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HYPONORMALITY OF THE SUM OF TOEPLITZ OPERATORS WITH NON–HARMONIC SYMBOL ON THE FOCK SPACE

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Abstract. In this paper, hyponormality of the Toeplitz operator on the Fock space is discussed. It shows that the Toeplitz operator with monomial non-harmonic symbol is hyponormal and observes that the sum of two hyponormal Toeplitz operators need not be hyponormal. The paper further derives the sufficient conditions for hyponormality of the sum of two Toeplitz operators corresponding to various non-harmonic symbols.

1. Introduction

Let $L^2(\mathbb{C},d\lambda)$ be the space of all Lebesgue measurable square integrable functions on the complex plane $\mathbb{C}$ with the norm given by

$$||f||^2 = \int_{\mathbb{C}} |f(z)|^2\, d\lambda(z) \;\text{ for } f \in L^2(\mathbb{C},d\lambda),$$

where $d\lambda$ is the Gaussian measure $d\lambda(z) = \frac{1}{\pi}e^{-|z|^2}dA(z),\ z\in\mathbb{C}$ and dA denote the area measure. Let $H(\mathbb{C})$ be the space of all entire functions on $\mathbb{C}$. Then the space $\mathscr{F}^2(\mathbb{C}) = L^2(\mathbb{C},d\lambda)\cap H(\mathbb{C})$ is called the Fock space. The space $\mathscr{F}^2(\mathbb{C})$ is a Hilbert space with the inner product

$$\langle f,g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}\,d\lambda(z) \;\text{ where } f,g\in\mathscr{F}^2(\mathbb{C}).$$

For non-negative integer n, let $e_n(z) = \frac{z^n}{\sqrt{n!}}$. Then the set $\{e_n\}_{n\geqslant 0}$ forms an orthonormal basis for $\mathscr{F}^2(\mathbb{C})$. For $z,w\in\mathbb{C}$, the reproducing kernel of the Fock space is given by $K_z(w) = e^{\bar{z}w}$ and the normalized reproducing kernel is given by $k_z(w) = \frac{K_z(w)}{||K_z(w)||} = e^{\bar{z}w-\frac{|z|^2}{2}}$. Let $P: L^2(\mathbb{C},d\lambda)\to\mathscr{F}^2(\mathbb{C})$ denote the orthogonal projection onto the Fock space and is given by

$$P(f(z)) = \langle Pf(w),K_z(w)\rangle = \int_{\mathbb{C}} e^{z\overline{w}} f(w)\,d\lambda(w).$$

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The Fock space was introduced independently by Segal [11] and Bargmann [1]. Berger and Coburn [2] studied Toeplitz operators on the $\mathscr{F}^2(\mathbb{C})$ and established the relation of Toeplitz operators to the Weyl operators in Quantum mechanics. In 2014, Zhu, Park and Cho [3] discussed the properties of Toeplitz operators defined on the Fock space. Isralowitz and Zhu [8] studied the operator when the symbol is of positive measure.

Let $L^\infty(\mathbb{C})$ be the Banach space of all essentially bounded Lebesgue measurable functions on the complex plane. For $\phi \in L^\infty(\mathbb{C})$, the Toeplitz operator $T_\phi : \mathscr{F}^2(\mathbb{C}) \to \mathscr{F}^2(\mathbb{C})$ is defined by $T_\phi(f) = P(\phi f) = \int_{\mathbb{C}} e^{z\overline{w}} \phi(w) f(w)\, d\lambda(w)$, for every $f \in \mathscr{F}^2(\mathbb{C})$. Let $\mathscr{A}$ be the set of all measurable functions ϕ such that $\phi f \in L^2(\mathbb{C}, d\lambda)$, for all $f \in \mathscr{F}^2(\mathbb{C})$. The Toeplitz operator is also well defined for the symbol $\phi \in \mathscr{A}$. For any complex numbers c and d, the Toeplitz operators with symbols $\phi, \psi \in \mathscr{A}$, satisfy the following:

$(i)\, T_{c\phi+d\psi} = cT_\phi + dT_\psi$

$(ii)\, T_{\overline{\phi}} = T_\phi{}^*$

$(iii)\, T_{\overline{\phi}} T_\psi = T_{\overline{\phi}\psi}$ where ϕ or ψ is analytic.

An operator T acting on a Hilbert space H with adjoint T^* is said to be hyponormal, if $||T^*u|| \leqslant ||Tu||$ for all $u \in H$. Hyponormal operators have been a keen area of interest for the researchers as these operators have profound applications in deriving their spectral properties. In this paper, we will study the hyponormality of the Toeplitz operator on the Fock space when its symbol belongs to the set $\mathscr{A}$. For the essentially bounded symbol, Cowen [4] completely characterised hyponormal Toeplitz operator in the setting of the Hardy space. The proof by Cowen was based on Sarason's [10] dilation theorem. In the setting of the Bergman space and the Fock space, there is no analogue to the Sarason theorem, so the characterization has been elusive. But substantial work has been done in the framework of the Bergman space by Hwang [7], Sadraoui [9], Curto and Čučković [5], when the symbol ϕ is a harmonic function. Subsequently, Wang and Zhao [13] established the necessary and sufficient conditions for the dual Toeplitz operators corresponding to the non-harmonic symbols to be hyponormal on the harmonic Bergman space. In 2019, Fleeman and Liaw [6] discussed the hyponormality of Toeplitz operator on the Bergman space with the non-harmonic symbol. Motivated by the work of Fleeman, we will be studying the hyponormality of the Toeplitz operator T_ϕ on the Fock space, when its symbol $\phi \in \mathscr{A}$ is a non-harmonic function. In Section 2, it is shown that the Toeplitz operator with monomial non-harmonic symbol is hyponormal on the Fock space. If a self-adjoint operator is perturbed by another hyponormal Toeplitz operator, the resultant need not be hyponormal. In Section 3, we focus on the hyponormality of the sum of various Toeplitz operators whose symbols are non-harmonic monomials (belonging to the set $\mathscr{A}$). The sufficient conditions are derived for the hyponormality of the Toeplitz operators whose symbols are two term non-harmonic polynomials. We present examples to show that a Toeplitz operator which is hyponormal on the Fock space need not be hyponormal on the Bergman space.

2. Monomial non-harmonic symbol

Consider the symbol $\phi = z^s$ (s is a non-negative integer) in the set $\mathscr{A}$, then the Toeplitz operator T_{z^s} is hyponormal as for $f \in \mathscr{F}^2(\mathbb{C})$, we have

$$||T_{z^s}f||^2 = ||Pz^s f||^2 = \int_{\mathbb{C}} |z^s f|^2 = \int_{\mathbb{C}} |\overline{z}^s f|^2 \geqslant \int_{\mathbb{C}} |P\overline{z}^s f|^2 = ||T_{z^s}^* f||^2.$$

Also, the Toeplitz operator $T_{|z|^2}$ is hyponormal being a self-adjoint operator. If a hyponormal operator is perturbed, the hyponormality is not preserved, in general. In the framework of Hardy space, Sadraoui [9] showed that $T_{z^m+\alpha\overline{z}^n}$ is hyponormal if and only if $|\alpha|$ is sufficiently small (depending upon the values of m and n). In [12], Simanek studied the effect of perturbation on the hyponormality of the Toeplitz operator in the setting of the Bergman space. To isolate the perturbations and understand their effect on the hyponormality of an operator, Fleeman and Liaw [6] gave the following result that discusses the behaviour of the self-commutator under the operator addition.

PROPOSITION 2.1. [6] *Let T and S be operators on a Hilbert space H. Then*

$$\begin{aligned}&\langle [(T+S)^*, T+S]u, u\rangle \\ &= \langle Tu, Tu\rangle - \langle T^*u, T^*u\rangle + 2Re[\langle Tu, Su\rangle - \langle T^*u, S^*u\rangle] + \langle Su, Su\rangle - \langle S^*u, S^*u\rangle\end{aligned}$$

for any $u \in H$.

In this section, we present the hyponormality of the Toeplitz operator on the Fock space, when its symbol is monomial non-harmonic function. Also, it is shown that, if a hyponormal Toeplitz operator is perturbed by another hyponormal symbol, the operator need not be hyponormal. The following lemma is instrumental in the subsequent results of the section.

LEMMA 2.2. *For any non-negative integers s and t,*

$$(i)\ \langle z^s, z^t\rangle = \begin{cases} s! & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$$

$$(ii)\ P(z^s\overline{z}^t) = \begin{cases} \dfrac{s!}{(s-t)!} z^{s-t} & \text{if } s \geqslant t \\ 0 & \text{if } s < t, \end{cases}$$

where $P : L^2 \to \mathscr{F}^2$ is the orthogonal projection.

Next theorem shows that for the monomial non-harmonic symbol, the Toeplitz operator is hyponormal in the Fock space.

THEOREM 2.3. *Consider $\phi = az^s\overline{z}^t \in \mathscr{A}, s \geqslant t$, where a is an arbitrary complex number. Then the Toeplitz operator T_ϕ is hyponormal.*

Proof. For an element $f(z)=\sum_{k=0}^{\infty} c_k z^k$ of the Fock space $\mathscr{F}^2(\mathbb{C})$ using $T_\phi^* = T_{\overline{\phi}}$ and Lemma 2.2, we get

$$\begin{aligned}
\langle [T_\phi^*, T_\phi] f, f\rangle &= \|T_\phi f\|^2 - \|T_\phi^* f\|^2 \\
&= \|P\sum_{k=0}^{\infty} a z^s \overline{z}^t c_k z^k\|^2 - \|P\sum_{k=0}^{\infty} \overline{a}\,\overline{z}^s z^t c_k z^k\|^2 \\
&= |a|^2\Big(\Big\|\sum_{k=0}^{\infty}\frac{(s+k)!}{(s+k-t)!}c_k z^{s+k-t}\Big\|^2 - \Big\|\sum_{k=s-t}^{\infty}\frac{(t+k)!}{(t+k-s)!}c_k z^{t+k-s}\Big\|^2\Big) \\
&= |a|^2\Big(\sum_{k=0}^{\infty}\frac{[(s+k)!]^2}{(s+k-t)!}|c_k|^2 - \sum_{k=s-t}^{\infty}\frac{[(t+k)!]^2}{(t+k-s)!}|c_k|^2\Big) \\
&= |a|^2\left(\sum_{k=0}^{s-t-1}\frac{[(s+k)!]^2}{(s+k-t)!}|c_k|^2 + \sum_{k=s-t}^{\infty}\Big[\frac{[(s+k)!]^2}{(s+k-t)!} - \frac{[(t+k)!]^2}{(t+k-s)!}\Big]|c_k|^2\right).
\end{aligned}$$

Consider the second term in the above expression,

$$\begin{aligned}
&\frac{[(s+k)!]^2}{(s+k-t)!} - \frac{[(t+k)!]^2}{(t+k-s)!} \\
&= k!\Big[(s+k)(s+k-1)\dots(k+1).(s+k)(s+k-1)\dots(s+k-t+1) \\
&\qquad - (t+k)(t+k-1)\dots(t+k-s+1).(t+k)(t+k-1)\dots(k+1)\Big]
\end{aligned}$$

Both expressions have the same number of terms, that is $s+t$. For $k \geqslant s-t$, the corresponding terms of the first expression are greater than the terms of the second expression. Thus, for all $f(z)=\sum_{k=0}^{\infty} c_k z^k \in \mathscr{F}^2(\mathbb{C})$, we have the following:

$$\sum_{k=0}^{s-t-1}\frac{[(s+k)!]^2}{(s+k-t)!}|c_k|^2 + \sum_{k=s-t}^{\infty}\Big[\frac{[(s+k)!]^2}{(s+k-t)!} - \frac{[(t+k)!]^2}{(t+k-s)!}\Big]|c_k|^2 \geqslant 0,$$

which concludes that $T_{az^s\overline{z}^t}$ is hyponormal for $a \in \mathbb{C}$. □

COROLLARY 2.4. *The self commutator, $[T^*_{z^s\overline{z}^t}, T_{z^s\overline{z}^t}]$ is a diagonal operator with the eigen values λ_k given by:*

$$\lambda_k = \begin{cases} \dfrac{[(s+k)!]^2}{k!\,(s+k-t)!} & \text{if } 0 \leqslant k < s-t \\[2ex] \dfrac{[(s+k)!]^2}{k!\,(s+k-t)!} - \dfrac{[(t+k)!]^2}{k!\,(t+k-s)!} & \text{if } k \geqslant s-t. \end{cases}$$

Proof. For any $z \in \mathbb{C}$, we have

$$\begin{aligned}
T^*_{z^s\overline{z}^t} T_{z^s\overline{z}^t}(z^k) &= T_{\overline{z}^s z^t} P z^{s+k}\overline{z}^t = T_{\overline{z}^s z^t}\left(\frac{(s+k)!}{(s+k-t)!}z^{s+k-t}\right) \\
&= \frac{(s+k)!}{(s+k-t)!}Pz^{s+k}\overline{z}^s = \frac{[(s+k)!]^2}{k!\,(s+k-t)!}z^k.
\end{aligned}$$

Similarly, one can show that:

$$T_{z^s\overline{z}^t}T^*_{z^s\overline{z}^t}(z^k)=\begin{cases}\dfrac{[(t+k)!]^2}{k!\,(t+k-s)!}z^k & \text{if } k\geqslant s-t\\ 0 & \text{otherwise.}\end{cases}$$

From the above two expressions, we can write self commutator as:

$$(T^*_{z^s\overline{z}^t}T_{z^s\overline{z}^t}-T_{z^s\overline{z}^t}T^*_{z^s\overline{z}^t})(z^k)=\begin{cases}\dfrac{[(s+k)!]^2}{k!\,(s+k-t)!}z^k & \text{if } 0\leqslant k<s-t\\ \dfrac{[(s+k)!]^2}{k!\,(s+k-t)!}z^k-\dfrac{[(t+k)!]^2}{k!\,(t+k-s)!}z^k & \text{if } k\geqslant s-t.\end{cases}$$

Thus, we conclude that $[T^*_{z^s\overline{z}^t},T_{z^s\overline{z}^t}]$ is a diagonal operator with the eigen values λ_k. □

When a symbol corresponding to a hyponormal Toeplitz operator is perturbed by a symbol corresponding to a self-adjoint operator, the resultant operator need not be hyponormal. The following example shows that $T_{z+B|z|^2}$ is not hyponormal for $|B|>1$.

EXAMPLE 2.5. The Toeplitz operator $T_{z+B|z|^2}$ is not hyponormal for $|B|>1$.

Proof. Consider $f(z)=\sum_{k=0}^{\infty}c_ke_k(z)\in\mathscr{F}^2(\mathbb{C})$, where $\{e_k\}$ is the orthonormal basis for the Fock space. Then,

$$T_zf=T_z\sum_{k=0}^{\infty}c_ke_k(z)=\sum_{k=0}^{\infty}\frac{c_kz^{k+1}}{\sqrt{k!}}=\sum_{k=0}^{\infty}c_k\sqrt{k+1}\,e_{k+1}(z), \tag{1}$$

$$T_{\overline{z}}f=P\sum_{k=0}^{\infty}\frac{c_kz^k\overline{z}}{\sqrt{k!}}=\sum_{k=1}^{\infty}c_k\sqrt{k}\,e_{k-1}(z), \tag{2}$$

$$T_{B|z|^2}f=P\sum_{k=0}^{\infty}Bc_k\frac{z^{k+1}\overline{z}}{\sqrt{k!}}=\sum_{k=0}^{\infty}Bc_k(k+1)\,e_k(z). \tag{3}$$

To compute the cross-terms, use equations (1), (2) and (3) to get the following:

$$\begin{aligned}\langle T_zf,T_{B|z|^2}f\rangle&=\langle\sum_{k=0}^{\infty}c_k\sqrt{k+1}\,e_{k+1}(z),\sum_{k=0}^{\infty}Bc_k(k+1)\,e_k(z)\rangle\\&=\sum_{k=0}^{\infty}\overline{B}c_k\overline{c_{k+1}}\sqrt{k+1}(k+2),\end{aligned} \tag{4}$$

$$\begin{aligned}\langle T_{\overline{z}}f,T_{\overline{B}|z|^2}f\rangle&=\langle\sum_{k=1}^{\infty}c_k\sqrt{k}\,e_{k-1}(z),\sum_{k=0}^{\infty}\overline{B}c_k(k+1)\,e_k(z)\rangle\\&=\sum_{k=0}^{\infty}B\overline{c_k}c_{k+1}\sqrt{k+1}(k+1).\end{aligned} \tag{5}$$

Using equations (4) and (5), the cross-terms are given by:

$$2\,Re\left[\langle T_z f, T_{B|z|^2} f\rangle - \langle T_{\overline{z}} f, T_{\overline{B}|z|^2} f\rangle\right]$$
$$= 2\,Re\left[\sum_{k=0}^{\infty}\sqrt{k+1}\left\{\overline{B}c_k\overline{c_{k+1}}(k+2) - B\overline{c_k}c_{k+1}(k+1)\right\}\right].$$

As $T_{B|z|^2}$ is a self-adjoint operator, we have $||T_{B|z|^2}f||^2 - ||T^*_{B|z|^2}f||^2 = 0$. Also note that, $||T_z f||^2 - ||T_{\overline{z}} f||^2 = \sum_{k=0}^{\infty}|c_k|^2$. Thus, to check for the hyponormality, we calculate the following:

$$\begin{aligned}
&\langle [T^*_{z+B|z|^2}, T_{z+B|z|^2}]f, f\rangle \\
&= ||T_z f||^2 - ||T_{\overline{z}} f||^2 + 2\,Re\left[\langle T_z f, T_{B|z|^2} f\rangle - \langle T_{\overline{z}} f, T_{\overline{B}|z|^2} f\rangle\right] \\
&= \sum_{k=0}^{\infty}|c_k|^2 + 2\,Re\left[\sum_{k=0}^{\infty}\sqrt{k+1}\left\{\overline{B}c_k\overline{c_{k+1}}(k+2) - B\overline{c_k}c_{k+1}(k+1)\right\}\right]. \qquad (6)
\end{aligned}$$

In particular, choose $f = c_0 e_0(z) + c_1 e_1(z)$, where $arg\, c_0 - arg\, c_1 = arg\, B - \pi$. Then, equation (6) can be written as:

$$\begin{aligned}
&\langle [T^*_{z+B|z|^2}, T_{z+B|z|^2}]f, f\rangle \\
&= |c_0|^2 + |c_1|^2 + 2\,Re\left[2\overline{B}c_o\overline{c_1} - B\overline{c_0}c_1\right] \\
&= |c_0|^2 + |c_1|^2 - 2|Bc_0c_1|.
\end{aligned}$$

For $|c_0| = |c_1|$, we have $\langle [T^*_{z+B|z|^2}, T_{z+B|z|^2}]f, f\rangle = 2|c_0|^2(1-|B|) < 0$, implying that $T_{z+B|z|^2}$ is not hyponormal for $|B| > 1$. □

3. Binomial non-harmonic symbol

In general, the sum of two hyponormal Toeplitz operators fails to be a hyponormal operator, but in this section, we find the sufficient conditions under which sum of the two hyponormal operators is again a hyponormal operator on the Fock space. Also, the sufficient conditions of hyponormality are discussed when the symbol is sum of the non-harmonic monomials corresponding to a hyponormal and co-hyponormal operator.

For non-negative integers, $s,t,p,$ and q with $s > t, p > q$ satisfying $s-t > p-q$, we define

$$\alpha_k = \begin{cases} \dfrac{(s+k)!\,(s+k-t+q)!}{(s+k-t)!} & \text{if } 0 \leqslant k < p-q \\[2ex] \dfrac{(s+k)!\,(s+k-t+q)!}{(s+k-t)!} - \dfrac{(q+k)!\,(s+k-p+q)!}{(q+k-p)!} & \text{if } k \geqslant p-q \end{cases} \qquad (7)$$

and

$$\beta_k = \begin{cases} 0 & \text{if } 0 \leqslant k < s-t-p+q \\ \alpha_{k-s+t+p-q} & \text{if } k \geqslant s-t-p+q \end{cases}$$

$$= \begin{cases} 0 & \text{if } 0 \leqslant k < s-t-p+q \\ \frac{(p+k)!\,(t+k+p-q)!}{(k+p-q)!} & \text{if } s-t-p+q \leqslant k < s-t \\ \frac{(p+k)!\,(t+k+p-q)!}{(p+k-q)!} - \frac{(t+k)!\,(t+k-s+p)!}{(t+k-s)!} & \text{if } k \geqslant s-t. \end{cases} \tag{8}$$

THEOREM 3.1. *Consider $T_\phi + T_\psi = T_{\phi+\psi}$ where the symbols $\phi = az^s\bar{z}^t$ and $\psi = bz^p\bar{z}^q$, $s > t$ and $p > q$ satisfying $s-t > p-q$, belong to the set $\mathscr{A}$. Then $T_{\phi+\psi}$ is hyponormal if the following conditions are satisfied:*

(i) *For $k \leqslant p-q-1$,*

$$\left|\frac{a}{b}\right| \frac{[(s+k)!]^2}{(s+k-t)!} + \left|\frac{b}{a}\right| \frac{[(p+k)!]^2}{(p+k-q)!} \geqslant \alpha_k + \beta_k,$$

(ii) *For $p-q \leqslant k \leqslant s-t-1$,*

$$\left|\frac{a}{b}\right| \frac{[(s+k)!]^2}{(s+k-t)!} + \left|\frac{b}{a}\right| \left[\frac{[(p+k)!]^2}{(p+k-q)!} - \frac{[(q+k)!]^2}{(q+k-p)!}\right] \geqslant \alpha_k + \beta_k,$$

(iii) *For $s-t \leqslant k$,*

$$\left|\frac{a}{b}\right| \left[\frac{[(s+k)!]^2}{(s+k-t)!} - \frac{[(t+k)!]^2}{(t+k-s)!}\right] + \left|\frac{b}{a}\right| \left[\frac{[(p+k)!]^2}{(p+k-q)!} - \frac{[(q+k)!]^2}{(q+k-p)!}\right] \geqslant \alpha_k + \beta_k.$$

Proof. For an arbitrary $f(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathscr{F}^2(\mathbb{C})$, we have $\phi f = \sum_{k=0}^{\infty} a\,c_k z^{s+k}\bar{z}^t$ and $\overline{\phi} f = \sum_{k=0}^{\infty} \bar{a}\,c_k z^{t+k}\bar{z}^s$. Lemma 2.2 implies that

$$P\phi f = \sum_{k=0}^{\infty} a\,c_k \frac{(s+k)!}{(s+k-t)!} z^{s+k-t} \quad \text{and} \quad P\overline{\phi} f = \sum_{k=s-t}^{\infty} \bar{a}\,c_k \frac{(t+k)!}{(t+k-s)!} z^{t+k-s}.$$

Similarly for the symbol ψ, we get

$$P\psi f = \sum_{k=0}^{\infty} b\,c_k \frac{(p+k)!}{(p+k-q)!} z^{p+k-q} \quad \text{and} \quad P\overline{\psi} f = \sum_{k=p-q}^{\infty} \bar{b}\,c_k \frac{(q+k)!}{(q+k-p)!} z^{q+k-p}.$$

The norms are given by

$$||P\phi f||^2 = \sum_{k=0}^{\infty} |a|^2 |c_k|^2 \frac{[(s+k)!]^2}{(s+k-t)!} \quad \text{and} \quad ||P\overline{\phi} f||^2 = \sum_{k=s-t}^{\infty} |a|^2 |c_k|^2 \frac{[(t+k)!]^2}{(t+k-s)!}, \tag{9}$$

$$||P\psi f||^2 = \sum_{k=0}^{\infty} |b|^2 |c_k|^2 \frac{[(p+k)!]^2}{(p+k-q)!} \quad \text{and} \quad ||P\overline{\psi} f||^2 = \sum_{k=p-q}^{\infty} |b|^2 |c_k|^2 \frac{[(q+k)!]^2}{(q+k-p)!}. \tag{10}$$

As $s-t>p-q$,

$$\begin{aligned}
&\langle P\phi f, P\psi f\rangle \\
&= \Big\langle \sum_{k=0}^{\infty} a c_k \frac{(s+k)!}{(s+k-t)!} z^{s+k-t}, \sum_{k=0}^{\infty} b c_k \frac{(p+k)!}{(p+k-q)!} z^{p+k-q} \Big\rangle \\
&= \Big\langle \sum_{k=0}^{\infty} a c_k \frac{(s+k)!}{(s+k-t)!} z^{s+k-t}, \sum_{k=s-t-p+q}^{\infty} b c_k \frac{(p+k)!}{(p+k-q)!} z^{p+k-q} \Big\rangle \\
&= \Big\langle \sum_{k=0}^{\infty} a c_k \frac{(s+k)!}{(s+k-t)!} z^{s+k-t}, \sum_{k=0}^{\infty} b c_{k+s-t-p+q} \frac{(s+k-t+q)!}{(s+k-t)!} z^{s+k-t} \Big\rangle \\
&= \sum_{k=0}^{\infty} a \overline{b} c_k \overline{c_{k+s-t-p+q}} \frac{(s+k)!(s+k-t+q)!}{(s+k-t)!} \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
&\langle P\overline{\phi} f, P\overline{\psi} f\rangle \\
&= \Big\langle \sum_{k=s-t}^{\infty} \overline{a} c_k \frac{(t+k)!}{(t+k-s)!} z^{t+k-s}, \sum_{k=p-q}^{\infty} \overline{b} c_k \frac{(q+k)!}{(q+k-p)!} z^{q+k-p} \Big\rangle \\
&= \Big\langle \sum_{k=p-q}^{\infty} \overline{a} c_{k+s-t-p+q} \frac{(s+k-p+q)!}{(q+k-p)!} z^{q+k-p}, \sum_{k=p-q}^{\infty} \overline{b} c_k \frac{(q+k)!}{(q+k-p)!} z^{q+k-p} \Big\rangle \\
&= \sum_{k=p-q}^{\infty} \overline{a} b \overline{c_k} c_{k+s-t-p+q} \frac{(q+k)!(s+k-p+q)!}{(q+k-p)!}. \qquad (12)
\end{aligned}$$

Thus, using (11) and (12), the cross term $Re[\langle P\phi f, P\psi f\rangle - \langle P\overline{\phi} f, P\overline{\psi} f\rangle]$ can be written as follows:

$$\begin{aligned}
&Re[\langle P\phi f, P\psi f\rangle - \langle P\overline{\phi} f, P\overline{\psi} f\rangle] \\
&= \begin{cases} \sum_{k=0}^{\infty} Re(a\overline{b} c_k \overline{c_{k+s-t-p+q}}) \frac{(s+k)!(s+k-t+q)!}{(s+k-t)!} & \text{if } 0 \leqslant k < p-q \\ \sum_{k=p-q}^{\infty} Re(\overline{a} b \overline{c_k} c_{k+s-t-p+q}) \left(\frac{(s+k)!(s+k-t+q)!}{(s+k-t)!} - \frac{(q+k)!(s+k-p+q)!}{(q+k-p)!} \right) & \text{if } k \geqslant p-q \end{cases} \\
&= \sum_{k=0}^{\infty} Re(a\overline{b} c_k \overline{c_{k+s-t-p+q}}) \alpha_k, \qquad (13)
\end{aligned}$$

where α_k is defined in (7). Further, using the inequality, $-2Re(uv) \leqslant 2|uv| \leqslant |u|^2 + |v|^2$, we can write

$$-2Re(a\overline{b} c_k \overline{c_{k+s-t-p+q}}) \leqslant 2|a\overline{b}|\,|c_k \overline{c_{k+s-t-p+q}}| \leqslant |ab| \left(|c_k|^2 + |c_{k+s-t-p+q}|^2 \right). \qquad (14)$$

For $T_{\phi+\psi}$ to be hyponormal, we need to show that $\langle [T^*_{\phi+\psi}, T_{\phi+\psi}] f, f\rangle \geqslant 0$. Using Proposition 2.1, it is sufficient to show that

$$||T_\phi f||^2 - ||T^*_\phi f||^2 + ||T_\psi f||^2 - ||T^*_\psi f||^2 \geqslant -2Re[\langle T_\phi f, T_\psi f\rangle - \langle T^*_\phi f, T^*_\psi f\rangle].$$

From the calculations done in equations (9) and (10), we have the following:

$$\begin{aligned}
&||T_\phi f||^2 - ||T_\phi^* f||^2 + ||T_\psi f||^2 - ||T_\psi^* f||^2 \\
&= ||P\phi f||^2 - ||P\overline{\phi} f||^2 + ||P\psi f||^2 - ||P\overline{\psi} f||^2 \\
&= \sum_{k=0}^{\infty} |a|^2 |c_k|^2 \frac{[(s+k)!]^2}{(s+k-t)!} - \sum_{k=s-t}^{\infty} |a|^2 |c_k|^2 \frac{[(t+k)!]^2}{(t+k-s)!} \\
&\quad + \sum_{k=0}^{\infty} |b|^2 |c_k|^2 \frac{[(p+k)!]^2}{(p+k-q)!} - \sum_{k=p-q}^{\infty} |b|^2 |c_k|^2 \frac{[(q+k)!]^2}{(q+k-p)!} \\
&= |a|^2 \left(\sum_{k=0}^{s-t-1} \frac{[(s+k)!]^2}{(s+k-t)!} |c_k|^2 + \sum_{k=s-t}^{\infty} \left[\frac{[(s+k)!]^2}{(s+k-t)!} - \frac{[(t+k)!]^2}{(t+k-s)!} \right] |c_k|^2 \right) \\
&\quad + |b|^2 \left(\sum_{k=0}^{p-q-1} \frac{[(p+k)!]^2}{(p+k-q)!} |c_k|^2 + \sum_{k=p-q}^{\infty} \left[\frac{[(p+k)!]^2}{(p+k-q)!} - \frac{[(q+k)!]^2}{(q+k-p)!} \right] |c_k|^2 \right).
\end{aligned} \tag{15}$$

Case I: Let $k \leqslant p-q-1$. Then from (15) and using the condition (i), we get

$$\begin{aligned}
&||T_\phi f||^2 - ||T_\phi^* f||^2 + ||T_\psi f||^2 - ||T_\psi^* f||^2 \\
&= |ab| \sum_{k=0}^{p-q-1} \left(\left|\frac{a}{b}\right| \frac{[(s+k)!]^2}{(s+k-t)!} + \left|\frac{b}{a}\right| \frac{[(p+k)!]^2}{(p+k-q)!} \right) |c_k|^2 \\
&\geqslant |ab| \sum_{k=0}^{p-q-1} (\alpha_k + \beta_k) |c_k|^2.
\end{aligned}$$

Case II: For $p-q \leqslant k \leqslant s-t-1$, applying condition (ii) in equation (15), we have

$$\begin{aligned}
&||T_\phi f||^2 - ||T_\phi^* f||^2 + ||T_\psi f||^2 - ||T_\psi^* f||^2 \\
&= |ab| \sum_{k=p-q}^{s-t-1} \left(\left|\frac{a}{b}\right| \frac{[(s+k)!]^2}{(s+k-t)!} + \left|\frac{b}{a}\right| \left[\frac{[(p+k)!]^2}{(p+k-q)!} - \frac{[(q+k)!]^2}{(q+k-p)!} \right] \right) |c_k|^2 \\
&\geqslant |ab| \sum_{k=p-q}^{s-t-1} (\alpha_k + \beta_k) |c_k|^2.
\end{aligned}$$

Case III: For $s-t \leqslant k$, the condition (iii) and equation (15) lead to the following:

$$\begin{aligned}
&||T_\phi f||^2 - ||T_\phi^* f||^2 + ||T_\psi f||^2 - ||T_\psi^* f||^2 \\
&= |ab| \sum_{k=s-t}^{\infty} \left|\frac{a}{b}\right| \left(\left[\frac{[(s+k)!]^2}{(s+k-t)!} - \frac{[(t+k)!]^2}{(t+k-s)!} \right] + \left|\frac{b}{a}\right| \left[\frac{[(p+k)!]^2}{(p+k-q)!} - \frac{[(q+k)!]^2}{(q+k-p)!} \right] \right) |c_k|^2 \\
&\geqslant |ab| \sum_{k=s-t}^{\infty} (\alpha_k + \beta_k) |c_k|^2.
\end{aligned}$$

From all the three cases, we get that

$$||T_\phi f||^2 - ||T_\phi^* f||^2 + ||T_\psi f||^2 - ||T_\psi^* f||^2 \geqslant |ab| \sum_{k=0}^{\infty} (\alpha_k + \beta_k)|c_k|^2.$$

Since β_k is zero for $k \leqslant s-t-p+q-1$, one can write

$$\begin{aligned}||T_\phi f||^2 - ||T_\phi^* f||^2 + ||T_\psi f||^2 - ||T_\psi^* f||^2 &\geqslant |ab| \left(\sum_{k=0}^{\infty} \alpha_k |c_k|^2 + \sum_{k=s-t-p+q}^{\infty} \beta_k |c_k|^2 \right) \\ &= |ab| \left(\sum_{k=0}^{\infty} \alpha_k |c_k|^2 + \sum_{k=s-t-p+q}^{\infty} \alpha_{k-s+t+p-q} |c_k|^2 \right) \\ &= |ab| \sum_{k=0}^{\infty} \alpha_k \left(|c_k|^2 + |c_{k+s-t-p+q}|^2 \right) \qquad (16)\end{aligned}$$

Computations from equations (14) and (16) show that $||T_\phi f||^2 - ||T_\phi^* f||^2 + ||T_\psi f||^2 - ||T_\psi^* f||^2 \geqslant -2Re[\langle P\phi f, P\psi f\rangle - \langle P\overline{\phi} f, P\overline{\psi} f\rangle]$ and hence from Proposition 2.1, $T_{\phi+\psi}$ is the hyponormal operator. $\square$

Consider the Toeplitz operator, T_ψ, where $\psi = b\overline{z}^p z^q \in \mathscr{A}$ with $p > q$. Then, its adjoint $T_\psi^* = T_{\overline{\psi}}$ is a hyponormal operator. In that case, T_ψ is called the co-hyponormal operator. In the next theorem, the hyponormality of Toeplitz operator is discussed when one of the term of binomial expression corresponds to the hyponormal operator and the other corresponds to the co-hyponormal operator. The main difference in this proof is in the calculation of cross-terms. As in the previous theorem, we will define

$$\alpha_k' = \frac{(s+k)!(p+k+s-t)!}{(s+k-t)!} - \frac{(p+k)!(s+k+p-q)!}{(p+k-q)!} \qquad (17)$$

and

$$\beta_k' = \begin{cases} 0 & \text{if } 0 \leqslant k \leqslant s-t+p-q-1 \\ \dfrac{(q+k)!\,(t+k-p+q)!}{(k-p+q)!} - \dfrac{(t+k)!\,(q+k-s+t)!}{(k-s+t)!} & \text{if } k \geqslant s-t+p-q. \end{cases} \qquad (18)$$

THEOREM 3.2. *Let* $\phi = az^s\overline{z}^t$, $\psi = b\overline{z}^p z^q \in \mathscr{A}$ *with* $s > t$ *and* $p > q$. *Then the Toeplitz operator* $T_{\phi+\psi}$ *is hyponormal if the following conditions are satisfied:*

(i) *For* $0 \leqslant k \leqslant \min(s-t, p-q) - 1$,

$$\alpha_k' \leqslant \left|\frac{a}{b}\right| \frac{[(s+k)!]^2}{(s+k-t)!} - \left|\frac{b}{a}\right| \frac{[(p+k)!]^2}{(p+k-q)!}.$$

(ii) *For* $\min(s-t, p-q) \leqslant k \leqslant \max(s-t, p-q) - 1$

$$\alpha_k' \leqslant \begin{cases} \left|\frac{a}{b}\right| \frac{[(s+k)!]^2}{(s+k-t)!} - \left|\frac{b}{a}\right| \left[\frac{[(p+k)!]^2}{(p+k-q)!} - \frac{[(q+k)!]^2}{(q+k-p)!}\right] & \textit{if } p-q < s-t \\ \left|\frac{a}{b}\right| \left[\frac{[(s+k)!]^2}{(s+k-t)!} - \frac{[(t+k)!]^2}{(t+k-s)!}\right] - \left|\frac{b}{a}\right| \left[\frac{[(p+k)!]^2}{(p+k-q)!}\right] & \textit{if } p-q > s-t. \end{cases}$$

(iii) *For* $\max(s-t,p-q) \leqslant k \leqslant s-t+p-q-1$

$$\alpha_k' \leqslant \left|\frac{a}{b}\right| \left[\frac{[(s+k)!]^2}{(s+k-t)!} - \frac{[(t+k)!]^2}{(t+k-s)!}\right] - \left|\frac{b}{a}\right| \left[\frac{[(p+k)!]^2}{(p+k-q)!} - \frac{[(q+k)!]^2}{(q+k-p)!}\right].$$

(iv) *For* $s-t+p-q \leqslant k$

$$\alpha_k' + \beta_k' \leqslant \left|\frac{a}{b}\right| \left[\frac{[(s+k)!]^2}{(s+k-t)!} - \frac{[(t+k)!]^2}{(t+k-s)!}\right] - \left|\frac{b}{a}\right| \left[\frac{[(p+k)!]^2}{(p+k-q)!} - \frac{[(q+k)!]^2}{(q+k-p)!}\right].$$

Proof. For $s>t, p>q$ and $f(z)=\sum_{k=0}^{\infty} c_k z^k \in \mathscr{F}^2(\mathbb{C})$, we have $\phi f = \sum_{k=0}^{\infty} a c_k z^{s+k}\overline{z}^t$ and $\psi f = \sum_{k=0}^{\infty} b c_k z^{q+k}\overline{z}^p$. Now proceeding as in the Theorem 3.1, we get the following:

$$\begin{aligned}
&||T_\phi f||^2 - ||T_\phi^* f||^2 + ||T_\psi f||^2 - ||T_\psi^* f||^2 \\
&= \sum_{k=0}^{\infty} |a|^2 |c_k|^2 \frac{[(s+k)!]^2}{(s+k-t)!} - \sum_{k=s-t}^{\infty} |a|^2 |c_k|^2 \frac{[(t+k)!]^2}{(t+k-s)!} \\
&\quad + \sum_{k=p-q}^{\infty} |b|^2 |c_k|^2 \frac{[(q+k)!]^2}{(q+k-p)!} - \sum_{k=0}^{\infty} |b|^2 |c_k|^2 \frac{[(p+k)!]^2}{(p+k-q)!} \\
&= |ab| \left(\left|\frac{a}{b}\right| \left\{ \sum_{k=0}^{s-t-1} \frac{[(s+k)!]^2}{(s+k-t)!} |c_k|^2 + \sum_{k=s-t}^{\infty} \left[\frac{[(s+k)!]^2}{(s+k-t)!} - \frac{[(t+k)!]^2}{(t+k-s)!}\right] |c_k|^2 \right\} \right. \\
&\quad \left. - \left|\frac{b}{a}\right| \left\{ \sum_{k=0}^{p-q-1} \frac{[(p+k)!]^2}{(p+k-q)!} |c_k|^2 + \sum_{k=p-q}^{\infty} \left[\frac{[(p+k)!]^2}{(p+k-q)!} - \frac{[(q+k)!]^2}{(q+k-p)!}\right] |c_k|^2 \right\} \right).
\end{aligned} \tag{19}$$

To get the cross-terms, we need to get the value of the following inner product

$$\begin{aligned}
&\langle T_\phi f, T_\psi f \rangle \\
&= \left\langle \sum_{k=0}^{\infty} a c_k \frac{(s+k)!}{(s+k-t)!} z^{s+k-t}, \sum_{k=p-q}^{\infty} b c_k \frac{(q+k)!}{(q+k-p)!} z^{q+k-p} \right\rangle \\
&= \left\langle \sum_{k=0}^{\infty} a c_k \frac{(s+k)!}{(s+k-t)!} z^{s+k-t}, \sum_{k=0}^{\infty} b c_{k+p-q} \frac{(p+k)!}{k!} z^{k} \right\rangle \\
&= \left\langle \sum_{k=0}^{\infty} a c_k \frac{(s+k)!}{(s+k-t)!} z^{s+k-t}, \sum_{k=s-t}^{\infty} b c_{k+p-q} \frac{(p+k)!}{k!} z^{k} \right\rangle \\
&= \left\langle \sum_{k=0}^{\infty} a c_k \frac{(s+k)!}{(s+k-t)!} z^{s+k-t}, \sum_{k=0}^{\infty} b c_{k+s-t+p-q} \frac{(p+k+s-t)!}{(k+s-t)!} z^{s+k-t} \right\rangle \\
&= \sum_{k=0}^{\infty} a\overline{b} c_k \overline{c_{k+s-t+p-q}} \frac{(s+k)!(p+k+s-t)!}{(s+k-t)!}.
\end{aligned} \tag{20}$$

Similarly, we can calculate

$$
\begin{aligned}
&\langle T_{\overline{\phi}} f, T_{\overline{\psi}} f\rangle \\
&= \left\langle \sum_{k=s-t}^{\infty} \overline{a} c_k \frac{(t+k)!}{(t+k-s)!} z^{t+k-s}, \sum_{k=0}^{\infty} \overline{b} c_k \frac{(p+k)!}{(p+k-q)!} z^{p+k-q} \right\rangle \\
&= \left\langle \sum_{k=0}^{\infty} \overline{a} c_{k+s-t} \frac{(s+k)!}{k!} z^{k}, \sum_{k=0}^{\infty} \overline{b} c_k \frac{(p+k)!}{(p+k-q)!} z^{p+k-q} \right\rangle \\
&= \left\langle \sum_{k=p-q}^{\infty} \overline{a} c_{k+s-t} \frac{(s+k)!}{k!} z^{k}, \sum_{k=0}^{\infty} \overline{b} c_k \frac{(p+k)!}{(p+k-q)!} z^{p+k-q} \right\rangle \\
&= \left\langle \sum_{k=0}^{\infty} \overline{a} c_{k+s-t+p-q} \frac{(s+k+p-q)!}{(p+k-q)!} z^{p+k-q}, \sum_{k=0}^{\infty} \overline{b} c_k \frac{(p+k)!}{(p+k-q)!} z^{p+k-q} \right\rangle \\
&= \sum_{k=0}^{\infty} \overline{a} b \overline{c_k} c_{k+s-t+p-q} \frac{(p+k)!(s+k+p-q)!}{(p+k-q)!}. \qquad (21)
\end{aligned}
$$

Thus, using (20) and (21), the cross term can be written as

$$
Re\left[\langle T_\phi f, T_\psi f\rangle - \langle T_{\overline{\phi}} f, T_{\overline{\psi}} f\rangle\right] = Re \sum_{k=0}^{\infty} a\overline{b} c_k \overline{c_{k+s-t+p-q}} \alpha'_k
$$

Case I: Let $0 \leqslant k \leqslant \min(s-t, p-q) - 1$, then from condition (i), we have

$$
\begin{aligned}
&||T_\phi f||^2 - ||T^*_\phi f||^2 + ||T_\psi f||^2 - ||T^*_\psi f||^2 \\
&= \sum_{k=0}^{\min(s-t,p-q)-1} |ab| \left(\left|\frac{a}{b}\right| \frac{[(s+k)!]^2}{(s+k-t)!} - \left|\frac{b}{a}\right| \frac{[(p+k)!]^2}{(p+k-q)!} \right) |c_k|^2 \\
&\geqslant \sum_{k=0}^{\min(s-t,p-q)-1} |ab|\, \alpha'_k |c_k|^2.
\end{aligned}
$$

Case II: Let $\min(s-t, p-q) \leqslant k \leqslant \max(s-t, p-q) - 1$. For the case $p-q < s-t$, we have $\min(s-t, p-q) = p-q$ and $\max(s-t, p-q) = s-t$. Thus from condition (ii), we get

$$
\begin{aligned}
&||T_\phi f||^2 - ||T^*_\phi f||^2 + ||T_\psi f||^2 - ||T^*_\psi f||^2 \\
&= \sum_{k=p-q}^{s-t-1} |ab| \left(\left|\frac{a}{b}\right| \frac{[(s+k)!]^2}{(s+k-t)!} - \left|\frac{b}{a}\right| \left[\frac{[(p+k)!]^2}{(p+k-q)!} - \frac{[(q+k)!]^2}{(q+k-p)!} \right] \right) |c_k|^2 \\
&\geqslant \sum_{k=p-q}^{s-t-1} |ab|\, \alpha'_k |c_k|^2.
\end{aligned}
$$

And if $p-q>s-t$, then $\min(s-t,p-q)=s-t$ and $\max(s-t,p-q)=p-q$. Again, condition (ii) implies,

$$\begin{aligned}&||T_\phi f||^2-||T_\phi^* f||^2+||T_\psi f||^2-||T_\psi^* f||^2\\&=\sum_{k=s-t}^{p-q-1}|ab|\left(\left|\frac{a}{b}\right|\left[\frac{[(s+k)!]^2}{(s+k-t)!}-\frac{[(t+k)!]^2}{(t+k-s)!}\right]\right.\\&\qquad\qquad\left.-\left|\frac{b}{a}\right|\left[\frac{[(p+k)!]^2}{(p+k-q)!}\right]\right)|c_k|^2\\&\geqslant\sum_{k=s-t}^{p-q-1}|ab|\,\alpha_k'|c_k|^2.\end{aligned}$$

Case III: Let $\max(s-t,p-q)\leqslant k\leqslant s-t+p-q-1$, then from condition (iii), we have

$$\begin{aligned}&||T_\phi f||^2-||T_\phi^* f||^2+||T_\psi f||^2-||T_\psi^* f||^2\\&=\sum_{k=\max(s-t,p-q)}^{s-t+p-q-1}|ab|\left(\left|\frac{a}{b}\right|\left[\frac{[(s+k)!]^2}{(s+k-t)!}-\frac{[(t+k)!]^2}{(t+k-s)!}\right]\right.\\&\qquad\qquad\left.-\left|\frac{b}{a}\right|\left[\frac{[(p+k)!]^2}{(p+k-q)!}-\frac{[(q+k)!]^2}{(q+k-p)!}\right]\right)|c_k|^2\\&\geqslant\sum_{k=\max(s-t,p-q)}^{s-t+p-q-1}|ab|\,\alpha_k'|c_k|^2.\end{aligned}$$

Case IV: Let $s-t+p-q\leqslant k$, then from condition (iv), we get

$$\begin{aligned}&||T_\phi f||^2-||T_\phi^* f||^2+||T_\psi f||^2-||T_\psi^* f||^2\\&=\sum_{s-t+p-q}^{\infty}|ab|\left(\left|\frac{a}{b}\right|\left[\frac{[(s+k)!]^2}{(s+k-t)!}-\frac{[(t+k)!]^2}{(t+k-s)!}\right]\right.\\&\qquad\qquad\left.-\left|\frac{b}{a}\right|\left[\frac{[(p+k)!]^2}{(p+k-q)!}-\frac{[(q+k)!]^2}{(q+k-p)!}\right]\right)|c_k|^2\\&\geqslant\sum_{s-t+p-q}^{\infty}|ab|\,\left(\alpha_k'+\beta_k'\right)|c_k|^2.\end{aligned}$$

Thus, in all the four cases, we get that $||T_\phi f||^2-||T_\phi^* f||^2+||T_\psi f||^2-||T_\psi^* f||^2$ is greater than equal to $\sum_{k=0}^{\infty}|ab|({\alpha'}_k+{\beta'}_k)|c_k|^2$. Proceeding as in Theorem 3.1, we get that Toeplitz operator $T_{\phi+\psi}$ is hyponormal. □

Given a hyponormal Toeplitz operator on the Fock space whose symbol is monomial, the next theorem gives a way to find a new hyponormal operator. For a fixed $r \in \mathbb{N}$, consider the symbol $\phi = z^{t+r}\overline{z}^t \in \mathscr{A}$, where t is an arbitrary natural number. Then by Theorem 2.3, T_ϕ is a hyponormal operator. In the next result, we show that for the symbol $\Psi = z^{t+r}\overline{z}^t + \frac{1}{2q+r} z^q \overline{z}^{q+r}$, the operator T_Ψ is also hyponormal provided $t > q$.

COROLLARY 3.3. *For every natural number t, there exist $q \in \mathbb{N}$ with $t > q$ such that the operator T_Ψ where $\Psi = z^{t+r}\overline{z}^t + \frac{1}{2q+r} z^q \overline{z}^{q+r} \in \mathscr{A}$ is hyponormal for a fixed $r \in \mathbb{N}$.*

Proof. In Theorem 3.2, take $s = t + r$, $p = q + r$, then $s - t = r = p - q$.

$$\alpha'_k = \frac{(t+k+r)!(q+k+2r)! - (q+k+r)!(t+k+2r)!}{(k+r)!}$$

and

$$\beta'_k = \begin{cases} 0 & \text{if } 0 \leqslant k \leqslant 2r-1 \\ \dfrac{(q+k)!\,(t+k-r)! - (t+k)!\,(q+k-r)!}{(k-r)!} & \text{if } k \geqslant 2r. \end{cases}$$

Note that $\alpha'_k < 0$, for $t > q$. Hence, β'_k is also negative for the chosen values.

Case I: Let $0 \leqslant k \leqslant r-1$. From condition (i) of the Theorem 3.2, we get that

$$\left|\frac{a}{b}\right| \frac{[(s+k)!]^2}{(s+k-t)!} - \left|\frac{b}{a}\right| \frac{[(p+k)!]^2}{(p+k-q)!} = \frac{(p+q)^2(t+k+r)!^2 - (q+k+r)!^2}{(k+r)!(p+q)} \geqslant 0.$$

Case II: Let $r \leqslant k \leqslant 2r-1$. In this case we will show that condition (iii) is satisfied. Let

$$\begin{aligned} \gamma'_k &= \left|\frac{a}{b}\right| \left[\frac{[(s+k)!]^2}{(s+k-t)!} - \frac{[(t+k)!]^2}{(t+k-s)!}\right] \\ &= (p+q)\left[\frac{[(t+k+r)!]^2}{(k+r)!} - \frac{[(t+k)!]^2}{(k-r)!}\right] \end{aligned}$$

and

$$\begin{aligned} \delta'_k &= \left|\frac{b}{a}\right| \left[\frac{[(p+k)!]^2}{(p+k-q)!} - \frac{[(q+k)!]^2}{(q+k-p)!}\right] \\ &= \frac{1}{(p+q)}\left[\frac{[(q+k+r)!]^2}{(k+r)!} - \frac{[(q+k)!]^2}{(k-r)!}\right] \end{aligned}$$

Consider,

$$\begin{aligned}
&\gamma_k' - (p+q)^2\delta_k' \\
&= (p+q)\left\{\frac{[(t+k+r)!]^2-[(q+k+r)!]^2}{(k+r)!} - \frac{[(t+k)!]^2-[(q+k)!]^2}{(k-r)!}\right\} \\
&= \frac{p+q}{(k-r)!(k+r)!}\Big\{(k-r)![(q+k+r)!]^2\{[(q+1+k+r)(q+2+k+r) \\
&\qquad \ldots(t+k+r)]^2-1\} - (k+r)![(q+k)!]^2\{[(q+1+k)(q+2+k) \\
&\qquad\qquad \ldots(t+k)]^2-1\}\Big\} \\
&= \frac{(p+q)[(q+k)!]^2}{(k+r)!}\Big\{\left[(q+k+1)\ldots(q+k+r)\right]^2\Big(\left[(q+1+k+r)\ldots\right. \\
&\qquad \left.(t+k+r)\right]^2-1\Big) - (k-r+1)(k-r+2)\ldots(k+r) \\
&\qquad\qquad \Big(\left[(q+1+k)\ldots(t+k)\right]^2-1\Big)\Big\} \geqslant 0.
\end{aligned}$$

This implies that $\gamma_k' - \delta_k' \geqslant 0 > \alpha_k'$, as required.

Case III: Let $k \geqslant 2r$. As in the previous case, we can show that $\gamma_k' \geqslant (p+q)^2\delta_k' \geqslant \delta_k' \geqslant \delta_k' + \alpha_k' + \beta_k'$.

Thus, by Theorem 3.2, we get that T_Ψ is a hyponormal operator. □

The following example presents how the above derived conditions can be applied to check the hyponormality of an operator. It may be observed that the bounds get substantially simplified under specific conditions.

EXAMPLE 3.4. Let $\phi = z^3\overline{z}^2 + \frac{1}{3}\overline{z}^2 z$. With the given choices of powers of z and $\overline{z}$, we have $r = 1$. The cases for $k = 0, 1$ are trivially satisfied. For $k \geqslant 2$, the values of $\gamma_k' = \dfrac{3[(k+2)!]^2(5k+9)}{(k+1)!}$, $\delta_k' = \dfrac{(k+1)!(3k+4)}{3}$, $\alpha_k' = -(k+2)(k+3)!$. From this, we get $\gamma_k' \geqslant \delta_k' - \alpha_k'$, as required.

The following remark shows that a hyponormal Toeplitz operator on the Fock space need not be hyponormal in the framework of the Bergman space.

REMARK 3.5. Consider the functions $\phi_1 = z^{21}\overline{z}^{20}$ and $\phi_2 = \frac{1}{3}\overline{z}^2 z$; $z \in \mathbb{D}$. Then for $f(z) = z^2$ in the Bergman space, we have $P\phi_1 f(z) = \frac{4}{24}z^3$, $P\phi_2 f(z) = \frac{1}{6}z$, $P\overline{\phi_1} f(z) = \frac{2}{23}z$ and $P\overline{\phi_2} f(z) = \frac{4}{5}z^3$, implying that $||T_{\phi_1+\phi_2}(z^2)||^2 = \frac{1}{48}$ and $||T_{\overline{\phi_1+\phi_2}}(z^2)||^2 = \frac{2166}{13225}$.

Thus, it follows that $||T_{\phi_1+\phi_2}(z^2)|| \leqslant ||T^*_{\phi_1+\phi_2}(z^2)||$, showing that $T_{\phi_1+\phi_2}$ is not hyponormal in the Bergman space.

But, $T_{\phi_1+\phi_2}$ satisfies the conditions of the Corollary 3.3 and hence is a hyponormal Toeplitz operator on the Fock space.

In the next example, we present a hyponormal operator on the Bergman space which is not hyponormal on the Fock space.

EXAMPLE 3.6. Consider the functions, $\phi_1 = z^2\overline{z}$ and $\phi_2 = \frac{1}{7}\overline{z}^4 z^3$.

Then by Theorem 10 [6], $T_{\phi_1+\phi_2}$ is hyponormal on the Bergman space. Whereas, it fails to be hyponormal on the Fock space.

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