^n 1/q_{2i},p}}^p &\geqslant C\sum_{j=k+3}^{N} 2^{j(n-\sum\limits_{i=1}^n \frac{1}{q_{2i}})p}2^{-j(n-\sum\limits_{i=1}^n \frac{1}{q_{2i}})p}|L(b,a)|^p\\ &= C\sum_{j=k+3}^{N}|L(b,a)|^p.\end{aligned}$$

Noting that $\|a\|_{H\dot{K}_{\vec{q}_2}^{n-\Sigma_{i=1}^n 1/q_{2i},p}} \leqslant 1$ and $\|v_1\|_{\dot{K}_{\vec{q}_2}^{n-\Sigma_{i=1}^n 1/q_{2i},p}} \leqslant C$, as a result of (ii), we have

$$\|v_4\|_{\dot{K}_{\vec{q}_2}^{n-\Sigma_{i=1}^n 1/q_{2i},p}} \leqslant C.$$

Consequently, when $N\to\infty$, then, $L(b,a)=0$, which implies b is a constant. □

Compared with the commutators, the multilinear singular integrals have better properties when $\alpha_1=-\sum_{i=1}^n 1/q_{1i}$. But when $\alpha_1=n-\sum_{i=1}^n 1/q_{1i}$, they have a similar property. The precise results are contained in the following two theorems.

THEOREM 3.7. *Suppose that function A has derivatives of order one in* $\mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)$, $n<\vec{q}<\infty$. *If* $1<\vec{q}_1$, $\vec{q}_2<\infty$ *and* $1/q_{2i}=1/q_i+1/q_{1i}$, *then* T^A *maps* $\dot{K}_{\vec{q}_2}^{-\Sigma_{i=1}^n 1/q_{2i},\infty}(\mathbb{R}^n)$ *continuously into* $\mathrm{CBMO}_{\vec{q}_2}(\mathbb{R}^n)$.

Proof. By Remark 2.2, this proof only needs to show that exists c_k such that

$$\frac{\left\|\left(T^A f-c_k\right)\chi_{B_k}\right\|_{\vec{q}_2}}{\|\chi_{B_k}\|_{\vec{q}_2}} \leqslant C\|f\|_{\dot{K}_{\vec{q}_1}^{-\Sigma_{i=1}^n 1/q_{1i},\infty}}.$$

We write $f_1=f\chi_{B_{k+4}}$ and $f_2=f-f_1$, taking $y_0\in E_{k+2}$, and let $c_k=T^A f_2(y_0)$, then

$$\begin{aligned}\frac{\left\|\left(T^A f-c_k\right)\chi_{B_k}\right\|_{\vec{q}_2}}{\|\chi_{B_k}\|_{\vec{q}_2}} &\leqslant \frac{\left\|\left(T^A f_1\right)\chi_{B_k}\right\|_{\vec{q}_2}}{\|\chi_{B_k}\|_{\vec{q}_2}}+\frac{\left\|\left(T^A f_2-T^A f_2(y_0)\right)\chi_{B_k}\right\|_{\vec{q}_2}}{\|\chi_{B_k}\|_{\vec{q}_2}}\\ &:= I_1+II_2.\end{aligned}$$

By the boundedness of T^A on mixed Lebesgue spaces, we have

$$I_1 \leqslant \frac{\left\|\left(T^A f_1\right)\chi_{B_k}\right\|_{\vec{q}_2}}{\|\chi_{B_k}\|_{\vec{q}_2}} \leqslant \|f\|_{\vec{q}_2}\|\chi_{B_k}\|_{\vec{q}_2}^{-1} \leqslant \|f\|_{\vec{q}_1}\|\chi_{B_{k+4}}\|_{\vec{q}}\|\chi_{B_k}\|_{\vec{q}_2}^{-1} \leqslant \|f\|_{\dot{K}_{\vec{q}_1}^{-\sum_{i=1}^n 1/q_{1i},\infty}}.$$

Let $A_k(y)=A(y)-m_{B_k}(\nabla A)y$, can easily check that $R_m(A;x,y)=R_m(A_k;x,z)$. By using inequality (7) and Hölder's inequality on mixed Lebesgue spaces, there holds

$$\begin{aligned}
\left|T^A f_2(y)-T^A f_2(y_0)\right| &\leqslant \left|\int_{\mathbb{R}^n}\left(\frac{R_m(A_k;y,z)}{|y-z|^{n+1}}+\frac{R_m(A_k;y_0,z)}{|y_0-z|^{n+1}}\right)f(z)dz\right| \\
&\leqslant \sum_{j=k+5}^{\infty}\int_{E_j}\left(\frac{|R_m(A_k;y,z)|}{|y-z|^{n+1}}+\frac{|R_m(A_k;y_0,z)|}{|y_0-z|^{n+1}}\right)|f(z)|dz \\
&\leqslant C\sum_{j=k+5}^{\infty}\int_{E_j}\frac{\left(\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}+|\nabla A(y)-m_{B_k}(\nabla A)|\right)}{|y-z|^{n}}|f(z)|dz \\
&\leqslant \sum_{j=k+5}^{\infty}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}2^{j\sum_{j=1}^{n}\frac{1}{q_i}}\|f\chi_j\|_{\vec{q}_2}2^{-jn}2^{j(n-\sum_{i=1}^{n}\frac{1}{q_i}-\sum_{i=1}^{n}\frac{1}{q_{2i}})} \\
&\leqslant \sum_{j=k+5}^{\infty}\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}2^{j\sum_{j=1}^{n}\frac{1}{q_i}}\|f\chi_j\|_{\vec{q}_1}2^{j\sum_{i=1}^{n}\frac{1}{q_i}}2^{-jn}2^{j(n-\sum_{i=1}^{n}\frac{1}{q_i}-\sum_{i=1}^{n}\frac{1}{q_{2i}})} \\
&\leqslant C\|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\|_{\dot{K}_{\vec{q}_1}^{-\sum_{i=1}^n\frac{1}{q_{1i}},\infty}}.
\end{aligned}$$

Hence,

$$I_2 \leqslant \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}\|f\|_{\dot{K}_{\vec{q}_1}^{-\sum_{i=1}^n 1/q_{1i},\infty}}.$$

Combining the estimates I_2 and I_2, the proof is completed. □

THEOREM 3.8. *Suppose that function A has derivatives of order one in $\mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)$, $n<\vec{q}<\infty$. If $0<p<\infty$, $1<q_1$, $q_2<\infty$ and $1/q_{2i}=1/q_i+1/q_{1i}$, then the following statements are equivalent:*

(i) *T^A maps $H\dot{K}_{\vec{q}_1}^{n-\sum_{i=1}^n 1/q_{1i},p}(\mathbb{R}^n)$ continuously into $\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}(\mathbb{R}^n)$.*

(ii) *A is a polynomial of degree no more than one. That is, $T^A\equiv 0$*

Proof. The proof is similar to that of Theorem 3.6, by using the atom decomposition of functions in Herz-Hardy spaces, we only need to consider the behavior of T^A acting on a central $(n-\sum_{i=1}^n 1/q_{1i},\vec{q}_1)$-atom. Let a be such an atom with support B_k. For $u\in E_{k+2}$, let

$$\mu_1(x)=\chi_{B_{k+4}}(x)T^A a(x),$$

$$\mu_2(x,u)=\chi_{(B_{k+4})^c}(x)\int_{B_k}\left(\frac{K(x-y)(A_k(x)-A_k(y))}{|x-y|}-\frac{K(x-u)(A_k(x)-A_k(u))}{|x-u|}\right)a(y)dy,$$

$$\mu_3(x,u) = \chi_{(B_{k+4})^c}(x) \int_{B_k} [K(x-y) - K(x-u)] \nabla A_k(y) a(y) dy$$

and

$$\mu_4(x,u) = \chi_{(B_{k+4})^c}(x) K(x-u) \int_{B_k} \nabla A_k(y) a(y) dy.$$

It is not difficult to check that

$$T^A a(x) = \mu_1(x) + \mu_2(x,u) - \mu_3(x,u) - \mu_4(x,u).$$

For μ_1, from the boundedness of operators T^A on mixed Lebesgue spaces, we have

$$\begin{aligned} \|\mu_1\|_{\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}} &\leqslant \left(\sum_{j=-\infty}^{k+2} 2^{j\left(n-\sum\limits_{i=1}^{n} \frac{1}{q_{2i}}\right)p} \left\|\chi_j \left(T^A a\right)\right\|_{\vec{q}_2}^p \right)^{1/p} \\ &\leqslant C \left(\sum_{j=-\infty}^{k+2} 2^{j\left(n-\sum\limits_{i=1}^{n} \frac{1}{q_{2i}}\right)p} \|a\|_{\vec{q}_1}^p \|\chi_k\|_{\vec{q}}^p \right)^{1/p} \\ &\leqslant C. \end{aligned}$$

For $x \in E_j$, $y \in B_k$ and $u \in B_k$, by Lemma 2.2, the size and difference condition of kernel of operators T^A yields that

$$\begin{aligned} \left| \frac{A_k(x) - A_k(y)}{|x-y|^{n+1}} - \frac{A_k(x) - A_k(u)}{|x-u|^{n+1}} \right| &\leqslant \frac{|K(x-y) - K(x-u)||A_k(x) - A_k(y)|}{|x-y|} \\ &\quad + \frac{|K(x-u)||A_k(y) - A_k(u)|}{|x-u|} \\ &\leqslant 2^{-j(n+1)} 2^k \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}. \end{aligned}$$

Thus, we have

$$\|\mu_2 \chi_j\|_{\vec{q}_2} \leqslant C 2^{-j(n+1) + j\sum\limits_{i=1}^{n} \frac{1}{q_{2i}} + k} \|a\|_{\vec{q}_2} \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}}.$$

Then, using similar estimates,

$$\|\mu_k\|_{\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}} \leqslant C \|\nabla A\|_{\mathrm{CBMO}_{\vec{q}}} \qquad (k = 2,3).$$

For ν_4, assume that $C_\gamma = \int_{B_k} \nabla A(y) a(y) dy$.

$$\begin{aligned} \|\mu_4\|_{\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}}^p &\geqslant C \sum_{j=k+3}^{N} 2^{j(n-\sum\limits_{i=1}^{n} \frac{1}{q_{2i}})p} 2^{-j(n-\sum\limits_{i=1}^{n} \frac{1}{q_{2i}})p} |C_\gamma|^p \\ &= C \sum_{j=k+3}^{N} |C_\gamma|^p. \end{aligned}$$

Noting that $\|a\|_{H\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}} \leqslant 1$ and $\|v_1\|_{\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}} \leqslant C$, as a result of (i), we have

$$\|v_4\|_{\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}} \leqslant C.$$

Then, when $N \to \infty$, we need $\int_{B_k} \nabla A(y)a(y)dy = 0$, thus, function A is a polynomial of degree not more than one. □

The above conclusions show that multilinear operators are bounded from mixed Herz spaces to $\mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)$ spaces. However, we will prove that some weak type estimates hold for these operators.

DEFINITION 3.3. Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < \vec{q} < \infty$. A measurable function f on $\mathbb{R}^n$ is said to belong to the homogeneous weak Herz spaces $W\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ if

$$\|f\|_{W\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|\chi_{E_\lambda}\|_{\vec{q}}^p \right\}^{1/p} < \infty,$$

where $E_\lambda = \{x \in E_k : \ |f(x)| > \lambda\}$.

THEOREM 3.9. *Let* $0 < p \leqslant 1$, $1 < \vec{q}_1$, $\vec{q}_2 < \infty$ *and* $1/q_{2i} = 1/q_i + 1/q_{1i}$.

(i) *Suppose that* $b \in \mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)$, $1 < \vec{q} < \infty$.

Then $[b,T]$ *maps* $H\dot{K}_{\vec{q}_1}^{n-\sum_{i=1}^n 1/q_{1i},p}(\mathbb{R}^n)$ *continuously into* $W\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}(\mathbb{R}^n)$;

(ii) *Suppose that* A *has derivatives of order one in* $\mathrm{CBMO}_{\vec{q}}(\mathbb{R}^n)$, $n < \vec{q} < \infty$. *Then* T^A *maps* $H\dot{K}_{\vec{q}_1}^{n-\sum_{i=1}^n 1/q_{1i},p}(\mathbb{R}^n)$ *continuously into* $W\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}(\mathbb{R}^n)$.

Proof. It is obvious that (ii) is just a direct corollary of (i). By the atom decomposition of Herz-Hardy spaces, we can write $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$ with each a_k a central $(n - \sum_{i=1}^n 1/q_{1i}, \vec{q}_1)$-atom supported on B_k and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. Write

$$E_{j\lambda} = \{x \in E_j : \ |[b,T]f(x)| > \lambda\}.$$

By Definition 3.2 and the inclusion relationship, there holds

$$\begin{aligned} E_{j\lambda} = &\left\{ x \in E_j : \left| \sum_{k=j-1}^{\infty} \lambda_k [b,T] a_k(x) \right| > \frac{\lambda}{3} \right\} \\ &\bigcup \left\{ x \in E_j : \left| \sum_{k=-\infty}^{j-2} \lambda_k \left(b(x) - b_{B_k}\right) T a_k(x) \right| > \frac{\lambda}{3} \right\} \\ &\bigcup \left\{ x \in E_j : \left| \sum_{k=-\infty}^{j-2} \lambda_k T\left(\left(b - b_{B_k}\right) a_k\right)(x) \right| > \frac{\lambda}{3} \right\} \\ = &E_{j\lambda}^1 + E_{j\lambda}^2 + E_{j\lambda}^3. \end{aligned}$$

Thus, by the above decomposition of the set $E_{j\lambda}$, we have

$$\begin{aligned}\|[b,T]f\|_{W\dot{K}_{\vec{q}_2}^{n-\sum_{i=1}^n 1/q_{2i},p}} &\leqslant \sup_{\lambda>0}\lambda\left[\sum_{j=-\infty}^{\infty}2^{j\left(n-\sum_{i=1}^n 1/q_{2i}\right)p}\left\|\chi_{E_{j\lambda}^1}\right\|_{\vec{q}_2}^p\right]^{1/p}\\ &\quad+C\sup_{\lambda>0}\lambda\left[\sum_{j=-\infty}^{\infty}2^{j\left(n-\sum_{i=1}^n 1/q_{2i}\right)p}\left\|\chi_{E_{j\lambda}^2}\right\|_{\vec{q}_2}^p\right]^{1/p}\\ &\quad+C\sup_{\lambda>0}\lambda\left[\sum_{j=-\infty}^{\infty}2^{j\left(n-\sum_{i=1}^n 1/q_{2i}\right)p}\left\|\chi_{E_{j\lambda}^3}\right\|_{\vec{q}_2}^p\right]^{1/p}\\ &:=G_1+G_2+G_3.\end{aligned}$$

From the boundedness of operators T on mixed Lebesgue space and Hölder's inequality on mixed Lebesgue spaces, and the size condition of the central $(n-\sum_{i=1}^n 1/q_{1i},\vec{q}_1)$-atoms, when $x\in E_j$, $y\in E_k$ and $k\geqslant j-1$, we obtain

$$|[b,T]a(x)|\lesssim 2^{-kn}\left(\left|b(x)-m_{B_k}(b)\right|+\|b\|_{\mathrm{CBMO}_{\vec{q}}}\right).$$

Hence,

$$\left\|[b,T]a(x)\chi_j\right\|_{\vec{q}_2}\lesssim 2^{-k(n-\sum_{i=1}^n \frac{1}{q_{1i}})}\|b\|_{\mathrm{CBMO}_{\vec{q}}}.$$

As a result,

$$\begin{aligned}G_1&\leqslant C\left[\sum_{j=-\infty}^{\infty}2^{j\left(n-\sum_{i=1}^n 1/q_{2i}\right)p}\left(\sum_{k=j-1}^{\infty}|\lambda_k|\left\|([b,T]a_k)\chi_j\right\|_{\vec{q}_2}\right)^p\right]^{1/p}\\ &\leqslant C\|b\|_{\mathrm{CBMO}_{\vec{q}}}\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p}.\end{aligned}$$

For G_2, we have

$$|Ta_k(x)|\leqslant\int_{B_k}\frac{|a_k(y)|}{|x-y|^n}dy\lesssim 2^{-jn},$$

which means

$$\begin{aligned}\left\|(b-m_{B_k}(b))Ta_k\,\chi_j\right\|_{\vec{q}_2}&\leqslant 2^{-jn+j\sum_{i=1}^n\frac{1}{q_{1i}}}\|b\|_{\mathrm{CBMO}_{\vec{q}}}\|\chi_j\|_{\vec{q}}\\ &\lesssim 2^{-j(n-\sum_{i=1}^n\frac{1}{q_{1i}})}\|b\|_{\mathrm{CBMO}_{\vec{q}}}\|\chi_j\|_{\vec{q}}.\end{aligned}$$

Thus

$$G_2\lesssim\|b\|_{\mathrm{CBMO}_{\vec{q}}}\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p}.$$

To estimate G_3, note that when $x \in E_j$,

$$\left|\sum_{k=-\infty}^{j-2} \lambda_k T\left(\left(b-b_{B_k}\right)a_k\right)(x)\right| \leqslant C2^{-jn}\|b\|_{\mathrm{CBMO}_{\vec{q}}} \sum_{k=-\infty}^{\infty} |\lambda_k|.$$

Take $j_0 \in \mathbb{Z}$ such that

$$2^{j_0 n} \leqslant 3C\lambda^{-1}\|b\|_{\mathrm{CBMO}_{\vec{q}}} \sum_{k=-\infty}^{\infty} |\lambda_k| < 2^{(j_0+1)n}.$$

Obviously, if $j \geqslant j_0+1$, the set

$$\left\{x \in E_j : \left|\sum_{k=-\infty}^{j-2} \lambda_k T\left(\left(b-b_{B_k}\right)a_k\right)(x)\right| > \frac{\lambda}{3}\right\}$$

is empty. Thus,

$$\begin{aligned} G_3 &\leqslant C\sup_{\lambda>0} \lambda \left(\sum_{j=-\infty}^{j_0} 2^{jn\left(1-\frac{1}{q_2}\right)p} \left|E_j\right|^{p/q_2}\right)^{1/p} \\ &\leqslant C\|b\|_{\mathrm{CBMO}_{\vec{q}}} \sum_{k=-\infty}^{\infty} |\lambda_k| \\ &\leqslant C\|b\|_{\mathrm{CBMO}_{\vec{q}}} \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p\right)^{1/p}. \end{aligned}$$

This proof is finished. □

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