Operators and Matrices
Volume 18, Number 1 (2024), 163–172 doi:10.7153/oam-2024-18-11

# JOINT SPECTRUM SHRINKING MAPS ON PROJECTIONS

WENHUA QIAN*, DANDAN XIAO, TANGHONG TAO, WENMING WU AND XIN YI

(*Communicated by L. Molnár*)

*Abstract.* Let $\mathscr{H}$ be a finite dimensional complex Hilbert space with dimension $n \geqslant 3$ and $\mathscr{P}(\mathscr{H})$ the set of projections on $\mathscr{H}$. Let $\varphi : \mathscr{P}(\mathscr{H}) \to \mathscr{P}(\mathscr{H})$ be a surjective map. We show that $\varphi$ shrinks the joint spectrum of any two projections if and only if it is induced by a semilinear automorphism on $\mathscr{H}$. In addition, $\varphi$ shrinks the joint spectrum of $I, P, Q$ for any two projections $P, Q \in \mathscr{P}(\mathscr{H})$ if and only if it is induced by a unitary or an anti-unitary. Assume that $\phi$ is a surjective map on the Grassmann space of rank one projections. We show that $\phi$ is joint spectrum shrinking for any $n$ rank one projections if and only if it is induced by a semilinear automorphism on $\mathscr{H}$. Moreover, for any $k > n$, $\phi$ is joint spectrum shrinking for any $k$ rank one projections if and only if it is induced by a unitary or an anti-unitary.

## 1. Introduction

The well-known Gleason-Kahane-Żelazko theorem ([8, 12]) states that a nonzero linear functional $\rho : \mathscr{A} \to \mathbb{C}$ on a unital complex Banach algebra $\mathscr{A}$ is an algebra homomorphism if and only if $\rho$ maps every element inside its spectrum. It is easy to verify that a nonzero linear functional $\rho$ on $\mathscr{A}$ is an algebra homomorphism if and only if $\rho$ is a Jordan homomorphism, that is, $\rho(I) = 1$ where $I$ is the unit of $\mathscr{A}$ and $\rho$ preserves the squares. Motivated by this classical result, in [13] Kaplansky asked whether a unital linear map $\varphi : \mathscr{A} \to \mathscr{B}$ between unital complex Banach algebras which shrinks spectrum (i.e., $\sigma(\varphi(A)) \subseteq \sigma(A)$, $\forall A \in \mathscr{A}$) is a Jordan homomorphism. Notice that a unital linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is spectrum shrinking if and only if it is invertibility preserving.

It is well-known that in general Kaplansky Problem has a negative answer. A counterexample can be found in [2]. A lot of work has been done on Kaplansky Problem by making additional assumptions (see [3, 10] for some survey). Aupetit conjectured that Kaplansky Problem has a positive answer when both Banach algebras are semi-simple and the map $\varphi$ is surjective and he confirmed this conjecture for von Neumann algebras [4]. This problem is still open, even for C$^*$-algebras [5, 9]. It was proved in

*Mathematics subject classification* (2020): Primary 47B49; Secondary 47A25.
*Keywords and phrases*: Joint spectrum preserving, joint spectrum shrinking, Kaplansky Problem, projections.
Qian (corresponding author) is supported in part by the Natural Science Foundation of Chongqing Science and Technology Commission (cstc2020jcyj-msxmX0723) and the Research Foundation of Chongqing Educational Committee (No. KJQN2021000529). Wu is supported in part by NSF of China (No. 11871127, No. 11971463).
* Corresponding author.


Paper OaM-18-11

[6] that the conjecture is true for $C^*$-algebras if in addition $\varphi$ is positive. In particular, some related maps on matrix algebras are also considered [7, 17].

Recall that [19] the joint spectrum of a tuple of operators $A_1, A_2, \dots, A_l$ acting on a Hilbert space $\mathscr{H}$ is the set

$$\sigma([A_1,\dots,A_l]) = \{(c_1,\dots,c_l) \in \mathbb{C}^l : c_1A_1 + \dots + c_lA_l \text{ is not invertible in } \mathscr{B}(\mathscr{H})\}.$$

It is an interesting issue to discuss the mapping which shrinks or preserves the joint spectrum of operators. It is easy to verify that a unital map $\varphi : \mathscr{A} \to \mathscr{B}$ is spectrum shrinking if and only if it shrinks the joint spectrum of the 2-tuple $[I,A]$ for any element $A \in \mathscr{A}$. Therefore according to Aupetit's results [4], we can obtain the form of the mapping preserving the joint spectrum of any two operators in $\mathscr{B}(\mathscr{H})$.

In this paper we will characterize the mappings which shrink or preserve the joint spectrum of a tuple of projections.

Assume that $\mathscr{H}$ is a Hilbert space with dimension $n < +\infty$. We first consider a surjective map $\varphi$ on the set $\mathscr{P}(\mathscr{H})$ of projections on $\mathscr{H}$ which shrinks the joint spectrum of any two projections. We first show that $\varphi$ leaves every Grassmann space invariant. By showing that the restriction of $\varphi$ on each Grassmann space is bijective, we get that $\varphi$ is bijective. A mathematical induction gives that $\varphi$ is determined by its action on rank $n-1$ projections and as a consequence we obtain that $\varphi$ is a lattice isomorphism. If $n = 2$, it is easy to verify that a surjective map $\varphi : \mathscr{P}(\mathscr{H}) \to \mathscr{P}(\mathscr{H})$ is joint spectrum shrinking for any two projections if and only if $\varphi$ is bijective with $\varphi(I) = I, \varphi(0) = 0$. Recall that a semilinear automorphism on $\mathscr{H}$ is a bijective transformation $S : \mathscr{H} \to \mathscr{H}$ such that $S(x+y) = S(x) + S(y), \forall\, x,y \in \mathscr{H}$ and there is an automorphism $f$ of $\mathbb{C}$ with $S(ax) = f(a)S(x), \forall\, a \in \mathbb{C}, x \in H$. If $n \geqslant 3$, some further calculations in Section 2 give the following result.

THEOREM 1.1. *Assume that* $3 \leqslant n(=\dim(\mathscr{H})) < +\infty$ *and* $\varphi : \mathscr{P}(\mathscr{H}) \to \mathscr{P}(\mathscr{H})$ *is a surjective map. Then the followings are equivalent.*

*(1)* $\varphi$ *shrinks the joint spectrum of any two projections;*

*(2)* $\varphi$ *preserves the joint spectrum of any two projections;*

*(3) there exists a semilinear automorphism* $S$ *on* $\mathscr{H}$ *such that* $\varphi(P)(\mathscr{H}) = S(P(\mathscr{H}))$.

Moreover, we consider a surjective map $\varphi$ on the set $\mathscr{P}(\mathscr{H})$ which shrinks the joint spectrum of $I,P,Q$ for any two projections $P,Q \in \mathscr{P}(\mathscr{H})$. We will further prove that $\varphi$ preserves the orthogonality of projections (i.e., $PQ = 0$ implies that $\varphi(P)\varphi(Q) = 0$) and obtain the following equivalent characterizations.

THEOREM 1.2. *Assume that* $3 \leqslant n(=\dim(\mathscr{H})) < +\infty$ *and* $\varphi : \mathscr{P}(\mathscr{H}) \to \mathscr{P}(\mathscr{H})$ *is a surjective map. Then the followings are equivalent.*

*(1)* $\varphi$ *shrinks the joint spectrum of* $I,P,Q$ *for any two projections* $P,Q \in \mathscr{P}(\mathscr{H})$*;*

*(2)* $\varphi$ *preserves the joint spectrum of* $I,P,Q$ *for any two projections* $P,Q \in \mathscr{P}(\mathscr{H})$*;*

(3) *there exists a unitary or anti-unitary* $U$ *such that* $\varphi(P) = U^*PU$, $\forall\, P \in \mathscr{P}(\mathscr{H})$.

We also investigate a surjective map $\phi$ on the set $\mathscr{P}_1(\mathscr{H})$ of rank one projections which shrinks the joint spectrum of any $n$ rank one projections. It is shown that $P_1 \vee P_2 \vee \ldots \vee P_n = I$ implies that $\phi(P_1) \vee \phi(P_2) \vee \ldots \vee \phi(P_n) = I$ for any $P_1, \ldots, P_n \in \mathscr{P}_1(\mathscr{H})$. It follows from the Fundamental Theorem of Projective Geometry that $\phi$ is induced by a semilinear automorphism on $\mathscr{H}$ and we obtain the following result.

THEOREM 1.3. *Assume that* $3 \leqslant n (= \dim(\mathscr{H})) < +\infty$ *and* $\phi : \mathscr{P}_1(\mathscr{H}) \to \mathscr{P}_1(\mathscr{H})$ *is a surjective map. Then the followings are equivalent.*

(1) $\phi$ *shrinks the joint spectrum of any* $n$ *rank one projections;*

(2) $\phi$ *preserves the joint spectrum of any* $n$ *rank one projections;*

(3) *there exists a semilinear automorphism* $S$ *on* $\mathscr{H}$ *such that* $\varphi(P)(\mathscr{H}) = S(P(\mathscr{H}))$.

Moreover, if $\phi : \mathscr{P}_1(\mathscr{H}) \to \mathscr{P}_1(\mathscr{H})$ is surjective and shrinks the joint spectrum of any $n+1$ projections, we can show that $\phi$ preserves the orthogonality of projections and obtain the following theorem.

THEOREM 1.4. *Assume that* $3 \leqslant n (= \dim(\mathscr{H})) < +\infty$ *and* $\phi : \mathscr{P}_1(\mathscr{H}) \to \mathscr{P}_1(\mathscr{H})$ *is a surjective map. Then the followings are equivalent.*

(1) *there exists* $k_0 \geqslant n+1$ *such that* $\phi$ *shrinks the joint spectrum of any* $k_0$ *projections;*

(2) *there exists* $k_0 \geqslant n+1$ *such that* $\phi$ *preserves the joint spectrum of any* $k_0$ *projections;*

(3) *for any* $k \geqslant n+1$, $\phi$ *shrinks the joint spectrum of any* $k$ *projections;*

(4) *for any* $k \geqslant n+1$, $\phi$ *preserves the joint spectrum of any* $k$ *projections;*

(5) *there exist a unitary or anti-unitary* $U$ *such that* $\phi(P) = U^*PU$, $\forall\, P \in \mathscr{P}_1(\mathscr{H})$.

## 2. Maps shrinking the joint spectrum of any two projections

Let $\mathscr{H}$ be a Hilbert space with dimension $n < +\infty$. Denote by $\mathscr{P}(\mathscr{H})$ and $\mathscr{P}_r(\mathscr{H})$ (i.e., the order $r$ Grassmann space) the set of projections and the set of rank $r$ projections on $\mathscr{H}$. In this section we assume that $\varphi : \mathscr{P}(\mathscr{H}) \to \mathscr{P}(\mathscr{H})$ is a surjective map which shrinks the joint spectrum of any two projections, i.e., $\sigma([\varphi(P), \varphi(Q)]) \subseteq \sigma([P,Q])$, $\forall\, P, Q \in \mathscr{P}(\mathscr{H})$.

LEMMA 2.1. $\varphi(I)=I, \varphi(0)=0$.

*Proof.* For any $Q\in\mathscr{P}(\mathscr{H})$, $(1,0)\notin\sigma([I,Q])$. Hence $(1,0)\notin\sigma([\varphi(I),\varphi(Q)])$ and by the surjection of $\varphi$ we have $\varphi(I)=I$. Since $(1,-1)\notin\sigma([I,0])$, we have $(1,-1)\notin\sigma([\varphi(I),\varphi(0)])=\sigma([I,\varphi(0)])$. Hence $\varphi(0)=0$. □

For any $P,Q\in\mathscr{P}(\mathscr{H})$, let $P\vee Q$ be the projection whose range is the sum of the ranges of $P,Q$ and $P\wedge Q$ the projection whose range is the intersection of the ranges of $P,Q$. Notice that these operations correspond to the operations on the lattice of subspaces of $H$. It is easy to verify that $P\vee Q=I$ if and only if $(1,1)\notin\sigma([P,Q])$. Thus the following lemma is obvious.

LEMMA 2.2. *Let $P,Q\in\mathscr{P}(\mathscr{H})$. If $P\vee Q=I$, then $\varphi(P)\vee\varphi(Q)=I$.*

By Theorem 2.1 in [18], if $P,Q\in\mathscr{P}(\mathscr{H})$ are nontrivial projections with $P\vee Q=I, P\wedge Q=0$, then either $\sigma([P,Q])=\mathbb{C}^2$ or $\sigma([P,Q])=\{(c_1,c_2)\in\mathbb{C}^2 : c_1c_2=0\}$.

LEMMA 2.3. *Let $P,Q\in\mathscr{P}(\mathscr{H})$. If $P\vee Q=I, P\wedge Q=0$, then $\varphi(P)\vee\varphi(Q)=I, \varphi(P)\wedge\varphi(Q)=0$.*

*Proof.* If $P=I,Q=0$ or $P=0,Q=I$, then Lemma 2.1 gives the result. Assume that $P,Q\in\mathscr{P}(\mathscr{H})\setminus\{0,I\}$. By $P\vee Q=I$ we have $(1,1)\notin\sigma([P,Q])$. Then it follows from Theorem 2.1 in [18] that $(1,-1)\notin\sigma([P,Q])$. Thus $\varphi(P)+\varphi(Q)$ and $\varphi(P)-\varphi(Q)$ are both invertible. Hence $\varphi(P)\vee\varphi(Q)=I, \varphi(P)\wedge\varphi(Q)=0$. □

In the following, we denote by $r(P)$ the rank of $P$ for any $P\in\mathscr{P}(\mathscr{H})$.

LEMMA 2.4. *Let $P,Q\in\mathscr{P}(\mathscr{H})$. If $r(P)=r(Q)$, then $r(\varphi(P))=r(\varphi(Q))$. Moreover, $\varphi(\mathscr{P}_k(\mathscr{H}))=\mathscr{P}_k(\mathscr{H}), \forall\, k\in\{0,1,2,\ldots,n\}$.*

*Proof.* Notice $\varphi(I)=I$, $\varphi(0)=0$. We may assume that $P,Q\in\mathscr{P}_k(\mathscr{H})$, where $k\in\{1,2,\ldots,n-1\}$.

We first assume that $r(P\wedge Q)=k-1$. It follows that $r(P\vee Q)=k+1$. Then there exist linearly independent vectors $x_1,x_2,\ldots,x_{k-1},\alpha,\beta\in\mathscr{H}$ such that $P$ is the projection onto the subspace generated by $x_1,x_2,\ldots,x_{k-1},\alpha$ and $Q$ is the projection onto the subspace generated by $x_1,x_2,\ldots,x_{k-1},\beta$. Take $R=P_1+(I-P\vee Q)$, where $P_1$ is the rank one projection onto $\mathbb{C}(\alpha+\beta)$. It follows that $P\vee R=I$, $P\wedge R=0$ and $Q\vee R=I$, $Q\wedge R=0$. By Lemma 2.3 we obtain that $\varphi(P)\vee\varphi(R)=I$, $\varphi(P)\wedge\varphi(R)=0$ and $\varphi(Q)\vee\varphi(R)=I$, $\varphi(Q)\wedge\varphi(R)=0$. Hence

$$r(\varphi(P))=n-r(\varphi(R))=r(\varphi(Q)).$$

Now assume that $r(P\wedge Q)=k-r$, where $1\leqslant r\leqslant k$. Then there exist linearly independent vectors $x_1,x_2,\ldots,x_{k-r},\alpha_1,\alpha_2,\ldots,\alpha_r,\beta_1,\beta_2,\ldots,\beta_r$ such that $P$ is the projection onto the subspace generated by $x_1,x_2,\ldots,x_{k-r},\alpha_1,\alpha_2,\ldots,\alpha_r$ and $Q$ is the projection onto the subspace generated by $x_1,x_2,\ldots,x_{k-r},\beta_1,\beta_2,\ldots,\beta_r$. Take $Q_0=P$,

$Q_r = Q$. For each $i \in \{1,2,\ldots,r-1\}$, let $Q_i$ be the projection onto the subspace generated by $x_1, x_2, \ldots, x_{k-r}, \beta_1, \ldots, \beta_i, \alpha_{i+1}, \ldots, \alpha_r$. It follows that $Q_0, Q_1, \ldots, Q_r \in \mathscr{P}_k(H)$ and $r(Q_i \wedge Q_{i+1}) = k-1$ for every $i \in \{0,1,\ldots,r-1\}$. Then the result of the previous paragraph implies that

$$r(\varphi(P)) = r(\varphi(Q)).$$

Hence there exists a map $g : \{0,1,2,\ldots,n\} \to \{0,1,2,\ldots,n\}$ such that $\varphi(\mathscr{P}_k(\mathscr{H})) \subseteq \mathscr{P}_{g(k)}(\mathscr{H}), \forall\, k \in \{0,1,2,\ldots,n\}$. By the fact that $\varphi$ is surjective, we obtain that $g$ is a bijection and $\varphi(\mathscr{P}_k(\mathscr{H})) = \mathscr{P}_{g(k)}(\mathscr{H}), \forall\, k \in \{0,1,2,\ldots,n\}$. In particular, by Lemma 2.3 we have $g(n-k) = n - g(k), \forall\, k \in \{0,1,2,\ldots,n\}$.

Clearly, $g(0) = 0$ and $g(n) = n$. Assume that $s = g(1) > 1$. Then $g(n-1) = n-s < n-1$. By the fact that $g$ is a bijection, there exists $l > 1$ such that $g(l) = 1$. Take two projections $P_1 \in \mathscr{P}_{n-1}(\mathscr{H})$, $P_2 \in \mathscr{P}_l(\mathscr{H})$ with $P_1 \vee P_2 = I$. It follows that $\varphi(P_1) \in \mathscr{P}_{n-s}(H)$, $\varphi(P_2) \in \mathscr{P}_1(H)$. Then $r(\varphi(P_1)) + r(\varphi(P_2)) < n$. Therefore $\varphi(P_1) \vee \varphi(P_2) \neq I$ and we obtain a contradiction according to Lemma 2.2. Hence $g(1) = 1$, $g(n-1) = n-1$. Continuing in this way, we have $\varphi(\mathscr{P}_k(\mathscr{H})) = \mathscr{P}_k(\mathscr{H})$, $\forall\, k \in \{0,1,2,\ldots,n\}$. $\square$

In the following we will show that the restriction of $\varphi$ on each Grassmann space $\mathscr{P}_k(\mathscr{H})$ is a bijection and thus $\varphi$ is a bijection. We first present a necessary lemma.

LEMMA 2.5. *Let $Q \in \mathscr{P}_{n-1}(\mathscr{H}), P \in \mathscr{P}(\mathscr{H})$. If $\varphi(P) \leqslant \varphi(Q)$, then $P \leqslant Q$. Moreover, $\varphi|_{\mathscr{P}_{n-1}(\mathscr{H})}$ is a bijection.*

*Proof.* By Lemma 2.4, $\varphi(Q) \in \mathscr{P}_{n-1}(\mathscr{H})$. Since $\varphi(P) \leqslant \varphi(Q)$, $\varphi(P) \vee \varphi(Q) \neq I$. By Lemma 2.2 and the fact that $Q \in \mathscr{P}_{n-1}(H)$, we have $P \leqslant Q$. It is easy to verify that $\varphi|_{\mathscr{P}_{n-1}(\mathscr{H})}$ is a bijection. $\square$

For convenience, we denote by $\Phi = \varphi|_{\mathscr{P}_{n-1}(\mathscr{H})}$ in the following proposition.

PROPOSITION 2.6. *Let $k \in \{1,2,\ldots,n\}$ and $P \in \mathscr{P}_{n-k}(\mathscr{H})$. Assume that $P' \in \mathscr{P}_{n-k}(\mathscr{H})$ with $\varphi(P') = P$. Then for any $k$ projections $Q_1, Q_2, \ldots, Q_k \in \mathscr{P}_{n-1}(\mathscr{H})$ with $P = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_k$, $P' = \wedge_{1 \leqslant i \leqslant k} \Phi^{-1}(Q_i)$. Moreover, $\varphi$ is a bijection.*

*Proof.* We prove the result by a mathematical induction on $k$. From Lemma 2.5, the result is true when $k = 1$. Assume that the result is true when $k = s$. Now let $k = s+1$ and assume that $Q_1, Q_2, \ldots, Q_s, Q_{s+1} \in \mathscr{P}_{n-1}(\mathscr{H})$ with $P = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_s \wedge Q_{s+1}$.

Take $P_1 = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_s$ and $P_2 = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_{s-1} \wedge Q_{s+1}$. Clearly $P_1, P_2$ are two different projections in $\mathscr{P}_{n-s}(\mathscr{H})$. By the assumption that the result is true when $k = s$, we have $\varphi|_{\mathscr{P}_{n-s}}(\mathscr{H})$ is a bijection and

$$\begin{aligned}
&(\varphi|_{\mathscr{P}_{n-s}(\mathscr{H})})^{-1}(P_1) = \Phi^{-1}(Q_1) \wedge \Phi^{-1}(Q_2) \wedge \ldots \wedge \Phi^{-1}(Q_s),\\
&(\varphi|_{\mathscr{P}_{n-s}(\mathscr{H})})^{-1}(P_2) = \Phi^{-1}(Q_1) \wedge \Phi^{-1}(Q_2) \wedge \ldots \wedge \Phi^{-1}(Q_{s-1}) \wedge \Phi^{-1}(Q_{s+1}). \quad (2.1)
\end{aligned}$$

By Lemma 2.5, $P' \leqslant \Phi^{-1}(Q_i)$ for each $i \in \{1,2,\ldots,s+1\}$. Hence $P' \leqslant (\varphi|_{\mathscr{P}_{n-s}(\mathscr{H})})^{-1}(P_1) \wedge (\varphi|_{\mathscr{P}_{n-s}(\mathscr{H})})^{-1}(P_2)$.

Since $\varphi|_{\mathscr{P}_{n-s}}$ is a bijection, $(\varphi|_{\mathscr{P}_{n-s}(\mathscr{H})})^{-1}(P_1) \neq (\varphi|_{\mathscr{P}_{n-s}(\mathscr{H})})^{-1}(P_2)$ and thus $r((\varphi|_{\mathscr{P}_{n-s}(\mathscr{H})})^{-1}(P_1) \wedge (\varphi|_{\mathscr{P}_{n-s}(\mathscr{H})})^{-1}(P_2)) \leqslant n-s-1 = r(P')$.

Therefore $P' = (\varphi|_{\mathscr{P}_{n-s}(\mathscr{H})})^{-1}(P_1) \wedge (\varphi|_{\mathscr{P}_{n-s}(\mathscr{H})})^{-1}(P_2) = \wedge_{1\leqslant i\leqslant s+1}\Phi^{-1}(Q_i)$ from (2.1). Hence $\varphi|_{\mathscr{P}_{n-s-1}}(\mathscr{H})$ is also a bijection. Moreover, $\varphi$ is a bijection. □

According to Proposition 2.6, we have the following corollary.

COROLLARY 2.7. *If $P, Q \in \mathscr{P}(H)$, then we have the following results.*

*(1) If $P \neq I$ and $\{Q_\lambda : \lambda \in \Omega\} \subseteq \mathscr{P}_{n-1}(\mathscr{H})$ with $\wedge_{\lambda\in\Lambda} Q_\lambda = P$, then $\varphi^{-1}(P) = \wedge_{\lambda\in\Omega}\varphi^{-1}(Q_\lambda)$;*

*(2) If $P \leqslant Q$, then $\varphi^{-1}(P) \leqslant \varphi^{-1}(Q)$;*

*Proof.* (1) Notice that $P \leqslant Q_\lambda$ for every $\lambda \in \Omega$. It follows from Lemma 2.5 that $\varphi^{-1}(P) \leqslant \varphi^{-1}(Q_\lambda), \forall\, \lambda \in \Omega$ and hence $\varphi^{-1}(P) \leqslant \wedge_{\lambda\in\Omega}\varphi^{-1}(Q_\lambda)$. Assume that $P \in \mathscr{P}_{n-k}(\mathscr{H})$. Then there exist $\lambda_1, \lambda_2, \ldots, \lambda_k \in \Omega$ such that $P = \wedge_{1\leqslant i\leqslant k} Q_{\lambda_i}$. It follows from Proposition 2.6 that $\varphi^{-1}(P) = \wedge_{1\leqslant i\leqslant k}\varphi^{-1}(Q_{\lambda_i})$. Hence

$$\varphi^{-1}(P) = \wedge_{\lambda\in\Omega}\varphi^{-1}(Q_\lambda).$$

(2) This is clear from (1). □

*Proof of Theorem* 1.1. It is clear that $(2) \Rightarrow (1)$ and we can easily verify that $(3) \Rightarrow (2)$. Now we only need to show that $(1) \Rightarrow (3)$. By Proposition 2.6, the restriction of $\varphi$ on $\mathscr{P}_1(H)$ is a bijection. Hence by a modification of the Fundamental Theorem of Projective Geometry(see Corollary 1.3 in [14]), the restriction of $\varphi^{-1}$ on $\mathscr{P}_1(\mathscr{H})$ is induced by a semilinear automorphism $T$ on $\mathscr{H}$. By Corollary 2.7, $\varphi^{-1}$ is order preserving, we obtain that $\varphi^{-1}(P)(\mathscr{H}) = T(P(\mathscr{H})),\ \forall\, P \in \mathscr{P}(\mathscr{H})$. Now the result follows by taking $S = T^{-1}$. □

## 3. Maps shrinking the joint spectrum of $I, P, Q$

Assume $\mathscr{H}$ is a finite dimensional Hilbert space with dimension $n \geqslant 3$. In this section we assume that $\varphi : \mathscr{P}(\mathscr{H}) \to \mathscr{P}(\mathscr{H})$ is a surjective map which shrinks the joint spectrum of the identity $I$ and any two projections, i.e., $\sigma([\varphi(I), \varphi(P), \varphi(Q)]) \subseteq \sigma([I, P, Q]),\ \forall\, P, Q \in \mathscr{P}(\mathscr{H})$. It is easy to verify that $\varphi$ also shrinks the joint spectrum of any two projections and thus $\varphi$ is also induced by a semilinear automorphism $S$ on $\mathscr{H}$ as in Theorem 1.1. In particular, $\varphi(I) = I, \varphi(0) = 0$.

Assume that $P, Q \in \mathscr{P}(\mathscr{H})$ with $PQ \neq QP$. Let $H = PQP - P \wedge Q$ and $\sigma(H)$ the spectrum of $H$ when it is viewed as a positive contraction on $(P - P \wedge Q)(\mathscr{H})$. By Corollary 3.1 in [18], $\{\lambda : (\lambda - 1)^2 \in \sigma(H)\} \subseteq \sigma(P+Q)$. Since $PQ \neq QP$, there exists $r \in \sigma(H) \cap (0,1)$, which implies that $(1, -\frac{1}{\sqrt{1+r}}, -\frac{1}{\sqrt{1+r}}) \in \sigma(P+Q)$. Then we have the following lemma.

LEMMA 3.1. *If* $P,Q \in \mathscr{P}(\mathscr{H})$ *such that* $PQ = QP$, *then* $\varphi(P)\varphi(Q) = \varphi(Q)\varphi(P)$.

*Proof.* Since $PQ = QP$, we have $(1,-t,-t) \notin \sigma([I,P,Q])$ for any $t \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. It follows that $(1,-t,-t) \notin \sigma([I,\varphi(P),\varphi(Q)])$ for any $t \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. It follows from Corollary 3.1 in [18] that $\varphi(P)\varphi(Q) = \varphi(Q)\varphi(P)$. □

LEMMA 3.2. *If* $P,Q \in \mathscr{P}(\mathscr{H})$ *such that* $PQ = 0$, *then* $\varphi(P)\varphi(Q) = 0$.

*Proof.* By Lemma 3.1, $\varphi(P)\varphi(Q) = \varphi(Q)\varphi(P)$. Notice that $PQ = 0$. It follows that $2I-(P+Q)$ is invertible and therefore $2I-(\varphi(P)+\varphi(Q))$ is invertible. Therefore $\varphi(P)\varphi(Q) = 0$. □

*Proof of Theorem 1.2.* It is clear that $(3) \Rightarrow (2) \Rightarrow (1)$. We only need to show that $(1) \Rightarrow (3)$. It follows from $(1)$ that $\varphi$ shrinks the joint spectrum of $I,P,Q$ for any $P,Q \in \mathscr{P}(\mathscr{H})$. Then it also shrinks the joint spectrum of any two projections on $\mathscr{H}$. By Theorem 1.1, $\varphi$ is induced by a semilinear automorphism $S$ on $\mathscr{H}$. By Lemma 3.2, $S$ preserves the orthogonality of vectors in $\mathscr{H}$. It follows from Proposition 4.2 in [15] that $S$ is a nonzero multiple of a unitary or an anti-unitary and the desired result follows. □

## 4. Joint spectrum shrinking maps on rank one projections

Assume that $n \geqslant 3$. In this section we assume that $\phi : \mathscr{P}_1(\mathscr{H}) \to \mathscr{P}_1(\mathscr{H})$ is a surjective map. It is easy to verify that for any positive integer $m < n$, the joint spectrum of any $m$ rank one projections $P_1,P_2,\ldots,P_m$ is $\mathbb{C}^m$. Therefore every map on $\mathscr{P}_1(\mathscr{H})$ preserves the joint spectrum of any $m$ rank one projections if $m < n$.

### 4.1. Maps shrinking the joint spectrum of any $n$ rank one projections

We start with a description of the joint spectrum of $n$ rank one projections.

LEMMA 4.1. *Let* $P_1,P_2,\ldots,P_n \in \mathscr{P}_1(\mathscr{H})$. *Then*

*(1) if* $P_1 \vee P_2 \vee \ldots \vee P_n \neq I$ *, then* $\sigma([P_1,P_2,\ldots,P_n]) = \mathbb{C}^n$*;*

*(2) if* $P_1 \vee P_2 \vee \ldots \vee P_n = I$*, then* $\sigma([P_1,P_2,\ldots,P_n]) = \{(c_1,c_2,\ldots,c_n) \in \mathbb{C}^n : c_1c_2\ldots c_n = 0\}$.

*Proof.* If $P_1 \vee P_2 \vee \ldots \vee P_n \neq I$, then the range of any linear combination of $P_1,P_2,\ldots,P_n$ is contained in the range of $P_1 \vee P_2 \vee \ldots \vee P_n$ and thus any linear combination of $P_1,P_2,\ldots,P_n$ is not invertible. Therefore $\sigma([P_1,P_2,\ldots,P_n]) = \mathbb{C}^n$.

On the other hand, assume that $P_1 \vee P_2 \vee \ldots \vee P_n = I$ and $c_1P_1 + c_2P_2 + \ldots + c_nP_n$ is not invertible. Then there exists a nonzero vector $\beta \in \mathscr{H}$ such that $c_1P_1\beta + c_2P_2\beta + \ldots + c_nP_n\beta = 0$. Hence $c_iP_i\beta = -c_1P_1\beta - \ldots - c_{i-1}P_{i-1}\beta - c_{i+1}P_{i+1}\beta - \ldots - c_nP_n\beta = 0$. By the fact that $P_1 \vee P_2 \vee \ldots \vee P_n = I$ we have $P_i \wedge (P_1 \vee \ldots \vee P_{i-1} \vee P_{i+1} \vee$

$\ldots \vee P_n) = 0$ for each $i \in \{1,2,\ldots,n\}$. If $c_1c_2\ldots c_n \neq 0$, then $P_1\beta = P_2\beta = \ldots = P_n\beta = 0$, which is a contradiction to that $P_1 \vee P_2 \vee \ldots \vee P_n = I$ and $\beta \neq 0$. Therefore $\sigma([P_1,P_2,\ldots,P_n]) \subseteq \{(c_1,c_2,\ldots,c_n) \in \mathbb{C}^n : c_1c_2\ldots c_n = 0\}$. It is obvious that $\{(c_1,c_2,\ldots,c_n) \in \mathbb{C}^n : c_1c_2\ldots c_n = 0\} \subseteq \sigma([P_1,P_2,\ldots,P_n])$. □

*Proof of Theorem 1.3.* (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is clear and we only need to verify that (1) $\Rightarrow$ (3). By Lemma 4.1, $P_1 \vee P_2 \vee \ldots \vee P_n = I$ implies that $\phi(P_1) \vee \phi(P_2) \vee \ldots \vee \phi(P_n) = I$ and $\phi$ is also a bijection. It follows from a modification of the Fundamental Theorem of Projective Geometry (see the arguments at Page 89 in [14]) that $\phi$ is induced by a semilinear automorphism on $\mathscr{H}$ and the desired result follows. □

REMARK 4.2. We refer to Example 3.5 in [14] for showing that the surjectivity of $\phi$ can not be omitted in the previous theorem.

### 4.2. Maps shrinking the joint spectrum of more than $n$ rank one projections

Now we assume that $\phi : \mathscr{P}_1(\mathscr{H}) \to \mathscr{P}_1(\mathscr{H})$ is a surjective map which shrinks the joint spectrum of $n+1$ projections. Notice that $\phi$ also shrinks the joint spectrum of any $n$ rank one projections. Then the previous subsection gives that $\phi$ is induced by a semilinear automorphism on $\mathscr{H}$. We follow a similar line as in Section 3 to show that $\phi$ preserves the orthogonality.

LEMMA 4.3. *Assume that $\phi : \mathscr{P}_1(\mathscr{H}) \to \mathscr{P}_1(\mathscr{H})$ is a surjective map which shrinks the joint spectrum of any $n+1$ projections. Then $\phi$ preserves the orthogonality.*

*Proof.* By way of contradiction, assume that $P, Q \in \mathscr{P}_1(\mathscr{H})$ such that $PQ = 0$ and $\phi(P)\phi(Q) \neq 0$. Take a unit vector $\xi \in \phi(P)\mathscr{H}$ such that $\phi(Q)\xi \neq 0$. Take a rank one projection $R$ with $\xi + \phi(Q)\xi$ in its range. Let $c = \|\phi(Q)\xi\| > 0$. It follows that

$$R\xi = \frac{\langle \xi, \xi + \phi(Q)\xi \rangle}{\langle \xi + \phi(Q)\xi, \xi + \phi(Q)\xi \rangle}(\xi + \phi(Q)\xi) = \frac{1+c^2}{1+3c^2}(\xi + \phi(Q)\xi)$$

and hence

$$(\phi(P) + \phi(Q) - \frac{1+3c^2}{1+c^2}R)\xi = 0. \tag{4.1}$$

Notice that $R \leqslant \phi(P) \vee \phi(Q)$. We have that $Ran(\phi(P) + \phi(Q) - \frac{1+3c^2}{1+c^2}R) \leqslant \phi(P) \vee \phi(Q)$, where $Ran(\phi(P) + \phi(Q) - \frac{1+3c^2}{1+c^2}R)$ denotes the range projection of $\phi(P) + \phi(Q) - \frac{1+3c^2}{1+c^2}R$. It follows from (4.1) that $(\phi(P) + \phi(Q) - \frac{1+3c^2}{1+c^2}R)\phi(P) = 0$ and therefore $Ran(\phi(P) + \phi(Q) - \frac{1+3c^2}{1+c^2}R) \leqslant \phi(P) \vee \phi(Q) - \phi(P)$, which implies that $r(Ran(\phi(P) + \phi(Q) - \frac{1+3c^2}{1+c^2}R)) \leqslant 1$. On the other hand, since $\phi(P), \phi(Q)$ are two distinguished rank one projections, we have that $\phi(P) + \phi(Q)$ has rank 2 and therefore $r(Ran(\phi(P)+$

$\phi(Q) - \frac{1+3c^2}{1+c^2}R)) \geqslant 1$. Hence we obtain that $r(Ran(\phi(P) + \phi(Q) - \frac{1+3c^2}{1+c^2}R)) = 1$. Take $P_3, P_4, \ldots, P_n \in \mathscr{P}_1(\mathscr{H})$ such that $P + Q + P_3 + \ldots + P_n = I$. Since $\frac{1+3c^2}{1+c^2} > 1$, $P + Q - \frac{1+3c^2}{1+c^2}\phi^{-1}(R) + P_3 + \ldots + P_n = I - \frac{1+3c^2}{1+c^2}\phi^{-1}(R)$ is invertible. Since $r(Ran(\phi(P) + \phi(Q) - \frac{1+3c^2}{1+c^2}R)) = 1$, $r(Ran(\phi(P) + \phi(Q) - \frac{1+3c^2}{1+c^2}R) + \phi(P_3) + \ldots + \phi(P_n))) \leqslant n-1$ and thus $\phi(P) + \phi(Q) - \frac{1+3c^2}{1+c^2}R + \phi(P_3) + \ldots + \phi(P_n)$ is not invertible. We obtain a contradiction. □

Now we can get the main result of this subsection.

*Proof of Theorem* 1.4. It is clear that $(5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (1)$ and $(4) \Rightarrow (2) \Rightarrow (1)$. In the following we only need to verify $(1) \Rightarrow (5)$.

Notice that $\phi$ also shrinks the joint spectrum of any $n$ rank one projections, it follows from the previous subsection that $\phi$ is induced by a semilinear automorphism $S$ on $\mathscr{H}$. By Lemma 4.3, $S$ preserves the orthogonality of vectors in $\mathscr{H}$. It follows from Proposition 4.2 in [15] that $S$ is a nonzero multiple of a unitary or an anti-unitary and the desired result follows. □

*Acknowledgements.* The authors would like to thank the anonymous referee for constructive criticisms and valuable comments, which leads us to relate this article to the Fundamental Theorem of Projective Geometry.

## REFERENCES


[1] F. V. ATKINSON, *Multiparameter eigenvalue problems*, Academic Press, New York-London, 1972.

[2] B. AUPETIT, *Propriétés spectrales des algèbres des Banach*, Lecture Notes in Mathematics **735**, Springer, 1979.

[3] B. AUPETIT, *Sur les transformations qui conservent le spectre*, Banach algebras **97** (Blaubeuren), de Gruyter, Berlin, 1998.

[4] B. AUPETIT, *Spectrum preserving linear mappings between Banach algebras or Jordan-Banach algebras*, J. London Math. Soc., 2000, **62** (2): 917–924.

[5] M. BREŠAR AND P. ŠEMRL, *An extension of the Gleason-Kahane-Żelazko theorem: A possible approach to Kaplansky's problem*, Expo. Math., 2008, **26** (3): 269–277.

[6] M. CHOI, D. HADWIN, E. NORDGREN, H. RADJAVI AND P. ROSENTHAL, *On positive linear maps preserving invertibility*, Journal of Functional Analysis, 1984, **59** (3): 462–469.

[7] A. FOŠNER AND P. ŠEMRL, *Additive maps on matrix algebras preserving invertibility or singularity*, Acta Mathematica Sinica, English Series, 2005, **21** (4): 681–684.

[8] A. M. GLEASON, *A characterization of maximal ideals*, J. Analyse Math., 1967, **19**: 171–172.

[9] L. A. HARRIS AND R. V. KADISON, *Affine mappings of invertible operators*, Proc. Amer. Math. Soc., 1996, **124**: 2415–2422.

[10] J. HOU AND P. ŠEMRL, *Linear maps preserving invertibility or related spectral properties*, Acta Mathematica Sinica, English Series, 2003, **19** (3): 473–484.

[11] E. JARLEBRING AND M. HOCHSTENBACH, *Polynomial two-parameter eigenvalue problems and matrix pencil methods for stability of delay-differential equations*, Linear Algebra Appl., 2009, **431** (3): 369–380.

[12] J. P. KAHANE AND W. ŻELAZKO, *A characterization of maximal ideals in commutative Banach algebras*, Studia Math., 1968, **29**: 339–343.

[13] I. KAPLANSKY, *Algebraic and analytic aspects of operator algebras*, Amer. Math. Soc., Providence, 1970.

[14] M. PANKOV, *Grassmannians of classical buildings*, World Scientific, 2010.

[15] M. PANKOV, *Wigner-Type Theorems for Hilbert Grassmannians*, London Mathematical Society Lecture Notes Series **460**, Cambridge University Press, 2020.
[16] B. D. SLEEMAN, *Multiparameter spectral theory in Hilbert spaces*, Res. Notes Math., vol. 22, Pitman, London, 1978.
[17] M. TOMAŠEVIĆ, *A variant of the Kaplansky problem for maps on positive matrices*, arXiv:2204.11622v1.
[18] W. WU, Y. JIANG, Y. RUAN AND W. QIAN, *The joint spectrum of a tuple of projections* (in Chinese), Sci. Sin. Math., 2021, **51**: 711–722.
[19] R. YANG, *Projective spectrum in Banach algebras*, Topol. Anal., 2009, **1**: 289–306.

(Received August 31, 2023)

*Wenhua Qian*
*School of Mathematical Sciences*
*Chongqing Normal University*
*Chongqing, China*
*e-mail:* whqian86@163.com

*Dandan Xiao*
*School of Mathematical Sciences*
*Chongqing Normal University*
*Chongqing, China*
*e-mail:* 1258940058@qq.com

*Tanghong Tao*
*School of Mathematical Sciences*
*Chongqing Normal University*
*Chongqing, China*
*e-mail:* 1482408506@qq.com

*Wenming Wu*
*School of Mathematical Sciences*
*Chongqing Normal University*
*Chongqing, China*
*e-mail:* wuwm@amss.ac.cn

*Xin Yi*
*School of Mathematical Sciences*
*Chongqing Normal University*
*Chongqing, China*
*e-mail:* 1648024603@qq.com