Operators
and
Matrices
Volume 18, Number 1 (2024), 191–203 doi:10.7153/oam-2024-18-13

RELATIVE RESIDUAL BOUNDS FOR EIGENVALUES IN GAPS OF THE ESSENTIAL SPECTRUM

ALBRECHT SEELMANN

(*Communicated by I. M. Spitkovsky*)

Abstract. The relative distance between eigenvalues of the compression of a not necessarily semibounded self-adjoint operator to a closed subspace and some of the eigenvalues of the original operator in a gap of the essential spectrum is considered. It is shown that this distance depends on the maximal angles between pairs of associated subspaces. This generalises results by Drmač in [Linear Algebra Appl. **244** (1996), 155–163] from matrices to not necessarily (semi)bounded operators.

1. Introduction and main results

Let H be a not necessarily semibounded self-adjoint operator in a Hilbert space $\mathscr{H}$ with bounded inverse. We denote by $\lambda_j \in (0,\infty)$ the j-th positive eigenvalue of H below $\inf\sigma_{\mathrm{ess}}(H)\cap(0,\infty)$, in increasing order and counting multiplicities, provided that this eigenvalue exists.

Let $\mathscr{U}$ be a finite dimensional subspace of $\operatorname{Dom}(H)$, and write $P_{\mathscr{U}}$ for the orthogonal projection onto $\mathscr{U}$. In the Hilbert space $\mathscr{U}$ we then consider the compression M of H to $\mathscr{U}$, that is, the self-adjoint operator

$$M = P_{\mathscr{U}} H|_{\mathscr{U}} : \mathscr{U} \to \mathscr{U},$$

with eigenvalues

$$\mu_1 \leqslant \ldots \leqslant \mu_m, \quad m = \dim \mathscr{U}.$$

Under suitable additional assumptions on $\mathscr{U}$, one expects at least some of the eigenvalues of M to be close to certain eigenvalues of H in a relative sense; cf. [7]. In order to make this precise, consider the finite dimensional subspaces

$$\mathscr{V} = \operatorname{Ran} H|_{\mathscr{U}} \quad \text{and} \quad \mathscr{W} = \operatorname{Ran} H^{-1}|_{\mathscr{U}},$$

and denote by P the (in general non-orthogonal) projection in $\mathscr{H}$ onto $\mathscr{V}$ along the orthogonal complement $\mathscr{W}^{\perp}$ of $\mathscr{W}$; it will be established in Lemma 3.3 below that P always exists and is given by $P = HP_{\mathscr{U}}H^{-1}$. The main result of this note now generalises Theorem 3 in [7] from matrices to the current setting of (unbounded) operators H.

Mathematics subject classification (2020): Primary 15A18; Secondary 47A20, 47A55, 47B15, 49Rxx.
Keywords and phrases: Relative distance between eigenvalues, gap of the essential spectrum, compression of a self-adjoint operator, variational values, maximal angle between closed subspaces.

Paper OaM-18-13

THEOREM 1.1. *Let H, λ_j, $\mathscr{U}$, M, μ_k, and P be as above, and suppose that* $\eta := \|P_{\mathscr{U}} - P\| < 1$. *Then:*

(a) *M is invertible.*

(b) *If numbers $m_0, m_1 \in \mathbb{N}$ with $m_0 \leqslant m_1 \leqslant \dim \mathscr{U}$ satisfy $\mu_{m_0} > 0$ and $\mu_{m_1} < (1-\eta)d$, where $d := \inf(\sigma_{\mathrm{ess}}(H) \cap (0,\infty)) \in (0,\infty]$, then H has at least $m_1 - m_0 + 1$ positive eigenvalues below d, counting multiplicities, and there are indices $j_{m_0} < \ldots < j_{m_1}$ with*

$$\frac{|\lambda_{j_k} - \mu_k|}{\lambda_{j_k}} \leqslant \eta \quad \textit{for all } m_0 \leqslant k \leqslant m_1. \tag{1.1}$$

Roughly speaking, Theorem 1.1 states that if $\eta < 1$, then *small enough* positive eigenvalues of M can be matched to certain positive eigenvalues of H with a suitable relative bound. Here, small enough refers to being well below a threshold close to the bottom of the positive essential spectrum of H, cf. parts (1) and (2) of Remark 1.2 below. As in [7,8,9], the proof of Theorem 1.1 relies on perturbing H into its diagonal part with respect to the decomposition $\operatorname{Ran} P_{\mathscr{U}} \oplus \operatorname{Ran}(I - P_{\mathscr{U}})$, which is reduced by $\mathscr{U}$ with corresponding part M, see Section 3 below. Note also that the subspace $\mathscr{U}$ is invariant (and then, in fact, reducing) for H if and only if $\mathscr{V} \subset \mathscr{U}$. In this case, one even has $\mathscr{V} = \mathscr{U} = \mathscr{W}$ and, therefore, $P = P_{\mathscr{U}}$, see Lemma 3.1 below. In this respect, the norm of the difference $P_{\mathscr{U}} - P$ can be regarded as an appropriate measure for how far $\mathscr{U}$ is off from being an invariant subspace for H. Also, if $H = H^{-1}$, then we have $\mathscr{V} = \mathscr{W}$ and, thus, $P = P_{\mathscr{V}} = P_{\mathscr{W}}$.

REMARK 1.2. (1) If H has no positive essential spectrum at all, that is, if $d = \infty$, then the condition $\mu_{m_1} < (1-\eta)d$ in part (b) of Theorem 1.1 is automatically satisfied and *all* positive eigenvalues of M can be matched to some positive eigenvalues of H, provided that $\eta < 1$.

(2) It is worth to note that the bound (1.1) together with $\mu_k < (1-\eta)d$ indeed entails $\lambda_{j_k} < d$. In this regard, it is a priori not possible to obtain in Theorem 1.1 analogous statements for eigenvalues $\mu_k \geqslant (1-\eta)d$. In fact, H may not even have correspondingly many positive eigenvalues below d.

(3) As already mentioned in [10, Remark 2.3], a bound of the form (1.1) also yields the relative bound

$$\frac{|\lambda_{j_k} - \mu_k|}{\mu_k} = \frac{\frac{|\lambda_{j_k} - \mu_k|}{\lambda_{j_k}}}{1 - \frac{\lambda_{j_k} - \mu_k}{\lambda_{j_k}}} \leqslant \frac{\eta}{1-\eta} \qquad \text{for all } m_0 \leqslant k \leqslant m_1.$$

(4) Upon replacing H and M by $-H$ and $-M$, respectively, one gets the analogous statement of Theorem 1.1 for negative eigenvalues in the gap of the essential spectrum.

(5) Similar statements regarding eigenvalues in gaps of the essential spectrum not containing zero are also possible (while still keeping the requirement of bounded invertibility of H), but this then requires a stronger assumption on the norm $\|P_{\mathscr{U}} - P\|$

depending on the gap under consideration, see Remark 3.8 below. The latter can, of course, formally be avoided with a suitable spectral shift of H (and M), but this then also affects the subspaces $\mathscr{V}$ and $\mathscr{W}$ and, thus, the projection P.

Let us now compare Theorem 1.1 to [7, Theorem 3] and comment on other related results in the literature.

REMARK 1.3. (1) If $J\colon \mathscr{K} \to \mathscr{H}$ is an isometry from some Hilbert space $\mathscr{K}$ with range $\mathscr{U}$, then the operator M is unitarily equivalent to J^*HJ. In this sense, the above setting is consistent with the framework of [7].

(2) It is easily seen that $P_{\mathscr{U}} - P = (P_{\mathscr{U}} - P_{\mathscr{U}}^{\perp})(P_{\mathscr{U}}(I-P) + P_{\mathscr{U}}^{\perp}P)$, where $P_{\mathscr{U}} - P_{\mathscr{U}}^{\perp}$ is unitary; cf. the proof of Lemma 3.5 below. In particular, we have $\|P_{\mathscr{U}} - P\| = \|P_{\mathscr{U}}(I-P) + P_{\mathscr{U}}^{\perp}P\|$. Taking into account parts (1) of this remark and of Remark 1.2, Theorem 1.1 therefore indeed contains [7, Theorem 3] as a special case and, thus, generalises it from matrices to (possibly unbounded) operators H.

(3) To the best of the author's knowledge, Theorem 1.1 is the first result of this kind applicable for gaps in the essential spectrum of not necessarily semibounded operators H. By contrast, for nonnegative operators H stronger results have been obtained in [10, 11] for eigenvalues below the essential spectrum. In particular, [10, Theorem 2.2] allows to consider subspaces $\mathscr{U}$ in the *form domain* of H and provides a stronger relative bound already in the case of matrices considered earlier in [7], cf. [7, Example 10].

The following result gives a geometric bound on the norm of the difference $P_{\mathscr{U}} - P$ in terms of the maximal angles between the pairs of subspaces $(\mathscr{U},\mathscr{V})$, $(\mathscr{U},\mathscr{W})$, and $(\mathscr{V},\mathscr{W})$. In this regard, it recovers Proposition 5 in [7] in the current setting. Recall that the *maximal angle* $\theta(\mathscr{M},\mathscr{N})$ between two closed subspaces $\mathscr{M}$ and $\mathscr{N}$ of $\mathscr{H}$ can be defined as

$$\theta(\mathscr{M},\mathscr{N}) = \arcsin(\|P_{\mathscr{M}} - P_{\mathscr{N}}\|),$$

see, e.g., [2, Definition 2.1].

THEOREM 1.4. *Let $\mathscr{U}$, $\mathscr{V}$, $\mathscr{W}$, and P be as in Theorem 1.1. Then*

$$\|P_{\mathscr{U}} - P\| \leqslant \min\bigl\{\sin\theta(\mathscr{U},\mathscr{V}), \sin\theta(\mathscr{U},\mathscr{W})\bigr\} + \tan\theta(\mathscr{V},\mathscr{W}).$$

The rest of this note is organised as follows: Section 2 presents a general perturbation result that addresses relative bounds for eigenvalues in gaps of the essential spectrum. In essence, it reproduces a result from [20] in an operator framework, but is proved here in an alternative way using the variational principle from [4, 5]. Section 3 then adds a geometric component in terms of the projections $P_{\mathscr{U}}$, $P_{\mathscr{V}}$, $P_{\mathscr{W}}$, and P that allows to infer from the general result in Section 2 the core result of this note, Theorem 3.6. The latter includes Theorem 1.1 as a particular case, while allowing the subspace $\mathscr{U}$ to have infinite dimension. A likewise more general version of Theorem 1.4, Theorem 3.11, is also proved in that section utilizing known results on maximal angles between closed subspaces.

2. Relative bounds for eigenvalues

In this section we prove a general residual bound for eigenvalues in gaps of the essential spectrum of self-adjoint operators, which lays the foundation for the proof of Theorem 1.1. The corresponding result essentially reproduces [20, Theorem 4.13] in the particular case of an operator framework; see also [16, Theorem 3.16] for the matrix case.

For a self-adjoint operator T, we denote by E_T the projection-valued spectral measure for T, and for $\gamma \in \mathbb{R}$ we write $\lambda_{\gamma,j}(T) = \lambda_j(T|_{\operatorname{Ran}\mathsf{E}_T((\gamma,\infty))}) \geqslant \gamma$, $j \in \mathbb{N}$, $j \leqslant \dim\operatorname{Ran}\mathsf{E}_T((\gamma,\infty))$, for the j-th standard variational value of the lower semibounded part $T|_{\operatorname{Ran}\mathsf{E}((\gamma,\infty))}$ of T. It agrees with the j-th eigenvalue of $T|_{\operatorname{Ran}\mathsf{E}((\gamma,\infty))}$ below its essential spectrum, in nondecreasing order and counting multiplicities, if this eigenvalue exists, and otherwise equals the bottom of the essential spectrum of $T|_{\operatorname{Ran}\mathsf{E}((\gamma,\infty))}$. In fact, if $\operatorname{Ran}\mathsf{E}((\gamma,\infty))$ is infinite dimensional, then $\lambda_{\gamma,j}(T) \to \inf(\sigma_{\mathrm{ess}}(T) \cap (\gamma,\infty)) \in [\gamma,\infty]$ as $j \to \infty$.

Let A be self-adjoint, and let V be symmetric with $\operatorname{Dom}(V) \supset \operatorname{Dom}(A)$. Suppose that for some constants $a \in \mathbb{R}$, $b \in [0,1)$ the operator $A_1 := a + b|A|$ is nonnegative and that $\|Vx\| \leqslant \|A_1 x\|$ for all $x \in \operatorname{Dom}(A)$. In particular, this gives $\|Vx\| \leqslant |a|\|x\| + b\|Ax\|$ for all $x \in \operatorname{Dom}(A)$, so that $B := A + V$ is self-adjoint on $\operatorname{Dom}(B) = \operatorname{Dom}(A)$ by the well-known Kato-Rellich theorem. The following result is used in Section 3 below only in the particular case where $a = 0$. However, the more general case of $a \in \mathbb{R}$ does not require much more efforts and is more in line with the mentioned guiding statement from [20].

PROPOSITION 2.1. *Let the interval (α,β) with $\beta - \alpha > 2a + b(|\alpha| + |\beta|)$ be in the resolvent set of A. Then:*

(a) *The interval $(\alpha + b|\alpha| + a, \beta - b|\beta| - a)$ belongs to the resolvent set of $B = A + V$.*

(b) *The subspace $\operatorname{Ran}\mathsf{E}_A((\alpha,\infty))$ has finite dimension if and only if $\operatorname{Ran}\mathsf{E}_B((\alpha + b|\alpha| + a,\infty))$ has finite dimension, and in this case $\dim\operatorname{Ran}\mathsf{E}_A((\alpha,\infty)) = \dim\operatorname{Ran}\mathsf{E}_B((\alpha + b|\alpha| + a,\infty))$ holds.*

(c) *We have*

$$|\lambda_{\alpha,j}(A) - \lambda_{\alpha+b|\alpha|+a,j}(B)| \leqslant a + b|\lambda_{\alpha,j}(A)|$$

for all $j \in \mathbb{N}$ with $j \leqslant \dim\operatorname{Ran}\mathsf{E}_A((\alpha,\infty))$.

(d) *With $d := \inf(\sigma_{\mathrm{ess}}(A) \cap (\alpha,\infty)) \in [\beta,\infty]$ we have*

$$d - b|d| - a \leqslant \inf\bigl(\sigma_{\mathrm{ess}}(B) \cap (\alpha + b|\alpha| + a,\infty)\bigr) \leqslant d + b|d| + a,$$

where the lower and upper bounds are interpreted as ∞ if $d = \infty$. In particular, the spectral part $\sigma(B) \cap (\alpha + b|\alpha| + a,\infty)$ is purely discrete if $\sigma(A) \cap (\alpha,\infty)$ is purely discrete.

For the convenience of the reader, a proof of Proposition 2.1 is presented below. Other than the approach in [20], which was based on analyticity properties, this proof alternatively relies on the minimax principle from [4, 5] for eigenvalues in gaps of the essential spectrum. The following proposition formulates a variant of this result tailored to the current situation; cf. also [6, 18].

PROPOSITION 2.2. ([5, Theorem 1]) *Let T be self-adjoint, and let Λ be an orthogonal projection in the same Hilbert space such that* $\operatorname{Dom}(T)$ *is invariant for* Λ. *With* $\mathscr{D}_+ := \operatorname{Dom}(T) \cap \operatorname{Ran}\Lambda$ *and* $\mathscr{D}_- := \operatorname{Dom}(T) \cap \operatorname{Ran}(I - \Lambda)$, *suppose that*

$$\nu := \sup_{\substack{x_- \in \mathscr{D}_- \\ \|x_-\|=1}} \langle x_-, Tx_- \rangle < \inf_{\substack{x_+ \in \mathscr{D}_+ \\ \|x_+\|=1}} \langle x_+, Tx_+ \rangle. \tag{2.1}$$

Then,

$$\lambda_{\nu,j}(T) = \inf_{\substack{\mathfrak{M} \subset \mathscr{D}_+ \\ \dim \mathfrak{M} = j}} \sup_{\substack{x \in \mathfrak{M} \oplus \mathscr{D}_- \\ \|x\|=1}} \langle x, Tx \rangle \tag{2.2}$$

for $j \in \mathbb{N}$ *with* $j \leqslant \dim \operatorname{Ran}\Lambda$, *and these describe all variational values of the lower semibounded part* $T|_{\operatorname{Ran} \mathsf{E}_T((\nu,\infty))}$ *of* T.

REMARK 2.3. The inequality (2.1) is usually called a *gap condition* for T. In [4, 5], the right-hand side of (2.1) is replaced by the possibly larger term

$$\inf_{x_+ \in \mathscr{D}_+ \setminus \{0\}} \sup_{x_- \in \mathscr{D}_-} \frac{\langle x_+ + x_-, T(x_+ + x_-) \rangle}{\|x_+ + x_-\|^2},$$

which agrees with the right-hand side of (2.2) for $j = 1$. In particular, the condition formulated by (2.1) is stricter than the corresponding one in [4,5]. However, it is exactly (2.1) that is verified in the proof of Proposition 2.1 below.

An implicit part of Proposition 2.2 is that under the hypotheses the subspace $\operatorname{Ran} \mathsf{E}_T((\nu,\infty))$ has finite dimension if and only if $\operatorname{Ran}\Lambda$ has, and, in this case, the two subspaces have the same dimension. Moreover, the interval $(\nu, \lambda_{\nu,1}(T))$ belongs to the resolvent set of T and, in particular, so does the interval (ν, ν'), where ν' denotes the right-hind side of (2.1), cf. Remark 2.3. With this is mind, we are ready to prove Proposition 2.1.

Proof of Proposition 2.1. We follow the general strategy of the proof of [16, Theorem 3.16]. Since by hypothesis A_1 is self-adjoint and nonnegative and V is symmetric with $\operatorname{Dom}(V) \supset \operatorname{Dom}(A_1) = \operatorname{Dom}(A)$ and $\|Vx\| \leqslant \|A_1 x\|$ for all $x \in \operatorname{Dom}(A)$, it follows from Löwner's theorem, see, e.g., [12, Theorem V.4.12], that

$$|\langle x, Vx \rangle| \leqslant \langle x, A_1 x \rangle \qquad \text{for all } x \in \operatorname{Dom}(A).$$

As a consequence, we have

$$A - A_1 \leqslant B \leqslant A + A_1 \tag{2.3}$$

in the sense of quadratic forms, where $\operatorname{Dom}(A \pm A_1) = \operatorname{Dom}(A) = \operatorname{Dom}(B)$.

In the notation of Proposition 2.2, we take $\Lambda = \mathsf{E}_A((\alpha,\infty)) = \mathsf{E}_A([\beta,\infty))$ and

$$\mathscr{D}_+ = \operatorname{Dom}(A) \cap \operatorname{Ran} \mathsf{E}_A([\beta,\infty)), \quad \mathscr{D}_- = \operatorname{Dom}(A) \cap \operatorname{Ran} \mathsf{E}_A((-\infty,\alpha]).$$

Define $f_\pm : \mathbb{R} \to \mathbb{R}$ by $f_\pm(t) = t \pm (a + b|t|)$, which both are continuous, bijective, and strictly increasing. Taking into account that $A \pm A_1 = f_\pm(A)$ by functional calculus, we then have

$$\langle x, (A - A_1)x \rangle \geqslant f_-(\beta)\|x\|^2 \quad \text{for} \quad x \in \mathscr{D}_+$$

and

$$\langle x, (A + A_1)x \rangle \leqslant f_+(\alpha)\|x\|^2 \quad \text{for} \quad x \in \mathscr{D}_-.$$

Moreover, the hypothesis on α and β guarantees that $f_-(\beta) > f_+(\alpha)$. In light of (2.3), for each of the choices $T \in \{A \pm A_1, B\}$ the gap condition (2.1) is therefore satisfied with

$$\sup_{\substack{x_- \in \mathscr{D}_- \\ \|x_-\|=1}} \langle x_-, Tx_- \rangle \leqslant f_+(\alpha) < f_-(\beta) \leqslant \inf_{\substack{x_+ \in \mathscr{D}_+ \\ \|x_+\|=1}} \langle x_+, Tx_+ \rangle,$$

so that Proposition 2.2 can be applied for all three choices. In particular, the interval $(f_+(\alpha), f_-(\beta))$ belongs to the resolvent set of all three operators $A \pm A_1$ and B. With $T = B$, this proves parts (a) and (b) of the claim. Furthermore, with $\gamma = f_+(\alpha)$ we obtain from (2.3) and the representation of the variational values in (2.2) that

$$\lambda_{\gamma,j}(A - A_1) \leqslant \lambda_{\gamma,j}(B) \leqslant \lambda_{\gamma,j}(A + A_1) \tag{2.4}$$

for all $j \in \mathbb{N}$, $j \leqslant \dim \operatorname{Ran} \mathsf{E}_A((\alpha,\infty))$. Here, we also have the representation $\lambda_{\gamma,j}(A - A_1) = \lambda_{f_-(\alpha),j}(f_-(A))$ since also the interval $(f_-(\alpha), f_-(\beta))$ belongs to the resolvent set of $f_-(A)$ by the spectral mapping theorem, as well as trivially $\lambda_{\gamma,j}(A + A_1) = \lambda_{f_+(\alpha),j}(f_+(A))$. In turn, again by the spectral mapping theorem, we have $\lambda_{f_\pm(\alpha),j}(f_\pm(A)) = f_\pm(\lambda_{\alpha,j}(A))$, so that we arrive at the representations $\lambda_{\gamma,j}(A \pm A_1) = f_\pm(\lambda_{\alpha,j}(A))$. Plugging the latter into (2.4) gives

$$\lambda_{\alpha,j}(A) - (a + b|\lambda_{\alpha,j}(A)|) \leqslant \lambda_{\gamma,j}(B) \leqslant \lambda_{\alpha,j}(A) + (a + b|\lambda_{\alpha,j}(A)|) \tag{2.5}$$

for all $j \in \mathbb{N}$ with $j \leqslant \dim \operatorname{Ran} \mathsf{E}_A((\alpha,\infty))$, which proves part (c) of the claim.

It remains to show part (d). If $\operatorname{Ran} \mathsf{E}_A((\alpha,\infty))$ has finite dimension, then also $\operatorname{Ran} \mathsf{E}_B((f_+(\alpha),\infty))$ has finite dimension by part (b) and there is nothing to prove. So, suppose that $\operatorname{Ran} \mathsf{E}_A((\alpha,\infty))$, and hence also $\operatorname{Ran} \mathsf{E}_B((\gamma,\infty))$, is infinite dimensional. The claim of part (d) then follows by taking in (2.5) the limit as $j \to \infty$. This completes the proof. $\square$

3. Geometric residual bounds and proof of main results

A large part of the considerations in this section also works under more general assumptions than the ones from Section 1. With this in mind, let H and λ_j be as in Section 1, and let $\mathscr{U}$ be a (not necessarily finite dimensional) closed subspace such that $\operatorname{Dom}(H)$ is invariant for the orthogonal projection $P_{\mathscr{U}}$ onto $\mathscr{U}$; this obviously includes the case where $\mathscr{U}$ is just a finite dimensional subspace of $\operatorname{Dom}(H)$ as in Section 1. Let M be the compression of H to $\mathscr{U}$, that is,

$$M = P_{\mathscr{U}} H|_{\mathscr{U}} \quad \text{with} \quad \operatorname{Dom}(M) = \operatorname{Dom}(H) \cap \mathscr{U} \subset \mathscr{U}, \tag{3.1}$$

as an operator in the Hilbert space $\mathscr{U}$. Finally, denote by $\mathscr{V}$ and $\mathscr{W}$ the closed subspaces

$$\mathscr{V} = \operatorname{Ran} H|_{\mathscr{U}} \quad \text{and} \quad \mathscr{W} = \overline{\operatorname{Ran} H^{-1}|_{\mathscr{U}}}.$$

We begin with the following elementary, essentially well-known lemma.

LEMMA 3.1. (a) $\operatorname{Dom}(H)$ *is invariant also for* $P_{\mathscr{U}}^{\perp} = I - P_{\mathscr{U}}$.

(b) $\operatorname{Dom}(H)$ *splits as*

$$\operatorname{Dom}(H) = (\operatorname{Dom}(H) \cap \mathscr{U}) \oplus (\operatorname{Dom}(H) \cap \mathscr{U}^{\perp}).$$

(c) M *is densely defined in* $\mathscr{U}$.

(d) *If* $\mathscr{U}$ *is invariant for* H, *then* $\mathscr{V} = \mathscr{W} = \mathscr{U}$.

Proof. (a) is clear, and (b) follows immediately from (a) and the identity $I = P_{\mathscr{U}} + P_{\mathscr{U}}^{\perp}$.

For part (c), let $u \in \mathscr{U}$. Since H is densely defined, we may choose a sequence (x_k) in $\operatorname{Dom}(H)$ that converges to u. Taking into account that $P_{\mathscr{U}}$ is bounded, the sequence (u_k) with $u_k = P_{\mathscr{U}} x_k \in \operatorname{Dom}(M)$ then converges to $P_{\mathscr{U}} u = u$ in $\mathscr{U}$, which proves the claim.

Finally, for part (d), suppose that $\mathscr{U}$ is invariant for H, that is, $\mathscr{V} \subset \mathscr{U}$. In view of part (c), a standard argument then shows that also $\mathscr{U}^{\perp}$ is invariant for H. Now, let $y \in \mathscr{U}$. By part (b), we may decompose $x := H^{-1}y \in \operatorname{Dom}(H)$ as $x = u + v$ with $u \in \operatorname{Dom}(H) \cap \mathscr{U}$ and $v \in \operatorname{Dom}(H) \cap \mathscr{U}^{\perp}$. Then, we have $Hu + Hv = Hx = y \in \mathscr{U}$, which by $Hu \in \mathscr{U}$ and $Hv \in \mathscr{U}^{\perp}$ implies that $Hv = 0$, so that $v = 0$ because H is invertible. We conclude that $H^{-1}y = u \in \mathscr{U}$ and $y = Hu \in \mathscr{V}$. Since $y \in \mathscr{U}$ was arbitrary and taking into account that $\mathscr{U}$ is closed, the former yields $\mathscr{W} \subset \mathscr{U}$, and the latter implies $\mathscr{U} \subset \mathscr{V}$, that is, $\mathscr{U} = \mathscr{V}$.

In order to show the remaining inclusion $\mathscr{U} \subset \mathscr{W}$, we observe that the invariance of $\mathscr{U}$ for H implies that $\operatorname{Dom}(H) \cap \mathscr{U} \subset \operatorname{Ran} H^{-1}|_{\mathscr{U}} \subset \mathscr{W}$. In view of part (c) and the closedness of $\mathscr{W}$, this shows that indeed $\mathscr{U} \subset \mathscr{W}$, which completes the proof. □

REMARK 3.2. The above reasoning for part (d) of Lemma 3.1 is essentially contained, at least in part, in the proof of Lemma 2.1 in [17]; cf. also Remark 2.3 and Lemma 2.4 in [19].

The next lemma proves the existence of the (not necessarily orthogonal) projection onto $\mathscr{V}$ along $\mathscr{W}^{\perp}$ by providing an explicit representation in terms of H and $P_{\mathscr{U}}$.

LEMMA 3.3. *The operator* $P = HP_{\mathscr{U}}H^{-1}$ *is the projection onto* $\mathscr{V}$ *along* $\mathscr{W}^{\perp}$, *that is,* P *is bounded with* $P^2 = P$ *and satisfies* $\operatorname{Ran} P = \mathscr{V}$ *and* $\operatorname{Ker} P = \mathscr{W}^{\perp}$.

Proof. Observe that $P = HP_{\mathscr{U}}H^{-1}$ is closed and everywhere defined, hence bounded by the closed graph theorem. It is then obvious that also $P^2 = P$. Finally, we have the identities $\operatorname{Ran} P = \operatorname{Ran}(HP_{\mathscr{U}}|_{\operatorname{Dom}(H)}) = \mathscr{V}$ as well as $\operatorname{Ker} P = \operatorname{Ker}(P_{\mathscr{U}}H^{-1}) = (\operatorname{Ran}(H^{-1}P_{\mathscr{U}}))^{\perp} = \mathscr{W}^{\perp}$. $\square$

REMARK 3.4. More generally, if L is a closed densely defined operator with bounded inverse such that $\operatorname{Dom}(L)$ is invariant for $P_{\mathscr{U}}$, then $LP_{\mathscr{U}}L^{-1}$ is the projection onto $\operatorname{Ran} L|_{\mathscr{U}}$ along $\operatorname{Ker}(P_{\mathscr{U}}L^{-1}) = (\operatorname{Ran}(L^{-*}|_{\mathscr{U}}))^{\perp}$.

In light of the domain splitting in part (b) of Lemma 3.1, we may define the diagonal and off-diagonal parts of H with respect to $\mathscr{U} \oplus \mathscr{U}^{\perp}$ as

$$H_{\mathrm{diag}} = P_{\mathscr{U}}HP_{\mathscr{U}} + P_{\mathscr{U}}^{\perp}HP_{\mathscr{U}}^{\perp}, \quad H_{\mathrm{off}} = P_{\mathscr{U}}HP_{\mathscr{U}}^{\perp} + P_{\mathscr{U}}^{\perp}HP_{\mathscr{U}}$$

with $\operatorname{Dom}(H_{\mathrm{diag}}) = \operatorname{Dom}(H) = \operatorname{Dom}(H_{\mathrm{off}})$; cf. also [8, 9]. In particular, we have the operator identity

$$H = H_{\mathrm{diag}} + H_{\mathrm{off}}.$$

Clearly, the subspace $\mathscr{U}$ reduces H_{diag} in the sense that H_{diag} is the direct sum of operators defined in $\mathscr{U}$ and $\mathscr{U}^{\perp}$, respectively, and M is the part of H_{diag} associated to $\mathscr{U}$. We now aim to apply Proposition 2.1 from the previous section with $A = H$ and $V = -H_{\mathrm{off}}$, so that $A + V = H_{\mathrm{diag}}$. To this end, we make the following elementary observation.

LEMMA 3.5. *We have*

$$H_{\mathrm{off}}H^{-1} = (P_{\mathscr{U}} - P_{\mathscr{U}}^{\perp})(P_{\mathscr{U}} - P) \tag{3.2}$$

with P as in Lemma 3.3.

Proof. We calculate

$$\begin{aligned} H_{\mathrm{off}}H^{-1} &= P_{\mathscr{U}}HP_{\mathscr{U}}^{\perp}H^{-1} + P_{\mathscr{U}}^{\perp}HP_{\mathscr{U}}H^{-1} = P_{\mathscr{U}}(I-P) + P_{\mathscr{U}}^{\perp}P \\ &= (P_{\mathscr{U}} - P_{\mathscr{U}}^{\perp})(P_{\mathscr{U}} - P). \quad \square \end{aligned}$$

Note that the factor $P_{\mathscr{U}} - P_{\mathscr{U}}^{\perp}$ on the right-hand side of (3.2) is self-adjoint and unitary and can therefore be ignored when it comes to estimating $H_{\mathrm{off}}H^{-1}$ in norm. With this in mind, we are now able to formulate and prove the core result of this note. Here, the particular case where $\mathscr{U}$ has finite dimension agrees with Theorem 1.1.

THEOREM 3.6. *Suppose that $\eta := \|P_{\mathscr{U}} - P\| < 1$ with $P = HP_{\mathscr{U}}H^{-1}$ as in Lemma 3.3.*

(a) *The operator M in (3.1) is self-adjoint and has a bounded inverse.*

(b) *With $d := \inf(\sigma_{\mathrm{ess}}(H) \cap (0,\infty)) \in (0,\infty]$ we have*

$$\inf\bigl(\sigma_{\mathrm{ess}}(M) \cap (0,\infty)\bigr) \geqslant (1-\eta)d.$$

(c) *Denote by* $(\mu_k)_{k\in J}$ *with* $J\subset\mathbb{N}$ *the (finite or infinite) collection of eigenvalues of* M *in the interval* $(0,(1-\eta)d)$, *in increasing order and counting multiplicities. Then, there is a family of indices* $j_k\in\mathbb{N}$, $k\in J$, *strictly increasing in* k, *such that for each* $k\in J$ *we have*

$$\frac{|\lambda_{j_k}-\mu_k|}{\lambda_{j_k}}\leqslant\eta.$$

Proof. In view of Lemma 3.5, we have $\|H_{\mathrm{off}}H^{-1}\|=\|P_{\mathscr{U}}-P\|=\eta<1$. In particular, this gives

$$\|H_{\mathrm{off}}x\|\leqslant\eta\|Hx\|=\eta\||H|x\|$$

for all $x\in\mathrm{Dom}(H)$. In the notation of Section 2, we may therefore take $A=H$ and $V=-H_{\mathrm{off}}$ with $a=0$ and $b=\eta\in[0,1)$. Moreover, since H has a bounded inverse, there are numbers $\alpha,\beta\in\mathbb{R}$ with $\alpha<0<\beta$ such that the interval (α,β) belongs to the resolvent set of H; in particular, we have

$$d=\inf(\sigma_{\mathrm{ess}}(H)\cap(0,\infty))=\inf(\sigma_{\mathrm{ess}}(H)\cap(\alpha,\infty))\geqslant\beta.$$

We observe that $2a+b(|\alpha|+|\beta|)=\eta(\beta-\alpha)<\beta-\alpha$, so that the hypotheses of Proposition 2.1 are satisfied. We conclude that $H_{\mathrm{diag}}=H-H_{\mathrm{off}}$ is self-adjoint and that the interval $((1-\eta)\alpha,(1-\eta)\beta)$ belongs to its resolvent set; in particular, H_{diag} has a bounded inverse. Moreover, part (d) of Proposition 2.1 gives

$$\inf(\sigma_{\mathrm{ess}}(H_{\mathrm{diag}})\cap((1-\eta)\alpha,\infty))\geqslant(1-\eta)d.$$

Since $\mathscr{U}$ reduces H_{diag} and M is the part of H_{diag} associated to $\mathscr{U}$, this proves (a) and (b).

Taking into account that each $\lambda_{\alpha,j}(H)$ is positive, it follows from part (c) of Proposition 2.1 that

$$\frac{|\lambda_{\alpha,j}(H)-\lambda_{(1-\eta)\alpha,j}(H_{\mathrm{diag}})|}{\lambda_{\alpha,j}(H)}\leqslant\eta \tag{3.3}$$

for all $j\in\mathbb{N}$ with $j\leqslant\dim\mathrm{Ran}\,\mathsf{E}_H((\alpha,\infty))$. In particular, this implies that $\lambda_{\alpha,j}(H)<d$ if $\lambda_{(1-\eta)\alpha,j}(H_{\mathrm{diag}})<(1-\eta)d$. Now, by definition of the μ_k there are indices j_k with $\lambda_{(1-\eta)\alpha,j_k}(H_{\mathrm{diag}})=\mu_k\in(0,(1-\eta)d)$ for all $k\in J$. Thus, $\lambda_{\alpha,j_k}(H)<d$ is the j_k-th positive eigenvalue of H below d, that is, $\lambda_{\alpha,j_k}(H)=\lambda_{j_k}$. Together with (3.3), this shows part (c) and, hence, completes the proof of the theorem. □

Let us collect some useful observations regarding part (b) of Theorem 3.6.

REMARK 3.7. (1) The proof of Theorem 3.6 gives

$$(1-\eta)d\leqslant\inf(\sigma_{\mathrm{ess}}(H_{\mathrm{diag}})\cap((1-\eta)\alpha,\infty))\leqslant\inf(\sigma_{\mathrm{ess}}(M)\cap((1-\eta)\alpha,\infty)),$$

and either inequality may a priori be strict. Thus, eigenvalues of M that are larger than (or equal to) $(1-\eta)d$ are not necessarily accessible via the variational values $\lambda_{(1-\eta)\alpha,j}(H_{\mathrm{diag}})$, and even for those that are accessible, we can no longer guarantee that the corresponding variational values $\lambda_{\alpha,j}(H)$ for H are smaller than d. The latter may therefore not correspond to eigenvalues of H.

(2) If $\mathscr{U}$ has finite dimension, then

$$\inf(\sigma_{\mathrm{ess}}(H_{\mathrm{diag}}) \cap ((1-\eta)\alpha,\infty)) = \inf(\sigma_{\mathrm{ess}}(H) \cap (\alpha,\infty)) = d.$$

Indeed, in this case $\operatorname{Ran}(P_{\mathscr{U}} - P)$ has finite dimension and, consequently, in view of Lemma 3.5, $H_{\mathrm{diag}}^{-1} - H^{-1} = H_{\mathrm{diag}}^{-1} H_{\mathrm{off}} H^{-1}$ is compact. Hence, $\sigma_{\mathrm{ess}}(H_{\mathrm{diag}}) = \sigma_{\mathrm{ess}}(H)$, see, e.g., [12, Theorem IV.5.35].

(3) Although $\inf(\sigma_{\mathrm{ess}}(H_{\mathrm{diag}}) \cap ((1-\eta)\alpha,\infty)) \leqslant (1+\eta)d$ by part (d) of Proposition 2.1, the term $\inf(\sigma_{\mathrm{ess}}(M) \cap ((1-\eta)\alpha,\infty))$ might a priori be a lot larger, for instance if H_{diag} has positive essential spectrum but M does not. In view of part (2) of this remark, this is the case, in particular, if H has positive essential spectrum and $\mathscr{U}$ has finite dimension.

The following remark addresses an extension of Theorem 3.6 to gaps of the essential spectrum of H that do not contain zero.

REMARK 3.8. The general form of Proposition 2.1 allows to obtain also similar statements as in Theorem 3.6 for eigenvalues in gaps of the essential spectrum not containing zero. More precisely, instead of the interval (α,β) in the proof of Theorem 3.6, we may consider any interval $(\tilde{\alpha},\tilde{\beta})$ belonging to the resolvent set of H such that $\eta = \|P_{\mathscr{U}} - P\|$ satisfies the (stronger) condition

$$\eta < \frac{\tilde{\beta}-\tilde{\alpha}}{|\tilde{\alpha}|+|\tilde{\beta}|}.$$

The terms $(1-\eta)\alpha$, $(1-\eta)\beta$, and $(1-\eta)d$ in the proof then just have to be replaced by $\tilde{\alpha}+\eta|\tilde{\alpha}|$, $\tilde{\beta}-\eta|\tilde{\beta}|$, and $\tilde{d}-\eta|\tilde{d}|$, respectively, where $\tilde{d}$ is given by $\tilde{d} = \inf(\sigma_{\mathrm{ess}}(H) \cap (\tilde{\alpha},\infty)) \geqslant \tilde{\beta}$.

Theorem 3.6 relies on the crucial condition $\|P_{\mathscr{U}} - P\| < 1$, so let us now address how the norm of $P_{\mathscr{U}} - P$ can be estimated. To this end, we may choose one of the alternative decompositions

$$P_{\mathscr{U}} - P = (P_{\mathscr{U}} - P_{\mathscr{V}}) + (P_{\mathscr{V}} - P) = (P_{\mathscr{U}} - P_{\mathscr{W}}) + (P_{\mathscr{W}} - P). \tag{3.4}$$

Here, the terms $P_{\mathscr{U}} - P_{\mathscr{V}}$ and $P_{\mathscr{U}} - P_{\mathscr{W}}$ correspond to sines of the operator angles associated to the pairs of subspaces $(\mathscr{U},\mathscr{V})$ and $(\mathscr{U},\mathscr{W})$, respectively. More precisely,

$$|P_{\mathscr{U}} - P_{\mathscr{V}}| = \sin\Theta(\mathscr{U},\mathscr{V}) \quad \text{and} \quad |P_{\mathscr{U}} - P_{\mathscr{W}}| = \sin\Theta(\mathscr{U},\mathscr{W}), \tag{3.5}$$

where $\Theta(\cdot,\cdot)$ denotes the operator angle associated with the respective subspaces, see, e.g., [14, Section 2] and the references cited therein for a discussion. In particular, the maximal angle introduced in Section 1 satisfies $\theta(\cdot,\cdot) = \|\Theta(\cdot,\cdot)\|$.

In order to address the other two terms, $P_{\mathscr{V}} - P$ and $P_{\mathscr{W}} - P$, we make the following considerations:

Since $\operatorname{Ran}(I_{\mathscr{H}} - P) = \operatorname{Ker} P = \mathscr{W}^{\perp}$, we obtain from $P + (I - P) = I$ that $P_{\mathscr{W}} P = P_{\mathscr{W}}$. Hence, the projection P can be represented with respect to the orthogonal decomposition $\mathscr{W} \oplus \mathscr{W}^{\perp}$ as the 2×2 block operator matrix

$$P = \begin{pmatrix} I_{\mathscr{W}} & 0 \\ X & 0 \end{pmatrix} \tag{3.6}$$

with $X := P_{\mathscr{W}}^{\perp} P|_{\mathscr{W}}$, interpreted as an operator from $\mathscr{W}$ to $\mathscr{W}^{\perp}$. In particular, $\mathscr{V} = \operatorname{Ran} P$ admits the graph subspace representation

$$\mathscr{V} = \{ f \oplus Xf \colon f \in \mathscr{W} \}. \tag{3.7}$$

Recall from [13, Corollary 3.4 and Remark 3.6] that consequently we have $\|P_{\mathscr{W}} - P_{\mathscr{V}}\| < 1$ and that X corresponds to the tangent of the operator angle associated to the subspaces $\mathscr{W}$ and $\mathscr{V}$, more precisely

$$\begin{pmatrix} |X| & 0 \\ 0 & |X^*| \end{pmatrix} = \tan \Theta(\mathscr{W}, \mathscr{V}). \tag{3.8}$$

Moreover, we have

$$P_{\mathscr{V}} = U P_{\mathscr{W}} U^*, \tag{3.9}$$

where U is the unitary operator given by the 2×2 block operator matrix

$$U = \begin{pmatrix} (I_{\mathscr{W}} + X^*X)^{-1/2} & -X^*(I_{\mathscr{W}^{\perp}} + XX^*)^{-1/2} \\ X(I_{\mathscr{W}} + X^*X)^{-1/2} & (I_{\mathscr{W}^{\perp}} + XX^*)^{-1/2} \end{pmatrix}. \tag{3.10}$$

A broader discussion on the operator angle and graph subspace representations can be found, for instance, in [15, Sections 1.3 and 1.5] and the references cited therein.

REMARK 3.9. The inequality $\|P_{\mathscr{W}} - P_{\mathscr{V}}\| < 1$ can alternatively also be verified as follows: Since the projection P onto $\mathscr{V}$ along $\mathscr{W}^{\perp}$ exists by Lemma 3.3, Proposition 1.6 in [3] yields that $\|P_{\mathscr{V}} P_{\mathscr{W}}^{\perp}\| < 1$. Taking into account that P^* is the projection onto $\mathscr{W}$ along $\mathscr{V}^{\perp}$, we obtain in the same way that $\|P_{\mathscr{V}}^{\perp} P_{\mathscr{W}}\| = \|P_{\mathscr{W}} P_{\mathscr{V}}^{\perp}\| < 1$. Using $\|P_{\mathscr{W}} - P_{\mathscr{V}}\| = \max\{\|P_{\mathscr{V}} P_{\mathscr{W}}^{\perp}\|, \|P_{\mathscr{V}}^{\perp} P_{\mathscr{W}}\|\}$, see, e.g., [1, Section 34], this gives $\|P_{\mathscr{W}} - P_{\mathscr{V}}\| < 1$.

LEMMA 3.10. *With* $X = P_{\mathscr{W}}^{\perp} P|_{\mathscr{W}} \colon \mathscr{W} \to \mathscr{W}^{\perp}$ *and* U *as in* (3.10) *we have*

$$P_{\mathscr{W}} - P = \begin{pmatrix} 0 & 0 \\ -X & 0 \end{pmatrix}$$

and

$$P_{\mathscr{V}} - P = U \begin{pmatrix} 0 & X^* \\ 0 & 0 \end{pmatrix} U^*.$$

Proof. The representation for $P_{\mathscr{W}} - P$ follows directly from (3.6). Moreover, using the identity $X^*(I_{\mathscr{W}^\perp} + XX^*)^{-1/2} = (I_{\mathscr{W}} + X^*X)^{-1/2}X^*$, the representation for $P_{\mathscr{V}} - P$ is verified from (3.6), (3.9), and (3.10) by plain multiplication of 2×2 block operator matrices. □

We now arrive at the following result, the particular case of which where $\mathscr{U}$ has finite dimension agrees with Theorem 1.4.

THEOREM 3.11. *We have*

$$\|P_{\mathscr{U}} - P\| \leqslant \min\big\{\sin\theta(\mathscr{U},\mathscr{V}), \sin\theta(\mathscr{U},\mathscr{W})\big\} + \tan\theta(\mathscr{V},\mathscr{W}).$$

Proof. From Lemma 3.10 and (3.8) we obtain that

$$\|P_{\mathscr{W}} - P\| = \|P_{\mathscr{V}} - P\| = \|X\| = \tan\theta(\mathscr{V},\mathscr{W}),$$

where for the last equality we used that $\|\tan\Theta(\mathscr{W},\mathscr{V})\| = \tan\theta(\mathscr{V},\mathscr{W})$. Combining the latter with (3.4) and (3.5) gives

$$\begin{aligned}\|P_{\mathscr{U}} - P\| &\leqslant \min\big\{\|P_{\mathscr{U}} - P_{\mathscr{V}}\|, \|P_{\mathscr{U}} - P_{\mathscr{W}}\|\big\} + \|X\|\\ &= \min\big\{\sin\theta(\mathscr{U},\mathscr{V}), \sin\theta(\mathscr{U},\mathscr{W})\big\} + \tan\theta(\mathscr{V},\mathscr{W}),\end{aligned}$$

which proves the claim. □

Acknowledgements. The author is grateful to Zlatko Drmač and Ivan Veselić for suggesting this research direction. He also thanks Krešimir Veselić for a helpful communication.

REFERENCES

[1] N. I. AKHIEZER, I. M. GLAZMAN, *Theory of Linear Operators in Hilbert Space*, Dover Publications, New York, 1993.

[2] S. ALBEVERIO, A. K. MOTOVILOV, *Sharpening the norm bound in the subspace perturbation theory*, Complex Anal. Oper. Theory **7** (2013), 1389–1416.

[3] A. BÖTTCHER, I. M. SPITKOVSKY, *A gentle guide to the basics of two projections theory*, Linear Algebra Appl. **432** (2010), 1412–1459.

[4] J. DOLBEAULT, M. J. ESTEBAN, E. SÉRÉ, *On the eigenvalues of operators with gaps. Application to Dirac Operators*, J. Funct. Anal. **174** (2000), 208–226.

[5] J. DOLBEAULT, M. J. ESTEBAN, E. SÉRÉ, *Corrigendum to "On the eigenvalues of operators with gaps. Application to Dirac Operators" [J. Funct. Anal. 174 (1) (2000) 208–226]*, J. Funct. Anal. **284** (2023), Paper No. 109651, 6 pp.

[6] J. DOLBEAULT, M. J. ESTEBAN, E. SÉRÉ, *Distinguished self-adjoint extension and eigenvalues of operators with gaps. Application to Dirac-Coulomb operators*, J. Spectr. Theory **13** (2023), 491–524.

[7] Z. DRMAČ, *On relative residual bounds for the eigenvalues of a Hermitian matix*, Linear Algebra Appl. **244** (1996), 155–163.

[8] L. GRUBIŠIĆ, *On eigenvalue and eigenvector estimates for nonnegative definite operators*, Siam J. Matrix Anal. Appl. **28** (2006), 1097–1125.

[9] L. GRUBIŠIĆ, I. NAKIĆ, *Error representation formula for eigenvalue approximations for positive definite operators*, Oper. Matrices **6** (2012), 793–808.

[10] L. GRUBIŠIĆ, K. VESELIĆ, *On Ritz approximations for positive definite operators I (theory)*, Linear Algebra Appl. **417** (2006), 397–422.

[11] L. GRUBIŠIĆ, K. VESELIĆ, *On weakly formulated Sylvester equations and applications*, Integral Equations Operator Theory **58** (2007), 175–204.

[12] T. KATO, *Perturbation Theory for Linear Operators*, Classics Math., Springer, Berlin, 1995.

[13] V. KOSTRYKIN, K. A. MAKAROV, A. K. MOTOVILOV, *Existence and uniqueness of solutions to the operator Riccati equation. A geometric approach*, In: Advances in Differential Equations and Mathematical Physics (Birmingham, AL, 2002), Contemp. Math., vol. 327, Amer. Math. Soc., Providence, RI, 2003, pp. 181–198.

[14] A. SEELMANN, *Notes on the* $\sin 2\Theta$ *theorem*, Integral Equations Operator Theory **79** (2014), 579–597.

[15] A. SEELMANN, *Perturbation theory for spectral subspaces*, Dissertation, Johannes Gutenberg-Universität Mainz, 2014.

[16] K. VESELIĆ, I. SLAPNIČAR, *Floating-point perturbations of Hermitian Matrices*, Linear Algebra Appl. **195** (1993), 81–116.

[17] K. A. MAKAROV, S. SCHMITZ, A. SEELMANN, *On invariant graph subspaces*, Integral Equations Operator Theory **85** (2016), 399–425.

[18] L. SCHIMMER, J. P. SOLOVEJ, S. TOKUS, *Friedrichs extension and min-max principle for operators with a gap*, Ann. Henri Poincaré **21** (2020), 327–357.

[19] C. TRETTER, C. WYSS, *Dichotomous Hamiltonians with unbounded entries and solutions of Riccati equations*, J. Evol. Equ. **14** (2014), 121–153.

[20] K. VESELIĆ, *Spectral perturbation bounds for selfadjoint operators. I*, Oper. Matrices **2** (2008), 307–339.

(Received September 13, 2023)

Albrecht Seelmann
Technische Universität Dortmund
Fakultät für Mathematik
D-44221 Dortmund, Germany
e-mail: albrecht.seelmann@tu-dortmund.de