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THE INHERITANCE OF m–COMPARISON FROM THE CONTAINING C*–ALGEBRA TO A LARGE SUBALGEBRA

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Abstract. Let A be a unital simple separable infinite dimensional stably finite C*-algebra and B be a large subalgebra of A. In this paper, we show that B has (strong tracial or tracial) m-comparison of positive elements if A has (strong tracial or tracial) m-comparison of positive elements.

1. Introduction

Large subalgebra was firstly defined by Phillips in [16], as an abstraction of Putnam subalgebra in [17], which plays a crucial role on the crossed products of minimal homeomorphisms (see e.g. [6, 12, 13, 15]). Subsequently, a stronger concept was introduced by Archey and Phillips [3], which is called centrally large subalgebra. Let A be a unital simple infinite dimensional C*-algebra and B be a (centrally) large subalgebra of A. A natural problem is which properties of B could be transferred to A or which properties of A could be inherited by B. Especially, if a property can pass from a C*-algebra to a large subalgebra and vice versa, we say the property is permanent for large subalgebras. Hereinafter, the property is permanent means the property is permanent for large subalgebras. Phillips [16] has shown some properties are permanent such as radius of comparison, finiteness and purely infiniteness. If B is a centrally large subalgebra of A, Archey and Phillips [3] proved that A has stable rank one if B has stable rank one. Moreover, Archey, Buck and Phillips [2] obtained tracially $\mathscr{Z}$-absorption is permanent if A is stably finite, and $\mathscr{Z}$-absorption is permanent if A and B are separable and nuclear in addition.

In the classification of separable simple nuclear C*-algebras, there are some regularity properties of the C*-algebras. Strict comparison of positive elements, $\mathscr{Z}$-absorption and finite nuclear dimension are three attractive regularity properties. Toms and Winter conjectured that the above three fundamental properties are equivalent for unital simple separable infinite dimensional nuclear C*-algebras (see [8, 22, 24]). Later, some other regularity properties are introduced to solve this conjecture (e.g. [11, 19, 23]). To show C*-algebras with finite nuclear dimension is $\mathscr{Z}$-absorbing, the definitions of m-comparison, tracial m-comparison and strong tracial m-comparison were

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introduced by Winter [23], where 0-comparison is strict comparison. Winter [23] proved that finite nuclear dimension can imply $\mathscr{Z}$-absorption for unital simple separable infinite dimensional C*-algebras. For the case where strict comparison implies $\mathscr{Z}$-absorption in the conjecture, firstly, H. Matui and Y. Sato [14] proved strict comparison and $\mathscr{Z}$-absorption are equivalent for unital separable simple infinite dimensional nuclear C*-algebras with finitely many extremal traces. Subsequently, influenced by this work, Kirchberg and Rørdam [11], Toms, White and Winter [21], Sato [20] extended this result almost at the same time. They proved the case independently under the weaker assumption that the extremal tracial boundary of the C*-algebra is compact and has finite covering dimension. In [11], Kirchberg and Rørdam gave the definitions of local weak comparison and weak comparison, and they proved that local weak comparison is equivalent to strict comparison for non-elementary unital simple separable stably finite nuclear C*-algebras with tracial simplex having finite (topological) dimensional closed extreme boundary. However, for general simple C*-algebras, whether strict comparison, m-comparison and (local) weak comparison are equivalent is still an open question.

Let A be a unital simple infinite dimensional separable stably finite C*-algebra and B be a large subalgebra of A. Phillips [16] proved A has strict comparison of positive elements if and only if B has strict comparison of positive elements. Fan, Fang and Zhao [9] proved that m-comparison (strong tracial m-comparison) of positive elements of B could be transferred to A, and they [25] proved that A has weak comparison if and only if B has weak comparison. To supplement and complete the inheritance of the comparison properties for large algebra, we consider whether (strong tracial or tracial) m-comparison of positive elements can be inherited by large subalgebras. To be precise, our main result in this paper is as follows:

THEOREM 1. *Let A be a unital simple infinite dimensional separable stably finite C*-algebra and B be a large subalgebra of A. If A has* (*strong tracial or tracial*) *m-comparison of positive elements, then B has* (*strong tracial or tracial*) *m-comparison of positive elements.*

The paper is organized as follows. First, we recall the definitions and known results about Cuntz subequivalence and large subalgebra in Section 2. Then we present the inheritance of m-comparison from the containing C*-algebra to large subalgebras in Section 3.

2. Preliminaries

In this paper, for a C*-algebra A, we use A_+ to denote the set of all positive elements in A and $M_\infty(A)$ to denote the algebraic inductive limit of system $(M_n(A))_{n=1}^{\infty}$. Let $K\otimes A$ denote the minimal tensor product of the set of all compact operators K and A; in fact, $K\otimes A$ is equal to the C*-algebraic inductive limit of system $(M_n(A))_{n=1}^{\infty}$. Besides, we always follow the identifications $A\subset M_n(A)\subset M_\infty(A)\subset K\otimes A$.

Let A be a C*-algebra, $a\in A_+$ and $\varepsilon>0$. $(a-\varepsilon)_+$ and $f_\varepsilon(a)$ denote the elements obtained by functional calculus evaluating with the functions $(t-\varepsilon)_+$ and $f_\varepsilon(t)$, where

$$(t-\varepsilon)_+ = \begin{cases} 0, & 0 \leqslant t \leqslant \varepsilon, \\ t-\varepsilon, & \varepsilon < t \leqslant \|a\|, \end{cases} \quad \text{and} \quad f_\varepsilon(t) = \begin{cases} \frac{t}{\varepsilon}, & 0 \leqslant t \leqslant \varepsilon, \\ 1, & \varepsilon < t \leqslant \|a\|, \end{cases} \tag{1}$$

(see Figure 1).

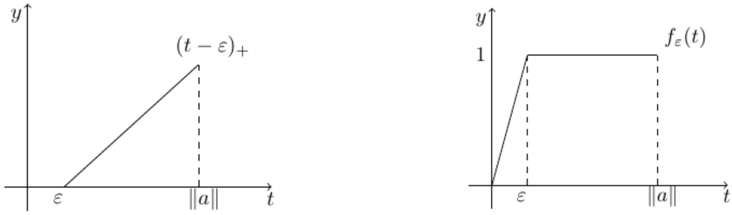

Figure 1: *Graph of* $(t-\varepsilon)_+$ *and* $f_\varepsilon(t)$

The following definitions of Cuntz subequivalence are originally introduced by Cuntz [5]; for more information, please see references [1] and [16].

DEFINITION 1. Let A be a C*-algebra and $a,b \in (K\otimes A)_+$.

(1) a is *Cuntz subequivalent* to b in A, written by $a \precsim_A b$, if there is a sequence $(v_k)_{k=1}^\infty$ in $K\otimes A$ such that $\lim\limits_{k\to\infty} v_k b v_k^* = a$.

(2) a and b are *Cuntz equivalent* in A, written by $a \sim_A b$, if $a \precsim_A b$ and $b \precsim_A a$. Denote $\langle a\rangle$ for the equivalence class of a.

(3) The *Cuntz semigroup* of A is

$$\mathrm{Cu}(A) = (K\otimes A)_+/\sim_A,$$

together with the operation $\langle a\rangle + \langle b\rangle = \langle a\oplus b\rangle$ and the partial order $\langle a\rangle \leqslant \langle b\rangle$ if $a \precsim_A b$.

(4) The semigroup

$$W(A) = M_\infty(A)_+/\sim_A$$

with the same operation and order as above.

In fact, if $a,b \in A_+$ and $a \precsim_A b$, then there exists a sequence $(v_k)_{k=1}^\infty$ exactly in A such that $\lim\limits_{k\to\infty} v_k b v_k^* = a$. Similarly, if $a,b \in M_n(A)_+$ (or $M_\infty(A)_+$) and $a \precsim_A b$, then $(v_k)_{k=1}^\infty$ can be taken exactly in $M_n(A)$ (or $M_\infty(A)$).

Next, we give some known facts about Cuntz subequivalence (see e.g. [10, 16, 18]).

LEMMA 1. *Let A be a C*-algebra.*

(1) *If $a,b \in A_+$, then the following statements are equivalent:*

(a) $a \precsim_A b$*;*

(b) $(a-\varepsilon)_+ \precsim_A b$ *for all* $\varepsilon > 0$*;*

(c) *for every* $\varepsilon > 0$, *there is* $\delta > 0$ *such that* $(a-\varepsilon)_+ \precsim_A (b-\delta)_+$.

(2) *Let* $\varepsilon > 0$ *and* $a, b \in A_+$. *If* $\|a-b\| < \varepsilon$, *then* $(a-\varepsilon)_+ \precsim_A b$.

(3) *Let* $a \in A_+$. *If* $f : [0, \|a\|] \to [0,\infty)$ *is a continuous function such that* $f(0) = 0$, *then* $f(a) \precsim_A a$.

(4) *Let* $a \in A_+$ *and* $\varepsilon_1, \varepsilon_2 > 0$, *then* $((a-\varepsilon_1)_+ - \varepsilon_2)_+ = (a-(\varepsilon_1+\varepsilon_2))_+$.

(5) *Let* $\varepsilon > 0$, $a \in A_+$ *and* $g \in A_+$ *with* $0 \leqslant g \leqslant 1$, *then*

$$(a-\varepsilon)_+ \precsim_A [(1-g)a(1-g)-\varepsilon]_+ \oplus g.$$

NOTATION 1. Let A be a unital C*-algebra.

(1) Denote $QT(A)$ to be the set of all normalized 2-quasitraces on A (see [1, Definition 2.31] and [4, II.1.1]).

(2) Define $d_\tau : M_\infty(A)_+ \to [0,\infty)$ by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}})$ for all $a \in M_\infty(A)_+$ and $\tau \in QT(A)$. Besides, we denote the same notation d_τ for the corresponding functions on $(K \otimes A)_+$, $\mathrm{Cu}(A)$ and $W(A)$. It follows that d_τ is well defined on $\mathrm{Cu}(A)$ and $W(A)$ by part of the proof in Proposition 4.2 of [7].

For a C*-algebra A, $QT(A)$ is compact if A is unital, $QT(A)$ is metrizable if A is separable, and $QT(A) \neq \emptyset$ if A is stably finite. According to Theorem II.2.2 of [4], for any $\tau \in QT(A)$, d_τ defines a lower semicontinuous function on A. Let $a \in A_+ \setminus \{0\}$. Then one could check that

$$d_\tau(a) = \lim_{n\to\infty} d_\tau\left(\left(a - \frac{1}{n}\right)_+\right).$$

Moreover, it is shown that d_τ defines a state on $W(A)$ by the proof of Theorem 2.32 in [1].

Now we recall the notion of large subalgebra and centrally large subalgebra defined in [16] and [3].

DEFINITION 2. Let A be a unital simple infinite dimensional C*-algebra. A unital subalgebra B of A is said to be *large* in A, if for every $m \in \mathbb{N} \setminus \{0\}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that

(1) $0 \leqslant g \leqslant 1$;

(2) $\|c_j - a_j\| < \varepsilon$ for $j = 1, 2, \ldots, m$;

(3) $(1-g)c_j \in B$ for $j = 1, 2, \ldots, m$;

(4) $g \precsim_B y$, $g \precsim_A x$;

(5) $\|(1-g)x(1-g)\| > 1-\varepsilon$.

B is said to be *centrally large* in A if we require that in addition

(6) $\|ga_j - a_j g\| < \varepsilon$ for $j = 1, 2, \ldots, m$.

A basic example of a large subalgebra is given in [16]. The next lemma is some basic properties about large subalgebras and centrally large subalgebras, which appears in [16, Lemma 4.8] and [3, Lemma 3.4].

LEMMA 2. *Let A be a unital simple infinite dimensional C*-algebra and B is a large subalgebra of A. Let $m,n\in\mathbb{N}\setminus\{0\}$, $a_1,a_2,\ldots,a_m\in A$, $b_1,b_2,\ldots,b_n\in A_+$, $\varepsilon>0$, $x\in A_+$ with $\|x\|=1$, and $y\in B_+\setminus\{0\}$. Then there are $c_1,c_2,\ldots,c_m\in A$, $d_1,d_2,\ldots,d_n\in A_+$ and $g\in B$ such that*

(1) $0\leqslant g\leqslant 1$;

(2) $\|c_j-a_j\|<\varepsilon$ *for* $j=1,2,\ldots,m$, $\|b_i-d_i\|<\varepsilon$ *for* $i=1,2,\ldots,n$;

(3) $\|c_j\|\leqslant\|a_j\|$ *for* $j=1,2,\ldots,m$, $\|b_i\|\leqslant\|d_i\|$ *for* $i=1,2,\ldots,n$;

(4) $(1-g)c_j\in B$ *for* $j=1,2,\ldots,m$, $(1-g)d_i(1-g)\in B$ *for* $i=1,2,\ldots,n$;

(5) $g\precsim_A x$, $g\precsim_B y$;

(6) $\|(1-g)x(1-g)\|>1-\varepsilon$.

If B is a centrally large subalgebra of A, then we have that in addition

(7) $\|ga_j-a_jg\|<\varepsilon$ *for* $j=1,2,\ldots,m$, $\|gb_i-b_ig\|<\varepsilon$ *for* $i=1,2,\ldots,n$.

3. Main results

In this section, we present our main results. First, we recall the definitions of (strong tracial or tracial) m-comparison of positive elements introduced by Winter in [23].

DEFINITION 3. Let A be a unital simple separable C*-algebra and $m\in\mathbb{N}$.

(1) A is said to have *m-comparison of positive elements*, if for any positive contractions a, $b_0,b_1,\ldots,b_m\in M_\infty(A)\setminus\{0\}$, we have

$$a\precsim b_0\oplus\cdots\oplus b_m$$

whenever $d_\tau(a)<d_\tau(b_i)$ for every $\tau\in QT(A)$ and $i=0,\ldots,m$.

(2) A is said to have *tracial m-comparison of positive elements*, if for any positive contractions $a,b_0,b_1,\ldots,b_m\in M_\infty(A)\setminus\{0\}$, we have

$$a\precsim b_0\oplus\cdots\oplus b_m$$

whenever $d_\tau(a)<\tau(b_i)$ for every $\tau\in QT(A)$ and $i=0,\ldots,m$.

(3) A is said to have *strong tracial m-comparison of positive elements*, if for any positive contractions $a,b\in M_\infty(A)\setminus\{0\}$, we have

$$a\precsim b$$

whenever $d_\tau(a)<\frac{1}{m+1}\tau(b)$ for every $\tau\in QT(A)$.

It is obvious that m-comparison of positive elements implies tracial m-comparison of positive elements and strong tracial m-comparison of positive elements implies tracial m-comparison of positive elements. According to Proposition 3.3 in [23], m-comparison and tracial m-comparison of positive elements are exactly equivalent for separable simple unital C*-algebras.

Next, we give some necessary lemmas.

LEMMA 3. [16, Lemma 6.13] *Let X be a compact Hausdorff space. If $\{f_n\} : X \to \mathbb{R} \cup \{\infty\}$ $(n \in \mathbb{N} \setminus \{0\})$ is a sequence of lower semicontinuous functions such that $f_1(x) \leqslant f_2(x) \leqslant \cdots$ for all $x \in X$, and $g : X \to \mathbb{R}$ is a continuous function such that $g(x) < \lim\limits_{n\to\infty} f_n(x)$ for all $x \in X$, then there is an integer $n_0 > 0$ such that $g(x) < f_{n_0}(x)$ for all $x \in X$.*

LEMMA 4. *Let A be a unital simple separable stably finite C*-algebra. Suppose that $a, b \in A_+$ and $\varepsilon > 0$. If $d_\tau(a) < d_\tau(b)$ for all $\tau \in QT(A)$, then there is $\delta > 0$ such that $d_\tau((a-\varepsilon)_+) < d_\tau((b-\delta)_+)$ for all $\tau \in QT(A)$.*

Proof. Let $f_\varepsilon(t)$ be a continuous function from $[0, \|a\|]$ to $[0,1]$ defined by (1). Then we have

$$d_\tau((a-\varepsilon)_+) \leqslant \tau(f_\varepsilon(a)) \leqslant d_\tau(a) < d_\tau(b)$$

for all $\tau \in QT(A)$.

Define $f : QT(A) \to \mathbb{R}$ and $f_n : QT(A) \to \mathbb{R}$ by $f(\tau) = \tau(f_\varepsilon(a))$ and $f_n(\tau) = d_\tau((b-\frac{1}{n})_+)$, then f is continuous and f_n is lower semicontinuous. Since $d_\tau(b) = \lim\limits_{n\to\infty} d_\tau((b-1/n)_+)$, then we have $\tau(f_\varepsilon(a)) < d_\tau(b) = \lim\limits_{n\to\infty} d_\tau((b-1/n)_+)$ for all $\tau \in QT(A)$. Lemma 3 implies that there exists an integer $n > 0$ such that $\tau(f_\varepsilon(a)) < d_\tau((b-\frac{1}{n})_+)$ for all $\tau \in QT(A)$. Let $\delta = \frac{1}{n}$; thus, we have $d_\tau((a-\varepsilon)_+) \leqslant \tau(f_\varepsilon(a)) < d_\tau((b-\delta)_+)$ for all $\tau \in QT(A)$. $\square$

Using the similar proof, replacing $d_\tau(b)$ with $\frac{1}{m+1}d_\tau(b)$, we have the same result as follows:

REMARK 1. Let A be a unital simple separable stably finite C*-algebra. Suppose that $a, b \in A_+$ and $\varepsilon > 0$. If $d_\tau(a) < \frac{1}{m+1}d_\tau(b)$ for all $\tau \in QT(A)$, then there is $\delta > 0$ such that $d_\tau((a-\varepsilon)_+) < \frac{1}{m+1}d_\tau((b-\delta)_+)$ for all $\tau \in QT(A)$.

LEMMA 5. [16, Lemma 2.7] *Let A be an infinite dimensional simple C*-algebra which is not of type I. Let $b \in A_+ \setminus \{0\}$, $\varepsilon > 0$ and $n \in \mathbb{N} \setminus \{0\}$. Then there are $c \in A_+$ and $y \in A_+ \setminus \{0\}$ such that*

$$n\langle (b-\varepsilon)_+\rangle \leqslant (n+1)\langle c\rangle \text{ and } \langle c\rangle + \langle y\rangle \leqslant \langle b\rangle$$

$\langle\cdot\rangle$ in $W(A)$.

Next, we prove m-comparison of positive elements could be inherited by a large subalgebra.

THEOREM 2. *Let A be a unital simple infinite dimensional separable stably finite C*-algebra. Let $B \subset A$ be a large subalgebra. If A has m-comparison of positive elements, then B has m-comparison of positive elements.*

Proof. Let $a, b_0, \ldots, b_m$ be positive contractions of $M_\infty(B) \setminus \{0\}$ such that $d_\tau(a) < d_\tau(b_i)$ for all $\tau \in QT(B)$; we need to show that $a \precsim_B b_0 \oplus b_1 \oplus \cdots \oplus b_m$. According to Lemma 1 (1), we only need to show $(a-\varepsilon)_+ \precsim_B b_0 \oplus b_1 \oplus \cdots \oplus b_m$ for all $\varepsilon > 0$. Without loss of generality, we may assume that $(a-\varepsilon)_+ \neq 0$. By Corollary 5.8 in [16], we have B is stably large in A, thus $M_n(B)$ is large in $M_n(A)$ for every positive integer n. Therefore, we may assume that $a, b_0, \ldots, b_m \in B$.

Since $d_\tau(a) < d_\tau(b_i)$ for all $\tau \in QT(B)$, Lemma 4 implies that there exists $\delta_i > 0$ such that $d_\tau((a-\frac{\varepsilon}{2})_+) < d_\tau((b_i - \delta_i)_+)$ for all $\tau \in QT(B)$ and every $i = 0, 1, \ldots, m$. Let $w_i(\tau) = d_\tau((b_i-\delta_i)_+) - d_\tau((a-\frac{\varepsilon}{2})_+) > 0$. Since we have assumed $(a-\varepsilon)_+ \neq 0$, we get $(a-\frac{\varepsilon}{2})_+ > (a-\varepsilon)_+ > 0$. Lemma 1.23 in [16] implies that $\inf_{\tau \in QT(B)} d_\tau((a-\frac{\varepsilon}{2})_+) > 0$. For each $i = 0, 1, \ldots, m$ and $\tau \in QT(B)$, we have

$$\frac{d_\tau((b_i-\delta_i)_+)}{w_i(\tau)} = \frac{d_\tau((b_i-\delta_i)_+)}{d_\tau((b_i-\delta_i)_+) - d_\tau((a-\frac{\varepsilon}{2})_+)} = \frac{\frac{d_\tau((b_i-\delta_i)_+)}{d_\tau((a-\frac{\varepsilon}{2})_+)}}{\frac{d_\tau((b_i-\delta_i)_+)}{d_\tau((a-\frac{\varepsilon}{2})_+)} - 1}.$$

Besides, by the proof of Lemma 4, we have $d_\tau((a-\frac{\varepsilon}{2})_+) \leqslant \tau(f_{\frac{\varepsilon}{2}}(a)) < d_\tau((b_i-\delta_i)_+)$ for all $\tau \in QT(B)$, where $f_{\frac{\varepsilon}{2}}(a)$ is defined by (1). Thus, we obtain

$$\frac{d_\tau((b_i-\delta_i)_+)}{d_\tau((a-\frac{\varepsilon}{2})_+)} \geqslant \frac{d_\tau((b_i-\delta_i)_+)}{\tau(f_{\frac{\varepsilon}{2}}(a))} > 1$$

for all $\tau \in QT(B)$. Since $d_\tau((b_i-\delta_i)_+)$ is a lower semicontinuous function and $\tau(f_{\frac{\varepsilon}{2}}(a))$ is a continuous function on $QT(B)$, we have $\frac{d_\tau((b_i-\delta_i)_+)}{\tau(f_{\frac{\varepsilon}{2}}(a))}$ is a lower semicontinuous function on $QT(B)$. Then $\frac{d_\tau((b_i-\delta_i)_+)}{\tau(f_{\frac{\varepsilon}{2}}(a))}$ has a minimum value on $QT(B)$ by the compactness of $QT(B)$. Denote the minimum value as m_i for each $i = 0, 1, \ldots, m$. Since $\frac{d_\tau((b_i-\delta_i)_+)}{\tau(f_{\frac{\varepsilon}{2}}(a))} > 1$ for all $i = 0, 1, \ldots, m$ and all $\tau \in QT(B)$, we have $\alpha = \min\{m_i : i = 1, 2, \ldots, m\} > 1$. Thus, we get $\frac{d_\tau((b_i-\delta_i)_+)}{d_\tau((a-\frac{\varepsilon}{2})_+)} \geqslant \alpha > 1$ for all $\tau \in QT(B)$ and all $i = 0, 1, \ldots, m$. Notice that $g(t) = \frac{t}{t-1}$ is a monotonically decreasing function, then we have

$$\frac{d_\tau((b_i-\delta_i)_+)}{w_i(\tau)} \leqslant \frac{\alpha}{\alpha - 1}$$

for all $i = 0, 1, \ldots, m$ and all $\tau \in QT(B)$.

Let $n > \frac{\alpha}{\alpha-1} - 1$. Then $n > \frac{d_\tau((b_i-\delta_i)_+)}{w_i(\tau)} - 1$ for all $i = 0, 1, \ldots, m$ and $\tau \in QT(B)$. As B is large in A, then B is a simple infinite dimensional C*-algebra by Proposition 5.2 and Proposition 5.5 in [16]. Since B is unital, we have B is not of type I. By Lemma

5, for above b_i, δ_i and n, there are $c_i \in B_+$ and $y_i \in B_+ \setminus \{0\}$ such that

$$n\langle (b_i - \delta_i)_+ \rangle \leqslant (n+1)\langle c_i \rangle, \tag{2}$$

$$\langle c_i \rangle + \langle y_i \rangle \leqslant \langle b_i \rangle \tag{3}$$

for $i = 0, 1, \dots, m$, where $\langle \cdot \rangle \in W(B)$. Then (2) implies that

$$\frac{n}{n+1} d_\tau((b_i - \delta_i)_+) \leqslant d_\tau(c_i)$$

for all $\tau \in QT(B)$. Therefore,

$$\begin{aligned}
& d_\tau(c_i) - d_\tau\left(\left(a - \frac{\varepsilon}{2}\right)_+\right) \\
& \geqslant \frac{n}{n+1} d_\tau((b_i - \delta_i)_+) - d_\tau\left(\left(a - \frac{\varepsilon}{2}\right)_+\right) \\
& = d_\tau((b_i - \delta_i)_+) - d_\tau\left(\left(a - \frac{\varepsilon}{2}\right)_+\right) - \frac{1}{n+1} d_\tau((b_i - \delta_i)_+) \\
& = w_i(\tau) - \frac{1}{n+1} d_\tau((b_i - \delta_i)_+) > 0
\end{aligned}$$

for all $\tau \in QT(B)$, where the last inequality is by the choice of n. Therefore, we have $d_\tau(c_i) > d_\tau((a - \frac{\varepsilon}{2})_+)$ for all $\tau \in QT(B)$. According to Proposition 6.9 in [16], we have the restriction map from $QT(A)$ to $QT(B)$ is a bijection. Thus, $d_\tau(c_i) > d_\tau((a - \frac{\varepsilon}{2})_+)$ for all $\tau \in QT(A)$. Since A has m-comparison of positive elements, we have

$$\left(a - \frac{\varepsilon}{2}\right)_+ \precsim_A c_0 \oplus \cdots \oplus c_m.$$

Let $c_0 \oplus \cdots \oplus c_m = c \in M_{m+1}(B)$, then there exists $v \in M_{m+1}(A)$ such that

$$\left\| v c v^* - \left(a - \frac{\varepsilon}{2}\right)_+ \right\| < \frac{\varepsilon}{4}. \tag{4}$$

Since $M_{m+1}(B)$ is a large subalgebra of $M_{m+1}(A)$, by Lemma 2, there exist $v_0 \in M_{m+1}(A)$ and $g \in M_{m+1}(B)$ such that

(i) $0 \leqslant g \leqslant 1$;

(ii) $\| v - v_0 \| < \frac{\varepsilon}{8\|c\|\|v\|+1}$;

(iii) $\| v_0 \| \leqslant \| v \|$;

(iv) $(1-g)v_0 \in M_{m+1}(B)$ and

(v) $g \precsim_B y_0 \oplus y_1 \oplus \cdots \oplus y_m$.

Then according to (ii) and (iii), we have

$$\| v_0 c v_0^* - v c v^* \| \leqslant \| v_0 c v_0^* - v_0 c v^* \| + \| v_0 c v^* - v c v^* \| < \frac{\varepsilon}{4}. \tag{5}$$

Combine with (4), we get

$$\left\| v_0 c v_0^* - \left(a - \frac{\varepsilon}{2}\right)_+ \right\| < \frac{\varepsilon}{2}.$$

Thus

$$\left\| (1-g) v_0 c ((1-g) v_0)^* - (1-g) \left(a - \frac{\varepsilon}{2}\right)_+ (1-g) \right\| < \frac{\varepsilon}{2}.$$

Therefore, Lemma 1 (2) implies

$$\left((1-g) \left(a - \frac{\varepsilon}{2}\right)_+ (1-g) - \frac{\varepsilon}{2} \right)_+ \precsim_B (1-g) v_0 c ((1-g) v_0)^* \precsim_B c. \tag{6}$$

Using Lemma 1 (4) at the first step, Lemma 1 (5) at the second step, (6) and (v) at the third step, (3) at the last step, one conclude that

$$\begin{aligned}
(a-\varepsilon)_+ &= \left(\left(a - \frac{\varepsilon}{2}\right)_+ - \frac{\varepsilon}{2} \right)_+ \\
&\precsim_B \left((1-g) \left(a - \frac{\varepsilon}{2}\right)_+ (1-g) - \frac{\varepsilon}{2} \right)_+ \oplus g \\
&\precsim_B c \oplus y_0 \oplus \cdots \oplus y_m \\
&\sim_B c_0 \oplus \cdots \oplus c_m \oplus y_0 \oplus \cdots \oplus y_m \\
&\precsim_B b_0 \oplus \cdots \oplus b_m,
\end{aligned}$$

that is, $a \precsim_B b_0 \oplus \cdots \oplus b_m$. Thus, we have proved B has m-comparison of positive elements. $\square$

Now we consider whether the tracial m-comparison of positive elements of A could be transferred to B. Since A is a simple separable unital C*-algebra, if A has tracial m-comparison, Proposition 3.3 in [23] implies that A has m-comparison. Then Theorem 2 implies that the large subalgebra B has m-comparison, thus, B has tracial m-comparison naturally. That is, we get the following corollary.

COROLLARY 1. *Let A be a unital simple infinite dimensional separable stably finite C*-algebra and $B \subset A$ be a large subalgebra. Suppose that A has tracial m-comparison of positive elements. Then B has tracial m-comparison of positive elements.*

Next, we show strong tracial m-comparison of positive elements could be deduced from the containing C*-algebra to a large subalgebra.

THEOREM 3. *Let A be a unital simple infinite dimensional separable stably finite C*-algebra. Let $B \subset A$ be a large subalgebra. If A has strong tracial m-comparison of positive elements, then B has strong tracial m-comparison of positive elements.*

Proof. Let a,b be positive contractions of $M_\infty(B)$ such that $d_\tau(a)<\frac{1}{m+1}\tau(b)$ for all $\tau\in QT(B)$. Without loss of generality, we may assume $a,b\in B_+$. We need to show that $(a-\varepsilon)_+\precsim_B b$ for all $\varepsilon>0$. Similarly, we assume $(a-\varepsilon)_+\neq 0$.

Since $d_\tau(a)<\frac{1}{m+1}\tau(b)\leqslant\frac{1}{m+1}d_\tau(b)$ for all $\tau\in QT(B)$, Remark 1 implies that there exists $\delta>0$ such that $d_\tau((a-\frac{\varepsilon}{4})_+)<\frac{1}{m+1}d_\tau((b-\delta)_+)$ for all $\tau\in QT(B)$. Let $w(\tau)=\frac{1}{m+1}d_\tau((b-\delta)_+)-d_\tau((a-\frac{\varepsilon}{4})_+)$. By the assumption of $(a-\varepsilon)_+\neq 0$, we have $(a-\frac{\varepsilon}{4})_+>(a-\varepsilon)_+>0$. Lemma 1.23 in [16] implies that $\inf_{\tau\in QT(B)}d_\tau((a-\frac{\varepsilon}{4})_+)>0$. Then we have

$$\frac{d_\tau((b-\delta)_+)}{(m+1)w(\tau)}=\frac{d_\tau((b-\delta)_+)}{d_\tau((b-\delta)_+)-(m+1)d_\tau((a-\frac{\varepsilon}{4})_+)}=\frac{\frac{d_\tau((b-\delta)_+)}{(m+1)d_\tau((a-\frac{\varepsilon}{4})_+)}}{\frac{d_\tau((b-\delta)_+)}{(m+1)d_\tau((a-\frac{\varepsilon}{4})_+)}-1}$$

for all $\tau\in QT(B)$. By the similar proof in Theorem 2, we have $\frac{d_\tau((b-\delta)_+)}{(m+1)d_\tau((a-\frac{\varepsilon}{4})_+)}$ has a lower bound $\beta>1$ and $\frac{d_\tau((b-\delta)_+)}{(m+1)w(\tau)}$ has an upper bound $\frac{\beta}{\beta-1}$ for all $\tau\in QT(B)$.

Let $n>\frac{\beta}{\beta-1}-1$. Then $n>\frac{d_\tau((b-\delta)_+)}{(m+1)w(\tau)}-1$ for all $\tau\in QT(B)$. By Proposition 5.2 and Proposition 5.5 in [16], we have B is simple and infinite dimensional. Since B is unital, B is not of type I. Hence, Lemma 5 implies that there are $c\in B_+$ and $y\in B_+\setminus\{0\}$ such that

$$n\langle(b-\delta)_+\rangle\leqslant(n+1)\langle c\rangle, \tag{7}$$
$$\langle c\rangle+\langle y\rangle\leqslant\langle b\rangle, \tag{8}$$

where $\langle\cdot\rangle\in W(B)$. Since d_τ defines a state on $W(B)$, (7) implies that $d_\tau(c)\geqslant\frac{n}{n+1}d_\tau((b-\delta)_+)$ for all $\tau\in QT(B)$. Therefore,

$$\begin{aligned}
&\frac{1}{m+1}d_\tau(c)-d_\tau\left(\left(a-\frac{\varepsilon}{4}\right)_+\right)\\
&\geqslant\frac{n}{n+1}\frac{1}{m+1}d_\tau((b-\delta)_+)-d_\tau\left(\left(a-\frac{\varepsilon}{4}\right)_+\right)\\
&=\frac{1}{m+1}d_\tau((b-\delta)_+)-d_\tau\left(\left(a-\frac{\varepsilon}{4}\right)_+\right)-\frac{1}{n+1}\frac{1}{m+1}d_\tau((b-\delta)_+)\\
&=w(\tau)-\frac{1}{n+1}\frac{1}{m+1}d_\tau((b-\delta)_+)>0
\end{aligned}$$

for all $\tau\in QT(B)$, where the last inequality is by the choice of n, that is, we have $d_\tau((a-\frac{\varepsilon}{4})_+)<\frac{1}{m+1}d_\tau(c)$ for all $\tau\in QT(B)$, and then by Remark 1, there exists δ_1 such that

$$d_\tau\left(\left(\left(a-\frac{\varepsilon}{4}\right)_+-\frac{\varepsilon}{4}\right)_+\right)<\frac{1}{m+1}d_\tau((c-\delta_1)_+)$$

for all $\tau\in QT(B)$. Let $f_{\delta_1}:[0,\|a\|]\to[0,1]$ defined as (1), then

$$d_\tau\left(\left(\left(a-\frac{\varepsilon}{4}\right)_+-\frac{\varepsilon}{4}\right)_+\right)<\frac{1}{m+1}d_\tau((c-\delta_1)_+)<\frac{1}{m+1}\tau(f_{\delta_1}(c))$$

for all $\tau \in QT(B)$. Since B is large subalgebra of A, $QT(A) \to QT(B)$ is a bijection, and thus,

$$d_\tau\left(\left(\left(a-\frac{\varepsilon}{4}\right)_+ - \frac{\varepsilon}{4}\right)_+\right) < \frac{1}{m+1}\tau(f_{\delta_1}(c))$$

for all $\tau \in QT(A)$. Since A has strong tracial m-comparison of positive elements, we have

$$\left(a-\frac{\varepsilon}{2}\right)_+ = \left(\left(a-\frac{\varepsilon}{4}\right)_+ - \frac{\varepsilon}{4}\right)_+ \precsim_A f_{\delta_1}(c).$$

It follows that there exists $v \in A$ such that

$$\left\| v f_{\delta_1}(c) v^* - \left(a-\frac{\varepsilon}{2}\right)_+ \right\| < \frac{\varepsilon}{4}. \tag{9}$$

Since B is large in A, there exist $v_0 \in A$ and $g \in B_+$ such that

(i) $0 \leqslant g \leqslant 1$;

(ii) $g \precsim_B y$;

(iii) $(1-g)v_0 \in B$;

(iv) $\| v_0 \| \leqslant \| v \|$ and

(v) $\| v - v_0 \| < \frac{\varepsilon}{8\|v\|+1}$.

By (iv) and (v), it follows that

$$\begin{aligned}\| v_0 f_{\delta_1}(c) v_0^* - v f_{\delta_1}(c) v^* \| &\leqslant \| v_0 f_{\delta_1}(c) v_0^* - v_0 f_{\delta_1}(c) v^* \| \\ &\quad + \| v_0 f_{\delta_1}(c) v^* - v f_{\delta_1}(c) v^* \| \\ &< \frac{\varepsilon}{4}.\end{aligned}$$

With (9) and $\| 1-g \| \leqslant 1$, we have

$$\begin{aligned}&\left\|(1-g)v_0 f_{\delta_1}(c) v_0^*(1-g) - (1-g)\left(a-\frac{\varepsilon}{2}\right)_+(1-g)\right\| \\ &\leqslant \| (1-g)v_0 f_{\delta_1}(c) v_0^*(1-g) - (1-g)v f_{\delta_1}(c) v^*(1-g) \| \\ &\quad + \left\|(1-g)v f_{\delta_1}(c) v^*(1-g) - (1-g)\left(a-\frac{\varepsilon}{2}\right)_+(1-g)\right\| \\ &< \frac{\varepsilon}{2}.\end{aligned} \tag{10}$$

Thus, Lemma 1 (2) and Lemma 1 (3) imply that

$$\begin{aligned}&\left((1-g)\left(a-\frac{\varepsilon}{2}\right)_+(1-g) - \frac{\varepsilon}{2}\right)_+ \\ &\precsim_B (1-g)v_0 f_{\delta_1}(c) v_0^*(1-g) \\ &\precsim_B f_{\delta_1}(c) \precsim_B c.\end{aligned} \tag{11}$$

Therefore, using Lemma 1 (4) at the first step, Lemma 1 (5) at the second step, (11), (ii) and (8) for the last three steps, we have

$$\begin{aligned}(a-\varepsilon)_+ &= \left(\left(a-\frac{\varepsilon}{2}\right)_+ - \frac{\varepsilon}{2}\right)_+ \\ &\precsim_B \left((1-g)\left(a-\frac{\varepsilon}{2}\right)_+ (1-g) - \frac{\varepsilon}{2}\right)_+ \oplus g \\ &\precsim_B c \oplus g \precsim_B c \oplus y \precsim_B b,\end{aligned}$$

which follows that $a \precsim_B b$. Thus, we have proved B has strong tracial m-comparison of positive elements. $\square$

With the results in [9], we have the following sufficient and necessary condition.

COROLLARY 2. *Let A be a unital simple infinite dimensional separable stably finite C*-algebra and B be a large subalgebra of A. A has* (*strong tracial or tracial*) *m-comparison of positive elements if and only if B has* (*strong tracial or tracial*) *m-comparison of positive elements.*

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