$\mathsf{D}^2 = \mathsf{H} + \frac{1}{4}$ WITH POINT INTERACTIONS

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Abstract. Let D and H be the self-adjoint, one-dimensional Dirac and Schrödinger operators in $L^2(\mathbb{R};\mathbb{C}^2)$ and $L^2(\mathbb{R};\mathbb{C})$ respectively. It is well known that, in absence of an external potential, the two operators are related through the equality $\mathsf{D}^2 = (\mathsf{H} + \frac{1}{4})\mathbb{1}$. We show that such a kind of relation also holds in the case of n-point singular perturbations: given any self-adjoint realization $\widehat{\mathsf{D}}$ of the formal sum $\mathsf{D} + \sum_{k=1}^{n} \gamma_k \delta_{y_k}$, we explicitly determine the self-adjoint realization $\widehat{\mathsf{H}}$ of $\mathsf{H}\mathbb{1} + \sum_{k=1}^{n}(\alpha_k \delta_{y_k} + \beta_k \delta'_{y_k})$ such that $\widehat{\mathsf{D}}^2 = \widehat{\mathsf{H}} + \frac{1}{4}$. The found correspondence preserves the subclasses of self-adjoint realizations corresponding to both the local and the separating boundary conditions. Some connections with supersymmetry are provided. The case of nonlocal boundary conditions allows the study of the relation $\mathsf{D}^2 = \mathsf{H} + \frac{1}{4}$ for quantum graphs with (at most) two ends; in particular, the square of the extension corresponding to Kirchhoff-type boundary conditions for the Dirac operator on the graph gives the direct sum of two Schrödinger operators on the same graph, one with the usual Kirchhoff boundary conditions and the other with a sort of reversed Kirchhoff ones.

1. Introduction

Let $L^2(\mathbb{R};\mathbb{C}^d)$ be the Hilbert space of $\mathbb{C}^d$-valued square integrable functions with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$; likewise, $H^2(\mathbb{R};\mathbb{C}^d) \subset H^1(\mathbb{R};\mathbb{C}^d) \subset C_b(\mathbb{R};\mathbb{C}^d)$ denote the Sobolev space on $\mathbb{R}$ of order 1 and 2 and the space of bounded continuous functions with values in $\mathbb{C}^d$ respectively. Whenever $d = 1$, we simply write $L^2(\mathbb{R})$, $H^k(\mathbb{R})$ and $C_b(\mathbb{R})$. In $L^2(\mathbb{R};\mathbb{C}^2)$ we consider the free self-adjoint Dirac operator D defined by

$$\mathsf{D} : H^1(\mathbb{R};\mathbb{C}^2) \subseteq L^2(\mathbb{R};\mathbb{C}^2) \to L^2(\mathbb{R};\mathbb{C}^2), \quad \mathsf{D} := -i\frac{d}{dx}\sigma_1 + \frac{1}{2}\sigma_3,$$

where σ_1 and σ_3 are the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Furthermore, we consider the free self-adjoint Schrödinger operator in $L^2(\mathbb{R})$

$$\mathsf{H} : H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \mathsf{H} := -\frac{d^2}{dx^2}.$$

It is well known and easy to check that in this free case there exists a relation between the two operators:

$$D^2 = \left(H + \frac{1}{4}\right)\mathbb{1}. \tag{1.1}$$

Here and below, we use the isomorphism $L^2(\mathbb{R};\mathbb{C}^2) \simeq L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and the identification $L\mathbb{1} \equiv L \oplus L$, L a linear operator in $L^2(\mathbb{R})$. More generally, in the following we use the shorthand notation $L\mathbb{1} \equiv L \oplus L$ for a linear operator $L : \text{dom}(L) \subseteq H_1 \to H_2$.

Notice that (1.1) entails a relation between the resolvent operators:

$$(-D+z)^{-1} = (D+z)\left(-H+z^2-\frac{1}{4}\right)^{-1}\mathbb{1}, \qquad z \in \mathbb{C} \setminus ((-\infty,-1/2]\cup[1/2,+\infty)). \tag{1.2}$$

The aim of this paper is to extend this connection between Dirac's and Schrödinger's operators to the case where D is perturbed by a sum of δ's potentials, equivalently, given any self-adjoint extension $D_{\Pi,\Theta}$ of the symmetric operator $D|C^\infty_{comp}(\mathbb{R}\setminus\{y_1,\ldots,y_n\};\mathbb{C}^2)$, we explicitly determine the couple $(\widehat{\Pi},\widehat{\Theta})$ such that

$$(D_{\Pi,\Theta})^2 = \left(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}} + \frac{1}{4}\right). \tag{1.3}$$

Here, we parametrize the self-adjoint extensions of $D|C^\infty_{comp}(\mathbb{R}\setminus\{y_1,\ldots,y_n\};\mathbb{C}^2)$ by couples (Π,Θ), $\Pi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ an orthogonal projector, $\Theta : \text{ran}(\Pi) \to \text{ran}(\Pi)$ a symmetric operator, and likewise $\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}$ denotes the self-adjoint extension of $H\mathbb{1}|C^\infty_{comp}(\mathbb{R}\setminus\{y_1,\ldots,y_n\};\mathbb{C}^2)$ corresponding to the couple $(\widehat{\Pi},\widehat{\Theta})$, $\widehat{\Pi} : \mathbb{C}^{4n} \to \widehat{\mathbb{C}}^{4n}$ an orthogonal projector, $\widehat{\Theta} : \text{ran}(\widehat{\Pi}) \to \text{ran}(\widehat{\Pi})$ a symmetric operator. Any operator of the kind $\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}$ is a self-adjoint realization of a singular perturbation of $H\mathbb{1}$ by a sum of δ's and δ''s potentials. As in the free case, the relation (1.3) entails another one for the resolvents:

$$(-D_{\Pi,\Theta}+z)^{-1} = (D_{\Pi,\Theta}+z)\left(-\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}+z^2-\frac{1}{4}\right)^{-1},$$

where $\pm z \in \rho(D_{\Pi,\Theta})$ if and only if $(z^2-\frac{1}{4}) \in \rho(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}})$; here, $\rho(L)$ denotes the resolvent set of the closed operator L.

The specific case here considered is an example of solution of the problem concerning the representation of the square of a singular perturbation of a self-adjoint operator A by a singular perturbation of A^2. This problem has been studied in [2]; however, in such a paper only the case $A > 0$ has been considered and the explicit examples there presented are limited to rank-one singular perturbations. The methods here used are different from the ones in [2], we do not use the resolvent formulae directly but instead use the self-adjointness domains.

In more detail, the content of the paper is the following. In Section 2 we build the whole families of the self-adjoint extensions of $D|C^\infty_{comp}(\mathbb{R}\setminus\{y_1,\ldots,y_n\};\mathbb{C}^2)$ and $H\mathbb{1}|C^\infty_{comp}(\mathbb{R}\setminus\{y_1,\ldots,y_n\};\mathbb{C}^2)$. Instead of using the standard von Neumann theory (see,

e.g., [1], [9], [18]), which gives a parametrization in terms of unitary operators between the defect spaces, we found more convenient to use the equivalent approach proposed in [24] and [25], which gives a parametrization in terms of couples (Π, Θ), where Π is an orthogonal projection and Θ is a self-adjoint operator in $\operatorname{ran}(\Pi)$; this allows for an easy writing of the corresponding resolvents. Then, in Section 3, by a comparison of the self-adjointness domains, we found the correspondence between the couple (Π, Θ) and $(\widehat{\Pi}, \widehat{\Theta})$ such that (1.3) holds. In order to enhance the reader intuition, we start with simplest case, where $n = 1$ and $\Pi = \mathbb{1}$ and then proceed step-by-step towards the most general case. Finally, in Section 4, we present various applications. In Subsection 4.1 we consider the subclass of self-adjoint extensions for the Dirac operator corresponding to local boundary conditions, i.e., to the ones which do not couple different points y_k and show that the corresponding extensions for the Schrödinger operator provide local boundary conditions as well. As a particular case of such a result, in Subsection 4.2 we consider the Gesztesy-Šeba realizations; they are the self-adjoint realizations of the Dirac operator with local point interactions corresponding, in the non relativistic limit, to Schrödinger operators with local point interactions either of δ-type or of δ'-type (see [19], [1, Appendix J], [15]). Then, in Subsection 4.3, we consider the subclass of self-adjoint extensions for the Dirac operator corresponding to separating (a.k.a. decoupling) boundary conditions, i.e., to the local ones for which, at any point, left limits are independent from right limits. This entails that the corresponding Dirac operator is the direct sum of self-adjoint Dirac operators D_k in $L^2(I_k, \mathbb{C}^2)$, where the I_k's are either the half-lines $(-\infty, y_1)$ and $(y_n, +\infty)$ or the bounded intervals (y_k, y_{k+1}); the same is true for the corresponding Schrödinger operator and $(D_k)^2 = \widehat{H}_k + \frac{1}{4}$. In Subsection 4.4, some connections with supersymmetry are discussed and a simple criterion of spontaneous supersymmetry breaking is provided (see [26], [3] and references therein for somehow different aspects of supersymmetry in presence of point interactions). In Subsection 4.5, we point out that our results, in the case of non local boundary conditions, allow the study of the connection between the square of the Dirac operator and the Schrödinger operator on quantum graphs with (at most) two ends. In particular, as an explicit example, we consider the Dirac operator on the eye graph with Kirchhoff-type boundary conditions at the vertices and show that its square is the direct sum of two Schrödinger operators on the same graph, one with Kirchhoff boundary conditions and the other with a sort of inverse Kirchhoff ones. These latter boundary conditions, like the Kirchhoff ones, reduce, in the case of the real line, to the free boundary conditions; this is consistent with (1.1). The procedure used for the eye graph can be extended, without substantial changes, to any kind of graph, thus showing that the property of conservation of Kirchhoff-like boundary conditions holds in general.

We presume that the results here presented can be extended to the more involved cases corresponding to extensions of symmetric operators with infinite deficiency indices as the 1-dimensional Dirac and Schrödinger operators with singular perturbations on discrete sets (see [21] and [15]) and the n-dimensional ($n = 2, 3$) Dirac and Schrödinger operators with singular perturbations on 1-codimensional surfaces (see, e.g., [5], [6], [16] and [8], [22]).

2. D **and** H **with point interactions**

Given a finite set of points $Y = \{y_1, \cdots, y_n\}$, $y_1 < y_2, \ldots < y_n$, we define

$$H^1(\mathbb{R} \setminus Y; \mathbb{C}^d) := H^1(I_0; \mathbb{C}^d) \oplus \cdots \oplus H^1(I_n; \mathbb{C}^d), \tag{2.1}$$

where,

$$I_0 := (-\infty, y_1), \quad I_1 := (y_1, y_2), \quad \ldots\ldots \quad I_{n-1} := (y_{n-1}, y_n), \quad I_n := (y_n, +\infty), \tag{2.2}$$

and

$$H^1(I_j; \mathbb{C}^d) := \{f \in L^2(I_j; \mathbb{C}^d) : f' \in L^2(I_j; \mathbb{C}^d)\}, \quad j = 0, \ldots, n.$$

Here and below, f' denotes the (distributional) derivative of f. Notice that the left and right limits $f(y_k^\pm)$ exists and are finite for any $f \in H^1(\mathbb{R} \setminus Y; \mathbb{C}^d)$. We define

$$H^2(\mathbb{R} \setminus Y; \mathbb{C}^d) := H^2(I_0; \mathbb{C}^d) \oplus \cdots \oplus H^2(I_n; \mathbb{C}^d),$$

where

$$H^2(I_j; \mathbb{C}^d) := \{f \in H^1(I_j; \mathbb{C}^d) : f'' \in L^2(I_j; \mathbb{C}^d)\}, \quad j = 0, \ldots, n.$$

Obviously,

$$H^2(\mathbb{R} \setminus Y; \mathbb{C}^d) \subset H^1(\mathbb{R} \setminus Y; \mathbb{C}^d) \subset L^2(\mathbb{R}; \mathbb{C}^d)$$

and $f \in H^2(\mathbb{R} \setminus Y; \mathbb{C}^d)$ implies $f' \in H^1(\mathbb{R} \setminus Y; \mathbb{C}^d)$. We simply write $H^k(\mathbb{R} \setminus Y)$, $k = 1, 2$, whenever $d = 1$. Next, we introduce the two bounded operators

$$\tau : H^1(\mathbb{R} \setminus Y; \mathbb{C}^2) \to \mathbb{C}^{2n}, \quad \tau\Psi := (\tau_{y_1}\Psi, \ldots, \tau_{y_n}\Psi), \quad \tau_y\Psi := \langle\Psi\rangle_y, \tag{2.3}$$

and

$$\widehat{\tau} : H^2(\mathbb{R} \setminus Y) \to \mathbb{C}^{2n}, \quad \widehat{\tau}\psi := (\widehat{\tau}_{y_1}\psi, \ldots, \widehat{\tau}_{y_n}\psi), \quad \widehat{\tau}_y\psi := \langle\psi\rangle_y \oplus \langle\psi'\rangle_y, \tag{2.4}$$

where

$$\langle f \rangle_y := \frac{1}{2}\left(f(y^-) + f(y^+)\right).$$

Clearly, $\langle f \rangle_{y_k} = f(y_k)$ whenever $f \in H^1(\mathbb{R}; \mathbb{C}^d) \subset C_b(\mathbb{R}; \mathbb{C}^d)$.

In this section, following the scheme proposed in [25] (for the equivalent approaches which use either von Neuman's theory or Boundary Triples theory, see, e.g., [18], [9] and [23, Sect. 4.1], [21], [15] respectively), we review the construction of the self-adjoint extensions of the closed symmetric operators

$$S := \mathsf{D}|\ker(\tau|H^1(\mathbb{R}; \mathbb{C}^2)), \quad \widehat{S} := \mathsf{H}|\ker(\widehat{\tau}|H^2(\mathbb{R})).$$

Both S and $\widehat{S}$ have defect indices $(2n, 2n)$; they are the closures of the symmetric operators

$$S^\circ := \mathsf{D}|C^\infty_{comp}(\mathbb{R} \setminus Y; \mathbb{C}^2), \quad \widehat{S}^\circ := \mathsf{H}|C^\infty_{comp}(\mathbb{R} \setminus Y).$$

Let $\widehat{g}_z(x-y)$ be the kernel of the free Schrödinger resolvent $(-\mathsf{H}+z)^{-1} = \left(\frac{d^2}{dx^2}+z\right)^{-1}$, with $z \in \rho(\mathsf{H}) = \mathbb{C} \setminus [0,+\infty)$, i.e.,

$$\widehat{g}_z(x) = \frac{e^{i\sqrt{z}|x|}}{2i\sqrt{z}}, \qquad \mathrm{Im}(\sqrt{z}) > 0. \tag{2.5}$$

By (1.2), setting $w_z := z^2 - \frac{1}{4}$, one then obtains the kernel $g_z(x-y)$ of the free Dirac resolvent $(-\mathsf{D}+z)^{-1}$, $z \in \rho(\mathsf{D}) = \mathbb{C} \setminus ((-\infty,-1/2] \cup [1/2,+\infty))$,

$$g_z(x) = (\mathsf{D}+z)\widehat{g}_{w_z}\mathbb{1} = \frac{e^{i\sqrt{w_z}|x|}}{2i} \begin{bmatrix} \zeta_z & \mathrm{sgn}(x) \\ \mathrm{sgn}(x) & \zeta_z^{-1} \end{bmatrix}, \tag{2.6}$$

where $\zeta_z := (\frac{1}{2}-z)/\sqrt{w_z}$ and $\mathrm{Im}(\sqrt{w_z}) > 0$. By such kernels, one gets that the bounded operators

$$G_z : \mathbb{C}^{2n} \to L^2(\mathbb{R};\mathbb{C}^2), \quad G_z := (\tau(-\mathsf{D}+\bar{z})^{-1})^*, \quad z \in \mathbb{C} \setminus ((-\infty,-1/2] \cup [1/2,+\infty)),$$

and

$$\widehat{G}_z : \mathbb{C}^{2n} \to L^2(\mathbb{R}), \quad \widehat{G}_z := (\widehat{\tau}(-\mathsf{H}+\bar{z})^{-1})^*, \quad z \in \mathbb{C} \setminus [0,+\infty),$$

represent as

$$[G_z\xi](x) = \sum_{k=1}^{n} g_z(y_k - x)\,\xi_k, \qquad \xi \equiv (\xi_1,\dots,\xi_n), \quad \xi_k \in \mathbb{C}^2.$$

and

$$[\widehat{G}_z\xi](x) = \sum_{k=1}^{n} (\widehat{g}_z(y_k-x)\,\xi_{k,1} + \widehat{g}_z'(y_k-x)\,\xi_{k,2}), \quad \xi \equiv ((\xi_{1,1},\xi_{1,2}),\dots,(\xi_{n,1},\xi_{n,2})).$$

Their adjoints

$$G_{\bar{z}}^* : L^2(\mathbb{R};\mathbb{C}^2) \to \mathbb{C}^{2n}, \qquad \widehat{G}_{\bar{z}}^* : L^2(\mathbb{R}) \to \mathbb{C}^{2n}$$

are given by

$$G_{\bar{z}}^*\Psi = ((G_{\bar{z}}^*\Psi)_1,\dots,(G_{\bar{z}}^*\Psi)_n), \qquad (G_{\bar{z}}^*\Psi)_k := \int_{\mathbb{R}} g_z(y_k - x)\Psi(x)\,dx$$

and

$$\widehat{G}_{\bar{z}}^*\psi = ((\widehat{G}_{\bar{z}}^*\Psi)_1,\dots,(\widehat{G}_{\bar{z}}^*\Psi)_n),$$

$$(\widehat{G}_{\bar{z}}^*\psi)_k := \left(\int_{\mathbb{R}} \widehat{g}_z(y_k - x)\psi(x)\,dx, \int_{\mathbb{R}} \widehat{g}_z'(y_k - x)\psi(x)\,dx \right).$$

Since

$$G_z\xi \in H^1(\mathbb{R}\setminus Y;\mathbb{C}^2) \quad \text{and} \quad \widehat{G}_z\xi \in H^2(\mathbb{R}\setminus Y),$$

both

$$\tau G_z : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \quad \text{and} \quad \widehat{\tau}\widehat{G}_z : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$$

are well defined and are represented by the two $n \times n$ block matrices with the 2×2 blocks

$$[\tau G_z]_{jk} = \frac{e^{i\sqrt{w_z}|y_k - y_j|}}{2i} \begin{bmatrix} \zeta_z & \mathrm{sgn}(y_k - y_j) \\ \mathrm{sgn}(y_k - y_j) & \zeta_z^{-1} \end{bmatrix}, \qquad (2.7)$$

$$[\widehat{\tau}\widehat{G}_z]_{jk} = \frac{e^{i\sqrt{z}|y_k - y_j|}}{2} \begin{bmatrix} (i\sqrt{z})^{-1} & \mathrm{sgn}(y_k - y_j) \\ -\mathrm{sgn}(y_k - y_j) & i\sqrt{z} \end{bmatrix}, \qquad (2.8)$$

where

$$\mathrm{sgn}(x) := \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ +1 & x > 0. \end{cases}$$

In the following, given an orthogonal projection $P : \mathbb{C}^d \to \mathbb{C}^d$, by a slight abuse of notation, we use the same symbol to denote both the surjection $P : \mathbb{C}^d \to \mathrm{ran}(P)$ and the injection $P : \mathrm{ran}(P) \to \mathbb{C}^d$.

THEOREM 2.1. *The sets of self-adjoint extensions of S and $\widehat{S}$ are both parametrized by couples (Π, Θ), where $\Pi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is an orthogonal projector and $\Theta : \mathrm{ran}(\Pi) \to \mathrm{ran}(\Pi)$ is symmetric. The extensions $D_{\Pi,\Theta}$ and $H_{\Pi,\Theta}$ have resolvents*

$$(-D_{\Pi,\Theta} + z)^{-1} = (-D + z)^{-1} + G_z \Pi (\Theta - \Pi \tau G_z \Pi)^{-1} \Pi G_{\bar{z}}^*, \qquad z \in \rho(D_{\Pi,\Theta}) \cap \rho(D)$$

$$(-H_{\Pi,\Theta} + z)^{-1} = (-H + z)^{-1} + \widehat{G}_z \Pi (\Theta - \Pi \widehat{\tau}\widehat{G}_z \Pi)^{-1} \Pi \widehat{G}_{\bar{z}}^*, \qquad z \in \rho(H_{\Pi,\Theta}) \cap \rho(H).$$

Moreover,

$$\mathrm{dom}(D_{\Pi,\Theta}) = \{\Psi \in L^2(\mathbb{R}; \mathbb{C}^2) : \Psi = \Psi_z + G_z \xi, \, \Psi_z \in H^1(\mathbb{R}; \mathbb{C}^2), \, \xi \in \mathrm{ran}(\Pi), \, \Pi \tau \Psi = \Theta \xi\}$$

$$(-D_{\Pi,\Theta} + z)\Psi = (-D + z)\Psi_z,$$

$$\mathrm{dom}(H_{\Pi,\Theta}) = \{\psi \in L^2(\mathbb{R}) : \psi = \psi_z + \widehat{G}_z \xi, \, \psi_z \in H^2(\mathbb{R}), \, \xi \in \mathrm{ran}(\Pi), \, \Pi \widehat{\tau}\psi = \Theta \xi\},$$

$$(-H_{\Pi,\Theta} + z)\psi = (-H + z)\psi_z;$$

such representations are z-independent and the decompositions of Ψ in $\mathrm{dom}(D_{\Pi,\Theta})$ and of ψ in $\mathrm{dom}(H_{\Pi,\Theta})$ are unique.

Proof. The statements regarding the resolvents and the actions of the extensions follow from [25, Theorem 2.1] with $\Gamma_{\Pi,\Theta}(z)$ there defined either as $\Gamma_{\Pi,\Theta}(z) := \Theta - \Pi \tau G_z \Pi$ or as $\Gamma_{\Pi,\Theta}(z) := \widehat{\Theta} - \Pi \widehat{\tau}\widehat{G}_z \Pi$.

As regards the operators domains, we give the proof only for $D_{\Pi,\Theta}$, since the one for $H_{\Pi,\Theta}$ is of the same kind. By the resolvent formula, one has

$$\mathrm{dom}(D_{\Pi,\Theta}) = \{\Psi \in L^2(\mathbb{R}; \mathbb{C}^2) : \Psi = \Psi_z + G_z \Pi (\Theta - \Pi \tau G_z \Pi)^{-1} \Pi \tau \Psi_z, \, \Psi_z \in H^1(\mathbb{R}; \mathbb{C}^2)\}.$$

Let us define $\xi_z := (\Theta - \Pi \tau G_z \Pi)^{-1} \Pi \tau \Psi_z \in \mathrm{ran}(\Pi)$; it is not difficult to check that ξ_z does not depend on z and so $\Psi = \Psi_z + G_z \xi$. Then

$$\Pi \tau \Psi - \Theta \xi = \Pi \tau \Psi_z + \Pi \tau G_z \xi - \Theta \xi = \Pi \tau \Psi_z - (\Theta - \Pi \tau G_z \Pi)\xi = 0. \quad \square$$

REMARK 2.2. Notice that the choice $\Pi = \mathbb{0}$ gives the self-adjoint extensions D and H. Therefore, in the following we always suppose $\Pi \neq \mathbb{0}$.

Since we want to extend the relation (1.1) to the case with point interactions, we also need to consider the self-adjoint extensions of $\widehat{S}^{\circ}\mathbb{1}$. There are no essential changes with respect to the case of $\mathbb{C}$-valued functions, the only relevant one being that the defect indices increase to $(4n, 4n)$. The result is of the same kind as in Theorem 2.1.

THEOREM 2.3. *The set of the self-adjoint extensions of $\widehat{S}\mathbb{1}$ is parametrized by couples $(\widehat{\Pi}, \widehat{\Theta})$, where $\widehat{\Pi} : \mathbb{C}^{4n} \to \mathbb{C}^{4n}$ is an orthogonal projector and $\widehat{\Theta} : \mathrm{ran}(\widehat{\Pi}) \to \mathrm{ran}(\widehat{\Pi})$ is symmetric. The extension $\widehat{\mathsf{H}}_{\widehat{\Pi}, \widehat{\Theta}}$ has resolvent*

$$(-\widehat{\mathsf{H}}_{\widehat{\Pi}, \widehat{\Theta}} + z)^{-1} = (-\mathsf{H} + z)^{-1} \mathbb{1} + (\widehat{G}_z \mathbb{1}) \widehat{\Pi} (\widehat{\Theta} - \widehat{\Pi}(\widehat{\tau}\widehat{G}_z \mathbb{1})\widehat{\Pi})^{-1} \widehat{\Pi}(\widehat{G}_{\bar{z}}^* \mathbb{1}),$$

$$z \in \rho(\widehat{\mathsf{H}}_{\widehat{\Pi}, \widehat{\Theta}}) \cap \rho(\mathsf{H}).$$

Moreover,

$$\mathrm{dom}(\widehat{\mathsf{H}}_{\widehat{\Pi}, \widehat{\Theta}})$$
$$= \{ \Psi \in L^2(\mathbb{R}; \mathbb{C}^2) : \Psi = \Psi_z + (\widehat{G}_z \mathbb{1})\widehat{\xi}, \ \Psi_z \in H^2(\mathbb{R}; \mathbb{C}^2), \ \widehat{\xi} \in \mathrm{ran}(\widehat{\Pi}), \ \widehat{\Pi}(\widehat{\tau}\mathbb{1})\Psi = \widehat{\Theta}\widehat{\xi} \},$$

$$(-\widehat{\mathsf{H}}_{\widehat{\Pi}, \widehat{\Theta}} + z)\Psi = (-\mathsf{H} + z)\mathbb{1}\Psi_z;$$

such representation is z-independent and the decomposition of Ψ in $\mathrm{dom}(\widehat{\mathsf{H}}_{\widehat{\Pi}, \widehat{\Theta}})$ is unique.

REMARK 2.4. By Theorems 2.1 and 2.3, if both $\widehat{\Pi}$ and $\widehat{\Theta}$ are block diagonal, i.e., $\widehat{\Pi} = \Pi_1 \oplus \Pi_2$ and $\widehat{\Theta} = \Theta_1 \oplus \Theta_1$, then

$$(-\widehat{\mathsf{H}}_{\Pi_1 \oplus \Pi_2, \Theta_1 \oplus \Pi_2} + z)^{-1} = (-\mathsf{H}_{\Pi_1, \Theta_1} + z)^{-1} \oplus (-\mathsf{H}_{\Pi_2, \Theta_2} + z)^{-1},$$

equivalently,

$$\widehat{\mathsf{H}}_{\Pi_1 \oplus \Pi_2, \Theta_1 \oplus \Pi_2} = \mathsf{H}_{\Pi_1, \Theta_1} \oplus \mathsf{H}_{\Pi_2, \Theta_2}.$$

In particular,

$$\widehat{\mathsf{H}}_{\Pi\mathbb{1}, \Theta\mathbb{1}} = \mathsf{H}_{\Pi, \Theta}\mathbb{1}.$$

REMARK 2.5. Since g_z is the fundamental solution of $-\mathsf{D} + z$, one has

$$(-\mathsf{D}_{\Pi, \Theta} + z)\Psi = (-\mathsf{D} + z)(\Psi - G_z \xi) = (-\mathsf{D} + z)\Psi - \sum_{k=1}^{n} \xi_k \delta_{y_k},$$

i.e.,

$$\mathsf{D}_{\Pi, \Theta}\Psi = \mathsf{D}\Psi + \sum_{k=1}^{n} \xi_k \delta_{y_k}, \quad \xi \equiv (\xi_1, \ldots, \xi_n),$$

where the action of D on $\Psi \in L^2(\mathbb{R};\mathbb{C}^2)$ is to be understood in distributional sense. Analogously,

$$\mathsf{H}_{\Pi,\Theta}\psi = \mathsf{H}\psi + \sum_{k=1}^{n}(\xi_{k,1}\delta_{y_k} + \xi_{k,2}\delta'_{y_k}), \quad \xi \equiv ((\xi_{1,1},\xi_{1,2}),\ldots,(\xi_{n,1},\xi_{n,2})),$$

$$\widehat{\mathsf{H}}_{\widehat{\Pi},\widehat{\Theta}}\Psi = \mathsf{H}\mathbb{1}\Psi + \sum_{k=1}^{n}(\widehat{\xi}_{k,1}\delta_{y_k} + \widehat{\xi}_{k,2}\delta'_{y_k}), \quad \widehat{\xi} \equiv ((\widehat{\xi}_{1,1},\widehat{\xi}_{1,2}),\ldots,(\widehat{\xi}_{n,1},\widehat{\xi}_{n,2})).$$

In the following, we use the abbreviated notations $\mathsf{D}_\Theta \equiv \mathsf{D}_{\mathbb{1},\Theta}$, $\mathsf{H}_\Theta \equiv \mathsf{H}_{\mathbb{1},\Theta}$, $\widehat{\mathsf{H}}_{\widehat{\Theta}} \equiv \widehat{\mathsf{H}}_{\mathbb{1},\widehat{\Theta}}$.

3. $\mathsf{D}^2 = \mathsf{H} + \frac{1}{4}$ with point interactions

We begin this section by providing an equivalent representation of the domains and actions of the self-adjoint operators we built in Section 2. In the next theorem and in the following,

$$\mathsf{D}_{\mathbb{R}\setminus Y} : \mathscr{D}'(\mathbb{R}\setminus Y;\mathbb{C}^2) \to \mathscr{D}'(\mathbb{R}\setminus Y;\mathbb{C}^2), \qquad \mathsf{H}_{\mathbb{R}\setminus Y} : \mathscr{D}'(\mathbb{R}\setminus Y) \to \mathscr{D}'(\mathbb{R}\setminus Y)$$

denote the free Dirac and Schrödinger operators in the space of distributions on $\mathbb{R}\setminus Y$; their restrictions to $H^1(\mathbb{R}\setminus Y;\mathbb{C}^2)$ and $H^2(\mathbb{R}\setminus Y)$ are $L^2(\mathbb{R};\mathbb{C}^2)$ and $L^2(\mathbb{R})$-valued respectively.

THEOREM 3.1. *Let* $\mathsf{D}_{\Pi,\Theta}$, $\mathsf{H}_{\Pi,\Theta}$ *and* $\widehat{\mathsf{H}}_{\widehat{\Pi},\widehat{\Theta}}$ *as in Section* 2. *Then*

$$\mathrm{dom}(\mathsf{D}_{\Pi,\Theta}) = \{\Psi \in H^1(\mathbb{R}\setminus Y;\mathbb{C}^2) : \rho\Psi \in \mathrm{ran}(\Pi), \Pi\tau\Psi = \Theta\rho\Psi\}, \quad \mathsf{D}_{\Pi,\Theta}\Psi = \mathsf{D}_{\mathbb{R}\setminus Y}\Psi,$$

$$\mathrm{dom}(\mathsf{H}_{\Pi,\Theta}) = \{\psi \in H^2(\mathbb{R}\setminus Y) : \widehat{\rho}\psi \in \mathrm{ran}(\Pi), \Pi\widehat{\tau}\psi = \Theta\widehat{\rho}\psi\}, \quad \mathsf{H}_{\Pi,\Theta}\psi = \mathsf{H}_{\mathbb{R}\setminus Y}\psi,$$

$$\mathrm{dom}(\widehat{\mathsf{H}}_{\widehat{\Pi},\widehat{\Theta}}) = \{\Psi \in H^2(\mathbb{R}\setminus Y;\mathbb{C}^2) : (\widehat{\rho}\mathbb{1})\Psi \in \mathrm{ran}(\widehat{\Pi}), \widehat{\Pi}(\widehat{\tau}\mathbb{1})\Psi = \widehat{\Theta}(\widehat{\rho}\mathbb{1})\Psi\},$$

$$\widehat{\mathsf{H}}_{\widehat{\Pi},\widehat{\Theta}}\Psi = (\mathsf{H}_{\mathbb{R}\setminus Y}\mathbb{1})\Psi,$$

where

$$\rho : H^1(\mathbb{R}\setminus Y;\mathbb{C}^2) \to \mathbb{C}^{2n}, \quad \rho\Psi := (\rho_{y_1}\Psi,\ldots,\rho_{y_n}\Psi), \qquad \rho_y\Psi := i\sigma_1[\Psi]_y,$$

$$\widehat{\rho} : H^2(\mathbb{R}\setminus Y) \to \mathbb{C}^{2n}, \qquad \widehat{\rho}\psi := (\widehat{\rho}_{y_1}\psi,\ldots,\widehat{\rho}_{y_n}\psi), \qquad \widehat{\rho}_y\psi := [\psi']_y \oplus [-\psi]_y,$$

$$[f]_y := f(y^+) - f(y^-).$$

Proof. Let $\Psi = \Psi_z + G_z\xi \in \mathrm{dom}(D_{\Pi,\Theta})$. One has $\Psi_z \in H^1(\mathbb{R};\mathbb{C}^2) \subset H^1(\mathbb{R}\setminus Y;\mathbb{C}^2)$ and $G_z\xi \in H^1(\mathbb{R}\setminus Y;\mathbb{C}^2)$; therefore, $\Psi \in H^1(\mathbb{R}\setminus Y;\mathbb{C}^2)$. By $[G_z\xi]_y = i\sigma_1\xi$, one gets $\rho G_z\xi = \xi$; furthermore, by $H^1(\mathbb{R};\mathbb{C}^2) \subset C_b(\mathbb{R};\mathbb{C}^2)$, one gets $\rho\Psi_z = 0$. Therefore,

$$\mathrm{dom}(D_{\Pi,\Theta}) \subseteq \mathscr{D} := \{\Psi \in H^1(\mathbb{R}\setminus Y;\mathbb{C}^2) : \rho\Psi \in \mathrm{ran}(\Pi), \ \Pi\tau\Psi = \Theta\rho\Psi\}.$$

By Remark 2.5, $D_{\Pi,\Theta}\Psi = D_{\mathbb{R}\setminus Y}\Psi$ for any $\Psi \in \mathrm{dom}(D_{\Pi,\Theta})$, i.e., $D_{\Pi,\Theta} \subset D_{\mathbb{R}\setminus Y}|\mathscr{D}$. Moreover, by integration by parts, $D_{\mathbb{R}\setminus Y}|\mathscr{D}$ is symmetric; hence, since $D_{\Pi,\Theta}$ is self-adjoint, one gets $D_{\Pi,\Theta} = D_{\mathbb{R}\setminus Y}|\mathscr{D}$.

The proofs for $H_{\Pi,\Theta}$ and $\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}$ are of the same kind, using the relation $\widehat{\rho}\widehat{G}_z\xi = \xi$. $\square$

REMARK 3.2. Notice that $\psi \in H^1(\mathbb{R}\setminus Y)$ belongs to $H^1(\mathbb{R})$ if and only if $[\psi]_{y_k} = 0$ for any k and consequently $\psi \in H^2(\mathbb{R}\setminus Y)$ belongs to $H^2(\mathbb{R})$ if and only if $[\psi]_{y_k} = [\psi']_{y_k} = 0$ for any k.

By Theorem 3.1 and by

$$(D_{\mathbb{R}\setminus Y})^2 = \left(H_{\mathbb{R}\setminus Y} + \frac{1}{4}\right)\mathbb{1},$$

given the couple (Π,Θ), one gets that the couple $(\widehat{\Pi},\widehat{\Theta})$ is such that

$$(D_{\Pi,\Theta})^2 = \widehat{H}_{\widehat{\Pi},\widehat{\Theta}} + \frac{1}{4}, \tag{3.1}$$

if and only if

$$\mathrm{dom}((D_{\Pi,\Theta})^2) = \mathrm{dom}(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}). \tag{3.2}$$

Therefore, exploiting the definitions of the operator domains in Theorem 3.1, there exists a couple $(\widehat{\Pi},\widehat{\Theta})$ for which (3.1) holds if and only if, given (Π,Θ), there exists $(\widehat{\Pi},\widehat{\Theta})$, $\widehat{\Pi}$ an orthogonal projector in $\mathbb{C}^{4n}$ and $\widehat{\Theta}$ symmetric in $\mathrm{ran}(\widehat{\Pi})$, such that

$$\begin{cases} \rho\Psi \oplus \rho D_{\mathbb{R}\setminus Y}\Psi \in \mathrm{ran}(\Pi \oplus \Pi) \\ (\Pi \oplus \Pi)\tau\Psi \oplus \tau D_{\mathbb{R}\setminus Y}\Psi = (\Theta \oplus \Theta)\rho\Psi \oplus \rho D_{\mathbb{R}\setminus Y}\Psi \end{cases} \iff \begin{cases} (\widehat{\rho}\mathbb{1})\Psi \in \mathrm{ran}(\widehat{\Pi}) \\ \widehat{\Pi}(\widehat{\tau}\mathbb{1})\Psi = \widehat{\Theta}(\widehat{\rho}\mathbb{1})\Psi. \end{cases} \tag{3.3}$$

3.1. Spectral correspondence

The relation (3.1) entails $\pm z \in \rho(D_{\Pi,\Theta})$ if and only if $z^2 - \frac{1}{4} \in \rho(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}})$, equivalently, $\pm\lambda \in \sigma(D_{\Pi,\Theta})$ if and only if $\lambda^2 - \frac{1}{4} \in \sigma(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}})$, and

$$(-D_{\Pi,\Theta} + z)^{-1} = (D_{\Pi,\Theta} + z)\left(-\widehat{H}_{\widehat{\Pi},\widehat{\Theta}} + \left(z^2 - \frac{1}{4}\right)\mathbb{1}\right)^{-1}. \tag{3.4}$$

Furthermore, since, by the invariance of the essential spectrum by finite-rank perturbations,

$$\sigma_{ess}(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}) = \sigma_{ess}(H\mathbb{1}) = [0,\infty), \qquad \sigma_{ess}(D_{\widehat{\Pi},\widehat{\Theta}}) = \sigma_{ess}(D) = \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, +\infty\right),$$

one gets

$$\lambda \in \sigma_{disc}(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}) \cap \left[-\frac{1}{4}, 0\right) \quad \Longleftrightarrow \quad \pm\left(\lambda + \frac{1}{4}\right)^{\frac{1}{2}} \in \sigma_{disc}(D_{\Pi,\Theta}).$$

By the resolvent formulae in Theorems 2.1 and 2.3,

$$\lambda \in \sigma_{disc}(D_{\Pi,\Theta}) \quad \Longleftrightarrow \quad \lambda \in (-1/2, 1/2) \quad \text{and} \quad \det(\Theta - \Pi\tau G_\lambda \Pi) = 0, \quad (3.5)$$

$$\lambda \in \sigma_{disc}(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}) \quad \Longleftrightarrow \quad \lambda \in (-\infty, 0) \quad \text{and} \quad \det(\widehat{\Theta} - \widehat{\Pi}(\widehat{\tau}\widehat{G}_\lambda \mathbb{1})\widehat{\Pi}) = 0. \quad (3.6)$$

Now, we solve (3.3) starting from the simplest case $n = 1$, $\Pi = \mathbb{1}$ and then proceeding step-by-step towards the most general case.

3.2. The case $n = 1$, $\Pi = \mathbb{1}$

By (3.3), given the 2×2 Hermitian matrix Θ, we need to find the 4×4 Hermitian matrix $\widehat{\Theta}$ such that

$$\begin{bmatrix} \tau_y \Psi \\ \tau_y D_{\mathbb{R}\setminus\{y\}} \Psi \end{bmatrix} = \begin{bmatrix} \Theta & \mathbb{0} \\ \mathbb{0} & \Theta \end{bmatrix} \begin{bmatrix} \rho_y \Psi \\ \rho_y D_{\mathbb{R}\setminus\{y\}} \Psi \end{bmatrix} \quad \Longleftrightarrow \quad (\widehat{\tau}_y \mathbb{1})\Psi = \widehat{\Theta}(\widehat{\rho}_y \mathbb{1})\Psi. \quad (3.7)$$

To solve (3.7), at first we look for the two invertible matrices M_1 and M_2 such that

$$(\widehat{\tau}_y \mathbb{1})\Psi = M_1 \begin{bmatrix} \tau_y \Psi \\ \tau_y D_{\mathbb{R}\setminus\{y\}} \Psi \end{bmatrix}, \qquad (\widehat{\rho}_y \mathbb{1})\Psi = M_2 \begin{bmatrix} \rho_y \Psi \\ \rho_y D_{\mathbb{R}\setminus\{y\}} \Psi \end{bmatrix}. \quad (3.8)$$

By direct calculations, one gets

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & i \\ 0 & 1 & 0 & 0 \\ -\frac{i}{2} & 0 & i & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} \frac{1}{2} & 0 & 1 & 0 \\ 0 & i & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \\ i & 0 & 0 & 0 \end{bmatrix}. \quad (3.9)$$

Therefore, (3.7) rewrites as

$$M_1^{-1}(\widehat{\tau}_y \mathbb{1})\Psi = (\Theta \oplus \Theta)M_2^{-1}(\widehat{\rho}_y \mathbb{1})\Psi \quad \Longleftrightarrow \quad (\widehat{\tau}_y \mathbb{1})\Psi = \widehat{\Theta}(\widehat{\rho}_y \mathbb{1})\Psi$$

and so the relation between $\widehat{\Theta}$ and Θ is given by

$$\widehat{\Theta} = M_1(\Theta \oplus \Theta)M_2^{-1}. \quad (3.10)$$

By

$$\widehat{\Theta} = \widehat{\Theta}^* \quad \Longleftrightarrow \quad M_1^* M_2(\Theta \oplus \Theta) = (\Theta \oplus \Theta)M_2^* M_1,$$

$\widehat{\Theta}$ is symmetric by the relations

$$M_1^* M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = M_2^* M_1 . \tag{3.11}$$

More explicitly, if

$$\Theta = \begin{bmatrix} a & b \\ \bar{b} & d \end{bmatrix}, \qquad a, d \in \mathbb{R}, \, b \in \mathbb{C} \,,$$

then $\widehat{\Theta}$ is represented by the Hermitian matrix

$$\widehat{\Theta} = \begin{bmatrix} 0 & -ib & 0 & -ia \\ i\bar{b} & d & id & 0 \\ 0 & -id & 0 & -i\bar{b} \\ ia & 0 & ib & -a \end{bmatrix} .$$

If $a = d = 0$ and $b \in \mathbb{R}$, i.e., if $\Theta = b\sigma_1$, then $\widehat{\Theta} = b(\sigma_2 \oplus \sigma_2) \equiv b\sigma_2 \mathbb{1}$, where σ_2 denotes the Pauli matrix

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

and, by Remark 2.4, the corresponding Schrödinger operator in $L^2(\mathbb{R}; \mathbb{C}^2)$ is block diagonal:

$$(D_{b\sigma_1})^2 = \left(H_{b\sigma_2} + \frac{1}{4} \right) \mathbb{1} . \tag{3.12}$$

3.3. The case $n = 1$, $\Pi \neq \mathbb{1}$

Here we take $\Pi : \mathbb{C}^2 \to \mathbb{C}^2$ a not trivial orthogonal projection, i.e., $\dim(\mathrm{ran}(\Pi)) = 1$, and $\Theta : \mathrm{ran}(\Pi) \to \mathrm{ran}(\Pi)$ identifies with the multiplication by $\theta \in \mathbb{R}$. By (3.8), (3.3) rewrites as

$$\begin{cases} (\widehat{\rho}_y \mathbb{1}) \Psi \in \mathrm{ran}(M_2(\Pi \oplus \Pi)) \\ M_2(\Pi \oplus \Pi) M_1^{-1}(\widehat{\tau}_y \mathbb{1}) \Psi = \theta(\widehat{\rho}_y \mathbb{1}) \Psi \end{cases} \quad \Longleftrightarrow \quad \begin{cases} (\widehat{\rho}_y \mathbb{1}) \Psi \in \mathrm{ran}(\widehat{\Pi}) \\ \widehat{\Pi}(\widehat{\tau}_y \mathbb{1}) \Psi = \widehat{\Theta}(\widehat{\rho}_y \mathbb{1}) \Psi . \end{cases} \tag{3.13}$$

Therefore, $\widehat{\Pi} : \mathbb{C}^4 \to \mathbb{C}^4$ is the orthogonal projection onto the 2-dimensional subspace

$$\mathrm{ran}(\widehat{\Pi}) = \mathrm{ran}(M_2(\Pi \oplus \Pi)) = \mathrm{ran}(M_2(\Pi \oplus \Pi) M_1^{-1}) \,,$$

i.e.,

$$\begin{aligned} \widehat{\Pi} &= M_2(\Pi \oplus \Pi)((\Pi \oplus \Pi) M_2^* M_2(\Pi \oplus \Pi))^{-1}(\Pi \oplus \Pi) M_2^* \\ &= M_2(\Pi \oplus \Pi)(M_2^* M_2)^{-1}(\Pi \oplus \Pi) M_2^* \\ &= (M_2(\Pi \oplus \Pi) M_2^{-1})(M_2(\Pi \oplus \Pi) M_2^{-1})^* . \end{aligned}$$

By (3.11), $M_2(\Pi \oplus \Pi) M_1^{-1}$ is symmetric. Hence, $\mathrm{ran}(\widehat{\Pi}) = \ker(M_2(\Pi \oplus \Pi) M_1^{-1})^\perp$ and the symmetric operator

$$M_2(\Pi \oplus \Pi) M_1^{-1} : \mathrm{ran}(\widehat{\Pi}) \to \mathrm{ran}(\widehat{\Pi})$$

is a bijection. Then, (3.13) gives

$$\widehat{\Theta} : \mathrm{ran}(\widehat{\Pi}) \to \mathrm{ran}(\widehat{\Pi}), \qquad \widehat{\Theta} := \theta(M_2(\Pi \oplus \Pi)M_1^{-1})^{-1}.$$

3.4. The case $n > 1$, $\Pi = \mathbb{1}$

In order to exploit the results from the $n = 1$ case, we introduce the unitary operator

$$U : \mathbb{C}^{4n} \to \mathbb{C}^{4n}, \qquad U(\xi_1, \xi_2, \dots, \xi_{2n}) := (\xi_1, \xi_{n+1}, \xi_2, \xi_{n+2}, \dots, \xi_n, \xi_{2n}), \quad \xi_k \in \mathbb{C}^2 . \tag{3.14}$$

By such a definition,

$$U(\tau\Psi \oplus \tau\mathsf{D}_{\mathbb{R}\backslash Y}\Psi) = \left(\begin{bmatrix} \tau_{y_1}\Psi \\ \tau_{y_1}\mathsf{D}_{\mathbb{R}\backslash Y}\Psi \end{bmatrix}, \dots, \begin{bmatrix} \tau_{y_n}\Psi \\ \tau_{y_n}\mathsf{D}_{\mathbb{R}\backslash Y}\Psi \end{bmatrix} \right),$$

$$U(\rho\Psi \oplus \rho\mathsf{D}_{\mathbb{R}\backslash Y}\Psi) = \left(\begin{bmatrix} \rho_{y_1}\Psi \\ \rho_{y_1}\mathsf{D}_{\mathbb{R}\backslash Y}\Psi \end{bmatrix}, \dots, \begin{bmatrix} \rho_{y_n}\Psi \\ \rho_{y_n}\mathsf{D}_{\mathbb{R}\backslash Y}\Psi \end{bmatrix} \right).$$

Therefore, setting

$$M_1^{\oplus} : \mathbb{C}^{4n} \to \mathbb{C}^{4n}, \qquad M_1^{\oplus} := M_1 \oplus \dots \oplus M_1 ,$$

$$M_2^{\oplus} : \mathbb{C}^{4n} \to \mathbb{C}^{4n}, \qquad M_2^{\oplus} := M_2 \oplus \dots \oplus M_2 ,$$

by (3.8), one gets

$$M_1^{\oplus}U(\tau\Psi \oplus \tau\mathsf{D}_{\mathbb{R}\backslash Y}\Psi) = ((\widehat{\tau}_{y_1}\mathbb{1})\Psi, \dots, (\widehat{\tau}_{y_n}\mathbb{1})\Psi) = U(\widehat{\tau}\mathbb{1})\Psi ,$$

$$M_2^{\oplus}U(\rho\Psi \oplus \rho\mathsf{D}_{\mathbb{R}\backslash Y}\Psi) = ((\widehat{\rho}_{y_1}\mathbb{1})\Psi, \dots, (\widehat{\rho}_{y_n}\mathbb{1})\Psi) = U(\widehat{\rho}\mathbb{1})\Psi$$

and so (3.3) rewrites as

$$U^*(M_1^{\oplus})^{-1}U(\widehat{\tau}\mathbb{1})\Psi = (\Theta \oplus \Theta)U^*(M_2^{\oplus})^{-1}U(\widehat{\rho}\mathbb{1})\Psi \quad \Longleftrightarrow \quad (\widehat{\tau}\mathbb{1})\Psi = \widehat{\Theta}(\widehat{\rho}\mathbb{1})\Psi .$$

This gives

$$\widehat{\Theta} = U^*M_1^{\oplus}U(\Theta \oplus \Theta)U^*(M_2^{\oplus})^{-1}U . \tag{3.15}$$

Such a operator $\widehat{\Theta}$ is symmetric by

$$U^*(M_1^{\oplus})^*M_2^{\oplus}U = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} = U^*(M_2^{\oplus})^*M_1^{\oplus}U . \tag{3.16}$$

The relations (3.16) generalize (3.11), since $U = \mathbb{1}$ whenever $n = 1$, and are a consequence of (3.11) itself and the definition (3.14).

3.5. The case $n > 1$, $\Pi \neq \mathbb{1}$

Finally, we consider the most general case. Using the unitary $U : \mathbb{C}^{4n} \to \mathbb{C}^{4n}$ as in the previous section, (3.3) rewrites as

$$\begin{cases} U^*(M_2^\oplus)^{-1}U(\widehat{\rho}\mathbb{1})\Psi \in \operatorname{ran}(\Pi \oplus \Pi) \\ (\Pi \oplus \Pi)U^*(M_1^\oplus)^{-1}U(\widehat{\tau}\mathbb{1})\Psi = (\Theta \oplus \Theta)U^*(M_2^\oplus)^{-1}U(\widehat{\rho}\mathbb{1})\Psi \end{cases}$$

$$\iff \quad \begin{cases} (\widehat{\rho}\mathbb{1})\Psi \in \operatorname{ran}(\widehat{\Pi}) \\ \widehat{\Pi}(\widehat{\tau}\mathbb{1})\Psi = \widehat{\Theta}(\widehat{\rho}\mathbb{1})\Psi. \end{cases} \tag{3.17}$$

This gives the orthogonal projector $\widehat{\Pi} : \mathbb{C}^{4n} \to \mathbb{C}^{4n}$, with $\dim(\operatorname{ran}(\widehat{\Pi})) = 2\dim(\operatorname{ran}(\Pi))$, such that

$$\operatorname{ran}(\widehat{\Pi}) = \operatorname{ran}(U^*M_2^\oplus U(\Pi \oplus \Pi)) = \operatorname{ran}(U^*M_2^\oplus U(\Pi \oplus \Pi)U^*(M_1^\oplus)^{-1}U), \tag{3.18}$$

i.e.,

$$\begin{aligned} \widehat{\Pi} &= (U^*M_2^\oplus U(\Pi \oplus \Pi))\big((U^*M_2^\oplus U(\Pi \oplus \Pi))^*(U^*M_2^\oplus U(\Pi \oplus \Pi))\big)^{-1}(U^*M_2^\oplus U(\Pi \oplus \Pi))^* \\ &= U^*M_2^\oplus U(\Pi \oplus \Pi)((U^*M_2^\oplus U)^*U^*M_2^\oplus U)^{-1}(\Pi \oplus \Pi)U^*(M_2^\oplus)^*U \\ &= \big(U^*M_2^\oplus U(\Pi \oplus \Pi)U^*(M_2^\oplus)^{-1}\big)\big(U^*M_2^\oplus U(\Pi \oplus \Pi)U^*(M_2^\oplus)^{-1}\big)^*, \end{aligned}$$

and $\widehat{\Theta} : \operatorname{ran}(\widehat{\Pi}) \to \operatorname{ran}(\widehat{\Pi})$,

$$\widehat{\Theta} := \big(U^*M_2^\oplus U(\Pi \oplus \Pi)U^*(M_1^\oplus)^{-1}U\big)^{-1}U^*M_2^\oplus U(\Theta \oplus \Theta)U^*(M_2^\oplus)^{-1}U. \tag{3.19}$$

By (3.16), $U^*M_2^\oplus U(\Pi \oplus \Pi)U^*(M_1^\oplus)^{-1}U$ is symmetric. Therefore, by

$$\operatorname{ran}(\widehat{\Pi}) = \operatorname{ran}(U^*M_2^\oplus U(\Pi \oplus \Pi)U^*(M_1^\oplus)^{-1}U) = \ker(U^*M_2^\oplus U(\Pi \oplus \Pi)U^*(M_1^\oplus)^{-1}U)^\perp,$$

the operator

$$U^*M_2^\oplus U(\Pi \oplus \Pi)U^*(M_1^\oplus)^{-1}U : \operatorname{ran}(\widehat{\Pi}) \to \operatorname{ran}(\widehat{\Pi})$$

is a bijection and $\widehat{\Theta}$ is well defined. To conclude, we have to show that $\widehat{\Theta}$ is symmetric. By (3.19) and by $U^*U = \mathbb{1}$,

$$\widehat{\Theta} \text{ is symmetric} \iff M_2^\oplus U(\Theta \oplus \Theta)(\Pi \oplus \Pi)U^*(M_1^\oplus)^{-1} \text{ is symmetric}$$

and so $\widehat{\Theta}$ is symmetric by (3.16).

REMARK 3.3. Let us point out that it is not necessary to determine $\widehat{\Pi}$ and $\widehat{\Theta}$ explicitly in order to write down the domain of $\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}$. Indeed, by (3.3),

$$\begin{aligned} \operatorname{dom}(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}) = \big\{\Psi \in H^2(\mathbb{R} \setminus Y; \mathbb{C}^2) : \rho\Psi \oplus \rho D_{\mathbb{R}\setminus Y}\Psi \in \operatorname{ran}(\Pi \oplus \Pi) \\ (\Pi\tau\Psi) \oplus (\Pi\tau D_{\mathbb{R}\setminus Y}\Psi) = (\Theta\rho\Psi) \oplus (\Theta\rho D_{\mathbb{R}\setminus Y}\Psi)\big\}. \end{aligned}$$

However, one needs to know $\widehat{\Pi}$ and $\widehat{\Theta}$ in order to write down the resolvent of $\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}$, according to Theorem 2.3.

The above representation of $\mathrm{dom}(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}})$ suggests an alternative way to build the self-adjoint extensions of $\widehat{S}°\mathbb{1} = \mathsf{H}\mathbb{1}|C^\infty_{comp}(\mathbb{R};\mathbb{C}^2)$: one can apply the results in [24] and [25] to $\mathsf{H}\mathbb{1}|\mathrm{ker}(\widetilde{\tau})$, where

$$\widetilde{\tau}: H^2(\mathbb{R};\mathbb{C}^2) \to \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}, \qquad \widetilde{\tau}\Psi := \tau\Psi \oplus \tau\mathsf{D}\Psi.$$

In that case, the family of self-adjoint extensions of $\widehat{S}°\mathbb{1}$ is represented by operators of the kind $\widetilde{H}_{\widetilde{\Pi},\widetilde{\Theta}}$, where $\widetilde{\Pi}$ is an orthogonal projector in $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ and $\widetilde{\Theta}$ is a symmetric operator in $\mathrm{ran}(\widetilde{\Theta})$. With respect to this parametrization, one has that $\mathsf{D}^2_{\Pi,\Theta} = \widetilde{H}_{\widetilde{\Pi},\widetilde{\Theta}} + \frac{1}{4}$ if and only if $\widetilde{\Pi} = \Pi \oplus \Pi$ and $\widetilde{\Theta} = \Theta \oplus \Theta$. Even if such a correspondence is more explicit than the one which uses the couple $(\widehat{\Pi},\widehat{\Theta})$, it has the drawback that it works with a representation of the family of self-adjoint extensions of $\widehat{S}°\mathbb{1}$ which is different from the usual one and which lacks of the analogous of the property $\mathsf{H}_{\Pi\mathbb{1},\Theta\mathbb{1}} = \mathsf{H}_{\Pi,\Theta}\mathbb{1}$. Therefore, in this paper we prefer to work with the family $\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}$.

REMARK 3.4. Suppose that for any $\Psi \equiv (\psi_1, \psi_2) \in \mathrm{dom}(\mathsf{D}_{\Pi,\Theta})$ one has

$$\begin{cases} \rho\Psi \in \mathrm{ran}(\Pi) \\ \Pi\tau\Psi = \Theta\rho\Psi \end{cases} \iff \begin{cases} B_1(\psi_1) = 0 \\ B_2(\psi_2) = 0, \end{cases}$$

with some linear operators $B_1 : H^1(\mathbb{R} \setminus Y) \to \mathbb{C}^{d_1}$ and $B_2 : H^1(\mathbb{R} \setminus Y) \to \mathbb{C}^{d_2}$. Then, by the representation of $\mathrm{dom}(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}})$ in Remark 3.3, there follows that the boundary conditions for $\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}$ rewrites as

$$\begin{cases} B_1(\psi_1) = 0 \\ B_1\left(-i\psi_2' + \frac{1}{2}\psi_1\right) = 0 \\ B_2(\psi_2) = 0 \\ B_2\left(-i\psi_1' - \frac{1}{2}\psi_2\right) = 0 \end{cases} \equiv \begin{cases} B_1(\psi_1) = 0 \\ B_2(\psi_1') = 0 \\ B_2(\psi_2) = 0 \\ B_1(\psi_2') = 0. \end{cases}$$

This entails

$$\mathrm{dom}(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}}) = \mathrm{dom}(\mathsf{H}_{1,2}) \oplus \mathrm{dom}(\mathsf{H}_{2,1}), \qquad (\mathsf{D}_{\Pi,\Theta})^2 = \left(\mathsf{H}_{1,2} + \frac{1}{4}\right) \oplus \left(\mathsf{H}_{2,1} + \frac{1}{4}\right),$$

where the self-adjoint operators $\mathsf{H}_{j,k} : \mathrm{dom}(\mathsf{H}_{j,k}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are defined by

$$\mathrm{dom}(\mathsf{H}_{1,2}) := \{\psi \in H^2(\mathbb{R} \setminus Y) : B_1(\psi) = 0, B_2(\psi') = 0\}, \quad \mathsf{H}_{1,2}\psi := \mathsf{H}_{\mathbb{R}\setminus Y}\psi.$$

$$\mathrm{dom}(\mathsf{H}_{2,1}) := \{\psi \in H^2(\mathbb{R} \setminus Y) : B_2(\psi) = 0, B_1(\psi') = 0\}, \quad \mathsf{H}_{2,1}\psi := \mathsf{H}_{\mathbb{R}\setminus Y}\psi.$$

4. Applications

4.1. Local boundary conditions

Here we consider the case corresponding to local boundary conditions for the Dirac operator, i.e., boundary conditions which do not couple the values of Ψ at different point. That means

$$\Pi = \Pi_1 \oplus \ldots \oplus \Pi_n, \qquad \Pi_k : \mathbb{C}^2 \to \mathbb{C}^2, \qquad 1 \leqslant k \leqslant n,$$

$$\Theta = \Theta_1 \oplus \ldots \oplus \Theta_n, \qquad \Theta_k : \operatorname{ran}(\Pi_k) \to \operatorname{ran}(\Pi_k), \qquad 1 \leqslant k \leqslant n.$$

In this case, by

$$U((\Pi_1 \oplus \ldots \oplus \Pi_n) \oplus (\Pi_1 \oplus \ldots \oplus \Pi_n))U^* = (\Pi_1 \oplus \Pi_1) \oplus \ldots \oplus (\Pi_n \oplus \Pi_n),$$

one gets, by (3.18),

$$\operatorname{ran}(\widehat{\Pi}) = \operatorname{ran}(U^*(M_2(\Pi_1 \oplus \Pi_1)M_1^{-1}) \oplus \ldots \oplus (M_2(\Pi_n \oplus \Pi_n)M_1^{-1})U)$$
$$= \operatorname{ran}(U^*(\widehat{\Pi}_1 \oplus \ldots \oplus \widehat{\Pi}_n)U),$$

where, in the case $\Pi_k \neq \mathbb{1}$, $\widehat{\Pi}_k : \mathbb{C}^4 \to \mathbb{C}^4$ is the orthogonal projector onto the 2-dimensional subspace

$$\operatorname{ran}(\widehat{\Pi}_k) = \operatorname{ran}(M_2(\Pi_k \oplus \Pi_k)M_1^{-1}), \tag{4.1}$$

otherwise $\widehat{\Pi}_k = \mathbb{1}$. Then, by (3.19) and by

$$U((\Pi_1 \Theta_1 \Pi_1 \oplus \ldots \oplus \Pi_n \Theta_n \Pi_n) \oplus (\Pi_1 \Theta_1 \Pi_1 \oplus \ldots \oplus \Pi_n \Theta_n \Pi_n))U^*$$
$$= (\Pi_1 \Theta_1 \Pi_1 \oplus \Pi_1 \Theta_1 \Pi_1) \oplus \ldots \oplus (\Pi_n \Theta_n \Pi_n \oplus \Pi_n \Theta_n \Pi_n),$$

one obtains

$$\widehat{\Theta} = U^*(\widehat{\Theta}_1 \oplus \ldots \oplus \widehat{\Theta}_n)U,$$

where, in the case $\Pi_k \neq \mathbb{1}$, $\Theta_k \in \mathbb{R}$,

$$\widehat{\Theta}_k : \operatorname{ran}(\widehat{\Pi}_k) \to \operatorname{ran}(\widehat{\Pi}_k), \qquad \widehat{\Theta}_k = \Theta_k M_1(\Pi_k \oplus \Pi_k)M_2^{-1},$$

otherwise,

$$\widehat{\Theta}_k : \mathbb{C}^4 \to \mathbb{C}^4, \qquad \widehat{\Theta}_k = M_1(\Theta_k \oplus \Theta_k)M_2^{-1}.$$

Therefore, the corresponding boundary conditions for $\widehat{\mathsf{H}}_{\widehat{\Pi},\widehat{\Theta}}$ are

$$\widehat{\rho}_{y_k}\Psi \in \operatorname{ran}(\widehat{\Pi}_k), \qquad \widehat{\Pi}_k \widehat{\tau}_{y_k}\Psi = \widehat{\Theta}_k \widehat{\rho}_{y_k}\Psi, \qquad 1 \leqslant k \leqslant n,$$

and so they are local as well.

4.2. Gesztesy-Šeba realizations

These two families of self-adjoint realizations of the Dirac operator with local point interactions correspond, in the non relativistic limit, to Schrödinger operators with local point interactions either of δ-type or of δ'-type (see [19], [1, Appendix J], [15]). The operators in the α-family have self-adjointness domains

$$\text{dom}(D_\alpha) = \{\Psi \equiv (\psi_1, \psi_2) \in H^1(\mathbb{R}) \oplus H^1(\mathbb{R} \setminus Y) : [\psi_2]_{y_k} = -i\alpha_k \psi_1(y_k), \, 1 \leqslant k \leqslant n\},$$
$$\alpha_k \in \mathbb{R}, \tag{4.2}$$

and the ones in the β-family have self-adjointness domains

$$\text{dom}(D_\beta) = \{\Psi \equiv (\psi_1, \psi_2) \in H^1(\mathbb{R} \setminus Y) \oplus H^1(\mathbb{R}) : [\psi_1]_{y_k} = -i\beta_k \psi_2(y_k), \, 1 \leqslant k \leqslant n\},$$
$$\beta_k \in \mathbb{R}. \tag{4.3}$$

Since the cases where all the α_k's or all the β_k's are equal to zero correspond to D, and the cases where there are $0 < m < n$ α_k's or β_k's which are zero reduce to the cases with $(n-m)$ point interactions, without loss of generality we can suppose that all the α_k's and β_k's are different from zero. By Theorem 3.1 and Remark 3.2, one has

$$D_\alpha = D_{\Pi^{(\alpha)}, \Theta^{(\alpha)}}, \quad \Pi^{(\alpha)} = \Pi_1^{(\alpha)} \oplus \ldots \oplus \Pi_n^{(\alpha)}, \quad \Theta^{(\alpha)} = \Theta_1^{(\alpha)} \oplus \ldots \oplus \Theta_n^{(\alpha)},$$

where

$$\Pi_k^{(\alpha)}(\xi_1, \xi_2) = (\xi_1, 0), \qquad \Theta_k^{(\alpha)} : \mathbb{C} \to \mathbb{C}, \quad \Theta_k^{(\alpha)} = \alpha_k^{-1}$$

and

$$D_\beta = D_{\Pi^{(\beta)}, \Theta^{(\beta)}}, \quad \Pi^{(\beta)} = \Pi_1^{(\beta)} \oplus \ldots \oplus \Pi_n^{(\beta)}, \quad \Theta^{(\beta)} = \Theta_1^{(\beta)} \oplus \ldots \oplus \Theta_n^{(\beta)},$$

where

$$\Pi_k^{(\beta)}(\xi_1, \xi_2) = (0, \xi_2), \qquad \Theta_k^{(\beta)} : \mathbb{C} \to \mathbb{C}, \quad \Theta_k^{(\beta)} = \beta_k^{-1}.$$

Therefore,

$$(D_\alpha)^2 = \widehat{H}_\alpha + \frac{1}{4},$$

where

$$\widehat{H}_\alpha = \widehat{H}_{\widehat{\Pi}^{(\alpha)}, \widehat{\Theta}^{(\alpha)}},$$

$$\text{ran}(\widehat{\Pi}^{(\alpha)}) = \text{ran}(U^*(\widehat{\Pi}_1^{(\alpha)} \oplus \ldots \oplus \widehat{\Pi}_n^{(\alpha)})U), \quad \widehat{\Theta}^{(\alpha)} = U^*(\widehat{\Theta}_1^{(\alpha)} \oplus \ldots \oplus \widehat{\Theta}_n^{(\alpha)})U,$$

$$\text{ran}(\widehat{\Pi}_k^{(\alpha)}) = \text{ran}(M_2(\Pi_k^{(\alpha)} \oplus \Pi_k^{(\alpha)})) = \mathbb{C} \oplus \{0\} \oplus \{0\} \oplus \mathbb{C} \equiv \mathbb{C}^2,$$

$$\widehat{\Theta}_k^{(\alpha)} = M_1(\Theta_k^{(\alpha)} \oplus \Theta_k^{(\alpha)})M_2^{-1} : \mathbb{C}^2 \to \mathbb{C}^2, \quad \widehat{\Theta}_k^{(\alpha)} = \frac{1}{\alpha_k} \begin{bmatrix} 0 & -i \\ i & -1 \end{bmatrix}$$

and

$$(D_\beta)^2 = \widehat{H}_\beta + \frac{1}{4},$$

where

$$\widehat{H}_\beta = \widehat{H}_{\widehat{\Pi}^{(\beta)}, \widehat{\Theta}^{(\beta)}},$$

$$\operatorname{ran}\big(\widehat{\Pi}^{(\beta)}\big) = \operatorname{ran}\big(U^*\big(\widehat{\Pi}_1^{(\beta)} \oplus \ldots \oplus \widehat{\Pi}_n^{(\beta)}\big)U\big), \quad \widehat{\Theta}^{(\beta)} = U^*\big(\widehat{\Theta}_1^{(\beta)} \oplus \ldots \oplus \widehat{\Theta}_n^{(\beta)}\big)U,$$

$$\operatorname{ran}(\widehat{\Pi}_k^{(\beta)}) = \operatorname{ran}(M_2(\Pi_k^{(\beta)} \oplus \Pi_k^{(\beta)})) = \{0\} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \{0\} \equiv \mathbb{C}^2,$$

$$\widehat{\Theta}_k^{(\beta)} = M_1(\Theta_k^{(\beta)} \oplus \Theta_k^{(\beta)})M_2^{-1} : \mathbb{C}^2 \to \mathbb{C}^2, \quad \widehat{\Theta}_k^{(\beta)} = \frac{1}{\beta_k}\begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}.$$

Hence,

$$\operatorname{dom}(\widehat{H}_\alpha) = \{\Psi \equiv (\psi_1, \psi_2) \in H^2(\mathbb{R}\setminus Y) : [\psi_1]_{y_k} = [\psi_2']_{y_k} = 0,$$
$$[\psi_1']_{y_k} = \alpha_k(\psi_1(y_k) - i\psi_2'(y_k)), \ [\psi_2]_{y_k} = -i\alpha_k\psi_1(y_k), \ 1 \leqslant k \leqslant n\},$$

and

$$\operatorname{dom}(\widehat{H}_\beta) = \{\Psi \equiv (\psi_1, \psi_2) \in H^2(\mathbb{R}\setminus Y) : [\psi_1']_{y_k} = [\psi_2]_{y_k} = 0,$$
$$[\psi_1]_{y_k} = -i\beta_k\psi_2(y_k), \ [\psi_2']_{y_k} = -\beta_k(\psi_2(y_k) + i\psi_1'(y_k)), \ 1 \leqslant k \leqslant n\}.$$

4.3. Separating boundary conditions

Let $n = 1$, and $\Pi = \mathbb{1}$. By $2\tau\Psi = \Psi(y^-) + \Psi(y^+)$ and $\rho\Psi = i\sigma_1(\Psi(y_+) - \Psi(y^+))$, the boundary condition $\tau\Psi = \Theta\rho\Psi$ rewrites as

$$(2i\Theta\sigma_1 + \mathbb{1})\Psi(y^-) = (2i\Theta\sigma_1 - \mathbb{1})\Psi(y^+).$$

If Θ is such that

$$\operatorname{ran}(2i\Theta\sigma_1 + \mathbb{1}) \cap \operatorname{ran}(2i\Theta\sigma_1 - \mathbb{1}) = \{0\}, \tag{4.4}$$

then $\tau\Psi = \Theta\rho\Psi$ is equivalent to the separating boundary conditions

$$(2i\Theta\sigma_1 + \mathbb{1})\Psi(y^-) = 0 \tag{4.5}$$

$$(2i\Theta\sigma_1 - \mathbb{1})\Psi(y^+) = 0. \tag{4.6}$$

By the equivalence of (4.4) with

$$\det(2i\sigma_1\Theta - \mathbb{1}) = 0, \tag{4.7}$$

one gets that (4.4) holds if and only if

$$\Theta = \Theta_{\omega,\alpha,\beta} := \frac{1}{2}\begin{bmatrix} \alpha & i\omega\sqrt{1+\alpha\beta} \\ -i\omega\sqrt{1+\alpha\beta} & \beta \end{bmatrix},$$
$$\omega \in \{-1, +1\}, \ \alpha, \beta \in \mathbb{R}, \ \alpha\beta \geqslant -1. \tag{4.8}$$

For such a Θ, the boundary conditions (4.5), (4.6) can be rewriten, whenever $\Psi \equiv (\psi_1, \psi_2)$, as

$$\psi_2(y^-) = i\eta^-_{\omega,\alpha,\beta}\,\psi_1(y^-), \tag{4.9}$$

$$\psi_2(y^+) = i\eta^+_{\omega,\alpha,\beta}\,\psi_1(y^+), \tag{4.10}$$

where

$$\eta^\pm_{\omega,\alpha,\beta} := \begin{cases} -\alpha^{-1}(\omega\sqrt{1+\alpha\beta}\pm 1) \\ \quad\equiv -\beta(\omega\sqrt{1+\alpha\beta}\mp 1)^{-1} & \alpha\neq 0,\ \omega\sqrt{1+\alpha\beta}\mp 1\neq 0 \\ \mp 2\alpha^{-1} & \alpha\neq 0,\ \beta=0,\ \omega=\pm 1 \\ \pm 2^{-1}\beta & \alpha=0,\ \omega=\mp 1 \\ \infty & \text{otherwise} \end{cases}$$

and the boundary condition $\psi_2(y^\pm) = i\infty\psi_1(y^\pm)$ is to be understood as $\psi_1(y^\pm) = 0$.

   Then

$$D_{\omega,\alpha,\beta} = D^-_{\omega,\alpha,\beta} \oplus D^+_{\omega,\alpha,\beta},$$

where $D_{\omega,\alpha,\beta} := D_{\Theta_{\omega,\alpha,\beta}}$ and the self-adjoint operators $D^-_{\omega,\alpha,\beta}$ and $D^+_{\omega,\alpha,\beta}$ denote the Dirac operators in $L^2((-\infty,y);\mathbb{C}^2)$ and $L^2((y,+\infty);\mathbb{C}^2)$, with boundary conditions (4.5) and (4.6) (equivalently, (4.9) and (4.10)) respectively; let us remark that separating boundary conditions of the kind (4.5), (4.6) (resp. (4.9), (4.10)) already appeared in [20, Prop. 2.2] (resp. in [7, Rem. 3.2]).

   Rewriting the boundary condition $\widehat{\tau}\Psi = \widehat{\Theta}_{\omega,\alpha,\beta}(\widehat{\rho}\mathbb{1})\Psi$ as

$$\big(2i\widehat{\Theta}_{\omega,\alpha,\beta}(\sigma_2\oplus\sigma_2)+\mathbb{1}\big)\widehat{\Psi}(y^-) = \big(2i\widehat{\Theta}_{\omega,\alpha,\beta}(\sigma_2\oplus\sigma_2)-\mathbb{1}\big)\widehat{\Psi}(y^+),$$

where $\widehat{\Psi} \equiv (\psi_1,\psi_1',\psi_2,\psi_2')$ and $\widehat{\Theta}_{\omega,\alpha,\beta}$ is defined by (3.10), i.e.,

$$\widehat{\Theta}_{\omega,\alpha,\beta} = \begin{bmatrix} 0 & \omega\sqrt{1+\alpha\beta} & 0 & -i\alpha \\ \omega\sqrt{1+\alpha\beta} & \beta & i\beta & 0 \\ 0 & -i\beta & 0 & -\omega\sqrt{1+\alpha\beta} \\ i\alpha & 0 & -\omega\sqrt{1+\alpha\beta} & -\alpha \end{bmatrix},$$

one can check that

$$\det\big(2i(\sigma_2\oplus\sigma_2)\widehat{\Theta}_{\omega,\alpha,\beta}-\mathbb{1}\big) = 0$$

and so, proceeding as above,

$$\mathrm{ran}\big(2i\widehat{\Theta}_{\omega,\alpha,\beta}(\sigma_2\oplus\sigma_2)+\mathbb{1}\big)\cap\mathrm{ran}\big(2i\widehat{\Theta}_{\omega,\alpha,\beta}(\sigma_2\oplus\sigma_2)-\mathbb{1}\big) = \{0\}.$$

Thus the separating boundary conditions

$$\big(2i\widehat{\Theta}_{\omega,\alpha,\beta}(\sigma_2\oplus\sigma_2)+\mathbb{1}\big)\widehat{\Psi}(y^-) = 0 \tag{4.11}$$

$$\big(2i\widehat{\Theta}_{\omega,\alpha,\beta}(\sigma_2\oplus\sigma_2)-\mathbb{1}\big)\widehat{\Psi}(y^+) = 0 \tag{4.12}$$

hold for $\widehat{H}_{\omega,\alpha,\beta} := \widehat{H}_{\widehat{\Theta}_{\omega,\alpha,\beta}}$ and

$$\widehat{H}_{\omega,\alpha,\beta} = \widehat{H}^-_{\omega,\alpha,\beta} \oplus \widehat{H}^+_{\omega,\alpha,\beta},$$

where the self-adjoint operators $\widehat{H}^-_{\omega,\alpha,\beta}$ and $\widehat{H}^+_{\omega,\alpha,\beta}$ denote the Schrödinger operator $-\frac{d^2}{dx^2}\mathbb{1}$ in $L^2((-\infty,y);\mathbb{C}^2)$ and $L^2((y,+\infty);\mathbb{C}^2)$, with boundary conditions (4.11) and (4.12) respectively. Furthermore,

$$(D^{\pm}_{\omega,\alpha,\beta})^2 = \widehat{H}^{\pm}_{\omega,\alpha,\beta} + \frac{\mathbb{1}}{4}.$$

By (4.9), (4.10) and by Remark 3.3, the separating boundary conditions (4.11), (4.12) for $\widehat{H}^{\pm}_{\omega,\alpha,\beta}$ rewrite, whenever $\Psi \equiv (\psi_1, \psi_2)$, as

$$\psi_2(y^{\pm}) = i\eta^{\pm}_{\omega,\alpha,\beta}\,\psi_1(y^{\pm}), \qquad i\eta^{\pm}_{\omega,\alpha,\beta}\,\psi'_2(y^{\pm}) = \psi'_1(y^{\pm}) + \eta^{\pm}_{\omega,\alpha,\beta}\,\psi_1(y^{\pm}).$$

In the case $n = 1$, $\Pi \neq \mathbb{1}$, the boundary conditions in $\mathrm{dom}(D_{\Pi,\Theta})$ give

$$(\Pi - \mathbb{1})\sigma_1 \Psi(y^-) = (\Pi - \mathbb{1})\sigma_1 \Psi(y^+), \qquad \Pi(2i\theta\sigma_1 + \mathbb{1})\Psi(y^-) = \Pi(2i\theta\sigma_1 - \mathbb{1})\Psi(y^+). \tag{4.13}$$

Since $\mathrm{ran}((\Pi - \mathbb{1})\sigma_1) = \mathrm{ran}(\Pi - \mathbb{1})$ and, by $\det(2i\theta\sigma_1 \pm \mathbb{1}) = 1 + 4\theta^2 \neq 0$, $\mathrm{ran}(\Pi(2i\theta\sigma_1 \pm \mathbb{1})) = \mathrm{ran}(\Pi)$, the relations (4.13) do not allow any separating boundary conditions.

By the $n = 1$ case, one immediately gets the family of separating and local boundary conditions: it suffices to take $\Theta_{\underline{\omega},\underline{\alpha},\underline{\beta}} := \Theta_{\omega_1,\alpha_1,\beta_1} \oplus \ldots \oplus \Theta_{\omega_n,\alpha_n,\beta_n}$. Then, using the abbreviated notations $D_{\underline{\omega},\underline{\alpha},\underline{\beta}} \equiv D_{\Theta_{\underline{\omega},\underline{\alpha},\underline{\beta}}}$ and $\widehat{H}_{\underline{\omega},\underline{\alpha},\underline{\beta}} \equiv \widehat{H}_{\widehat{\Theta}_{\underline{\omega},\underline{\alpha},\underline{\beta}}}$, where $\widehat{\Theta}_{\underline{\omega},\underline{\alpha},\underline{\beta}} := U^*(\widehat{\Theta}_{\omega_1,\alpha_1,\beta_1} \oplus \ldots \oplus \widehat{\Theta}_{\omega_n,\alpha_n,\beta_n})U$ and $\widehat{\Theta}_{\omega_k,\alpha_k,\beta_k}$ is defined by (3.10), i.e., $\widehat{\Theta}_{\omega_k,\alpha_k,\beta_k} := M_1(\Theta_{\omega_k,\alpha_k,\beta_k} \oplus \Theta_{\omega,\alpha,\beta})M_2^{-1}$, one obtains

$$D_{\underline{\omega},\underline{\alpha},\underline{\beta}} = D^-_{\omega_1,\alpha_1,\beta_1} \oplus D_{\omega_{1,2},\alpha_{1,2},\beta_{1,2}} \oplus D_{\omega_{2,3},\alpha_{2,3},\beta_{2,3}} \oplus \ldots \oplus D_{\omega_{n-1,n},\alpha_{n-1,n},\beta_{n-1,n}} \oplus D^+_{\omega_n,\alpha_n,\beta_n}$$

and

$$\widehat{H}_{\underline{\omega},\underline{\alpha},\underline{\beta}} = \widehat{H}^-_{\omega_1,\alpha_1,\beta_1} \oplus \widehat{H}_{\omega_{1,2},\alpha_{1,2},\beta_{1,2}} \oplus \widehat{H}_{\omega_{2,3},\alpha_{2,3},\beta_{2,3}} \oplus \ldots \oplus \widehat{H}_{\omega_{n-1,n},\alpha_{n-1,n},\beta_{n-1,n}} \oplus \widehat{H}^+_{\omega_n,\alpha_n,\beta_n}.$$

Here $D_{\omega_{k-1,k},\alpha_{k-1,k},\beta_{k-1,k}}$ denotes the self-adjoint Dirac operator in $L^2((y_{k-1},y_k);\mathbb{C}^2)$ with boundary conditions of the kind (4.6) at y_{k-1} (with parameters $\omega_{k-1}, \alpha_{k-1}, \beta_{k-1}$) and of the kind (4.5) at y_k (with parameters $\omega_k, \alpha_k, \beta_k$); $\widehat{H}_{\omega_{k-1,k},\alpha_{k-1,k},\beta_{k-1,k}}$ is defined in a similar way, using the boundary conditions (4.11) and (4.12). Furthermore,

$$(D_{\omega_{k-1,k},\alpha_{k-1,k},\beta_{k-1,k}})^2 = \widehat{H}_{\omega_{k-1,k},\alpha_{k-1,k},\beta_{k-1,k}} + \frac{\mathbb{1}}{4}, \qquad 1 \leqslant k \leqslant n.$$

4.4. Supersymmetry

Since

$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = 0 = \sigma_3\sigma_2 + \sigma_2\sigma_3,$$

one has

$$\sigma_2 D_{\mathbb{R}\setminus Y} + D_{\mathbb{R}\setminus Y}\sigma_2 = 0.$$

Therefore, if (Π,Θ) is such that

$$\begin{cases} \rho\Psi \in \mathrm{ran}(\Pi) \\ \Pi\tau\Psi = \Theta\rho\Psi \end{cases} \implies \begin{cases} \rho\sigma_2\Psi \in \mathrm{ran}(\Pi) \\ \Pi\tau\sigma_2\Psi = \Theta\rho\sigma_2\Psi, \end{cases} \tag{4.14}$$

then σ_2 anti-commutes with $D_{\Pi,\Theta}$ and so, by (3.1), the system

$$\left(\widehat{H}_{\widehat{\Pi},\widehat{\Theta}} + \frac{\mathbb{1}}{4}, \sigma_2, D_{\Pi,\Theta} \right) \tag{4.15}$$

has supersymmetry (see, e.g., [4, Chapter 1], [17, Section 6.3]).

By

$$\langle \sigma_2\Psi \rangle_y = \sigma_2\langle\Psi\rangle_y, \qquad [\sigma_2\Psi]_y = \sigma_2[\Psi]_y,$$

and by $\sigma_1\sigma_2 = -\sigma_2\sigma_1$, one gets

$$\tau\sigma_2\Psi = \sigma_2^{\oplus}\tau\Psi, \qquad \rho\sigma_2\Psi = -\sigma_2^{\oplus}\rho\Psi, \qquad \sigma_2^{\oplus} := \sigma_2 \oplus \ldots \oplus \sigma_2.$$

Therefore, (4.14) holds whenever

$$\Pi\sigma_2^{\oplus} - \sigma_2^{\oplus}\Pi = 0 = \Theta\sigma_2^{\oplus} + \sigma_2^{\oplus}\Theta. \tag{4.16}$$

Given a couple (Π,Θ) which satisfies (4.16), let us further suppose that

$$\det(\Theta + \Pi\tau G_0\Pi) \neq 0. \tag{4.17}$$

Then, by (3.5), zero is not an eigenvalue of $D_{\Pi,\Theta}$, i.e., the system (4.15) has no supersymmetric state and there is a spontaneous supersymmetry breaking (see, e.g., [4, Section 1.8]).

In the case $n = 1$, the solutions of (4.16) are found immediately:

if $\Pi \neq \mathbb{1}$, then $\Pi = \Pi_{\pm} := |v_{\pm}\rangle\langle v_{\pm}|$ and $\Theta = 0$, where $v_{\pm}$, $|v_{\pm}| = 1$, solves $\sigma_2 v_{\pm} = \pm v_{\pm}$;

if $\Pi = \mathbb{1}$, then $\Theta = \Theta_{a,b} := b\sigma_1 + a\sigma_3$, $a, b \in \mathbb{R}$.

Since, by (2.7), $\tau G_0 = -\frac{1}{2}\sigma_3$, $\det(\Theta_{a,b} + \tau G_0) = 0$ if and only if $b = 0$ and $a = \frac{1}{2}$. Therefore, for any $(a,b) \in \mathbb{R}^2 \setminus \{(\frac{1}{2},0)\}$ the system (4.15) with $\Pi = \mathbb{1}$ and $\Theta = \Theta_{a,b}$ has no supersymmetric state and there is a spontaneous supersymmetry breaking.

Notice that once the solution of (4.16) in known in the $n = 1$ case, then the set of solutions for the case of $n > 1$ local point interactions is readily obtained: $\Pi = \Pi_1 \oplus \ldots \oplus \Pi_n$ and $\Theta = \Theta_1 \oplus \ldots \oplus \Theta_n$, where (Π_k,Θ_k) is equal either to $(\Pi_{\pm},0)$ or to $(\mathbb{1},\Theta_{a_k,b_k})$.

4.5. Quantum graphs

Since $D_{\Pi,\Theta}$ is a generic self-adjoint extension of

$$S = D|C^\infty_{comp}(\mathbb{R} \setminus Y; \mathbb{C}^2) = (D|C^\infty_{comp}(I_0; \mathbb{C}^2)) \oplus \cdots \oplus (D|C^\infty_{comp}(I_n; \mathbb{C}^2)),$$

the nonlocal extensions of S provide the self-adjoint realizations of the Dirac operator on a quantum graph with the two ends $\bar{I}_0 = (-\infty, y_1]$ and $\bar{I}_n = [y_n, +\infty)$ and the $(n-1)$ edges $\bar{I}_1 = [y_1, y_2], \ldots, \bar{I}_{n-1} = [y_{n-1}, y_n]$; the boundary conditions corresponding to the couple (Π, Θ) specify the connectivity of the graph. The case of a compact graph can be obtained by imposing separating boundary conditions at the two ends. Likewise, the nonlocal extensions of $H\mathbb{1}|C^\infty_{comp}(\mathbb{R} \setminus Y; \mathbb{C}^2)$ provide self-adjoint realizations of the Schrödinger operator on a quantum graph with two ends and $(n-1)$ edges. For an introduction to the theory of quantum graphs we refer to the book [10] and the many references there; however, let us point out that our way of building the self-adjoint realizations on the graph is not the standard one.

As an explicit example, let us consider the Dirac operator on the eye graph (see [14, Section III.D]). Therefore, we choose the subclass of boundary conditions for the Dirac operator in $L^2(G; \mathbb{C}^2)$, $G = (-\infty, y_1] \sqcup [y_1, y_2] \sqcup [y_2, y_3] \sqcup [y_3, +\infty)$, connecting $\Psi(y_1^-)$ with both $\Psi(y_1^+)$ and $\Psi(y_2^+)$ and connecting $\Psi(y_3^-)$ with both $\Psi(y_2^-)$ and $\Psi(y_3^-)$. Such kind of boundary conditions give to G the topology of a circle with two ends.
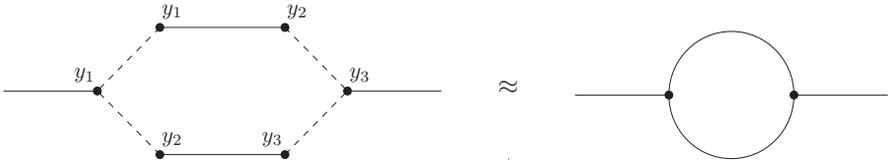
Furthermore, we restrict to Kirchhoff-type boundary conditions, meaning that we select the ones which, in the non relativistic limit, correspond to Kirchhoff (or free) boundary conditions for the Schrödinger operator in $L^2(G)$ (see [11], [12], [13]). Therefore, we require, for $\Psi \equiv (\psi_1, \psi_2)$ in the self-adjointness domain,

$$\begin{cases} \psi_1(y_1^-) = \psi_1(y_1^+) = \psi_1(y_2^+) \\ \psi_2(y_1^-) - \psi_2(y_1^+) - \psi_2(y_2^+) = 0 \\ \psi_1(y_2^-) = \psi_1(y_3^-) = \psi_1(y_3^+) \\ \psi_2(y_2^-) + \psi_2(y_3^-) - \psi_2(y_3^+) = 0. \end{cases} \tag{4.18}$$

These boundary conditions rewrite as

$$\begin{cases} [\psi_1]_{y_1} = [\psi_1]_{y_3} = 0 \\ [\psi_2]_{y_1} = -\psi_2(y_2^+) \\ [\psi_2]_{y_3} = \psi_2(y_2^-) \\ \langle \psi_1 \rangle_{y_1} = \psi_1(y_2^+) \\ \langle \psi_1 \rangle_{y_3} = \psi_1(y_2^-) \end{cases} \equiv \begin{cases} [\psi_1]_{y_1} = [\psi_1]_{y_3} = 0 \\ [\psi_2]_{y_1} + [\psi_2]_{y_2} + [\psi_2]_{y_3} = 0 \\ \langle \psi_1 \rangle_{y_1} - \langle \psi_1 \rangle_{y_3} = [\psi_1]_{y_2} \\ \langle \psi_1 \rangle_{y_1} - 2\langle \psi_1 \rangle_{y_2} + \langle \psi_1 \rangle_{y_3} = 0 \\ 2\langle \psi_2 \rangle_{y_2} = [\psi_2]_{y_3} - [\psi_2]_{y_1}. \end{cases} \tag{4.19}$$

The relations $[\psi_1]_{y_1} = [\psi_1]_{y_3} = [\psi_2]_{y_1} + [\psi_2]_{y_2} + [\psi_2]_{y_3} = 0$ in (4.19) coincide with $\rho\Psi \in \mathrm{ran}(\Pi)$, where the orthogonal projector $\Pi : \mathbb{C}^6 \to \mathbb{C}^6$ is represented by the matrix

$$\Pi = \frac{1}{3}\begin{bmatrix} 2 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.20}$$

Then, one can easily check that the other relations in (4.19) are equivalent to $\Pi\tau\Psi = \Theta\rho\Psi$ whenever Θ, a symmetric operator in the 3-dimensional subspace $\mathrm{ran}(\Pi)$, is represented, as a symmetric linear operator in $\mathbb{C}^6$ preserving $\mathrm{ran}(\Pi)$, by the Hermitian matrix

$$\Theta = \frac{i}{2}\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.21}$$

Therefore, using $L^2(G;\mathbb{C}^2) \equiv L^2(\mathbb{R};\mathbb{C}^2)$, one gets $\mathsf{D}_\mathsf{K} \equiv \mathsf{D}_{\Pi,\Theta}$, where Π and Θ are as in (4.20) and (4.21) respectively and where D_K denotes the Dirac operator on the eye graph with the Kirchhoff boundary conditions at the two vertices. Then, by Remark 3.4 and by (4.18), the Schrödinger operator $\widehat{\mathsf{H}}_{\widehat{\Pi},\widehat{\Theta}}$ satisfies both the boundary conditions K and K_*, where

$$\mathsf{K} \equiv \begin{cases} \psi_1(y_1^-) = \psi_1(y_1^+) = \psi_1(y_2^+) \\ \psi_1'(y_1^-) - \psi_1'(y_1^+) - \psi_1'(y_2^+) = 0 \\ \psi_1(y_2^-) = \psi_1(y_3^-) = \psi_1(y_3^+) \\ \psi_1'(y_2^-) + \psi_1'(y_3^-) - \psi_1'(y_3^+) = 0 \end{cases} \qquad \mathsf{K}_* \equiv \begin{cases} \psi_2'(y_1^-) = \psi_2'(y_1^+) = \psi_2'(y_2^+) \\ \psi_2(y_1^-) - \psi_2(y_1^+) - \psi_2(y_2^+) = 0 \\ \psi_2'(y_2^-) = \psi_2'(y_3^-) = \psi_2'(y_3^+) \\ \psi_2(y_2^-) + \psi_2(y_3^-) - \psi_2(y_3^+) = 0. \end{cases} \tag{4.22}$$

This gives

$$(\mathsf{D}_\mathsf{K})^2 = \left(\mathsf{H}_\mathsf{K} + \frac{1}{4}\right) \oplus \left(\mathsf{H}_{\mathsf{K}_*} + \frac{1}{4}\right), \tag{4.23}$$

where H_K is the Schrödinger operator in $L^2(G)$ with the boundary conditions K and $\mathsf{H}_{\mathsf{K}_*}$ is the Schrödinger operator in $L^2(G)$ with the boundary conditions K_*. The boundary conditions K coincide with the usual Kirchhoff ones (see [10, eq. (1.4.4)]) while the boundary conditions K_*, are as sort of reversed Kirchhoff ones (named "homogeneous δ' vertex conditions" in [12]) given by the exchange $\psi \leftrightarrow \psi'$. The boundary conditions K_*, like the K ones, give, in the case of the real line, the free Schrödinger operator; thus, (4.23) is consistent with (1.1). Furthermore, the Schrödinger operator $\mathsf{H}_\mathsf{K} \oplus \mathsf{H}_{\mathsf{K}_*}$ appears in the nonrelativistic limit of D_K, see [12, Proposition 1.3].

The arguments in the previous example extend to any graph: by Remark 3.4, to the Kirchhoff-type boundary conditions for the Dirac operator D_K on the graph, i.e., to

$$\begin{cases} \psi_1 \text{ continuous at any vertex } v \\ \sum_v^{\pm} \psi_2(v) = 0 \text{ for any vertex } v, \end{cases}$$

correspond, for the Schrödinger operators H_K and H_{K_*} such that (4.23) holds, the boundary conditions

$$K \equiv \begin{cases} \psi \text{ continuous at any vertex } v \\ \sum_v^{\pm} \psi'(v) = 0 \text{ for any vertex } v \end{cases} \qquad K_* \equiv \begin{cases} \psi' \text{ continuous an any vertex } v \\ \sum_v^{\pm} \psi(v) = 0 \text{ for any vertex } v. \end{cases}$$

Here, $\sum_v^{\pm} f(v)$ means the sum over all the points $y_k \in Y$ corresponding to the vertex v with the sign convention

$$f(v) := \begin{cases} -f(y_k^+) & y_k \text{ is at the left end of the interval/half-line} \\ +f(y_k^-) & y_k \text{ is at the right end of the interval/half-line}. \end{cases}$$

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