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THE ALEKSANDROV PROBLEM AND THE TINGLEY PROBLEM FOR EXPANSIVE AND NONEXPANSIVE OPERATORS IN p–NORMED SPACES

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Abstract. In this paper several positive answers are given to the Aleksandrov type problems and the Tingley type problems for some expansive and nonexpansive operators between a real p-normed space and a real q-normed space $(0 < p,q \leqslant 1)$. On the basis of the characteristics of p-normed spaces, the notion of isometry is generalized to the case of with some parameters. It is obtained that some operators of distance preserving can become isometries, and some isometric operators can be extended from the unit sphere to the whole space.

1. Introduction

Let X and Y be two real normed spaces. Aleksandrov [3] in 1970 proposed the following problem: under what conditions is an operator $T : X \to Y$ preserving unit distance an isometry? Tingley [30] in 1987 proposed the following problem: let T_0 be a bijective isometry between the unit spheres $S(X)$ and $S(Y)$ of X,Y respectively. Is it true that T_0 extends to a linear (bijective) isometry T from X to Y?

The foregoing Aleksandrov problem and Tingley problem are all related to isometric operators. Solving the two problems has always aroused extensive attentions. In the process of solving we can explore the properties of operators through the geometric structure of spaces. It is really impressive the development of machinery and technics that the two problems have led to. During the past two decades, many mathematicians have been working on these topics, in particular, the two problems have been solved in positive for many concrete classical normed spaces (see [5, 7, 8, 9, 10, 15, 20, 21, 25, 28, 29, 33] and the references therein). It is worth mentioning that a number of publications have covered the two problems in the F-spaces or n-normed spaces. Several counterexamples in [2, 17] illustrate that there does not exist any isometric operator in some specific form of spaces. A series of positive answers have been given in n-normed spaces or p-normed spaces (see [6, 11, 12, 14, 16, 17, 18, 26, 27, 32, 35, 37]).

Expansive operators and nonexpansive operators are the two kinds of operators that are closest to isometries. On the basis of the characteristics of p-normed spaces,

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in this paper we generalize the notions of expansive, nonexpansive and isometric operators to the case of with some parameters (see Definition 2.1). We will systematically consider the following Tingley type and Aleksandrov type problems for r-isometries between a real p-normed space X and a real q-normed space Y $(0 < p,q \leqslant 1)$ which have never been studied (to the best of our knowledge):

PROBLEM A. Let T_0 be a r-isometry between the unit spheres $S(X)$ and $S(Y)$ of X,Y respectively. What assumptions warrant the conclusion that T_0 extends to an r-isometry T from X to Y?

PROBLEM B. Under what conditions is an operator $T: X \to Y$ which satisfies the distance power r preserving property, an r-isometry?

As a result, we obtain several positive answers to Problem A in Section 3 and to Problem B in Section 4: some r isometric operators can be extended from the unit sphere to the whole space, and some operators of distance power r preserving can become r-isometries.

2. Preliminaries and some lemmas

Throughout the paper, X denotes a linear space over $\mathbb{R}$ with origin θ, where $\mathbb{R}$ is the field of real numbers. By $\mathbb{N}$ we mean the set of all nonnegative integers. A p-norm on X is a nonnegative real-valued functional $\|\cdot\|_p$ on X with $0 < p \leqslant 1$, satisfying the following conditions:

(a) $\|x\|_p = 0$ if and only if $x = \theta$;

(b) $\|\lambda x\|_p = |\lambda|^p \|x\|_p$, for all $x \in X$, $\lambda \in \mathbb{R}$;

(c) $\|x+y\|_p \leqslant \|x\|_p + \|y\|_p$, for all $x,y \in X$.

A linear space X endowed with a p-norm is called a p-normed space and is denoted by $(X,\|\cdot\|_p)$. If $p = 1$, then it is a normed space, and the norm is denoted by $\|\cdot\|$. A normed space $(X,\|\cdot\|)$ is strictly convex means that, whenever $x,y \in X$ and $\|x+y\| = \|x\| + \|y\|$, one of the two vectors must be a nonnegative real multiple of the other. A p-normed space $(X,\|\cdot\|_p)$ is also a metric linear space with a translation invariant metric d_X, where d_X is defined by $d_X(x,y) = \|x-y\|_p$ for $x,y \in X$. The class of p-normed spaces $(0 < p \leqslant 1)$ is an important generalization of classical normed spaces, and it has a rich topological and geometrical structure. If $0 < p < 1$, then the p-norm is nonhomogeneous, and the unit ball with center θ may not necessarily be a convex set. This determines that there are many differences between a p-normed space $(0 < p < 1)$ and a normed space (see for examples [4, 13, 23, 34, 36]).

In the sequel, $S(X)$ and $B(X)$ denote the unit sphere and the unit ball (with center θ) of a p-normed space X, respectively.

Vogt [31] generalized the notion of isometry to the case of with a gauge function. Similarly, on the basis of the characteristics of p-normed spaces, we will need the following definitions.

DEFINITION 2.1. Let (X,d_X) and (Y,d_Y) be two metric spaces, and $r,\sigma > 0$. An operator $T: X \to Y$ is said to be r-nonexpansive if

$$d_Y(Tx,Ty) \leqslant [d_X(x,y)]^r, \quad \text{for all } x,y \in X; \tag{2.1}$$

and to be r-expansive if

$$d_Y(Tx,Ty) \geqslant [d_X(x,y)]^r, \quad \text{for all } x,y \in X. \tag{2.2}$$

T is said to be an r-isometry if equality holds in (2.1) or (2.2) for all $x,y \in X$. If $r = 1$, then T is nonexpansive, expansive and isometric in the usual sense, respectively. T is said to be locally r-nonexpansive if the inequality (2.1) holds for $x,y \in X$ with $d_X(x,y) \leqslant \alpha$, where α is a positive constant. An operator $T : X \to Y$ is said to satisfy the distance σ power r preserving property if

$$d_X(x,y) = \sigma \ \Rightarrow \ d_Y(Tx,Ty) = \sigma^r, \quad \text{for all } x,y \in X.$$

In the implication above, if $\sigma = 1$, then T satisfies the distance one preserving property. $T : X \to Y$ is said to satisfy the strong distance σ power r preserving property if

$$d_X(x,y) = \sigma \ \Leftrightarrow \ d_Y(Tx,Ty) = \sigma^r, \quad \text{for all } x,y \in X.$$

DEFINITION 2.2. Let X and Y be two linear spaces, and $r > 0$. An operator $T : X \to Y$ is said to be positively r-homogeneous if

$$T(\alpha x) = \alpha^r Tx, \quad \text{for all } x \in X \text{ and } \alpha \geqslant 0.$$

If $r = 1$, then T is positively homogeneous in the usual sense.

LEMMA 2.1. *Let $r > 0$, $\alpha \geqslant 0$ and $\beta \geqslant 0$. Then $(\alpha+\beta)^r \leqslant \alpha^r + \beta^r \Leftrightarrow 0 < r \leqslant 1$, and $|\alpha - \beta|^r \leqslant |\alpha^r - \beta^r| \Leftrightarrow r \geqslant 1$.*

LEMMA 2.2. *Let X be a p-normed space $(0 < p \leqslant 1)$. Let $y_0 \in B(X)$, $x_0 \in X$ and $x_0 \neq y_0$. Then there exist $z_0 \in S(X)$ and $\lambda_0 \in (0,1]$ such that $y_0 = \lambda_0 z_0 + (1 - \lambda_0)x_0$.*

Proof. If $y_0 \in S(X)$, then we take $z_0 = y_0$ and $\lambda_0 = 1$, and so the assertion holds. Now we suppose that $\|y_0\|_p < 1$ and $z(t) = y_0 + t(y_0 - x_0)$ for $t \in [0,+\infty)$. Then $\|z(t)\|_p$ is continuous on $[0,+\infty)$. Since $x_0 \neq y_0$ and $\|z(t)\|_p \geqslant t^p \|y_0 - x_0\|_p - \|y_0\|_p$, we have $\lim\limits_{t\to+\infty} \|z(t)\|_p = +\infty$. Thus, from $\|z(0)\|_p = \|y_0\|_p < 1$ and the intermediate value theorem it follows that there exists $t_0 \in (0,+\infty)$ such that $\|z(t_0)\|_p = 1$. By taking $z_0 = z(t_0)$ and $\lambda_0 = \frac{1}{1+t_0}$, we have $y_0 = \lambda_0 z_0 + (1-\lambda_0)x_0$, which is the desired equality. $\square$

LEMMA 2.3. *Let X be a p-normed space $(0 < p \leqslant 1)$, $r > 0$ and $rp \geqslant 1$. Then for any $y \in B(X)$ and $x \in X$ with $x \neq y$, there holds*

$$\sup_{u \in S(X)} \big| \|x-u\|_p - \|u-y\|_p \big|^r \leqslant \|x-y\|_p^r \leqslant \sup_{u \in S(X)} \left(\|x-u\|_p^r - \|u-y\|_p^r \right).$$

Proof. For any $u \in S(X)$, we have $\big| \|x-u\|_p - \|u-y\|_p \big|^r \leqslant \|x-y\|_p^r$, and hence

$$\sup_{u \in S(X)} \big| \|x-u\|_p - \|u-y\|_p \big|^r \leqslant \|x-y\|_p^r.$$

On the other hand, by Lemma 2.2 there exist $u_0 \in S(X)$ and $\lambda_0 \in (0,1]$ such that $y = \lambda_0 u_0 + (1-\lambda_0)x$. This implies that $\|x-y\|_p^r = \lambda_0^{pr}\|x-u_0\|_p^r$ and $\|u_0-y\|_p^r = (1-\lambda_0)^{pr}\|x-u_0\|_p^r$. Thus, by Lemma 2.1 we have

$$\begin{aligned}\sup_{u\in S(X)}\left(\|x-u\|_p^r - \|u-y\|_p^r\right) &\geqslant \|x-u_0\|_p^r - \|u_0-y\|_p^r \\ &= \frac{1-(1-\lambda_0)^{pr}}{\lambda_0^{pr}}\|x-y\|_p^r \\ &\geqslant \|x-y\|_p^r,\end{aligned}$$

which is the desired inequality. □

REMARK 2.1. For Lemma 2.3 to hold with $r=1$, we need $p=1$. Thus for a normed space X we have

$$\|x-y\| = \sup_{u\in S(X)}\left(\|x-u\| - \|u-y\|\right),\ \ \forall x\in X,\ \forall y \in B(X),\ x\neq y.$$

LEMMA 2.4. *Let X be a p-normed space and Y a q-normed space $(0<p,q\leqslant 1)$. Let $r>0$ and $rp\geqslant 1$. If $T : X \to Y$ is a locally r-nonexpansive operator, then T is r-nonexpansive.*

Proof. Since $T : X\to Y$ is locally r-nonexpansive, we have

$$\|Tx-Ty\|_q \leqslant \|x-y\|_p^r \ \text{ for } x,y\in X,\ \text{with } \|x-y\|_p \leqslant \alpha \text{ for certain } \alpha>0. \quad (2.3)$$

Now we suppose $\|x-y\|_p > \alpha$. Then there exists $n_0\in\mathbb{N}$ with $n_0\geqslant 2$ such that $(n_0-1)^p\alpha < \|x-y\|_p \leqslant n_0^p\alpha$. Taking $x_i = x + \frac{i}{n_0}(y-x)$, $i=0,1,\cdots,n_0$, we have $\|x_i - x_{i-1}\|_p = \frac{\|x-y\|_p}{n_0^p} \leqslant \alpha$. From (2.3) it follows that

$$\|Tx_i - Tx_{i-1}\|_q \leqslant \|x_i - x_{i-1}\|_p^r = \frac{\|y-x\|_p^r}{n_0^{pr}}.$$

Hence

$$\begin{aligned}\|Ty-Tx\|_q = \|Tx_{n_0} - Tx_0\|_q &= \left\|\sum_{i=1}^{n_0}(Tx_i - Tx_{i-1})\right\|_q \leqslant \sum_{i=1}^{n_0}\|Tx_i - Tx_{i-1}\|_q \\ &\leqslant \sum_{i=1}^{n_0}\|x_i - x_{i-1}\|_p^r = \sum_{i=1}^{n_0}\frac{\|y-x\|_p^r}{n_0^{pr}} \\ &= n_0^{1-pr}\|y-x\|_p^r \leqslant \|y-x\|_p^r,\end{aligned}$$

which shows that T is r-nonexpansive. □

LEMMA 2.5. *Let X be a p-normed space and Y a q-normed space $(0 < p,q \leqslant 1)$. Let $T : X \to Y$ be an r-nonexpansive operator, where $rp \geqslant 1$. If T satisfies the distance σ power r preserving property, then $\|Tx-Ty\|_q = \|x-y\|_p^r$ for $x,y \in X$ with $\|x-y\|_p \leqslant \sigma$.*

Proof. If $x = y \in X$, then from r-nonexpansion of T we have $Tx = Ty$. For $x,y \in X$, and $x \neq y$, we set $c = \|x-y\|_p$, and prove that

$$\|Tx-Ty\|_q = \|x-y\|_p^r \text{ if } \|x-y\|_p = c \leqslant \sigma. \tag{2.4}$$

Clearly, (2.4) holds in the cases of $c = 0$ and $c = \sigma$, since T satisfies the distance σ power r preserving property. Without loss of generality, we suppose $0 < c = \|x-y\|_p < \sigma$. If $\|Tx-Ty\|_q < \|x-y\|_p^r$, then by taking $z = x + \left(\frac{\sigma}{c}\right)^{\frac{1}{p}}(y-x)$, we have $\|z-x\|_p = \sigma$ and $\|z-y\|_p = \left\|(x-y)\left[1-\left(\frac{\sigma}{c}\right)^{\frac{1}{p}}\right]\right\|_p = \left(\sigma^{\frac{1}{p}} - c^{\frac{1}{p}}\right)^p$. Since T is r-nonexpansive and distance σ power r preserving, by Lemma 2.1 we can deduce that

$$\begin{aligned}
\sigma^r &= \|Tz-Tx\|_q \leqslant \|Tz-Ty\|_q + \|Ty-Tx\|_q \\
&< \|z-y\|_p^r + \|x-y\|_p^r = \left(\sigma^{\frac{1}{p}} - c^{\frac{1}{p}}\right)^{pr} + c^r \\
&\leqslant \sigma^r - c^r + c^r = \sigma^r,
\end{aligned}$$

a contradiction. Hence (2.4) holds, which shows the assertion. □

3. Extension of isometries

THEOREM 3.1. *Let X be a p-normed space and Y a q-normed space $(0 < p,q \leqslant 1)$. Let $r > 0, rp \geqslant 1$ and $s = \frac{q}{pr}$. Let $T_0 : S(X) \to S(Y)$ be an r-isometry (not necessarily surjective) satisfying*

$$\|T_0x - \lambda T_0y\|_q \leqslant \|x-\lambda^s y\|_p^r, \ \textit{for } x,y \in S(X) \textit{ and } \lambda \in [0,1]. \tag{3.1}$$

Then T_0 can be extended to be an r-isometry T of X into Y and T is positively s^{-1}-homogeneous.

Proof. Suppose that $x,y \in S(X)$ and $\lambda \in [0,1]$. Since T_0 is an r-isometry, from (3.1) we have $\|T_0x - T_0u\|_q = \|x-u\|_p^r$ and $\|T_0u - \lambda T_0y\|_q \leqslant \|u-\lambda^s y\|_p^r$ for all $u \in S(X)$. This implies that

$$\|x-u\|_p^r - \|u-\lambda^s y\|_p^r \leqslant \|T_0x - T_0u\|_q - \|T_0u - \lambda T_0y\|_q \leqslant \|T_0x - \lambda T_0y\|_q. \tag{3.2}$$

By Lemma 2.3, the inequality (3.2) yields

$$\|x-\lambda^s y\|_p^r \leqslant \sup_{u \in S(X)} \left(\|x-u\|_p^r - \|u-\lambda^s y\|_p^r\right) \leqslant \|T_0x - \lambda T_0y\|_q. \tag{3.3}$$

Combining (3.1) and (3.3) we can infer that

$$\|T_0x - \lambda T_0y\|_q = \|x-\lambda^s y\|_p^r, \ \text{ for } x,y \in S(X) \text{ and } \lambda \in [0,1]. \tag{3.4}$$

Now we define an operator $T : X \to Y$ by

$$Tx = \begin{cases} \|x\|_p^{\frac{r}{q}} T_0\left(\dfrac{x}{\|x\|_p^{\frac{1}{p}}}\right), & x \neq \theta; \\ \theta, & x = \theta. \end{cases}$$

Then for $\alpha > 0$ we have $T(\alpha x) = \alpha^{\frac{pr}{q}} Tx = \alpha^{s^{-1}} Tx$, i.e., T is positively s^{-1}-homogeneous. It is evident that $\|Tx\|_q = \|x\|_p^r$ for all $x \in X$. Evidently, $Tx = T_0 x$ for all $x \in S(X)$, i.e., T is an extension of T_0. For any $x, y \in X$, if $x = \theta$ or $y = \theta$, then $\|Tx - Ty\|_q = \|x - y\|_p^r$; if $x \neq \theta$ and $y \neq \theta$, without loss of generality we can assume that $\|y\|_p \leqslant \|x\|_p$ and $\lambda = \left(\frac{\|y\|_p}{\|x\|_p}\right)^{\frac{r}{q}}$, then from (3.4) we have

$$\begin{aligned} \|Tx - Ty\|_q &= \|x\|_p^r \left\| T_0\left(\frac{x}{\|x\|_p^{\frac{1}{p}}}\right) - \left(\frac{\|y\|_p}{\|x\|_p}\right)^{\frac{r}{q}} T_0\left(\frac{y}{\|y\|_p^{\frac{1}{p}}}\right) \right\|_q \\ &= \|x\|_p^r \left\| \frac{x}{\|x\|_p^{\frac{1}{p}}} - \left(\frac{\|y\|_p}{\|x\|_p}\right)^{\frac{rs}{q}} \frac{y}{\|y\|_p^{\frac{1}{p}}} \right\|_p^r \\ &= \|x - y\|_p^r. \end{aligned}$$

Therefore T is an r-isometry. This completes the proof. □

THEOREM 3.2. *Let X be a p-normed space and Y a q-normed space $(0 < p, q \leqslant 1)$. Let $0 < r \leqslant q$ and $s = \frac{q}{pr}$. Let $T_0 : S(X) \to S(Y)$ be a surjectively r-isometric operator satisfying*

$$\|T_0 x - \lambda T_0 y\|_q \geqslant \|x - \lambda^s y\|_p^r, \ \textit{for } x, y \in S(X) \textit{ and } \lambda \in [0,1]. \tag{3.5}$$

Then T_0 can be extended to be an r-isometry T of X onto Y and T is positively s^{-1}-homogeneous.

Proof. Since $T_0 : S(X) \to S(Y)$ is an r-isometric operator, we derive that T_0 is an injection. Since T_0 is also surjective, we see that T_0 is bijective. Thus, $S_0 = T_0^{-1} : S(Y) \to S(X)$ is an r^{-1}-isometry. Let $\mu = \lambda^s$ and $t = s^{-1}$. From (3.5) we have

$$\|S_0 x - \mu S_0 y\|_p \leqslant \|x - \mu^t y\|_q^{\frac{1}{r}}, \ \text{for } x, y \in S(Y) \text{ and } \mu \in [0,1].$$

By Theorem 3.1, S_0 can be extended to be a bijective r^{-1}-isometry S of Y onto X and S is positively t^{-1}-homogeneous. Setting $T = S^{-1}$, then as an extension of T_0, T is r-isometric from X onto Y and is positively s^{-1}-homogeneous. This completes the proof. □

THEOREM 3.3. *Let X be a p-normed space and Y a q-normed space $(0 < p, q \leqslant 1)$. Let $r > 0, rp \geqslant 1$ and $s = \frac{q}{pr}$. Let $T : X \to Y$ be an r-nonexpansive operator* *(not necessarily surjective) such that the restriction $T|_{S(X)}$ is an r-isometry. If T is positively s^{-1}-homogeneous, then T is a r-isometry.*

Proof. Firstly, we verify that

$$\|Tx-Ty\|_q = \|x-y\|_p^r, \quad \text{for } x \in S(X) \text{ and } y \in B(X). \tag{3.6}$$

Assume that (3.6) does not hold. Since T is r-nonexpansive and $T|_{S(X)} : S(X) \to S(Y)$ is a r-isometry, there exist $x_0, y_0 \in X$ with $\|x_0\|_p = 1$ and $\|y_0\|_p < 1$ such that

$$\|Tx_0 - Ty_0\|_q < \|x_0 - y_0\|_p^r.$$

By Lemma 2.2, there exist $z_0 \in S(X)$ and $\lambda_0 \in (0,1]$ such that $y_0 = \lambda_0 z_0 + (1-\lambda_0)x_0$. From Lemma 2.1 it follows that

$$\begin{aligned} \|x_0 - z_0\|_p^r &\geqslant \left[\lambda_0^{pr} + (1-\lambda_0)^{pr}\right] \|x_0 - z_0\|_p^r = \|x_0 - y_0\|_p^r + \|y_0 - z_0\|_p^r \\ &> \|Tx_0 - Ty_0\|_q + \|Ty_0 - Tz_0\|_q \geqslant \|Tx_0 - Tz_0\|_q = \|x_0 - z_0\|_p^r, \end{aligned}$$

which is a contradiction.

Next, we will prove that

$$\|Tx-Ty\|_q = \|x-y\|_p^r, \quad \text{for } x, y \in B(X). \tag{3.7}$$

Keeping in mind that (3.6) holds, without loss of generality, we suppose that $\|x\|_p < 1$ and $\|y\|_p < 1$ with $y \neq x$. By Lemma 2.2, there exist $z \in S(X)$ and $\mu \in (0,1]$ such that $y = \mu z + (1-\mu)x$. Thus, by Lemma 2.1 we have

$$\begin{aligned} \|x-z\|_p^r &\geqslant [\mu^{pr} + (1-\mu)^{pr}]\, \|x-z\|_p^r = \|x-y\|_p^r + \|y-z\|_p^r \\ &\geqslant \|Tx-Ty\|_q + \|Ty-Tz\|_q \geqslant \|Tx-Tz\|_q. \end{aligned} \tag{3.8}$$

For $x, y \in B(X)$ and $z \in S(X)$, by using (3.6) we get

$$\|x-z\|_p^r = \|Tx-Tz\|_q \quad \text{and} \quad \|y-z\|_p^r = \|Ty-Tz\|_q. \tag{3.9}$$

Combining (3.8) and (3.9) we deduce that $\|Tx-Ty\|_q = \|x-y\|_p^r$, namely (3.7) holds.

Finally, we show that T is a r-isometry. In fact, for $x, y \in X$ there is $\alpha > 0$ such that $\alpha x, \alpha y \in B(X)$. Thus, by (3.7) and the positive s^{-1}-homogeneity of T we conclude that

$$\|Tx-Ty\|_q = \alpha^{-qs^{-1}} \|T(\alpha x) - T(\alpha y)\|_q = \alpha^{-qs^{-1}} \|\alpha x - \alpha y\|_p^r = \|x-y\|_p^r.$$

This completes the proof. $\square$

THEOREM 3.4. *Let X be a p-normed space and Y a q-normed space $(0 < p, q \leqslant 1)$. Let $0 < r \leqslant q$ and $s = \frac{q}{pr}$. Let $T : X \to Y$ be an r-expansive operator such that the restriction $T|_{S(X)}$ is a surjective r-isometry. If T is positively s^{-1}-homogeneous, then T is an r-isometry.*

Proof. Since $T|_{S(X)}$ is surjective and T is positively s^{-1}-homogeneous, we see that T is surjective. Taking into account the fact that T is r-expansive, we deduce that

T is also injective, and so it is invertible. Hence $T^{-1}: Y \to X$ is r^{-1}-nonexpansive such that the restriction $T^{-1}|_{S(Y)}$ is a r^{-1}-isometry. It is evident that $q \cdot \frac{1}{r} \geqslant 1$. Set $t = \frac{p}{q \cdot \frac{1}{r}} = \frac{pr}{q} = s^{-1}$. Then T^{-1} is positively t^{-1}-homogeneous. Using Theorem 3.3, we get that T^{-1} is an r^{-1}-isometry. Therefore $T: X \to Y$ is an r-isometry. $\square$

Taking $r = \frac{1}{p}$ in Theorems 3.1 and $r = q$ in Theorems 3.2, from Theorems 3.1 and 3.2 we obtain the following consequences, respectively.

COROLLARY 3.5. *Let X be a p-normed space and Y a q-normed space $(0 < p, q \leqslant 1)$. Let $T_0 : S(X) \to S(Y)$ be a p^{-1}-isometry (not necessarily surjective) satisfying*

$$\|T_0x - \lambda T_0y\|_q \leqslant \|x - \lambda^q y\|_p^{\frac{1}{p}}, \text{ for } x, y \in S(X) \text{ and } \lambda \in [0,1].$$

Then T_0 can be extended to be a p^{-1}-isometry T of X into Y and T is positively $\frac{1}{q}$-homogeneous.

COROLLARY 3.6. *Let X be a p-normed space and Y a q-normed space $(0 < p, q \leqslant 1)$. Let $T_0 : S(X) \to S(Y)$ be a surjectively q-isometric operator satisfying*

$$\|T_0x - \lambda T_0y\|_q \geqslant \|x - \lambda^{\frac{1}{p}} y\|_p^q, \text{ for } x, y \in S(X) \text{ and } \lambda \in [0,1].$$

Then T_0 can be extended to be a q-isometry T of X onto Y and T is positively p-homogeneous.

In Theorem 3.1, if $r = 1$, then we have $p = 1$. In Theorem 3.2, if $r = 1$, then we have $q = 1$. Thus, from Theorems 3.1 and 3.2 we obtain the following consequences, respectively.

COROLLARY 3.7. *Let X be a normed space and Y a q-normed space $(0 < q \leqslant 1)$. Let $T_0 : S(X) \to S(Y)$ be an isometry (not necessarily surjective) satisfying*

$$\|T_0x - \lambda T_0y\|_q \leqslant \|x - \lambda^q y\|, \text{ for } x, y \in S(X) \text{ and } \lambda \in [0,1].$$

Then T_0 can be extended to be an isometry T of X into Y and T is positively $\frac{1}{q}$-homogeneous.

COROLLARY 3.8. *Let X be a p-normed space $(0 < p \leqslant 1)$ and Y a normed space. Let $T_0 : S(X) \to S(Y)$ be a surjectively isometric operator satisfying*

$$\|T_0x - \lambda T_0y\| \geqslant \|x - \lambda^{\frac{1}{p}} y\|_p, \text{ for } x, y \in S(X) \text{ and } \lambda \in [0,1].$$

Then T_0 can be extended to be an isometry T of X onto Y and T is positively p-homogeneous.

REMARK 3.1. The well-known Mazur-Ulam theorem [19] states that any surjective isometry T between two real normed spaces with $T(\theta) = \theta$ must be linear. According to the Mazur-Ulam theorem, we can see that the operator in Theorem 3.2 is linear in the case of $p=q=r=1$. From the following Corollary 3.9 we can see that the operator in Theorem 3.1 in the case of $p=q=r=1$ is linear if Y is strictly convex. Theorem 3.1 in the case of $p=q=r=1$ and Lemma 2.1 in [10] are the results under different conditions: in Theorem 3.1 (and Corollary 3.9) in the case of $p=q=r=1$, the operator $T_0: S(X) \to S(Y)$ may not necessarily be surjective, but Y is strictly convex; in Lemma 2.1 in [10], the operator may necessarily be surjective. Another formal difference between the two is $\lambda \in [0,1]$ and $\lambda > 0$. Since the norm is continuous, condition $\lambda = 0$ can be omitted. Since the unit sphere is symmetric, condition $\lambda > 1$ can be omitted. Considering the inverse operator in conditions in Theorem 3.2, we see that Theorem 3.2 in the case of $p=q=r=1$ and Lemma 2.1 in [10] are consistent.

COROLLARY 3.9. *Let X be a p-normed space $(0<p\leqslant 1)$ and Y a strictly convex normed space. Let $T_0: S(X)\to S(Y)$ be a p^{-1}-isometric operator (not necessarily surjective) satisfying*

$$\|T_0x-\lambda T_0y\| \leqslant \|x-\lambda y\|_p^{\frac{1}{p}} \text{ for } x,y\in S(X) \text{ and } \lambda\in[0,1].$$

Then T_0 can be extended to be a linear p^{-1}-isometry T of X into Y.

Proof. Let $x,y\in X$ and $z=x+y$. Then

$$\left\|\frac{z}{2}-x\right\|_p^{\frac{1}{p}} = \left\|\frac{z}{2}-y\right\|_p^{\frac{1}{p}} = \frac{1}{2}\|x-y\|_p^{\frac{1}{p}}.$$

Since T is a p^{-1}-isometry by Corollary 3.5 $(q=1)$, we have

$$\left\|T\left(\frac{z}{2}\right)-Tx\right\| = \left\|T\left(\frac{z}{2}\right)-Ty\right\| = \frac{1}{2}\|Tx-Ty\|, \tag{3.10}$$

which follows that

$$\left\|Tx-T\left(\frac{z}{2}\right)\right\| + \left\|T\left(\frac{z}{2}\right)-Ty\right\| = \left\|Tx-T\left(\frac{z}{2}\right)+T\left(\frac{z}{2}\right)-Ty\right\|.$$

Since Y is strictly convex, there exists $\gamma>0$ such that

$$Tx-T\left(\frac{z}{2}\right) = \gamma\left[T\left(\frac{z}{2}\right)-Ty\right].$$

From (3.10) implies that $\gamma=1$, and thus $T\left(\frac{x+y}{2}\right)=\frac{Tx+Ty}{2}$. By Corollary 3.5 $(q=1)$, T is positively homogeneous. So, we can infer that T is additive and $T(\theta)=\theta$. Also, from $\theta=T(x-x)=T(x)+T(-x)$ we have $T(-x)=-T(x)$, which follows that T is homogeneous. Therefore, T is linear. This completes the proof. □

REMARK 3.2. If $p=1$, then the result in Corollary 3.9 was given by Yang et al [37].

REMARK 3.3. In Corollary 3.9, if T_0 is a p^{-1}-isometric bijection, then by using a generalization of the Mazur-Ulam theorem ([1, 22, 24]) we see that the condition of strictly convex space can be omitted.

REMARK 3.4. A counterexample given in [2] is that L_p and L_q are not isometric, where $0 < p < q \leqslant 1$. It was pointed out in [17] that, for two normed spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$, if $(Y, \|\cdot\|)$ is a strictly convex and $0 < p < 1$, then there does not exist any isometry from $(X, \|\cdot\|)$ to $(Y, \|\cdot\|^p)$. In our discussions, we consider r-isometry with $rp \geqslant 1$ or $0 < r \leqslant q$, avoid the above two situations.

4. Isometries of distance preserving

THEOREM 4.1. *Let X be a p-normed space $(0 < p \leqslant 1)$ and Y a strictly convex normed space. Let $T : X \to Y$ be an r-nonexpansive operator. If T satisfies the distance σ power r preserving property and $rp \geqslant 1$, then T is an r-isometry.*

Proof. By Lemma 2.5 $(q = 1)$, we have

$$\|Tx - Ty\| = \|x - y\|_p^r \text{ if } \|x - y\|_p \leqslant \sigma \text{ for } x, y \in X. \tag{4.1}$$

In order to reach the desired conclusion, it is enough to prove that

$$\|Tx - Ty\| = \|x - y\|_p^r \text{ if } \|x - y\|_p \leqslant 2^{np}\sigma \text{ for all } n \in \mathbb{N} \text{ and } x, y \in X. \tag{4.2}$$

Evidently, from (4.1) we see that (4.2) holds in the case of $n = 0$. We now assume that (4.2) holds in the case of $n = k$ and $\|x - y\|_p \leqslant 2^{(k+1)p}\sigma$ for $x, y \in X$. From (4.1) we can suppose $a = \|x - y\|_p > \sigma$, which does not restrict the generality. Taking $w = \frac{x+y}{2}$ and $u = w + \left(\frac{\sigma}{a}\right)^{\frac{1}{p}}(x - w)$ we have successively

$$\|x - w\|_p = \|y - w\|_p = \frac{\|x - y\|_p}{2^p} \leqslant 2^{kp}\sigma, \tag{4.3}$$

$$\|u - w\|_p = \frac{\sigma}{a}\|x - w\|_p = \frac{\sigma}{a} \cdot \frac{a}{2^p} = \frac{\sigma}{2^p} < \sigma, \tag{4.4}$$

$$\begin{aligned}\|u - x\|_p^{\frac{1}{p}} &= \left\|(w - x)\left[1 - \left(\frac{\sigma}{a}\right)^{\frac{1}{p}}\right]\right\|_p^{\frac{1}{p}} = \|w - x\|_p^{\frac{1}{p}}\left[1 - \left(\frac{\sigma}{a}\right)^{\frac{1}{p}}\right] \\ &= \|w - x\|_p^{\frac{1}{p}} - \|w - u\|_p^{\frac{1}{p}} < 2^k\sigma^{\frac{1}{p}}.\end{aligned} \tag{4.5}$$

In view of the inductive assumption, from (4.3), (4.4) and (4.5) we infer that

$$\|Tw - Tx\|^{\frac{1}{rp}} = \|Tw - Tu\|^{\frac{1}{rp}} + \|Tu - Tx\|^{\frac{1}{rp}}. \tag{4.6}$$

Since $rp \geqslant 1$, it follows (4.6) and Lemma 2.1 that $\|Tw - Tx\| \geqslant \|Tw - Tu\| + \|Tu - Tx\|$. Combining it with $\|Tw - Tx\| \leqslant \|Tw - Tu\| + \|Tu - Tx\|$, we obtain

$$\|Tw - Tx\| = \|Tw - Tu\| + \|Tu - Tx\|. \tag{4.7}$$

Since Y is strictly convex, from (4.7) it follows that there exists $\beta > 0$ such that

$$Tu - Tx = \beta(Tw - Tu), \tag{4.8}$$

and so $\|Tu-Tx\| = \beta\|Tw-Tu\|$. Moreover, in view of the inductive assumption, from (4.7) we deduce that

$$\|w-x\|_p^r = \|w-u\|_p^r + \|u-x\|_p^r \text{ and } \|u-x\|_p^r = \beta\|w-u\|_p^r. \tag{4.9}$$

Thus, (4.9) and (4.4) yield $\beta = \left(\frac{a}{\sigma}\right)^r - 1$, and (4.8) means that

$$Tx = (1+\beta)Tu - \beta Tw. \tag{4.10}$$

Likewise, taking $v = w + \left(\frac{\sigma}{a}\right)^{\frac{1}{p}}(y-w)$, in the similar way as above, we can infer that

$$Ty = (1+\beta)Tv - \beta Tw, \tag{4.11}$$

where $\beta = \left(\frac{a}{\sigma}\right)^r - 1$. Keeping in mind that T satisfies the distance σ power r preserving property, from the equalities $\|u-v\| = \sigma$, (4.10) and (4.11) we conclude that

$$\|Tx-Ty\| = (1+\beta)\|Tu-Tv\| = (1+\beta)\sigma^r = a^r = \|x-y\|_p^r, \tag{4.12}$$

which shows that (4.2) holds in the case of $n = k+1$. By induction, the proof is completed. $\square$

COROLLARY 4.2. *Let X be a p-normed space $(0 < p \leqslant 1)$ and Y a strictly convex normed space. Let $T : X \to Y$ be a locally r-nonexpansive operator. If T satisfies the distance σ power r preserving property and $rp \geqslant 1$, then T is an r-isometry.*

Proof. It follows from Theorem 4.1 and Lemma 2.4. $\square$

REMARK 4.1. If $\sigma = 1$ and $r = \frac{1}{p}$, then the result in Corollary 4.2 was given by Ma [17].

THEOREM 4.3. *Let X be a strictly convex normed space and Y a q-normed space $(0 < q \leqslant 1)$. Let $T : X \to Y$ be a surjectively r-expansive operator and $r \leqslant q$. If T satisfies the strong distance σ power r preserving property, then T is an r-isometry.*

Proof. Since $T : X \to Y$ is r-expansive, namely

$$\|Tx-Ty\|_q \geqslant \|x-y\|^r, \text{ for } x, y \in X, \tag{4.13}$$

we can infer that T is an injection. Considering that T is surjective, we claim that T is an invertible operator. From (4.13) we see that $S = T^{-1} : Y \to X$ satisfies

$$\|Sx-Sy\| \leqslant \|x-y\|_q^{\frac{1}{r}}, \text{ for } x, y \in Y.$$

This means that S is $\frac{1}{r}$-nonexpansive and $\frac{q}{r} \geqslant 1$. Moreover, in view of the hypothesis, we deduce that S satisfies the strong distance σ power $\frac{1}{r}$ preserving property. By Theorem 4.1, S is an $\frac{1}{r}$-isometry. Therefore T is an r-isometry. This completes the proof. $\square$

REMARK 4.2. The condition $r \leqslant q$ in Theorem 4.3 is necessary. Let X, Y be normed spaces (namely $q = 1$) and $r > 1$. We now point out that there does not exist any r-isometry from X to Y. Assume that T is an r-isometry from X to Y, $x, y \in X$ with $x \neq y$, then we can infer that

$$\begin{aligned}\|x-y\|^r = \|Tx - Ty\| &\leqslant \left\|Tx - T\left(\frac{x+y}{2}\right)\right\| + \left\|T\left(\frac{x+y}{2}\right) - Ty\right\| \\ &= \left\|x - \frac{x+y}{2}\right\|^r + \left\|\frac{x+y}{2} - y\right\|^r = 2^{1-r}\|x-y\|^r,\end{aligned}$$

which contradicts with $2^{1-r} < 1$.

THEOREM 4.4. *Let X and Y be two normed spaces. Let $T: X \to Y$ be a surjectively expansive operator. If T satisfies the strong distance σ preserving property, and*

$$\|x-y\| < \sigma \ \Rightarrow\ \|Tx-Ty\| < \sigma, \ \textit{for}\ x, y \in X, \tag{4.14}$$

then T is an isometry.

Proof. We first prove that

$$\|Tx-Ty\| \leqslant \sigma \ \Rightarrow\ \|Tx-Ty\| \leqslant \|x-y\|, \ \text{for}\ x, y \in X. \tag{4.15}$$

Clearly, since T satisfies the strong distance σ preserving property, (4.15) holds in the case of $\|Tx-Ty\| = \sigma$. Without loss of generality, we suppose $\|Tx-Ty\| < \sigma$ and $Tx \neq Ty$. Take $u = Tx + \frac{\sigma}{b}(Ty - Tx)$, where $b = \|Tx - Ty\|$. Since T is surjective, there exists $z \in X$ such that $u = Tz$, namely $Tz = Tx + \frac{\sigma}{b}(Ty - Tx)$. Thus, we have $\|Tz - Tx\| = \sigma$ and $\|Tz - Ty\| = \left\|(Tx-Ty)\left[1 - \frac{\sigma}{b}\right]\right\| = \sigma - b$. Since T is expansive and strong distance σ preserving, we obtain

$$\begin{aligned}\|Tx-Ty\| = b = \sigma - \|Tz-Ty\| &\leqslant \sigma - \|z-y\| \\ &= \|z-x\| - \|z-y\| \leqslant \|x-y\|.\end{aligned}$$

Hence (4.15) holds. Again, taking into account that T is expansive and strong distance σ preserving, we have

$$\|x-y\| = \sigma \ \Leftrightarrow\ \|Tx-Ty\| = \sigma; \ \text{for}\ x, y \in X. \tag{4.16}$$

$$\|Tx-Ty\| \leqslant \sigma \ \Rightarrow\ \|x-y\| \leqslant \sigma, \ \text{for}\ x, y \in X. \tag{4.17}$$

Combining (4.14), (4.16) and (4.17) we deduce that

$$\|x-y\| \leqslant \sigma \ \Leftrightarrow\ \|Tx-Ty\| \leqslant \sigma, \ \text{for}\ x, y \in X. \tag{4.18}$$

Thus, (4.18) and (4.15) yield

$$\|x-y\| \leqslant \sigma \ \Rightarrow\ \|Tx-Ty\| \leqslant \|x-y\|, \ \text{for}\ x, y \in X,$$

which means that T is locally nonexpansive. By Lemma 2.4, T is nonexpansive in the whole space. Finally, keeping in mind that T is expansive, we conclude that T is an isometry. This completes the proof. $\square$

THEOREM 4.5. *Let X be a p-normed space and Y a q-normed space $(0 < p,q \leqslant 1)$. Let $T : X \to Y$ be a locally r-nonexpansive operator, where $rp \geqslant 1$. If T satisfies the distance $n\sigma$ power r preserving property for each $n \in \mathbb{N}$, then T is an r-isometry.*

Proof. Since T is locally r-nonexpansive, and $rp \geqslant 1$, by Lemma 2.4 we see that T is r-nonexpansive. Since T satisfies the distance σ power r preserving property, by Lemma 2.5 we have

$$\|Tx - Ty\|_q = \|x - y\|_p^r \text{ if } \|x - y\|_p \leqslant \sigma. \tag{4.19}$$

Now we suppose that $x, y \in X$, $\|x - y\|_p > \sigma$, and set $\|x - y\|_p = b$. Then there exists $n \in \mathbb{N}$ such that $n\sigma < \|x - y\|_p \leqslant (n+1)\sigma$. If $\|Tx - Ty\|_q < \|x - y\|_p^r$, then by taking $w = x + (n+1)^{\frac{1}{p}} \left(\frac{\sigma}{b}\right)^{\frac{1}{p}} (y - x)$, we have $\|w - x\|_p = (n+1)\sigma$ and

$$\|w - y\|_p = \left\| (x-y)\left[1 - (n+1)^{\frac{1}{p}} \left(\frac{\sigma}{b}\right)^{\frac{1}{p}}\right] \right\|_p = \left[(n+1)^{\frac{1}{p}} \sigma^{\frac{1}{p}} - b^{\frac{1}{p}}\right]^p. \tag{4.20}$$

Since $\|Tw - Ty\|_q \leqslant \|w - y\|_p^r$, from (4.20) it follows that

$$\begin{aligned}(n+1)^r \sigma^r &= \|w - x\|_p^r = \|Tw - Tx\|_q \leqslant \|Tw - Ty\|_q + \|Ty - Tx\|_q \\ &< \|w - y\|_p^r + \|x - y\|_p^r = \left[(n+1)^{\frac{1}{p}} \sigma^{\frac{1}{p}} - b^{\frac{1}{p}}\right]^{pr} + b^r \\ &\leqslant (n+1)^r \sigma^r - b^r + b^r = (n+1)^r \sigma^r,\end{aligned}$$

which is a contradiction. Hence we have $\|Tx - Ty\|_q = \|x - y\|_p^r$ if $\|x - y\|_p > \sigma$. Combining it with (4.19), we conclude that T is an r-isometry. This completes the proof. $\square$

COROLLARY 4.6. *Let X and Y be two normed spaces. Let $T : X \to Y$ be a locally nonexpansive and surjective operator. If T satisfies the distance σ preserving property and*

$$\|x - y\| > \sigma \Rightarrow \|Tx - Ty\| > \sigma, \text{ for } x, y \in X, \tag{4.21}$$

then T is an isometry.

Proof. Since T is locally nonexpansive and surjective, by Lemma 2.4 $(r = p = q = 1)$ we see that T is nonexpansive and bijective. For each $n \in \mathbb{N}$ $(n \geqslant 2)$ and $x, y \in X$ with $\|x - y\| = n\sigma$, we will prove that $\|Tx - Ty\| = n\sigma$. By Lemma 2.5 $(r = p = q = 1)$ we have

$$\|Tx - Ty\| = \|x - y\| \text{ if } \|x - y\| \leqslant \sigma. \tag{4.22}$$

If $\|Tx - Ty\| < n\sigma$, then by taking $u_i = Tx + \frac{i}{n}(Ty - Tx)$, $i = 0, 1, \cdots, n$, we get $\|u_i - u_{i-1}\| = \frac{1}{n}\|Tx - Ty\| < \sigma$. Since T is a bijection, there is $z_i \in X$ such that $Tz_i = u_i$

$(i=0,1,\cdots,n)$, $z_0=x$ and $z_n=y$. From $\|Tz_i-Tz_{i-1}\|<\sigma$ and (4.21) we infer that $\|z_i-z_{i-1}\|\leqslant\sigma$. It follows from (4.22) that $\|z_i-z_{i-1}\|=\|Tz_i-Tz_{i-1}\|<\sigma$, and so

$$\|y-x\|=\left\|\sum_{i=1}^{n}(z_i-z_{i-1})\right\|\leqslant\sum_{i=1}^{n}\|z_i-z_{i-1}\|<n\sigma,$$

which is a contradiction. Hence T satisfies the distance σ power preserving property for each $n\in\mathbb{N}$. By Theorem 4.5 $(r=p=q=1)$, T is an isometry. $\square$

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REFERENCES

[1] M. AGHAJANI, *On the Mazur-Ulam theorem*, Analysis Math. **44** (2018), 401–407.

[2] F. ALBIAC, *Nonlinear structure of some classical quasi-Banach spaces and F-spaces*, J. Math. Anal. Appl. **340** (2008), 1312–1325.

[3] A. D. ALEKSANDROV, *Mappings of families of sets*, Dokl. Akad. Nauk SSSR **190** (3) (1970), 502–505.

[4] A. BAYOUMI, *Foundations of Complex Analysis in Non Locally Convex Spaces–Function Theory Without Convexity Conditions*, Mathematics Studied **193**, North Holland, Amsterdam, 2003.

[5] L. CHENG AND Y. DONG, *On a generalized Mazur-Ulam question: extension of isometries between unit spheres of Banach spaces*, J. Math. Anal. Appl. **377** (2011), 464–470.

[6] H. Y. CHU, C. G. PARK AND W. G. PARK, *The Alexandrov problem in linear 2-normed spaces*, J. Math. Anal. Appl. **289** (2004), 666–672.

[7] G.-G. DING, *On isometric extension problem between two unit spheres*, Sci. China Ser. A **52** (2009), 2069–2083.

[8] G.-G. DING, *On the linearly isometric extension problem* (in Chinese), Scientia Sinica Mathematica **45** (2015), 1–8.

[9] G.-G. DING AND J.-Z. LI, *Isometric extension problem between strictly convex two-dimensional normed spaces*, Acta Math. Sin. (Engl. Ser.) **35** (2019), 513–518.

[10] X.-N. FANG AND J.-H. WANG, *On extension of isometries between the unit spheres of normed space E and $C(\Omega)$*, Acta Math. Sin. (Engl. Ser.) **22** (2006), 1819–1824.

[11] J.-M. GAO, *On the Alexandrov problem of distance preserving mapping*, J. Math. Anal. Appl. **352** (2009), 583–590.

[12] X.-J. HUANG AND D.-N. TAN, *Mappings of preserving n-distance one in n-normed spaces*, Aequat. Math. **92** (2018), 401–413.

[13] N. J. KALTON, *Quasi-Banach spaces*, Handbook of the Geometry of Banach spaces, Elsevier, Amsterdam, 2003.

[14] H. KHODAEI AND A. MOHAMMADI, *Generalizations of Alesandrov problem and Mazur-Ulam theorem for two-isometries and two-expansive mappings*, Commun. Korean Math. Soc. **34** (2019), 771–782.

[15] C.-W. LEUNG, C.-K. NG AND N.-C. WONG, *On a variant of Tingley's problem for some function spaces*, J. Math. Anal. Appl. **496**: 124800 (2021), 1–16.

[16] L. LI AND W.-Y. REN, *Isometries on the quasi-Banach spaces L_p $(0<p<1)$*, Acta Math. Sin. (Engl. Ser.) **26** (2010), 1519–1524.

[17] Y.-M. MA, *The Aleksandrov problem for unit distance preserving mapping*, Acta. Math. Sci. (Ser. B) **20** (2000), 359–364.

[18] Y.-M. MA, *The Aleksandrov-Benz-Rassias problem on linear n-normed spaces*, Monatsh. Math. **180** (2016), 305–316.

[19] S. MAZUR AND S. ULAM, *Sur les transformations isométriques d'espaces vectoriels normés*, C. R. Acad. Sci. Paris **194** (1932), 946–948.

[20] B. MIELNIK AND T. M. RASSIAS, *On the Aleksandrov problem of conservative distance*, Proc. Amer. Math. Soc. **116** (1992), 1115–1118.

[21] T. M. RASSIAS, *On the A. D. Aleksandrov problem of conservative distances and the Mazur-Ulam theorem*, Nonlinear Anal. **47** (2001), 2597–2608.

[22] T. M. RASSIAS, *Properties of isometric mappings*, J. Math. Anal. Appl. **235** (1999), 108–121.

[23] S. ROLEWICZ, *Metric Linear Spaces*, PWN-Polich Scientific Publishers, Warszawa, 1985.

[24] S. ROLEWICZ, *A generalization of the Mazur-Ulam theorem*, Studia Math. **31** (1968), 501–505.

[25] H. RUI, *On the maps preserving the equality of distance*, J. Math. Anal. Appl. **343** (2008), 1161–1165.

[26] D.-N. TAN, *Nonexpansive mapping and expansive mappings on the unit spheres of some F-spaces*, Bull. Aust. Math. Soc. **82** (2010), 22–30.

[27] D.-N. TAN, *On extension of isometries on the unit spheres of L^p-spaces for $0 < p \leqslant 1$*, Nonlinear Anal. **74** (2011), 6981–6987.

[28] D.-N. TAN, *Phase-isometries on the unit sphere of $C(K)$*, Ann. Funct. Anal. **12**: 15 (2021), 1–14.

[29] D.-N. TAN AND X.-J. HUANG, *Phase-isometries on real normed spaces*, J. Math. Anal. Appl. **488**: 124058 (2020), 1–17.

[30] D. TINGLEY, *Isometries of the unit sphere*, Geom. Dedicata **22** (1987), 371–378.

[31] A. VOGT, *Maps which preserve equality of distance*, Stud. Math. **45** (1973), 43–48.

[32] J. WANG, *On extension of isometries between unit spheres of AL_p-spaces $(0 < p < \infty)$*, Proc. Amer. Math. Soc. **132** (2004), 2899–2909.

[33] S.-M. WANG AND W.-Y. REN, *On the Aleksandrov-Rassias problem of distance preserving mappings*, Bull. Aust. Math. Soc. **91** (2015), 464–470.

[34] J.-Z. XIAO AND Y. LU, *Some fixed point theorems for s-convex subsets in p-normed spaces based on measures of noncompactness*, J. Fixed Point Theory Appl. **20**: 2 (2018), 1–22.

[35] J.-Z. XIAO AND Y. LU, *Extension of isometries between the unit spheres of p-normed spaces*, Ann. Funct. Anal. **13**: 25 (2022), 1–10.

[36] J.-Z. XIAO AND X.-H. ZHU, *Some fixed point theorems for s-convex subsets in p-normed spaces*, Nonlinear Anal. **74** (2011), 1738–1748.

[37] X.-Z. YANG, Z.-B. HOU AND X.-H. FU, *On linear extension of isometries between the unit spheres of β-normed spaces*, Acta Math. Sinica (Chin. Ser.) **48** (2005), 1199–1202.

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