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ON THE BEREZIN NUMBER OF OPERATOR MATRICES

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Abstract. Scalar quantities associated with Hilbert-space operators have attracted the attention of numerous researchers due to their role in understanding the geometry of the C^*-algebra of bounded linear operators on a Hilbert space. In this paper, we explore the Berezin number of operator matrices, and present several new relations that simulate the existing relations between the numerical radius and the operator norm.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on a non trivial complex Hilbert space $\mathscr{H}$ with inner product $\langle \cdot, \cdot \rangle$ and associated operator norm $\|\cdot\|$. An operator $A \in \mathscr{B}(\mathscr{H})$ is called positive if $\langle Ax, x \rangle \geqslant 0$ for all $x \in \mathscr{H}$, and we then write $A \geqslant 0$. The absolute value of A is denoted by $|A|$, that is $|A| = (A^*A)^{\frac{1}{2}}$, where A^* stands for the adjoint of A.

For $A \in \mathscr{B}(\mathscr{H})$, the numerical range of A is defined by

$$W(A) = \{\langle Ax, x \rangle : x \in \mathscr{H}, \|x\| = 1\},$$

while the numerical radius is defined as

$$\omega(A) = \sup_{z \in W(A)} |z|.$$

It is well known that the norm $\|\cdot\|$ and the numerical radius $\omega(\cdot)$ are equivalent, where one has the two-sided inequality [15]

$$\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|.$$

This inequality has been studied extensively in the literature, where numerous researchers attempted finding sharper bounds for each of the two inequalities. We refer the reader to [2, 10, 14, 17, 22, 25, 26, 28, 30, 31] as a sample of references treated this and other numerical radius inequalities. When discussing scalar quantities associated with

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a Hilbert space operator, the spectral radius $r(\cdot)$ deserves mentioning. We recall that for $A \in \mathscr{B}(\mathscr{H})$, we have

$$r(A) := \sup_{z \in \sigma(A)} |z|,$$

where $\sigma(A)$ is the spectrum of A.

Another interesting scalar quantity that is associated with a Hilbert-space operator is the so called Berezin number. In the following few lines, we remind the reader of this significant number.

Let Ω be a nonempty set. A functional Hilbert space $\mathscr{H} = \mathscr{H}(\Omega)$ is a Hilbert space of complex valued functions, which has the property that point evaluations are continuous, i.e., for each $\lambda \in \Omega$ the map $f \longmapsto f(\lambda)$ is a continuous linear functional on $\mathscr{H}$. The Riesz representation theorem ensures that for each $\lambda \in \Omega$ there exists a unique element $k_\lambda \in \mathscr{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathscr{H}$. The set $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of the space $\mathscr{H}$. If $\{e_n\}_{n \geqslant 0}$ is an orthonormal basis for a functional Hilbert space $\mathscr{H}$, then the reproducing kernel of $\mathscr{H}$ is given by $k_\lambda(z) = \sum_{n=0}^{+\infty} \overline{e_n(\lambda)} e_n(z)$ (see, e.g., [16, 27]). For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathscr{H}$. Let A be a bounded linear operator on $\mathscr{H}$. The Berezin symbol of A, which was firstly introduced by Berezin [11] is the function $\tilde{A}$ on Ω defined by

$$\tilde{A}(\lambda) := \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle.$$

The Berezin set and the Berezin number of the operator A are defined respectively by

$$\mathbf{Ber}(A) := \{\langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle : \lambda \in \Omega\}$$

and

$$\mathbf{ber}(A) := \sup\{|\langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle| : \lambda \in \Omega\}.$$

It is clear that the Berezin symbol $\tilde{A}$ is the bounded function on Ω whose value lies in the numerical range of the operator A and hence for any $A \in \mathscr{B}(\mathscr{H})$,

$$\mathbf{Ber}(A) \subset W(A) \quad \text{and} \quad \mathbf{ber}(A) \leqslant \omega(A).$$

Thus the Berezin number is strongly connected with the numerical radius and the operator norm.

Other immediate properties of the the Berezin number of an operator A can be stated as follows [20]

(i) $\mathbf{ber}(A) \leqslant \|A\|$.

(ii) $\mathbf{ber}(\alpha A) = |\alpha| \, \mathbf{ber}(A)$ for all $\alpha \in \mathbb{C}$.

(iii) $\mathbf{ber}(A+B) \leqslant \mathbf{ber}(A) + \mathbf{ber}(B)$ for all $A, B \in \mathscr{B}(\mathscr{H})$.

At this point, we should remark that, in general, the Berezin number does not define a norm. However, if $\mathscr{H}$ is a reproducing kernel Hilbert space of analytic functions, (for instance on the unit disc $D = \{z \in \mathbb{C} : |z < 1|\}$), then $\mathbf{ber}(\cdot)$ defines a norm on $\mathscr{B}(\mathscr{H}(D))$ (see [19, 20]).

The Berezin symbol has been studied in details for Toeplitz and Hankel operators on Hardy and Bergman spaces. A nice property of the Berezin symbol is mentioned next. If $\tilde{A}(\lambda) = \tilde{B}(\lambda)$ for all $\lambda \in \Omega$, then $A = B$. Therefore, the Berezin symbol uniquely determines the operator. We refer the reader to [6, 7, 12, 16, 32, 33] as a sample where the reader can find some progress on the study of Berezin symbol and Berezin number.

Notice that the operator norm is defined by $\|A\| = \sup\{|\langle Ax,y\rangle| : x,y \in \mathscr{H}, \|x\| = \|y\| = 1\}$, and the numerical radius is defined by $\omega(A) = \sup\{|\langle Ax,x\rangle| : x \in \mathscr{H}, \|x\| = 1\}$. A similar connection is available between the Berezin number and the Berezin norm. The Berezin norm of A is defined by

$$\|A\|_{ber} := \sup\left\{\left|\left\langle A\hat{k}_\lambda, \hat{k}_\mu\right\rangle\right| : \lambda, \mu \in \Omega\right\},$$

where $\hat{k}_\lambda, \hat{k}_\mu$ are normalized reproducing kernels for λ, μ, respectively. Clearly from the definition, Berezin norm satisfies the following properties:

(i) $\mathbf{ber}(A) \leqslant \|A\|_{ber}$.

(ii) $\|A\|_{ber} \leqslant \|A\|$.

(iii) $\|A^*\|_{ber} = \|A\|_{ber}$.

The direct sum of two copies of $\mathscr{H}$ is denoted by $\mathscr{H}^2 := \mathscr{H} \oplus \mathscr{H}$. If $A,B,C,D \in \mathscr{B}(\mathscr{H})$, then the operator matrix $T := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be considered as an operator in $\mathscr{B}(\mathscr{H} \oplus \mathscr{H})$, which is defined by $Tx = \begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{pmatrix}$ for every vector $x = (x_1, x_2) \in \mathscr{H} \oplus \mathscr{H}$.

Operator matrices and their properties and inequalities have received a considerable attention in the literature, as one can see in [1, 4, 8, 10, 18]; where operator norm and numerical radius inequalities were studied and applied. Very recently, some inequalities for the Berezin number of operator matrices have been presented in [5, 9].

Our goal in this paper is to present Berezin number inequalities for operator matrices in a way that complements the existing inequalities for the Berezin number, the numerical radius and the operator norm.

However, to achieve our goals, we will need some lemmas as follows. These lemmas are needed for two reasons. First, some of them will be used to be able to accomplish our proofs. Second, other lemmas will be used for comparison reasons so that the reader comprehends the full picture behind the results.

LEMMA 1.1. [8] *Let $A,B \in \mathscr{B}(\mathscr{H})$. Then*

(i) $\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leqslant \max\{\mathbf{ber}(A), \mathbf{ber}(B)\}$.

(ii) $\mathbf{ber}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leqslant \frac{\|A\|+\|B\|}{2}$. *In particular,*

(iii) $\mathbf{ber}\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) \leqslant \|A\|$.

LEMMA 1.2. [8] *Let* $A,B,C,D \in \mathscr{B}(\mathscr{H})$*. Then*

$$\mathbf{ber}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \omega\left(\begin{bmatrix} \mathbf{ber}(A) & \|B\| \\ \|C\| & \mathbf{ber}(D) \end{bmatrix}\right).$$

LEMMA 1.3. [18] *Let* $A,B,C,D \in \mathscr{B}(\mathscr{H})$*. Then*

$$r\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant r\left(\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right).$$

Next, we need the following technical lemma which can be easily verified.

LEMMA 1.4. *If* $a,b,c,d \in \mathbb{R}^{+}$*, then*

$$r\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{2}\left(a+d+\sqrt{(a-d)^{2}+4bc}\right).$$

LEMMA 1.5. [23] *Let* $A,B \in \mathscr{B}(\mathscr{H})$*. Then*
(i) *If* A *is self-adjoint, then* $\omega(A) = r(A)$*.*
(ii) $r(AB) = r(BA)$*.*

LEMMA 1.6. [7] *Let* $A \in \mathscr{B}(\mathscr{H})$*. Then*

$$\mathbf{ber}(A) = \sup_{\theta\in\mathbb{R}} \mathbf{ber}\left(\Re\left(e^{i\theta}A\right)\right),$$

where $\Re(A) = \frac{A+A^{*}}{2}$*.*

LEMMA 1.7. [12] *Let* $A \in \mathscr{B}(\mathscr{H})$ *be a positive operator. Then*

$$\|A\|_{ber} = \mathbf{ber}(A).$$

LEMMA 1.8. [21] *Let* $A \in \mathscr{B}(\mathscr{H})$*. Then*

$$|\langle Ax,y\rangle|^{2} \leqslant |\langle |A|x,x\rangle|\,|\langle |A^{*}|y,y\rangle|,$$

foll all $x,y \in \mathscr{H}$*.*

LEMMA 1.9. [21] *Let* $T \in \mathscr{B}(\mathscr{H})$ *and let* f *and* g *be non-negative continuous functions on* $[0,+\infty)$ *such that* $f(t)g(t) = t$ *for all* $t \in [0,+\infty)$*. Then*

$$|\langle Tx,y\rangle|^{2} \leqslant \left\langle f^{2}(|T|)x,x\right\rangle \left\langle g^{2}(|T^{*}|)y,y\right\rangle,$$

for all $x,y \in \mathscr{H}$*.*

LEMMA 1.10. [24] *If* f *is a convex function on a real interval* J *containing the spectrum of the self-adjoint operator* A*, then for any unit vector* $x \in \mathscr{H}$*,*

$$f(\langle Ax,x\rangle) \leqslant \langle f(A)x,x\rangle.$$

LEMMA 1.11. [13] *Let* $x,y,z \in \mathscr{H}$ *with* $\|z\| = 1$. *Then*

$$|\langle x,z\rangle \langle z,y\rangle| \leqslant \frac{1}{2}\left(\|x\|\,\|y\| + |\langle x,y\rangle|\right).$$

LEMMA 1.12. [3, Theorem 3.5] *Let* $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathscr{B}(\mathscr{H}\oplus\mathscr{H})$ *and let* f *and* g *be non-negative continuous functions on* $[0,+\infty)$ *such that* $f(t)g(t) = t \ (t \geqslant 0)$. *Then, for all non-negative nondecreasing convex functions* h *on* $[0,+\infty)$,

$$h(\omega(T)) \leqslant \frac{1}{2}\max\left\{\left\|h\left(f^2(|A|)\right) + h\left(g^2(|A^*|)\right)\right\|, \left\|h\left(f^2(|B|)\right) + h\left(g^2(|B^*|)\right)\right\|\right\}.$$

LEMMA 1.13. [29, Theorem 3.1] *Let* $A,B \in \mathscr{B}(\mathscr{H})$. *Then*

$$\omega\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|A\| + \|AA^* + BB^*\|^{\frac{1}{2}}\right).$$

LEMMA 1.14. [10, Theorem 2.2] *Let* $A,B,C,D \in \mathscr{B}(\mathscr{H})$. *Then*

$$\begin{aligned}\omega^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant{}& \max\left\{\omega^2(A), \omega^2(D)\right\} + \omega^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \\ &+ \frac{1}{2}\max\left\{\left\||A|^2 + |B^*|^2\right\|, \left\||D|^2 + |C^*|^2\right\|\right\} + \omega\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right).\end{aligned}$$

LEMMA 1.15. [10, Theorem 2.3] *Let* $A,B,C,D \in \mathscr{B}(\mathscr{H})$. *Then*

$$\begin{aligned}\omega^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant{}& \max\left\{\omega^2(A), \omega^2(D)\right\} + \frac{1}{4}\max\left\{\left\||C|^2 + |B^*|^2\right\|, \left\||B|^2 + |C^*|^2\right\|\right\} \\ &+ \max\left\{\omega(A), \omega(D)\right\}\max\left\{\||C| + |B^*|\|, \||B| + |C^*|\|\right\} \\ &+ \frac{1}{2}\max\left\{\omega(BC), \omega(CB)\right\}.\end{aligned}$$

LEMMA 1.16. [4, Theorem 2.8] *Let* $A,B,C,D \in \mathscr{B}(\mathscr{H})$. *Then*

$$\omega\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\omega(A) + 2\omega(D) + \sqrt{t^2\omega^2(A) + \|B\|^2} + \sqrt{(1-t)^2\omega^2(A) + \|C\|^2}\right)$$

for $t \in [0,1]$.

LEMMA 1.17. [4, Theorem 2.7] *Let* $A,B,C,D \in \mathscr{B}(\mathscr{H})$. *Then*

$$\omega\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|A\| + 2\omega(D) + \sqrt{\|t^2AA^* + BB^*\|} + \sqrt{\left\|(1-t)^2AA^* + C^*C\right\|}\right)$$

for $t \in [0,1]$.

2. Main results

In this section we present our results. In the first result, we present an inequality that governs the relation between the Berezin number and the Berezin norm, in a way similar to the relation between the numerical radius and the operator norm as in Lemma 1.12. This further explains the similarity between the different concepts.

THEOREM 2.1. *Let $A,B \in \mathscr{B}(\mathscr{H})$ and let f and g be non-negative continuous functions on $[0,+\infty)$ such that $f(t)g(t) = t$ $(t \geqslant 0)$. Then, for all non-negative non-decreasing convex functions h on $[0,+\infty)$,*

$$h\left(\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)\right) \leqslant \frac{1}{2}\max\left\{\left\|h\left(f^2(|A|)\right)+h\left(g^2(|A^*|)\right)\right\|_{ber},\right.$$
$$\left.\left\|h\left(f^2(|B|)\right)+h\left(g^2(|B^*|)\right)\right\|_{ber}\right\}.$$

Proof. Put $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. For any $(\lambda_1,\lambda_2) \in \Omega\times\Omega$, let $\hat{k}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel in $\mathscr{H}\oplus\mathscr{H}$. Then, in view of Lemma 1.9, we have

$$\begin{aligned}\left|\left\langle T\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| &\leqslant \left\langle f^2(|T|)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{1}{2}}\left\langle g^2(|T^*|)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{1}{2}} \\ &\leqslant \frac{\left\langle f^2(|T|)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\left\langle g^2(|T^*|)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle}{2} \\ &\quad \text{(by the arithmetic-geometric mean inequality).}\end{aligned}$$

Therefore, we infer that

$$\begin{aligned}&h\left(\left|\left\langle T\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\right) \\ &\leqslant h\left(\frac{\left\langle f^2(|T|)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle+\left\langle g^2(|T^*|)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle}{2}\right) \\ &\leqslant \frac{1}{2}\left(h\left(\left\langle f^2(|T|)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right)+h\left(\left\langle g^2(|T^*|)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right)\right) \\ &\quad \text{(by the convexity of } h\text{)} \\ &\leqslant \frac{1}{2}\left(\left(\left\langle h\left(f^2(|T|)\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right)+\left(\left\langle h\left(g^2(|T^*|)\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right)\right) \\ &\quad \text{(by Lemma 1.10)} \\ &= \frac{1}{2}\left\langle\begin{bmatrix} h\left(f^2(|A|)\right)+h\left(g^2(|A^*|)\right) & 0 \\ 0 & h\left(f^2(|B|)\right)+h\left(g^2(|B^*|)\right)\end{bmatrix}\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle.\end{aligned}$$

Taking the supremum over all $\hat{k}_{(\lambda_1,\lambda_2)} \in \mathscr{H} \oplus \mathscr{H}$ with $\left\|\hat{k}_{(\lambda_1,\lambda_2)}\right\| = 1$, we obtain

$$\begin{aligned}
& h(\mathbf{ber}(T)) \\
& \leqslant \mathbf{ber}\left(\begin{bmatrix} h\left(f^2(|A|)\right) + h\left(g^2(|A^*|)\right) & 0 \\ 0 & h\left(f^2(|B|)\right) + h\left(g^2(|B^*|)\right) \end{bmatrix}\right) \\
& \leqslant \frac{1}{2} \max\left\{\mathbf{ber}\left(h\left(f^2(|A|)\right) + h\left(g^2(|A^*|)\right)\right), \mathbf{ber}\left(h\left(f^2(|B|)\right) + h\left(g^2(|B^*|)\right)\right)\right\} \\
& \quad \text{(by Lemma 1.1 (i))} \\
& = \frac{1}{2} \max\left\{\left\|h\left(f^2(|A|)\right) + h\left(g^2(|A^*|)\right)\right\|_{ber}, \left\|h\left(f^2(|B|)\right) + h\left(g^2(|B^*|)\right)\right\|_{ber}\right\} \\
& \quad \text{(by Lemma 1.7).}
\end{aligned}$$

Thus,

$$h(\mathbf{ber}(T)) \leqslant \frac{1}{2} \max\left\{\left\|h\left(f^2(|A|)\right) + h\left(g^2(|A^*|)\right)\right\|_{ber}, \left\|h\left(f^2(|B|)\right) + h\left(g^2(|B^*|)\right)\right\|_{ber}\right\},$$

which completes the proof. □

The following corollary follows from Theorem 2.1.

COROLLARY 2.2. *Let $A, B \in \mathscr{B}(\mathscr{H})$, $r \geqslant 1$ and $\alpha \in [0,1]$. Then*

$$\mathbf{ber}^r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leqslant \frac{1}{2} \max\left\{\left\||A|^{2r\alpha} + |A^*|^{2r(1-\alpha)}\right\|_{ber}, \left\||B|^{2r\alpha} + |B^*|^{2r(1-\alpha)}\right\|_{ber}\right\}.$$

Proof. The result follows immediately from Theorem 2.1 for $h(t) = t^r$, $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$. □

For $r = 1$ and $\alpha = \frac{1}{2}$ in Corollary 2.2, we get the following corollary.

COROLLARY 2.3. *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then*

$$\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leqslant \frac{1}{2} \max\left\{\||A| + |A^*|\|_{ber}, \||B| + |B^*|\|_{ber}\right\}.$$

In the following we obtain an upper bound for the Berezin number of operator matrix $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ in a way similar to the relation between the numerical radius and the operator norm of the same operator matrix, as in Lemma 1.13.

THEOREM 2.4. *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then*

$$\mathbf{ber}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|A\| + \|AA^* + BB^*\|^{\frac{1}{2}}\right).$$

Proof. Let $T = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$. We have

$$\begin{aligned} 2\mathbf{ber}\left(\Re\left(e^{i\theta}T\right)\right) &= \mathbf{ber}\left(e^{i\theta}T + e^{-i\theta}T^*\right) \\ &= \mathbf{ber}\left(\begin{bmatrix} e^{i\theta}A + e^{-i\theta}A^* & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) \\ &\leqslant \omega\left(\begin{bmatrix} e^{i\theta}A + e^{-i\theta}A^* & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} e^{i\theta}A + e^{-i\theta}A^* & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) \quad \text{(by Lemma 1.5 (i))} \\ &= r\left(\begin{bmatrix} A^* & e^{i\theta}I \\ B^* & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 \\ A & B \end{bmatrix}\right). \end{aligned}$$

Using the commutative property of the spectral radius, we get

$$\begin{aligned} 2\mathbf{ber}\left(\Re\left(e^{i\theta}T\right)\right) &\leqslant r\left(\begin{bmatrix} A^* & e^{i\theta}I \\ B^* & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 \\ A & B \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} e^{-i\theta}I & 0 \\ A & B \end{bmatrix}\begin{bmatrix} A^* & e^{i\theta}I \\ B^* & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} e^{-i\theta}A^* & I \\ AA^* + BB^* & e^{i\theta}A \end{bmatrix}\right) \\ &\leqslant r\left(\begin{bmatrix} \|A\| & 1 \\ \|AA^* + BB^*\| & \|A\| \end{bmatrix}\right) \quad \text{(by Lemma 1.3)} \\ &= \|A\| + \|AA^* + BB^*\|^{\frac{1}{2}} \quad \text{(by Lemma 1.4)}. \end{aligned}$$

Therefore, we have

$$\mathbf{ber}\left(\Re\left(e^{i\theta}T\right)\right) \leqslant \frac{1}{2}\left(\|A\| + \|AA^* + BB^*\|^{\frac{1}{2}}\right).$$

Taking the supremum over all $\theta \in \mathbb{R}$ in the above inequality, and noting Lemma 1.6, we get the desired result. □

Our next result presents new relations for the Berezin number, in a way similar to Lemma 1.14, where the numerical radius was treated.

THEOREM 2.5. *Let $A, B, C, D \in \mathscr{B}(\mathscr{H})$. Then*

$$\begin{aligned} \mathbf{ber}^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant\ & \mathbf{ber}^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + \mathbf{ber}\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right) \\ & + \max\left\{\mathbf{ber}^2(A), \mathbf{ber}^2(D)\right\} \\ & + \frac{1}{2}\max\left\{\|A^*A + BB^*\|_{ber}, \|CC^* + D^*D\|_{ber}\right\}. \end{aligned}$$

Proof. Put $M = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, $N = \begin{bmatrix} BB^* & 0 \\ 0 & CC^* \end{bmatrix}$, $P = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$, $Q = \begin{bmatrix} A^*A & 0 \\ 0 & D^*D \end{bmatrix}$ and $R = \begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}$. Notice that $MP = R$ and $Q + N = \begin{bmatrix} A^*A + BB^* & 0 \\ 0 & D^*D + CC^* \end{bmatrix}$. Also, $MM^* = N$ and $PP^* = Q$.

Let $\hat{k}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel in $\mathscr{H} \oplus \mathscr{H}$. Then, we have

$$\begin{aligned}
&\left|\left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right|^2 \\
&= \left|\left\langle (P+M)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right|^2 \\
&= \left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle + \left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right|^2 \\
&\leqslant \left(\left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right| + \left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right|\right)^2 \\
&= \left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right|^2 + \left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right|^2 \\
&\quad + 2\left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right| \\
&\leqslant \mathbf{ber}^2(P) + \mathbf{ber}^2(M) + 2\left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right| \\
&= \mathbf{ber}^2(P) + \mathbf{ber}^2(M) + 2\left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \left\langle \hat{k}_{(\lambda_1,\lambda_2)}, M^*\hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right| \\
&\leqslant \mathbf{ber}^2(P) + \mathbf{ber}^2(M) + \left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)}, M^*\hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right| \\
&\quad + \left\|P\hat{k}_{(\lambda_1,\lambda_2)}\right\| \left\|M^*\hat{k}_{(\lambda_1,\lambda_2)}\right\| \quad \text{(by Lemma 1.11)} \\
&= \mathbf{ber}^2(P) + \mathbf{ber}^2(M) + \left|\left\langle MP\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right| \\
&\quad + \sqrt{\left\langle P^*P\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \left\langle MM^*\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle} \\
&= \mathbf{ber}^2(P) + \mathbf{ber}^2(M) + \left|\left\langle R\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right| \\
&\quad + \sqrt{\left\langle Q\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \left\langle N\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle} \\
&\leqslant \mathbf{ber}^2(P) + \mathbf{ber}^2(M) + \left|\left\langle R\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right| \\
&\quad + \frac{1}{2}\left(\left\langle Q\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle + \left\langle N\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right) \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&\leqslant \mathbf{ber}^2(P) + \mathbf{ber}^2(M) + \mathbf{ber}(R) + \frac{1}{2}\left\langle (Q+N)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \\
&\leqslant \mathbf{ber}^2(P) + \mathbf{ber}^2(M) + \mathbf{ber}(R) + \frac{1}{2}\mathbf{ber}(Q+N) \\
&\leqslant \mathbf{ber}^2(P) + \mathbf{ber}^2(M) + \mathbf{ber}(R) + \frac{1}{2}\max\{\mathbf{ber}(A^*A + BB^*), \mathbf{ber}(CC^* + D^*D)\} \\
&\quad \text{(by Lemma 1.1 (i))}
\end{aligned}$$

$$= \max\left\{\mathbf{ber}^2(A), \mathbf{ber}^2(D)\right\} + \mathbf{ber}^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + \mathbf{ber}\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right)$$
$$+\frac{1}{2}\max\{\|A^*A+BB^*\|_{ber}, \|CC^*+D^*D\|_{ber}\} \quad \text{(by Lemma 1.7).}$$

Thus,

$$\left|\left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle\right|^2$$
$$\leqslant \max\left\{\mathbf{ber}^2(A), \mathbf{ber}^2(D)\right\} + \mathbf{ber}^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$$
$$+\mathbf{ber}\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right) + \frac{1}{2}\max\{\|A^*A+BB^*\|_{ber}, \|D^*D+CC^*\|_{ber}\}.$$

Taking the supremum over all $\hat{k}_{(\lambda_1,\lambda_2)} \in \mathscr{H} \oplus \mathscr{H}$ with $\left\|\hat{k}_{(\lambda_1,\lambda_2)}\right\| = 1$ in the above inequality, we get the desired result. □

As an immediate consequence of Theorem 2.5, we have the following.

COROLLARY 2.6. *Let $A, B \in B(\mathscr{H})$. Then*

$$\mathbf{ber}^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \leqslant \|B\|^2 + \|BA\|^2 + \mathbf{ber}^2(A) + \frac{1}{2}\|A^*A+BB^*\|_{ber}.$$

Proof. Using Theorem 2.5 and Lemma 1.1, we have

$$\mathbf{ber}^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \leqslant \mathbf{ber}^2\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) + \mathbf{ber}\left(\begin{bmatrix} 0 & BA \\ BA & 0 \end{bmatrix}\right) + \max\left\{\mathbf{ber}^2(A), \mathbf{ber}^2(A)\right\}$$
$$+\frac{1}{2}\max\{\|A^*A+BB^*\|_{ber}, \|A^*A+BB^*\|_{ber}\}$$
$$\leqslant \|B\|^2 + \|BA\|^2 + \mathbf{ber}^2(A) + \frac{1}{2}\|A^*A+BB^*\|_{ber}. \quad \square$$

In the next theorem we obtain a new upper bound for a 2×2 operator matrix, similar to the numerical radius bound found in Lemma 1.15.

THEOREM 2.7. *Let $A, B, C, D \in \mathscr{B}(\mathscr{H})$. Then*

$$\mathbf{ber}^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$$
$$\leqslant \max\left\{\mathbf{ber}^2(A), \mathbf{ber}^2(D)\right\} + \frac{1}{4}\max\left\{\left\||C|^2+|B^*|^2\right\|_{ber}, \left\||B|^2+|C^*|^2\right\|_{ber}\right\}$$
$$+\max\{\mathbf{ber}(A), \mathbf{ber}(D)\}\max\{\||C|+|B^*|\|_{ber}, \||B|+|C^*|\|_{ber}\}$$
$$+\frac{1}{2}\max\{\mathbf{ber}(BC), \mathbf{ber}(CB)\}.$$

Proof. Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, we put $M = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $N = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Let $\hat{k}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel in $\mathscr{H} \oplus \mathscr{H}$. Then we have

$$\begin{aligned}
&\left|\left\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \\
&= \left|\left\langle (M+N)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \\
&= \left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle + \left\langle N\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \\
&\leqslant \left(\left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| + \left|\left\langle N\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\right)^2 \\
&= \left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 + 2\left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \left\langle N\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
&\quad + \left|\left\langle N\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \\
&\leqslant \left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 + 2\left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
&\quad \times \sqrt{\left\langle |N|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \left\langle |N^*|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle} \\
&\quad + \frac{1}{2}\left(\sqrt{\left\langle |N|^2\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \left\langle |N^*|^2\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle} + \left|\left\langle N\hat{k}_{(\lambda_1,\lambda_2)}, N^*\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\right) \\
&\quad \text{(by Lemma 1.8 and Lemma 1.11)} \\
&\leqslant \left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \\
&\quad + \left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \left(\left\langle |N|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle + \left\langle |N^*|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right) \\
&\quad + \frac{1}{4}\left(\left\langle |N|^2\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle + \left\langle |N^*|^2\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right) + \frac{1}{2}\left|\left\langle N^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 + \left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \left\langle (|N| + |N^*|)\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
&\quad + \frac{1}{4}\left\langle \left(|N|^2 + |N^*|^2\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle + \frac{1}{2}\left|\left\langle N^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
&\leqslant \mathbf{ber}^2(M) + \mathbf{ber}(M)\,\mathbf{ber}(|N| + |N^*|) + \frac{1}{4}\mathbf{ber}\left(|N|^2 + |N^*|^2\right) + \frac{1}{2}\mathbf{ber}\left(N^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\left\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^2 \leqslant \mathbf{ber}^2(M) + \mathbf{ber}(M)\,\mathbf{ber}(|N| + |N^*|) + \frac{1}{4}\mathbf{ber}\left(|N|^2 + |N^*|^2\right) \\
+ \frac{1}{2}\mathbf{ber}\left(N^2\right).
\end{aligned}$$

Taking the supremum over all $\hat{k}_{(\lambda_1,\lambda_2)} \in \mathscr{H} \oplus \mathscr{H}$ with $\left\|\hat{k}_{(\lambda_1,\lambda_2)}\right\| = 1$ in the above inequality, we get

$$\mathbf{ber}^2(T) \leqslant \mathbf{ber}^2(M) + \mathbf{ber}(M)\,\mathbf{ber}(|N| + |N^*|) + \frac{1}{4}\mathbf{ber}\left(|N|^2 + |N^*|^2\right) + \frac{1}{2}\mathbf{ber}\left(N^2\right).$$

Therefore,

$$\begin{aligned}\mathbf{ber}^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant\ & \mathbf{ber}^2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + \mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right)\mathbf{ber}\left(\begin{bmatrix} |C|+|B^*| & 0 \\ 0 & |B|+|C^*| \end{bmatrix}\right) \\ & + \frac{1}{4}\mathbf{ber}\left(\begin{bmatrix} |C|^2+|B^*|^2 & 0 \\ 0 & |B|^2+|C^*|^2 \end{bmatrix}\right) + \frac{1}{2}\mathbf{ber}\left(\begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix}\right).\end{aligned}$$

Using Lemma 1.1 (i), we obtain

$$\begin{aligned}\mathbf{ber}^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant\ & \max\left\{\mathbf{ber}^2(A), \mathbf{ber}^2(D)\right\} \\ & + \max\{\mathbf{ber}(A), \mathbf{ber}(D)\}\max\{\mathbf{ber}(|C|+|B^*|), \mathbf{ber}(|B|+|C^*|)\} \\ & + \frac{1}{4}\max\left\{\mathbf{ber}\left(|C|^2+|B^*|^2\right), \mathbf{ber}\left(|B|^2+|C^*|^2\right)\right\} \\ & + \frac{1}{2}\max\{\mathbf{ber}(BC), \mathbf{ber}(CB)\}.\end{aligned}$$

Applying Lemma 1.7, we obtain

$$\begin{aligned}\mathbf{ber}^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant\ & \max\left\{\mathbf{ber}^2(A), \mathbf{ber}^2(D)\right\} \\ & + \frac{1}{4}\max\left\{\left\||C|^2+|B^*|^2\right\|_{ber}, \left\||B|^2+|C^*|^2\right\|_{ber}\right\} \\ & + \max\{\mathbf{ber}(A), \mathbf{ber}(D)\}\max\{\||C|+|B^*|\|_{ber}, \||B|+|C^*|\|_{ber}\} \\ & + \frac{1}{2}\max\{\mathbf{ber}(BC), \mathbf{ber}(CB)\}.\end{aligned}$$

This completes the proof. □

Several consequences of Theorem 2.7 can be derived. If $A = D$ and $B = C$ in Theorem 2.7, we get the following corollary.

COROLLARY 2.8. *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then*

$$\begin{aligned}\mathbf{ber}^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \leqslant\ & \mathbf{ber}^2(A) + \frac{1}{4}\left\||B|^2+|B^*|^2\right\|_{ber} + \mathbf{ber}(A)\||B|+|B^*|\|_{ber} \\ & + \frac{1}{2}\mathbf{ber}\left(B^2\right).\end{aligned}$$

If $A = 0$ in Corollary 2.8, then we get the following simpler form.

COROLLARY 2.9. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$\mathbf{ber}^2\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) \leqslant \frac{1}{4}\left\||B|^2+|B^*|^2\right\|_{ber} + \frac{1}{2}\mathbf{ber}\left(B^2\right).$$

REMARK 2.10. Using the fact $\mathbf{ber}(X) \leqslant \|X\|_{ber} \leqslant \|X\|$ for every $X \in \mathscr{B}(\mathscr{H})$, it follows that

$$\begin{aligned}\mathbf{ber}^2\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) &\leqslant \frac{1}{4}\left\||B|^2+|B^*|^2\right\|_{ber}+\frac{1}{2}\mathbf{ber}\left(B^2\right)\\ &\leqslant \frac{1}{4}\left\||B|^2+|B^*|^2\right\|+\frac{1}{2}\left\|B^2\right\|\\ &\leqslant \frac{1}{4}\left\||B|^2\right\|+\frac{1}{4}\left\||B^*|^2\right\|+\frac{1}{2}\|B\|^2\\ &= \|B\|^2.\end{aligned}$$

Thus,

$$\mathbf{ber}\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) \leqslant \left(\frac{1}{4}\left\||B|^2+|B^*|^2\right\|_{ber}+\frac{1}{2}\mathbf{ber}\left(B^2\right)\right)^{\frac{1}{2}} \leqslant \|B\|.$$

This is a non-trivial improvement of Lemma 1.1 (iii).
If $A = D = 0$ in Theorem 2.7, then we get the following result.

COROLLARY 2.11. *Let* $B, C \in \mathscr{B}(\mathscr{H})$. *Then*

$$\begin{aligned}\mathbf{ber}^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leqslant &\frac{1}{4}\max\left\{\left\||C|^2+|B^*|^2\right\|_{ber},\left\||B|^2+|C^*|^2\right\|_{ber}\right\}\\ &+\frac{1}{2}\max\left\{\mathbf{ber}(BC),\mathbf{ber}(CB)\right\}.\end{aligned}$$

REMARK 2.12. Using the fact $\mathbf{ber}(X) \leqslant \|X\|_{ber} \leqslant \|X\|$ for every $X \in \mathscr{B}(\mathscr{H})$

$$\begin{aligned}\mathbf{ber}^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) &\leqslant \frac{1}{4}\max\left\{\left\||C|^2+|B^*|^2\right\|_{ber},\left\||B|^2+|C^*|^2\right\|_{ber}\right\}\\ &\quad+\frac{1}{2}\max\left\{\mathbf{ber}(BC),\mathbf{ber}(CB)\right\}\\ &\leqslant \frac{1}{4}\left(\|B\|^2+\|C\|^2+2\|B\|\|C\|\right)\\ &= \frac{1}{4}\left(\|B\|+\|C\|\right)^2.\end{aligned}$$

Thus,

$$\mathbf{ber}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|B\|+\|C\|\right).$$

We note that the inequality in Corollary 2.9 refines Lemma 1.1 (ii).

Now, we are in the position to state an upper bound for the Berezin norm of a 2×2 operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ defined on $\mathscr{H} \oplus \mathscr{H}$, in similar form to the existing bound for the numerical radius appearing in Lemma 1.16.

THEOREM 2.13. *Let $A,B,C,D \in \mathscr{B}(\mathscr{H})$ and let $t \in [0,1]$. Then*

$$\mathbf{ber}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\mathbf{ber}(A) + 2\mathbf{ber}(D) + \sqrt{t^2\mathbf{ber}^2(A) + \|B\|^2} + \sqrt{(1-t)^2\,\mathbf{ber}^2(A) + \|C\|^2}\right).$$

Proof. Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. We have

$$\begin{aligned}
2\mathbf{ber}\left(\Re\left(e^{i\theta}T\right)\right) &= \mathbf{ber}\left(e^{i\theta}T + e^{-i\theta}T^*\right) \\
&= \mathbf{ber}\left(\begin{bmatrix} e^{i\theta}A + e^{-i\theta}A^* & e^{i\theta}B + e^{-i\theta}C^* \\ e^{i\theta}C + e^{-i\theta}B^* & e^{i\theta}D + e^{-i\theta}D^* \end{bmatrix}\right) \\
&\leqslant \mathbf{ber}\left(\begin{bmatrix} t\left(e^{i\theta}A + e^{-i\theta}A^*\right) & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) \\
&\quad + \mathbf{ber}\left(\begin{bmatrix} (1-t)\left(e^{i\theta}A + e^{-i\theta}A^*\right) & e^{-i\theta}C^* \\ e^{i\theta}C & 0 \end{bmatrix}\right) \\
&\quad + \mathbf{ber}\left(\begin{bmatrix} 0 & 0 \\ 0 & e^{i\theta}D + e^{-i\theta}D^* \end{bmatrix}\right) \\
&\leqslant \mathbf{ber}\left(\begin{bmatrix} t\left(e^{i\theta}A + e^{-i\theta}A^*\right) & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) \\
&\quad + \mathbf{ber}\left(\begin{bmatrix} (1-t)\left(e^{i\theta}A + e^{-i\theta}A^*\right) & e^{-i\theta}C^* \\ e^{i\theta}C & 0 \end{bmatrix}\right) + 2\mathbf{ber}(D) \\
&\leqslant \omega\left(\begin{bmatrix} t\mathbf{ber}\left(e^{i\theta}A + e^{-i\theta}A^*\right) & \|B\| \\ \|B\| & 0 \end{bmatrix}\right) \\
&\quad + \omega\left(\begin{bmatrix} (1-t)\mathbf{ber}\left(e^{i\theta}A + e^{-i\theta}A^*\right) & \|C\| \\ \|C\| & 0 \end{bmatrix}\right) + 2\mathbf{ber}(D) \\
&\quad \text{(by Lemma 1.2)} \\
&= \omega\left(\begin{bmatrix} 2t\mathbf{ber}\left(\Re\left(e^{i\theta}A\right)\right) & \|B\| \\ \|B\| & 0 \end{bmatrix}\right) \\
&\quad + \omega\left(\begin{bmatrix} 2(1-t)\mathbf{ber}\left(\Re\left(e^{i\theta}A\right)\right) & \|C\| \\ \|C\| & 0 \end{bmatrix}\right) + 2\mathbf{ber}(D) \\
&\leqslant \omega\left(\begin{bmatrix} 2t\mathbf{ber}(A) & \|B\| \\ \|B\| & 0 \end{bmatrix}\right) \\
&\quad + \omega\left(\begin{bmatrix} 2(1-t)\mathbf{ber}(A) & \|C\| \\ \|C\| & 0 \end{bmatrix}\right) + 2\mathbf{ber}(D) \\
&= r\left(\begin{bmatrix} 2t\mathbf{ber}(A) & \|B\| \\ \|B\| & 0 \end{bmatrix}\right) + r\left(\begin{bmatrix} 2(1-t)\mathbf{ber}(A) & \|C\| \\ \|C\| & 0 \end{bmatrix}\right) \\
&\quad + 2\mathbf{ber}(D) \quad \text{(by Lemma 1.5 (i))}
\end{aligned}$$

$$= \mathbf{ber}(A) + \sqrt{t^2\mathbf{ber}^2(A) + \|B\|^2} + \sqrt{(1-t)^2\mathbf{ber}^2(A) + \|C\|^2}$$
$$+ 2\mathbf{ber}(D) \quad \text{(by Lemma 1.4)}.$$

Hence

$$\mathbf{ber}\left(\Re\left(e^{i\theta}T\right)\right)$$
$$\leqslant \frac{1}{2}\left(\mathbf{ber}(A) + 2\mathbf{ber}(D) + \sqrt{t^2\mathbf{ber}^2(A) + \|B\|^2} + \sqrt{(1-t)^2\mathbf{ber}^2(A) + \|C\|^2}\right).$$

Taking the supremum over all $\theta \in \mathbb{R}$ in the above inequality we get the desired result, by using Lemma 1.6. □

REMARK 2.14. Theorem 2.13 has recently been proved by Bakherad et al. in [5, Theorem 3.5]. Our approach here is different from theirs.

Next, an analogue of Lemma 1.17 is stated.

THEOREM 2.15. *Let* $A, B, C, D \in \mathscr{B}(\mathscr{H})$ *and let* $t \in [0,1]$. *Then*

$$\mathbf{ber}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|A\| + 2\mathbf{ber}(D) + \sqrt{\|t^2AA^* + BB^*\|} + \sqrt{\left\|(1-t)^2AA^* + C^*C\right\|}\right).$$

Proof. Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. We have

$$\begin{aligned}
2\mathbf{ber}\left(\Re\left(e^{i\theta}T\right)\right) &= \mathbf{ber}\left(e^{i\theta}T + e^{-i\theta}T^*\right) \\
&= \mathbf{ber}\left(\begin{bmatrix} e^{i\theta}A + e^{-i\theta}A^* & e^{i\theta}B + e^{-i\theta}C^* \\ e^{i\theta}C + e^{-i\theta}B^* & e^{i\theta}D + e^{-i\theta}D^* \end{bmatrix}\right) \\
&\leqslant \mathbf{ber}\left(\begin{bmatrix} t\left(e^{i\theta}A + e^{-i\theta}A^*\right) & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) \\
&\quad + \mathbf{ber}\left(\begin{bmatrix} (1-t)\left(e^{i\theta}A + e^{-i\theta}A^*\right) & e^{-i\theta}C^* \\ e^{i\theta}C & 0 \end{bmatrix}\right) \\
&\quad + \mathbf{ber}\left(\begin{bmatrix} 0 & 0^* \\ 0 & e^{i\theta}D + e^{-i\theta}D^* \end{bmatrix}\right) \\
&\leqslant \mathbf{ber}\left(\begin{bmatrix} t\left(e^{i\theta}A + e^{-i\theta}A^*\right) & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) \\
&\quad + \mathbf{ber}\left(\begin{bmatrix} (1-t)\left(e^{i\theta}A + e^{-i\theta}A^*\right) & e^{-i\theta}C^* \\ e^{i\theta}C & 0 \end{bmatrix}\right) + 2\mathbf{ber}(D) \\
&\leqslant \omega\left(\begin{bmatrix} t\left(e^{i\theta}A + e^{-i\theta}A^*\right) & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) \\
&\quad + \omega\left(\begin{bmatrix} (1-t)\left(e^{i\theta}A + e^{-i\theta}A^*\right) & e^{-i\theta}C^* \\ e^{i\theta}C & 0 \end{bmatrix}\right) + 2\mathbf{ber}(D)
\end{aligned}$$

$$= r\left(\begin{bmatrix} t\left(e^{i\theta}A+e^{-i\theta}A^*\right) & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) + r\left(\begin{bmatrix} (1-t)\left(e^{i\theta}A+e^{-i\theta}A^*\right) & e^{-i\theta}C^* \\ e^{i\theta}C & 0 \end{bmatrix}\right) + 2\mathbf{ber}(D),$$

where we have used Lemma 1.5 (i) to obtain the last equality. Thus, we have

$$2\mathbf{ber}\left(\Re\left(e^{i\theta}T\right)\right) = r\left(\begin{bmatrix} tA^* & e^{i\theta}I \\ B^* & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 \\ tA & B \end{bmatrix}\right) + r\left(\begin{bmatrix} (1-t)A^* & e^{i\theta}I \\ e^{2i\theta}C & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 \\ (1-t)A & e^{-2i\theta}C^* \end{bmatrix}\right) + 2\mathbf{ber}(D).$$

Now using the second part of Lemma 1.5, we obtain

$$\begin{aligned} 2\mathbf{ber}\left(\Re\left(e^{i\theta}T\right)\right) &= r\left(\begin{bmatrix} e^{-i\theta}I & 0 \\ tA & B \end{bmatrix}\begin{bmatrix} tA^* & e^{i\theta}I \\ B^* & 0 \end{bmatrix}\right) \\ &\quad + r\left(\begin{bmatrix} e^{-i\theta}I & 0 \\ (1-t)A & e^{-2i\theta}C^* \end{bmatrix}\begin{bmatrix} (1-t)A^* & e^{i\theta}I \\ e^{2i\theta}C & 0 \end{bmatrix}\right) + 2\mathbf{ber}(D) \\ &\leqslant r\left(\begin{bmatrix} t\|A\| & 1 \\ \|t^2AA^*+BB^*\| & t\|A\| \end{bmatrix}\right) \\ &\quad + r\left(\begin{bmatrix} (1-t)\|A\| & 1 \\ \left\|(1-t)^2AA^*+C^*C\right\| & (1-t)\|A\| \end{bmatrix}\right) + 2\mathbf{ber}(D) \\ &\quad \text{(by Lemma 1.3)} \\ &= \|A\| + 2\mathbf{ber}(D) + \sqrt{\|t^2AA^*+BB^*\|} + \sqrt{\left\|(1-t)^2AA^*+C^*C\right\|} \\ &\quad \text{(by Lemma 1.4).} \end{aligned}$$

Consequently,

$$\mathbf{ber}\left(\Re\left(e^{i\theta}T\right)\right) \leqslant \frac{1}{2}\left(\|A\|+2\mathbf{ber}(D)+\sqrt{\|t^2AA^*+BB^*\|}+\sqrt{\left\|(1-t)^2AA^*+C^*C\right\|}\right).$$

Taking the supremum over all $\theta \in \mathbb{R}$ in the above inequality we get desired result. □

As a consequence of Theorem 2.15, we have the following well-known result.

COROLLARY 2.16. *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then*

$$\mathbf{ber}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|A\|+\|B\|\right).$$

Proof. Appling Theorem 2.15, we have

$$\begin{aligned}\mathbf{ber}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\leqslant \frac{1}{2}\left(\|AA^*\|^{\frac{1}{2}} + \|B^*B\|^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\left(\|A\| + \|B\|\right).\end{aligned}$$

Hence, we get the desired inequality. □

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