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INNER PRODUCT INEQUALITIES THROUGH CARTESIAN DECOMPOSITION WITH APPLICATIONS TO NUMERICAL RADIUS INEQUALITIES

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Abstract. This paper intends to show several inner product inequalities using the Cartesian decomposition of the operator. We utilize the obtained results to get norm and numerical radius inequalities. Our results extend and improve some earlier inequalities. Among other inequalities, it is revealed that if T is a $n\times n$ complex matrix with the imaginary part $\Im T=\frac{T-T^*}{2\mathrm{i}}$, then

$$\frac{1}{2}\max\left(\left\|TT^*-\mathrm{i}\Im T^2\right\|^{\frac{1}{2}},\left\|T^*T+\mathrm{i}\Im T^2\right\|^{\frac{1}{2}}\right)\leqslant\omega(T)$$

which is a significant improvement of the classical inequality $\frac{1}{2}\|T\|\leqslant\omega(T)$.

1. Introduction

In a complex Hilbert space $\mathscr{H}$ with the inner product $\langle\cdot,\cdot\rangle$, we denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$ as $\mathscr{B}(\mathscr{H})$. In the case when $\dim\mathscr{H}=n$, we identify $\mathscr{B}(\mathscr{H})$ with the matrix algebra $\mathscr{M}_n$ of all $n\times n$ matrices with entries in the complex field $\mathbb{C}$. For any $T\in\mathscr{B}(\mathscr{H})$, we can write $T=A+\mathrm{i}B$ in which $A=\Re T=\frac{T+T^*}{2}$ and $B=\Im T=\frac{T-T^*}{2\mathrm{i}}$ are self-adjoint operators. This is the so-called Cartesian decomposition of T. For any $T\in\mathscr{B}(\mathscr{H})$, we can define its numerical radius and the operator norm, respectively represented by $\omega(T)=\sup_{\|x\|=1}|\langle Tx,x\rangle|$ and $\|T\|=\sup_{\|x\|=1}\|Tx\|$. Two important inequalities for the usual operator norm and numerical radius are that

$$\|T^n\|\leqslant\|T\|^n \text{ and } \omega(T^n)\leqslant\omega^n(T);\ n=1,2,\ldots.$$

If T is normal, meaning $T^*T=TT^*$, it is widely known that $\omega(T)=\|T\|$. However, this equality fails for non-normal operators. Instead, we can establish the following inequality for any $T\in\mathscr{B}(\mathscr{H})$:

$$\frac{1}{2}\|T\|\leqslant\omega(T)\leqslant\|T\|. \tag{1.1}$$

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This inequality is significant because it approximates the numerical radius $\omega(T)$ in terms of the more computationally manageable quantity $\|T\|$.

As a result, researchers have been focusing on sharpening this and other inequalities for the numerical radius, as found in [5, 10, 12, 13, 14, 17]. Below, we list some results regarding the inequality (1.1).

Kittaneh [16, Theorem 1] proposed an improvement of (1.1) in the following manner:

$$\frac{1}{4}\left\||T|^2+|T^*|^2\right\| \leqslant \omega^2(T) \leqslant \frac{1}{2}\left\||T|^2+|T^*|^2\right\|.$$

In [14, Corollary 3.4], the previouse inequality was improved as follows:

$$\omega(T) \leqslant \frac{1}{2}\sqrt{\left\||T|^2+|T^*|^2\right\|+2\omega(|T|\,|T^*|)}. \tag{1.2}$$

After that, in [20, Corollary 2.8], inequality (1.2) was refined:

$$\omega(T) \leqslant \frac{1}{2}\sqrt{\left\||T|^2+|T^*|^2\right\|+\|\,|T|\,|T^*|+|T^*|\,|T|\,\|}. \tag{1.3}$$

Inequality (1.3) can be written in the following arrangement:

$$\omega(T) \leqslant \frac{1}{2}\sqrt{\left\||T|^2+|T^*|^2\right\|+2\,\|\Re(|T|\,|T^*|)\|}.$$

Here, we point out that inequalities (1.2) and (1.3) have been proved and generalized separately in [2] and [3].

This paper aims to demonstrate considerable inequalities for inner products through the operator's Cartesian decomposition. The results are then applied to obtain inequalities for norm and numerical radius. Furthermore, our research improves and generalizes earlier established inequalities.

In order to accomplish these aims, we will require the following facts:

(I) (Mixed Schwarz inequality [11, pp. 75–76]) For any $T \in \mathscr{B}(\mathscr{H})$ and $x,y \in \mathscr{H}$,

$$|\langle Tx,y\rangle|^2 \leqslant \left\langle |T|^{2\nu}x,x\right\rangle \left\langle |T^*|^{2(1-\nu)}y,y\right\rangle;\ (\nu \in [0,1]). \tag{1.4}$$

(II) [7, (2.26)] For any $x,y,z \in \mathscr{H}$,

$$|\langle z,x\rangle|^2+|\langle z,y\rangle|^2 \leqslant \|z\|^2 \max\left(\|x\|^2,\|y\|^2\right)+|\langle x,y\rangle|. \tag{1.5}$$

(III) (Buzano inequality [4]) For any $x,y,z \in \mathscr{H}$,

$$|\langle z,x\rangle|\,|\langle z,y\rangle| \leqslant \frac{\|z\|^2}{2}\left(|\langle x,y\rangle|+\|x\|\,\|y\|\right). \tag{1.6}$$

(IV) (Arithmetic-geometric mean inequality for the usual operator norm [1]) For any $S,T \in \mathscr{B}(\mathscr{H})$,

$$\|ST\| \leqslant \frac{1}{2}\left\||S|^2+|T^*|^2\right\|. \tag{1.7}$$

2. Inner product inequalities

We start this section with an uncomplicated comment regarding inequality (1.5). More precisely, the following remark shows the inequality (1.5) holds for any orthogonal projection.

REMARK 2.1. Assume that $P : \mathscr{H} \to \mathscr{H}$ is a contraction operator; namely, it satisfies the condition $\|P\| \leqslant 1$. If we replace z by Pz, in (1.5), we obtain

$$\begin{aligned} |\langle Pz,x\rangle|^2 + |\langle Pz,y\rangle|^2 &\leqslant \|Pz\|^2 \max\left(\|x\|^2, \|y\|^2\right) + |\langle x,y\rangle| \\ &\leqslant \|P\|^2 \|z\|^2 \max\left(\|x\|^2, \|y\|^2\right) + |\langle x,y\rangle| \\ &\leqslant \|z\|^2 \max\left(\|x\|^2, \|y\|^2\right) + |\langle x,y\rangle| . \end{aligned}$$

The following theorem suggests an upper bound for $|\langle Tx,y\rangle|$ using polar decomposition.

THEOREM 2.1. *Let $S,T \in \mathscr{B}(\mathscr{H})$. Then*

$$|\langle (S+\mathrm{i}T)x,y\rangle|^2 \leqslant \max\left(\|S^*y\|^2, \|T^*y\|^2\right) + |\langle TS^*y,y\rangle| + 2\,|\langle Sx,y\rangle|\,|\langle Tx,y\rangle|,$$

for any unit vectors $x,y \in \mathscr{H}$. If $T \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $T = A+\mathrm{i}B$, then

$$|\langle Tx,y\rangle|^2 \leqslant \max\left(\|Ay\|^2, \|By\|^2\right) + |\langle BAy,y\rangle| + 2\,|\langle Ax,y\rangle|\,|\langle Bx,y\rangle| . \tag{2.1}$$

Proof. Taking $x = S^*y$, $y = T^*y$ and $z = x$ with $\|x\| = \|y\| = 1$, in (1.5), we get

$$\begin{aligned} |\langle Sx,y\rangle|^2 + |\langle Tx,y\rangle|^2 &= |\langle x,S^*y\rangle|^2 + |\langle x,T^*y\rangle|^2 \\ &\leqslant \max\left(\|S^*y\|^2, \|T^*y\|^2\right) + |\langle S^*y,T^*y\rangle| . \end{aligned}$$

Therefore,

$$\begin{aligned} |\langle (S+T)x,y\rangle|^2 &= |\langle Sx,y\rangle + \langle Tx,y\rangle|^2 \\ &\leqslant (|\langle Sx,y\rangle| + |\langle Tx,y\rangle|)^2 \quad \text{(by the triangle inequality)} \\ &= |\langle Sx,y\rangle|^2 + |\langle Tx,y\rangle|^2 + 2\,|\langle Sx,y\rangle|\,|\langle Tx,y\rangle| \\ &\leqslant \max\left(\|S^*y\|^2, \|T^*y\|^2\right) + |\langle TS^*y,y\rangle| + 2\,|\langle Sx,y\rangle|\,|\langle Tx,y\rangle|, \end{aligned}$$

i.e.,

$$|\langle (S+T)x,y\rangle|^2 \leqslant \max\left(\|S^*y\|^2, \|T^*y\|^2\right) + |\langle TS^*y,y\rangle| + 2\,|\langle Sx,y\rangle|\,|\langle Tx,y\rangle| . \tag{2.2}$$

We get the desired inequality by replacing T by $\mathrm{i}T$ in the inequality (2.2). $\square$

Inequality (2.1) can be stated in the following form:

COROLLARY 2.1. *Let $T \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $T = A + \mathrm{i}B$. Then*

$$|\langle Tx,y\rangle|^2 \leqslant \frac{1}{2}\left(\left\langle\left(|A|^2+|B|^2\right)y,y\right\rangle + \left|\left\langle\left(|A|^2-|B|^2\right)y,y\right\rangle\right|\right) + |\langle BAy,y\rangle| + 2|\langle Ax,y\rangle||\langle Bx,y\rangle|,$$

for any unit vectors $x,y \in \mathscr{H}$.

Proof. We have

$$\begin{aligned}
&|\langle Tx,y\rangle|^2 \\
&\leqslant \max\left(\|Ay\|^2,\|By\|^2\right) + |\langle BAy,y\rangle| + 2|\langle Ax,y\rangle||\langle Bx,y\rangle| \\
&= \frac{1}{2}\left(\|Ay\|^2+\|By\|^2+\left|\|Ay\|^2-\|By\|^2\right|\right) + |\langle BAy,y\rangle| + 2|\langle Ax,y\rangle||\langle Bx,y\rangle| \\
&= \frac{1}{2}\left(\left\langle|A|^2y,y\right\rangle + \left\langle|B|^2y,y\right\rangle + \left|\left\langle|A|^2y,y\right\rangle - \left\langle|B|^2y,y\right\rangle\right|\right) + |\langle BAy,y\rangle| \\
&\quad + 2|\langle Ax,y\rangle||\langle Bx,y\rangle| \\
&= \frac{1}{2}\left(\left\langle\left(|A|^2+|B|^2\right)y,y\right\rangle + \left|\left\langle\left(|A|^2-|B|^2\right)y,y\right\rangle\right|\right) + |\langle BAy,y\rangle| + 2|\langle Ax,y\rangle||\langle Bx,y\rangle|,
\end{aligned}$$

as wished. □

The next theorem provides an upper bound for the product of two operators.

THEOREM 2.2. *Let $A,B \in \mathscr{B}(\mathscr{H})$. Then*

$$|\langle B^*Ax,x\rangle|^2 \leqslant \frac{1}{2}\left(\max\left(\left\||A|^2x\right\|^2,\left\||B|^2x\right\|^2\right) + \left|\left\langle|B|^2|A|^2x,x\right\rangle\right|\right),$$

for any unit vector $x \in \mathscr{H}$.

Proof. Taking $x = |A|^2x$, $y = |B|^2x$, and $z = x$, in (1.5), we get

$$\left|\left\langle x,|A|^2x\right\rangle\right|^2 + \left|\left\langle x,|B|^2x\right\rangle\right|^2 \leqslant \max\left(\left\||A|^2x\right\|^2,\left\||B|^2x\right\|^2\right) + \left|\left\langle|A|^2x,|B|^2x\right\rangle\right|. \quad (2.3)$$

So,

$$\begin{aligned}
2|\langle B^*Ax,x\rangle|^2 &= 2|\langle Ax,Bx\rangle|^2 \\
&\leqslant 2\|Ax\|^2\|Bx\|^2 \quad \text{(by the Cauchy-Schwarz inequality)} \\
&= 2\langle Ax,Ax\rangle\langle Bx,Bx\rangle \\
&= 2\langle A^*Ax,x\rangle\langle B^*Bx,x\rangle \\
&= 2\left\langle|A|^2x,x\right\rangle\left\langle|B|^2x,x\right\rangle
\end{aligned}$$

$$\leqslant \left\langle |A|^2x,x\right\rangle^2+\left\langle |B|^2x,x\right\rangle^2$$
(by the arithmetic-geometric mean inequality)
$$=\left|\left\langle x,|A|^2x\right\rangle\right|^2+\left|\left\langle x,|B|^2x\right\rangle\right|^2$$
$$\leqslant \max\left(\left\||A|^2x\right\|^2,\left\||B|^2x\right\|^2\right)+\left|\left\langle |A|^2x,|B|^2x\right\rangle\right| \quad \text{(by (2.3))}$$
$$=\max\left(\left\||A|^2x\right\|^2,\left\||B|^2x\right\|^2\right)+\left|\left\langle |B|^2|A|^2x,x\right\rangle\right|.$$

Consequently,

$$|\langle B^*Ax,x\rangle|^2\leqslant \frac{1}{2}\left(\max\left(\left\||A|^2x\right\|^2,\left\||B|^2x\right\|^2\right)+\left|\left\langle |B|^2|A|^2x,x\right\rangle\right|\right),$$

as desired. □

As a consequence of Theorem 2.2, we have:

COROLLARY 2.2. *Let $T\in\mathscr{B}(\mathscr{H})$ and let $0\leqslant\nu\leqslant 1$. Then*

$$|\langle Tx,x\rangle|^2\leqslant \frac{1}{2}\left(\max\left(\left\||T|^{2\nu}x\right\|^2,\left\||T^*|^{2(1-\nu)}x\right\|^2\right)+\left|\left\langle |T^*|^{2(1-\nu)}|T|^{2\nu}x,x\right\rangle\right|\right),$$

for any unit vector $x\in\mathscr{H}$.

Proof. Letting $B^*=U|T|^{1-\nu}$ and $A=|T|^{\nu}$, in Theorem 2.2, we reach

$$|\langle Tx,x\rangle|^2\leqslant \frac{1}{2}\left(\max\left(\left\||T|^{2\nu}x\right\|^2,\left\|U|T|^{2(1-\nu)}U^*x\right\|^2\right)\right.$$
$$\left.+\left|\left\langle |T|^{2\nu},U|T|^{2(1-\nu)}U^*x\right\rangle\right|\right)$$
$$=\frac{1}{2}\left(\max\left(\left\||T|^{2\nu}x\right\|^2,\left\||T^*|^{2(1-\nu)}x\right\|^2\right)+\left|\left\langle |T|^{2\nu}x,|T^*|^{2(1-\nu)}x\right\rangle\right|\right)$$
(by [9, Theorem 4 (ii), p. 58])
$$=\frac{1}{2}\left(\max\left(\left\||T|^{2\nu}x\right\|^2,\left\||T^*|^{2(1-\nu)}x\right\|^2\right)+\left|\left\langle |T^*|^{2(1-\nu)}|T|^{2\nu}x,x\right\rangle\right|\right),$$

as required. □

Next, we obtain another upper bound for $|\langle Tx,y\rangle|$ using polar decompostion.

THEOREM 2.3. *Let $S,T\in\mathscr{B}(\mathscr{H})$. Then for any $0\leqslant\nu\leqslant 1$,*

$$|\langle (S+\mathrm{i}T)x,y\rangle|\leqslant\sqrt{\left\langle \left(|S|^{2\nu}+|T|^{2\nu}\right)x,x\right\rangle}\sqrt{\left\langle \left(|S^*|^{2(1-\nu)}+|T^*|^{2(1-\nu)}\right)y,y\right\rangle},$$

for any unit vectors $x,y \in \mathscr{H}$. *If* $T \in \mathscr{B}(\mathscr{H})$ *with the Cartesian decomposition* $T = A + \mathrm{i}B$, *then*

$$|\langle Tx,y\rangle| \leqslant \sqrt{\left\langle \left(|A|^{2v}+|B|^{2v}\right)x,x\right\rangle \left\langle \left(|A|^{2(1-v)}+|B|^{2(1-v)}\right)y,y\right\rangle}.$$

Proof. Let $x,y \in \mathscr{H}$ be unit vectors. Then

$$\begin{aligned}
|\langle (S+\mathrm{i}T)x,y\rangle| &= |\langle Sx,y\rangle + \mathrm{i}\langle Tx,y\rangle| \\
&\leqslant |\langle Sx,y\rangle| + |\langle Tx,y\rangle| \quad \text{(by the triangle inequality)} \\
&\leqslant \sqrt{\left\langle |S|^{2v}x,x\right\rangle \left\langle |S^*|^{2(1-v)}y,y\right\rangle} + \sqrt{\left\langle |T|^{2v}x,x\right\rangle \left\langle |T^*|^{2(1-v)}y,y\right\rangle} \\
&\qquad \text{(by (1.4))} \\
&\leqslant \sqrt{\left\langle |S|^{2v}x,x\right\rangle + \left\langle |T|^{2v}x,x\right\rangle}\sqrt{\left\langle |S^*|^{2(1-v)}y,y\right\rangle + \left\langle |T^*|^{2(1-v)}y,y\right\rangle} \\
&\qquad \text{(by the Cauchy-Schwarz inequality)} \\
&= \sqrt{\left\langle \left(|S|^{2v}+|T|^{2v}\right)x,x\right\rangle}\sqrt{\left\langle \left(|S^*|^{2(1-v)}+|T^*|^{2(1-v)}\right)y,y\right\rangle},
\end{aligned}$$

i.e.,

$$|\langle (S+\mathrm{i}T)x,y\rangle| \leqslant \sqrt{\left\langle \left(|S|^{2v}+|T|^{2v}\right)x,x\right\rangle}\sqrt{\left\langle \left(|S^*|^{2(1-v)}+|T^*|^{2(1-v)}\right)y,y\right\rangle},$$

as expected. □

3. Norm and numerical radii inequalities

This section derives several inequalities for the usual operator norm and numerical radii. The first result is the improvement of [21, Theorem 2.1].

PROPOSITION 3.1. *Let* $S,T \in \mathscr{B}(\mathscr{H})$. *Then*

$$\begin{aligned}
\|S+T\|^2 \leqslant\ & \frac{1}{2}\min\left(\left\||S|^2+|T|^2\right\| + \left\||S|^2-|T|^2\right\|, \left\||S^*|^2+|T^*|^2\right\| + \left\||S^*|^2-|T^*|^2\right\|\right) \\
&+ \min\left(\omega\left(T^*S\right), \omega\left(TS^*\right)\right) + 2\|S\|\,\|T\|.
\end{aligned}$$

Proof. It follows from (2.2) that

$$\begin{aligned}
|\langle (S+T)x,y\rangle|^2 &\leqslant \frac{1}{2}\left(\left\langle \left(|S^*|^2+|T^*|^2\right)y,y\right\rangle + \left|\left\langle \left(|S^*|^2-|T^*|^2\right)y,y\right\rangle\right|\right) \\
&\quad + |\langle TS^*y,y\rangle| + 2|\langle Sx,y\rangle|\,|\langle Tx,y\rangle| \\
&\leqslant \frac{1}{2}\left(\left\||S^*|^2+|T^*|^2\right\| + \left\||S^*|^2-|T^*|^2\right\|\right) + \omega\left(TS^*\right) + 2\|S\|\,\|T\|.
\end{aligned}$$

Now, by taking supremum over all unit vectors $x \in \mathscr{H}$, we obtain

$$\|S+T\|^2 \leqslant \frac{1}{2}\left(\left\||S^*|^2+|T^*|^2\right\|+\left\||S^*|^2-|T^*|^2\right\|\right)+\omega(TS^*)+2\|S\|\|T\|. \quad (3.1)$$

If we substitute S and T by S^* and T^*, in (3.1), we deduce

$$\begin{aligned}\|S+T\|^2 &= \|S^*+T^*\|^2 \\ &\leqslant \frac{1}{2}\left(\left\||S|^2+|T|^2\right\|+\left\||S|^2-|T|^2\right\|\right)+\omega(T^*S)+2\|S^*\|\|T^*\| \quad (3.2)\\ &= \frac{1}{2}\left(\left\||S|^2+|T|^2\right\|+\left\||S|^2-|T|^2\right\|\right)+\omega(T^*S)+2\|S\|\|T\|.\end{aligned}$$

We conclude the desired result by combining two inequalities (3.1) and (3.2). □

A refinement of [21, Corollary 2.1] is given in the following.

PROPOSITION 3.2. *Let $S,T \in \mathscr{B}(\mathscr{H})$. Then*

$$\begin{aligned}\omega^2(S+T) \leqslant& \frac{1}{2}\min\left(\left\||S|^2+|T|^2\right\|+\left\||S|^2-|T|^2\right\|, \left\||S^*|^2+|T^*|^2\right\|+\left\||S^*|^2-|T^*|^2\right\|\right)\\ &+\min\left(\omega(T^*S),\omega(TS^*)\right)+2\omega(S)\omega(T).\end{aligned}$$

Proof. Letting $y=x$, in (2.2), we observe that

$$\begin{aligned}|\langle(S+T)x,x\rangle|^2 \leqslant& \frac{1}{2}\left(\left\langle\left(|S^*|^2+|T^*|^2\right)x,x\right\rangle+\left|\left\langle\left(|S^*|^2-|T^*|^2\right)x,x\right\rangle\right|\right)+|\langle TS^*x,x\rangle|\\ &+2|\langle Sx,x\rangle||\langle Tx,x\rangle|\\ \leqslant& \frac{1}{2}\left(\left\||S^*|^2+|T^*|^2\right\|+\left\||S^*|^2-|T^*|^2\right\|\right)+\omega(TS^*)+2\omega(S)\omega(T),\end{aligned}$$

which implies

$$\omega^2(S+T) \leqslant \frac{1}{2}\left(\left\||S^*|^2+|T^*|^2\right\|+\left\||S^*|^2-|T^*|^2\right\|\right)+\omega(TS^*)+2\omega(S)\omega(T).$$

If we substitute S and T by S^* and T^*, in the above inequality, we infer

$$\omega^2(S+T) \leqslant \frac{1}{2}\left(\left\||S|^2+|T|^2\right\|+\left\||S|^2-|T|^2\right\|\right)+\omega(T^*S)+2\omega(S)\omega(T).$$

Now, the result follows by incorporating these two inequalities. □

REMARK 3.1. The case $S=T$, in Proposition 3.2, recovers the second inequality in (1.1).

The following result is a consequence of Theorem 2.2.

COROLLARY 3.1. *Let $A,B \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega^2(B^*A) \leqslant \frac{1}{2}\left(\max\left(\|A\|^4, \|B\|^4\right) + \omega\left(|B|^2|A|^2\right)\right).$$

The following theorem proposes an upper bound for the numerical radii of the product of two operators.

THEOREM 3.1. *Let $A,B \in \mathscr{B}(\mathscr{H})$. Then for any $r,s \geqslant 1$,*

$$\omega(B^*A) \leqslant \sqrt{\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|^{\frac{1}{r}}\left\|\frac{|A|^{2s}+|B|^{2s}}{2}\right\|^{\frac{1}{s}}}.$$

Proof. It has been shown in [6, Corollary 4] that

$$\left\|\frac{B^*A+A^*B}{2}\right\| \leqslant \sqrt{\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|^{\frac{1}{r}}\left\|\frac{|A|^{2s}+|B|^{2s}}{2}\right\|^{\frac{1}{s}}},$$

which can be written as

$$\|\mathfrak{R}(B^*A)\| \leqslant \sqrt{\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|^{\frac{1}{r}}\left\|\frac{|A|^{2s}+|B|^{2s}}{2}\right\|^{\frac{1}{s}}}.$$

Replacing A by $e^{i\theta}A$, we receive

$$\left\|\mathfrak{R}e^{i\theta}(B^*A)\right\| \leqslant \sqrt{\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|^{\frac{1}{r}}\left\|\frac{|A|^{2s}+|B|^{2s}}{2}\right\|^{\frac{1}{s}}}.$$

Now taking supremum over $\theta \in \mathbb{R}$, we infer that

$$\omega(B^*A) \leqslant \sqrt{\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|^{\frac{1}{r}}\left\|\frac{|A|^{2s}+|B|^{2s}}{2}\right\|^{\frac{1}{s}}},$$

due to $\sup_{\theta\in\mathbb{R}}\left\|\mathfrak{R}e^{i\theta}T\right\| = \omega(T)$ [22]. □

REMARK 3.2. The case $s=r$, in Theorem 3.1, reduces to (see [6, Theorem 1])

$$\omega^r(B^*A) \leqslant \frac{1}{2}\left\||A|^{2r}+|B|^{2r}\right\|.$$

By applying the same approach as in the proof of Corollary 2.2, we can write from Theorem 3.1 that:

COROLLARY 3.2. *Let $T \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega(T) \leqslant \sqrt{\left\| \frac{|T|^{2rv} + |T^*|^{2r(1-v)}}{2} \right\|^{\frac{1}{r}} \left\| \frac{|T|^{2sv} + |T^*|^{2s(1-v)}}{2} \right\|^{\frac{1}{s}}}; \ (r,s \geqslant 1, 0 \leqslant v \leqslant 1).$$

REMARK 3.3. The case $s = r$, in Corollary 3.2, reduces to (see [8, Theorem 1])

$$\omega^r(T) \leqslant \frac{1}{2} \left\| |T|^{2rv} + |T^*|^{2r(1-v)} \right\|.$$

It is easy to see that if $T = A + \mathrm{i}B$ is the Cartesian decomposition of $T \in \mathscr{B}(\mathscr{H})$, then

$$\|T\| \leqslant \|A\| + \|B\|.$$

Closely related to the above inequality, one may state the following result, which is a direct consequence of Theorem 2.3.

COROLLARY 3.3. *Let $S, T \in \mathscr{B}(\mathscr{H})$ be two self-adjoint operators. Then for any $0 \leqslant v \leqslant 1$,*

$$\|S + \mathrm{i}T\| \leqslant \sqrt{\left\| |S|^{2v} + |T|^{2v} \right\| \left\| |S|^{2(1-v)} + |T|^{2(1-v)} \right\|}.$$

If $T \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $T = A + \mathrm{i}B$, then

$$\|T\| \leqslant \sqrt{\left\| |A|^{2v} + |B|^{2v} \right\| \left\| |A|^{2(1-v)} + |B|^{2(1-v)} \right\|}.$$

REMARK 3.4. Corollary 3.3 says that if $S, T \in \mathscr{B}(\mathscr{H})$ are self-adjoint operators, then

$$\|S + \mathrm{i}T\| \leqslant \| \, |S| + |T| \, \|.$$

This can be compared with the following inequality for positive operators S, T and unitarily invariant norm $\|\cdot\|_u$ (see [18, (3.8)])

$$\|S + \mathrm{i}T\|_u \leqslant \|S + T\|_u.$$

REMARK 3.5. Letting $v = \frac{1}{2}$ in Corollary 3.3 to get

$$\|S + \mathrm{i}T\| \leqslant \| \, |S| + |T| \, \| \leqslant \|S\| + \|T\|$$

where the second inequality is obvious by the triangle inequality. By substituting $S = \Re T$ and $T = \Im T$, we deduce

$$\begin{aligned} \|T\| &\leqslant \frac{1}{2} \left\| \sqrt{TT^* + T^*T + 2\Re T^2} + \sqrt{TT^* + T^*T - 2\Re T^2} \right\| \\ &\leqslant \|\Re T\| + \|\Im T\|. \end{aligned}$$

COROLLARY 3.4. *Let $T \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $T = A + \mathrm{i}B$. Then for any $0 \leqslant \nu \leqslant 1$,*

$$\omega(T) \leqslant \frac{1}{2}\left\| |A|^{2\nu} + |A|^{2(1-\nu)} + |B|^{2\nu} + |B|^{2(1-\nu)} \right\|.$$

Proof. Letting $y = x$, in Theorem 2.3, we can write

$$\begin{aligned} |\langle Tx, x\rangle| &\leqslant \sqrt{\left\langle \left(|A|^{2\nu} + |B|^{2\nu}\right)x, x\right\rangle \left\langle \left(|A|^{2(1-\nu)} + |B|^{2(1-\nu)}\right)x, x\right\rangle} \\ &\leqslant \frac{1}{2}\left\langle \left(|A|^{2\nu} + |A|^{2(1-\nu)} + |B|^{2\nu} + |B|^{2(1-\nu)}\right)x, x\right\rangle \\ &\leqslant \frac{1}{2}\left\| |A|^{2\nu} + |A|^{2(1-\nu)} + |B|^{2\nu} + |B|^{2(1-\nu)} \right\|, \end{aligned}$$

where the second inequality follows from the arithmetic-geometric mean inequality. Taking supremum over all unit vectors $x \in \mathscr{H}$ produces the desired result. □

REMARK 3.6. From [15, Corollary 2.4], we know that

$$\omega(T) \leqslant \| \, |A| + |B| \, \|. \tag{3.3}$$

Thus, Corollary 3.4 is an extension of (3.3).

Another corresponding result can be stated as follows.

PROPOSITION 3.3. *Let $T \in \mathscr{M}_n$. Then*

$$\|T\|^2 \leqslant \left\|TT^* - \mathrm{i}\Im T^2\right\| \leqslant \left\|TT^* - 2\mathrm{i}\Im T^2\right\|.$$

Proof. We know that [19, Corollary 2.5] for any $A, B \in \mathscr{M}_n$

$$\begin{aligned} \left\|(A+B)(A+B)^*\right\| &\leqslant \|AA^* + BB^* + 2AB^*\| \\ &\leqslant \left\|(A-B)(A-B)^* + 4AB^*\right\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|A+B\|^2 &\leqslant \|AA^* + BB^* + 2AB^*\| \\ &\leqslant \left\|(A-B)(A-B)^* + 4AB^*\right\|. \end{aligned}$$

If we replace B by $\mathrm{i}B$, we get

$$\begin{aligned} \|A+\mathrm{i}B\|^2 &\leqslant \|AA^* + BB^* - 2\mathrm{i}AB^*\| \\ &\leqslant \left\|(A-\mathrm{i}B)(A-\mathrm{i}B)^* - 4\mathrm{i}AB^*\right\|. \end{aligned}$$

Now, if $T = A + \mathrm{i}B$ is the Cartesian decomposition of $T \in \mathscr{M}_n$, then

$$\begin{aligned}\|T\|^2 &= \|A+\mathrm{i}B\|^2\\ &\leqslant \left\|A^2+B^2-2\mathrm{i}AB\right\|\\ &= \frac{1}{2}\left\|2TT^*+(T^*)^2-T^2\right\|\\ &= \left\|TT^*-\mathrm{i}\Im T^2\right\|\\ &\leqslant \left\|(A-\mathrm{i}B)(A-\mathrm{i}B)^*-4\mathrm{i}AB\right\|\\ &= \left\|TT^*+(T^*)^2-T^2\right\|\\ &= \left\|TT^*-2\mathrm{i}\Im T^2\right\|,\end{aligned}$$

i.e.,

$$\|T\|^2 \leqslant \left\|TT^*-\mathrm{i}\Im T^2\right\| \leqslant \left\|TT^*-2\mathrm{i}\Im T^2\right\|. \quad \square$$

REMARK 3.7. Notice that

$$\begin{aligned}\left\|TT^*-\mathrm{i}\Im T^2\right\| &= \left\|A^2+B^2-2\mathrm{i}AB\right\|\\ &= \left\|A(A+\mathrm{i}B)^*+(A+\mathrm{i}B)(-\mathrm{i}B)\right\|\\ &\leqslant \left\|A(A+\mathrm{i}B)^*\right\|+\|(A+\mathrm{i}B)(-\mathrm{i}B)\|\\ &\leqslant \|A\|\,\|A+\mathrm{i}B\|+\|A+\mathrm{i}B\|\,\|-\mathrm{i}B\|\\ &= \|\Re T\|\,\|T\|+\|T\|\,\|\Im T\|\\ &= \|T\|\left(\|\Re T\|+\|\Im T\|\right).\end{aligned}$$

Thus, by Proposition 3.3, we infer that

$$\|T\| \leqslant \left\|TT^*-\mathrm{i}\Im T^2\right\|^{\frac{1}{2}} \leqslant \|\Re T\|+\|\Im T\|.$$

REMARK 3.8. It is easy to follow that

$$\|\Re T\|, \|\Im T\| \leqslant \omega(T),$$

which implies

$$\frac{1}{4}\left\|TT^*-\mathrm{i}\Im T^2\right\| \leqslant \omega^2(T). \tag{3.4}$$

If we replace T by T^* in (3.4), and use the fact that $\omega(T) = \omega(T^*)$, we obtain

$$\frac{1}{4}\left\|T^*T+\mathrm{i}\Im T^2\right\| \leqslant \omega^2(T). \tag{3.5}$$

Therefore, by (3.4) and (3.5), we deduce

$$\frac{1}{4}\max\left(\left\|TT^*-\mathrm{i}\Im T^2\right\|, \left\|T^*T+\mathrm{i}\Im T^2\right\|\right) \leqslant \omega^2(T). \tag{3.6}$$

Of course (3.6), is better than $\frac{1}{2}\|T\| \leqslant \omega(T)$.

Declarations

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REFERENCES

[1] R. BHATIA, F. KITTANEH, *On the singular values of a product of operators*, SIAM J. Matrix Anal. Appl., **11** (1990), 272–277.

[2] P. BHUNIA, K. PAUL, *New upper bounds for the numerical radius of Hilbert space operators*, Bull. Sci. Math., **167** (2021), Paper No. 102959, 11 pp.

[3] P. BHUNIA, K. PAUL, *Refinement of numerical radius inequalities of complex Hilbert space operators*, Acta Sci. Math. (Szeged), (2023). https://doi.org/10.1007/s44146-023-00070-1

[4] M. L. BUZANO, *Generalizzazione della diseguaglianza di Cauchy–Schwarz'*, Rend. Sem. Mat. Univ. Politech. Torino., **31** (1971/73), 405–409; (1974) (in Italian).

[5] M. AL-DOLAT, I. JARADAT, AND B. AL-HUSBAN, *A novel numerical radius upper bounds for* 2×2 *operator matrices*, Linear Multilinear Algebra., **70** (6) (2022), 1173–1184.

[6] S. S. DRAGOMIR, *Power inequalities for the numerical radius of a product of two operators in Hilbert spaces*, Sarajevo J. Math., **5** (18) (2009), 269–278.

[7] S. S. DRAGOMIR, *Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces*, Linear Algebra Appl., **419** (2006), 256–264.

[8] M. EL-HADDAD, F. KITTANEH, *Numerical radius inequalities for Hilbert space operators. II*, Studia Math., **182** (2) (2007), 133–140.

[9] T. FURUTA, *Invitation to Linear Operators*, Taylor and Francis, London, 2001.

[10] I. H. GÜMÜŞ, H. R. MORADI, M. SABABHEH, *On positive and positive partial transpose matrices*, Electron. J. Linear Algebra., **38** (2022), 792–802.

[11] P. R. HALMOS, *A Hilbert Space Problem Book*, 2nd ed., Springer, New York, 1982.

[12] M. HASSANI, M. E. OMIDVAR, AND H. R. MORADI, *New estimates on numerical radius and operator norm of Hilbert space operators*, Tokyo J. Math., **44** (2) (2021), 439–449.

[13] O. HERZALLAH, F. KITTANEH, AND K. SHEBRAWI, *Numerical radius inequalities for certain* 2×2 *operator matrices*, Integr. Equ. Oper. Theory., **71** (2011), 129–147.

[14] Z. HEYDARBEYGI, M. SABABHEH, AND H. R. MORADI, *A convex treatment of numerical radius inequalities*, Czech Math J., **72** (2022), 601–614.

[15] F. KITTANEH, *Numerical radius inequalities associated with the Cartesian decomposition*, Math. Inequal. Appl., **18** (3) (2015), 915–922.

[16] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Studia Math., **168** (1) (2005), 73–80.

[17] F. KITTANEH, H. R. MORADI, AND M. SABABHEH, *Sharper bounds for the numerical radius*, Linear Multilinear Algebra. https://doi.org/10.1080/03081087.2023.2177248.

[18] M. LIN, D. ZHOU, *Norm inequalities for accretive-dissipative operator matrices*, J. Math. Anal. Appl., **407** (2013), 436–442.

[19] J. S. MATHARU, J. S. AUJLA, *Some inequalities for unitarily invariant norms*, Linear Algebra Appl., **436** (2012), 1623–1631.

[20] F. P. NAJAFABADI, H. R. MORADI, *Advanced refinements of numerical radius inequalities*, International Journal of Mathematical Modelling & Computations., **11** (4) (2021), 1–10.

[21] S. TAFAZOLI, H. R. MORADI, S. FURUICHI, AND P. HARIKRISHNAN, *Further inequalities for the numerical radius of Hilbert space operators*, J. Math. Inequal., **13** (4) (2019), 955–967.

[22] T. YAMAZAKI, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math. 178 (2007), 83–89.

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