Operators
and
Matrices
Volume 18, Number 1 (2024), 205–234
doi:10.7153/oam-2024-18-14

ON SINGULARITIES OF LABELED GRAPH C^*–ALGEBRAS

DEBENDRA P. BANJADE, AMY CHAMBERS AND MENASSIE EPHREM

(*Communicated by L. Molnár*)

Abstract. Given a directed graph E and a labeling $\mathscr{L}$, one forms the labeled graph C^*-algebra by taking a weakly left–resolving labeled space $(E,\mathscr{L},\mathscr{B})$ and considering a universal generating family of partial isometries and projections.

In this paper, given a labeled space $(E,\mathscr{L},\mathscr{B})$, we provide a process in which one can build a "larger" desingularized labeled space $(F,\mathscr{L}_F,\mathscr{B}_F)$ whose graph F essentially maintains the loop structure of the original graph E and such that the unitization of $C^*(E,\mathscr{L},\mathscr{B})$ is a full corner of $C^*(F,\mathscr{L}_F,\mathscr{B}_F)$.

1. Introduction

Since the early 1970's graphs have been used as a tool to study a large class of C^*-algebras. In 1980, in [10], Enomoto and Watatani introduced the notion of C^*-algebras associated to directed graphs represented by adjacency matrices, and the theory of graph algebras has been rigorously developed in subsequent years. In [13], Kumjian, Pask, Raeburn and Renault defined the graph groupoid of a countable row–finite directed graph with no sinks and showed that the C^*-algebra of this groupoid coincided with a universal C^*-algebra generated by partial isometries satisfying relations naturally generalizing those given in [6].

Since that time, many people have worked on generalizing these results to arbitrary directed graphs and beyond, including higher rank graphs, ultragraphs, and labeled graphs.

After the introduction of ultragraphs by Tomforde in [14], Bates and Pask, in [3], introduced a new class of C^*-algebras called C^*-algebras of labeled graphs. Later, in a series of papers (along with Carlsen) [4, 2], they provided some classifications of these algebras, including computations of their K-theories.

For a directed graph, a singular vertex is simply a vertex that emits infinitely many edges or none at all. In their paper [7] Drinen and Tomforde presented a way to desingularize a directed graph, by presenting a larger graph whose C^*-algebra contains the C^*-algebra of the original graph as a full corner.

In recent years several works have been done on labeled graph C^*-algebras, such as the ideal structures (gauge–invariant, primitive etc.). However, in most of these

Mathematics subject classification (2020): 46L05, 46L35, 46L55.
Keywords and phrases: Labeled graph, directed graph, Cuntz–Krieger algebra, labeled graph C^*-algebra.

Paper OaM-18-14

works two restrictive assumptions are regularly made: the graph would have to have no sinks and the labeled space would have to be set–finite (see [11], [9]). In this work, given a labeled space $(E,\mathscr{L},\mathscr{B})$ where $\mathscr{B}$ may contain "singular" sets, we build a "larger" desingularized labeled space $(F,\mathscr{L}_F,\mathscr{B}_F)$ with a C^*-algebra $C^*(F,\mathscr{L}_F,\mathscr{B}_F)$ containing $C^*(E,\mathscr{L},\mathscr{B})$; and that the unitization of $C^*(E,\mathscr{L},\mathscr{B})$ is a full corner of $C^*(F,\mathscr{L}_F,\mathscr{B}_F)$ (see Theorem 4.18).

REMARK 1.1. In [7], Drinen and Tomforde describe a desingularization process for directed graphs, which allows for the extension of certain results involving graph C^*-algebras. The constructions and results described in this paper are highly inspired by this work, as well as Tomforde's later paper, [14], in which he skillfully generalizes the Drinen-Tomforde Desingularization to ultragraphs. However, the constructions and proofs contained in this paper are significantly different from those works, primarily because each projection in the set of generating projections of the C^*-algebra of a directed graph or an ultragraph corresponds to a vertex of the graph or ultragraph, whereas each projection in the set of generating projections of the C^*-algebra of a labeled graph corresponds to a set of vertices.

A directed graph $E=(E^0,E^1,s,r)$ consists of a countable set E^0 of vertices and E^1 of edges, and maps $s,r:E^1\to E^0$ identifying the source (origin) and the range (terminus) of each edge. The graph is row–finite if each vertex emits at most finitely many edges. A vertex is a sink if it is not a source of any edge. A path is a sequence of edges $e_1e_2\dots e_n$ with $r(e_i)=s(e_{i+1})$ for each $i=1,2,\dots,n-1$. An infinite path is a sequence $e_1e_2\dots$ of edges with $r(e_i)=s(e_{i+1})$ for each i.

For a finite path $p=e_1e_2\dots e_n$, we define $s(p):=s(e_1)$ and $r(p):=r(e_n)$. For an infinite path $p=e_1e_2\dots$, we define $s(p):=s(e_1)$. We use the following notations:

$tE^1=\{e\in E^1 : s(e)=t\}$.

$E^*:=\bigcup_{n=0}^{\infty}E^n$, where E^n := $\{p : p$ is a path of length $n\}$.

$E^{**}:=E^*\cup E^\infty$, where E^∞ is the set of infinite paths.

The paper is organized as follows. In section 2 we develop some terminologies for labeled graphs. In section 3 we describe labeled graph C^*-algebras. In section 4 we first introduce the notion of a singular set, the labeled graph equivalent of a singular vertex for graphs, we then construct a desingularized labeled graph $(F,\mathscr{L}_F)$, build a labeled space $(F,\mathscr{L}_F,\mathscr{B}_F)$, and present the main result.

The desingularization process is a two step process. Given a labeled graph $(E,\mathscr{L})$, we first desingularize the labeled graph by removing the existing edges, adding new edges, and labeling. This creates a new labeled space $(F,\mathscr{L}_F)$. For the second step, given a set $\mathscr{B}\subseteq 2^{E^0}$, we create a new set $\mathscr{B}_F$ and build a new labeled space $(F,\mathscr{L}_F,\mathscr{B}_F)$. A reader who would like to jump ahead and see the desingulaization process may read example 4.4 and the subsequent remarks.

2. Preliminaries

Let $E=(E^0,\ E^1,\ s,\ r)$ be a directed graph and let $\mathscr{A}$ be a countable alphabet (a countable set of colors). A labeling is a function $\mathscr{L}: E^1 \longrightarrow \mathscr{A}$. Without loss of generality, we will assume that $\mathscr{A}=\mathscr{L}(E^1)$. The pair $(E,\mathscr{L})$ is called a labeled graph.

Given a labeled graph $(E,\mathscr{L})$, we extend the labeling function $\mathscr{L}$ canonically to the sets E^* and E^∞ as follows. Using $\mathscr{A}^n$ for the set of words of length n, $\mathscr{L}$ is defined from E^n into $\mathscr{A}^n$ as $\mathscr{L}(e_1e_2\dots e_n)=\mathscr{L}(e_1)\mathscr{L}(e_2)\dots\mathscr{L}(e_n)$. Similarly, for $p=e_1e_2\dots\in E^\infty$, $\mathscr{L}(p)=\mathscr{L}(e_1)\mathscr{L}(e_2)\dots\in\mathscr{A}^\infty$.

Following a tradition, we use $\mathscr{L}^*(E):=\bigcup_{n=1}^{\infty}\mathscr{L}(E^n)$, and $\mathscr{L}^\infty(E):=\mathscr{L}(E^\infty)$.

For a word $\alpha=a_1a_2\dots a_n\in\mathscr{L}^n(E)$, we write

$$s(\alpha):=\{s(p): p\in E^n,\ \mathscr{L}(p)=\alpha\}$$

and

$$r(\alpha):=\{r(p): p\in E^n,\ \mathscr{L}(p)=\alpha\}.$$

Similarly for $\alpha=a_1a_2\dots\in\mathscr{L}^\infty(E)$,

$$s(\alpha):=\{s(p): p\in E^\infty,\ \mathscr{L}(p)=\alpha\}$$

Each of these sets is a subset of E^0. The use of s and r for an edge/path verses a label/word should be clear from the context.

A labeled graph $(E,\mathscr{L})$ is said to be left–resolving if for each $v\in E^0$ the function $\mathscr{L}: r^{-1}(v)\to\mathscr{A}$ is injective. In other words, no two edges pointing to the same vertex are labeled the same.

Let $\mathscr{B}$ be a non-empty subset of 2^{E^0}. Given a set $A\in\mathscr{B}$ we write $\mathscr{L}(AE^1)$ for the set $\{\mathscr{L}(e): e\in E^1 \text{ and } s(e)\in A\}$.

For a set $A\in\mathscr{B}$ and a word $\alpha\in\mathscr{L}^n(E)$ we define the relative range of α with respect to A as

$$r(A,\alpha):=\{r(p): p\in E^*,\ \mathscr{L}(p)=\alpha \text{ and } s(p)\in A\}.$$

We say $\mathscr{B}$ is closed under relative ranges if $r(A,\alpha)\in\mathscr{B}$ for any $A\in\mathscr{B}$ and any $\alpha\in\mathscr{L}(E^n)$.

$\mathscr{B}$ is said to be *accommodating* if

1. $r(\alpha)\in\mathscr{B}$ for each $\alpha\in\mathscr{L}^*(E)$
2. $\mathscr{B}$ is closed under relative ranges
3. $\mathscr{B}$ is closed under finite intersections and unions.

If $\mathscr{B}$ is accommodating for $(E,\mathscr{L})$, the triple $(E,\mathscr{L},\mathscr{B})$ is called a labeled space. For trivial reasons, we will assume that $\mathscr{B}\neq\{\emptyset\}$

A labeled space $(E,\mathscr{L},\mathscr{B})$ is called *weakly left–resolving* if for any $A,\ B\in\mathscr{B}$ and any $\alpha\in\mathscr{L}^*(E)$

$$r(A\cap B,\alpha)=r(A,\alpha)\cap r(B,\alpha).$$

3. Labeled graph C^*-algebras

DEFINITION 3.1. Let $(E,\mathscr{L},\mathscr{B})$ be a weakly left–resolving labeled space. A representation of $(E,\mathscr{L},\mathscr{B})$ in a C^*-algebra consists of projections $\{p_A : A\in\mathscr{B}\}$, and partial isometries $\{s_a : a\in\mathscr{A}\}$ that satisfy the following Cuntz-Krieger type relations.

(CK-1) If $A,\ B\in\mathscr{B}$, then $p_Ap_B = p_{A\cap B}$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$, and $p_\phi = 0$.

(CK-2) For any $a,\ b\in\mathscr{A}$, $s_a^*s_b = p_{r(a)}\delta_{a,b}$.

(CK-3) For any $a\in\mathscr{A}$ and $A\in\mathscr{B}$, $s_a^*p_A = p_{r(A,a)}s_a^*$.

(CK-4) For $A\in\mathscr{B}$ with $\mathscr{L}(AE^1)$ finite, and $A\cap B=\emptyset$ for all $B\in\mathscr{B}$ satisfying $B\subseteq E^0_{sink}$, we have

$$p_A = \sum_{a\in\mathscr{L}(AE^1)} s_a p_{r(A,a)} s_a^*.$$

The *labeled graph C^*-algebra* is the C^*-algebra generated by a universal representation of $(E,\mathscr{L},\mathscr{B})$. For a word $\mu = a_1\cdots a_n$ we write s_μ to mean $s_{a_1}\cdots s_{a_n}$. One easily checks from the relations that $s_\mu^*s_\mu = p_{r(\mu)}$ and that $s_\nu^*s_\mu = 0$ unless one of μ, ν extends the other. In this case, e.g. if $\mu = \nu\alpha$, we have $s_\nu^*s_\mu = p_{r(\nu)}s_\alpha$.

Using ε to denote the empty word, we find that

$$C^*(E,\mathscr{L},\mathscr{B}) = \overline{\operatorname{span}}\left\{s_\mu p_A s_\nu^* : \mu,\ \nu\in\mathscr{L}(E^*)\cup\{\varepsilon\} \text{ and } A\in\mathscr{B}\right\}.$$

Here we use s_ε to denote the unit element of the multiplier algebra of $C^*(E,\mathscr{L},\mathscr{B})$.

Given a weakly left–resolving labeled space $(E,\mathscr{L},\mathscr{B})$, we say that the labeled space is set–finite if, for any $A\in\mathscr{B}$, the set $s^{-1}(A)$ is finite.

Notice that if E has no sinks and $(E,\mathscr{L},\mathscr{B})$ is set–finite, then for any $A\in\mathscr{B}$ we get

$$p_A = \sum_{a\in\mathscr{L}(AE^1)} s_a p_{r(A,a)} s_a^*. \tag{CK-5}$$

4. Desingularization

We begin with the definition of a singular set in $(E,\mathscr{L},\mathscr{B})$.

DEFINITION 4.1. Let $(E,\mathscr{L},\mathscr{B})$ be a labeled space. A set $A\in\mathscr{B}$ is said to be a singular set if A contains a sink or $\{a\in\mathscr{A} : s(a)\in A\}$ is not finite.

The main step of the desingularization procedure does not depend on the labeled space. We start with a labeled graph $(E,\mathscr{L})$ and build a labeled graph $(F,\mathscr{L}_F)$, where the graph F has no sinks. With this construction, given a labeled space $(E,\mathscr{L},\mathscr{B})$ that may contain singular sets, we construct a larger labeled space $(F,\mathscr{L}_F,\mathscr{B}_F)$ whose C^*-algebra has the desired properties.

Suppose $(E,\mathscr{L})$ is a labeled graph with the labeling set $\mathscr{A}$. Since the labeling function $\mathscr{L}: E^1 \longrightarrow \mathscr{A}$ is onto and E^1 is countable, the set $\mathscr{A}$ must be countable. Let $\mathscr{A} = \{a_1, a_2, \ldots, a_N\}$, where N is the cardinality of $\mathscr{A}$, which could be finite or countably infinite. Let $\mathscr{L}(tE^1) = \{a \in \mathscr{A} : t \in s(a)\}$.

We describe the construction of the desingularization of a labeled graph and follow with the definition. It may be useful to take a look at Example 4.4 when going through Definition 4.2.

Starting with the set of vertices E^0, we add vertices and create new edges and labelings as follows:

- For each vertex $t \in E^0$:
  - If t is a sink, attach an infinite path to t with vertices $v_t^1, v_t^2, \ldots$ and corresponding edges labeled $b_1, b_2, \ldots$ (for each sink, the labeling of this added path will be the same).
  - If t is not a sink then let
  $$k_t = \begin{cases} \max\{i : a_i \in \mathscr{L}(tE^1)\} & \text{if the set } \mathscr{L}(tE^1) \text{ is finite} \\ \infty & \text{if the set } \mathscr{L}(tE^1) \text{ is not finite} \end{cases}$$
  If $k_t < \infty$, attach a path of length k_t to t with vertices $v_t^1, v_t^2, \ldots, v_t^{k_t}$ and corresponding edges labeled $b_1, b_2, \ldots, b_{k_t}$. If $k_t = \infty$, attach an infinite path to t with vertices $v_t^1, v_t^2, \ldots$ and corresponding edges labeled $b_1, b_2, \ldots$.
- For each $i = 1 \ldots N$, if $a_i \in \mathscr{L}(tE^1)$, look at the set of edges $\mathscr{L}^{-1}(a_i) \cap tE^1$. For each $e \in \mathscr{L}^{-1}(a_i) \cap tE^1$, remove the edge e and add an edge from vertex v_t^i to $r(e)$ and label this edge as c_i. Each edge labeled a_i is now replaced by a path labeled $b_1 b_2 \ldots b_i c_i$.

DEFINITION 4.2. The desingularization of the labeled graph $(E,\mathscr{L})$ with labeling $\mathscr{A}$ is the labeled graph $(F,\mathscr{L}_F)$ with labeling $\mathscr{A}_F$ having vertex and labeling sets:

$$F^0 = E^0 \cup \bigcup_{t \in E^0} \{v_t^i : i = 1, \ldots, k_t\}$$

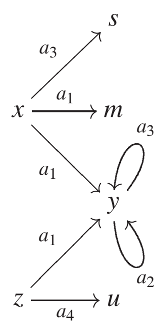
and

$$\mathscr{A}_F := B \cup C$$

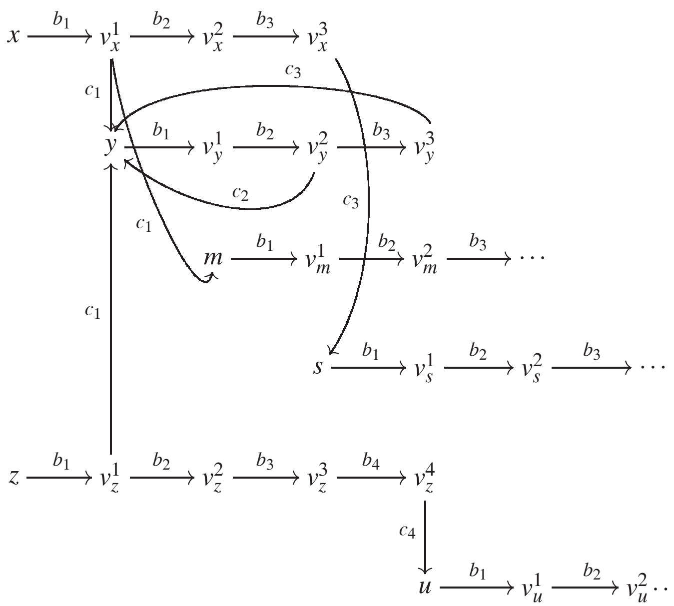
where $B := \{b_1, b_2, \ldots \; b_l\}$ with $l = \max\{i : a_i \in \mathscr{A}\}$ if E has no sinks and the labeling set $\mathscr{A}$ is finite or $l = \infty$ otherwise, and $C = \{c_1, c_2, \ldots, c_N\}$ where $N = \max\{i : a_i \in \mathscr{A}\}$ if the labeling set $\mathscr{A}$ is finite or $N = \infty$ if the set of labels is infinite. The set of edges as well as the source, range, and labeling maps are determined by the construction.

REMARK 4.3. This process of desingularization creates a one-to-one map from the set $\mathscr{L}^{-1}(a_i)$ onto $\mathscr{L}_F^{-1}(b_1 b_2 \ldots b_i c_i)$. Denote this map by ϕ. Let $\beta_i = b_1 b_2 \ldots b_i$ and $\gamma_i = b_1 \ldots b_i c_i = \beta_i c_i$. Thus $\phi : \mathscr{L}^{-1}(a_i) \longrightarrow \mathscr{L}_F^{-1}(\gamma_i)$ is one-to-one, onto, and source and range preserving (at the level of graphs). Moreover $s(a_i) = s(\gamma_i) \subseteq E^0$ and $r(a_i) = r(\gamma_i) \subseteq E^0$.

EXAMPLE 4.4. Consider the following labeled graph $(E,\mathscr{L})$.

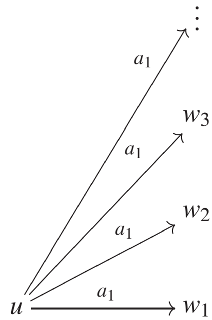


After replacing these edges, the labeled graph $(F,\mathscr{L}_F)$ will look like:

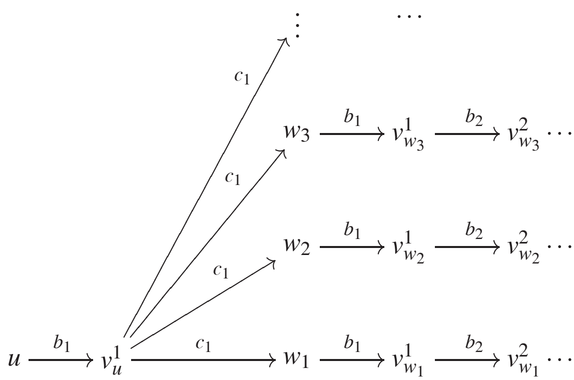


Notice that each vertex in E^0 is a source of the label b_1 and b_1 only.

EXAMPLE 4.5. Consider the following labeled graph $(E,\mathscr{L})$.

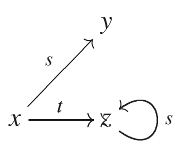


Then $(F,\mathscr{L}_F)$ becomes:



REMARK 4.6. Given a labeled graph $(E,\mathscr{L})$, the construction of $(F,\mathscr{L}_F)$ is not necessarily unique, and different constructions do not necessarily yield isomorphic labeled graphs. This is due to the fact that $\mathscr{A}$ may be ordered in different ways. We will demonstrate this fact using the following example.

EXAMPLE 4.7. Consider the following labeled graph $(E,\mathscr{L})$, where $\mathscr{A}=\{s,t\}$.



Ordering $\mathscr{A}$ as $\mathscr{A} = \{a_1 = s, a_2 = t\}$ the labeled graph $(F, \mathscr{L}_F)$ will be:

$$\begin{tikzcd}
x \arrow[r, "b_1"] & v_x^1 \arrow[r, "b_2"] \arrow[d, "c_1"'] & v_x^2 \arrow[dd, bend left, "c_2"] & & \\
& y \arrow[r, "b_1"] & v_y^1 \arrow[r, "b_2"] & v_y^2 \arrow[r, "b_3"] & \cdots \\
& & z \arrow[r, "b_1"] & v_z^1 \arrow[l, bend left, "c_1"] &
\end{tikzcd}$$

Ordering $\mathscr{A}$ as $\mathscr{A} = \{a_1 = t, a_2 = s\}$ the labeled graph $(F, \mathscr{L}_F)$ will be:

$$\begin{tikzcd}
x \arrow[r, "b_1"] & v_x^1 \arrow[r, "b_2"] \arrow[dd, "c_1"'] & v_x^2 \arrow[d, "c_2"'] & & & \\
& & y \arrow[r, "b_1"] & v_y^1 \arrow[r, "b_2"] & v_y^2 \arrow[r, "b_3"] & \cdots \\
& z \arrow[r, "b_1"] & v_z^1 \arrow[r, "b_2"] & v_z^2 \arrow[ll, bend left, "c_2"] & &
\end{tikzcd}$$

REMARK 4.8. Let $(F, \mathscr{L}_F)$ be a desingularization of a given labeled space $(E, \mathscr{L})$ and let ϕ be the function described in Remark 4.3.

1. $E^0 \subsetneq F^0$.
2. Each edge e in E^1 is now replaced by a (unique) path $\phi(e)$ in F, sharing the same source and range.
3. If $e_1 e_2 \dots e_n$ is a loop in E then $\phi(e_1)\phi(e_2)\dots\phi(e_n)$ is the corresponding loop in F, with base point in E^0; and this is a one-to-one map from the collection of loops in E onto the collection of loops in F with base points in E^0.
4. For any i, $r(b_i) = r(\beta_i)$, and $r(b_i) \cap E^0 = \emptyset$.
5. For any non empty subset A of E^0 and any i, $\mathscr{L}_F(AF^1) = \{b_1\}$, $\emptyset \subsetneq \mathscr{L}_F(r(A, \beta_i)F^1) \subseteq \{b_{i+1},\ c_i\}$, and $r(A, \gamma_i) = r(A, a_i)$.

Suppose a labeled space $(E, \mathscr{L}, \mathscr{B})$ is given. To help us build a desingularized labeled space $(F, \mathscr{L}_F, \mathscr{B}_F)$ we will define two new collections of subsets of F^0. Let

$$\mathscr{C} := \{r(A, \beta_i) : A \in \mathscr{B}, i \in \mathbb{N}\} \cup \{\emptyset\},$$

$$\mathscr{D} := \{r(\beta_i) : i \in \mathbb{N}\} \cup \{\emptyset\} = \{r(b_i) : i \in \mathbb{N}\} \cup \{\emptyset\}.$$

One can easily see that each of the sets $\mathscr{C}$ and $\mathscr{D}$ is closed under finite intersection. We provide a few other properties of the elements of $\mathscr{C}$ and $\mathscr{D}$ in the lemma below.

LEMMA 4.9.

1. *For any $A \in \mathscr{B}$ and any $C \in \mathscr{C}$, $A \cap C = \emptyset$.*

2. *For any $A \in \mathscr{B}$ and any $D \in \mathscr{D}$, $A \cap D = \emptyset$.*

3. *If $C_1 = r(A_1, \beta_i)$ and $C_2 = r(A_2, \beta_j)$ then*

$$C_1 \cap C_2 = \begin{cases} \emptyset & \text{if } \ i \neq j \\ r(A_1 \cap A_2, \beta_i) & \text{if } \ i = j \end{cases}$$

   Hence $C_1 \cap C_2 \in \mathscr{C}$.

   Moreover $r(A_1, \beta_i) \cup r(A_2, \beta_j) = r((A_1 \cup A_2), \beta_i)$ if $i = j$. This is useful for avoiding redundancies of the β_i's when writing the union of elements of $\mathscr{C}$.

4. *If $D_1 = r(b_i)$ and $D_2 = r(b_j)$ then*

$$D_1 \cap D_2 = \begin{cases} \emptyset & \text{if } \ i \neq j \\ r(b_i) & \text{if } \ i = j \end{cases}$$

   Hence $D_1 \cap D_2 \in \mathscr{D}$.

   This also implies that the collection $\{r(b_i)\}$ is pairwise disjoint. This is useful for avoiding redundancies of the b_i's when writing the union of elements of $\mathscr{D}$.

5. *If $C = r(A, \beta_i) \in \mathscr{C}$ and $D = r(b_j) \in \mathscr{D}$ then*

$$C \cap D = \begin{cases} \emptyset & \text{if } \ i \neq j \\ C & \text{if } \ i = j \end{cases}$$

   Hence $C \cap D \in \mathscr{C}$.

   Moreover $r(A_1, \beta_i) \cup r(b_j) = r(b_i)$ if $i = j$. This is useful for avoiding redundancies of the indices when writing the union of elements of $\mathscr{C}$ with elements of $\mathscr{D}$.

DEFINITION 4.10. Given a labeled space $(E, \mathscr{L}, \mathscr{B})$, let $(F, \mathscr{L}_F)$ be a desingularization of the labeled graph $(E, \mathscr{L})$. Define $\mathscr{B}_F$ as:

$$\mathscr{B}_F := \{A \cup (\cup_{i=1}^{n} C_i) \cup \left(\cup_{j=1}^{m} D_j\right) : A \in \mathscr{B},\ C_i \in \mathscr{C},\ D_j \in \mathscr{D}\}.$$

For an element $M = A \cup (\cup C_i) \cup (\cup D_j)$, where $C_i = r(A_i, \beta_{r_i})$, $D_j = r(\beta_{r_j})$, we can assume that the collection of indices $\{r_i, r_j\}$ is pairwise unequal, that is, no two β_{r_i}, β_{r_j} are of equal length. This makes the set $A \cup (\cup C_i) \cup (\cup D_j)$ a disjoint union, which helps significantly for showing that $(F,\mathscr{L}_F,\mathscr{B}_F)$ is weakly left–resolving and other proofs down the line.

Now, let $\alpha\in\mathscr{L}^*(F)$, then

$$r(M,\alpha)=\begin{cases} r(A,\alpha) & \text{if } \alpha=b_1\ldots \\ r(C_i,\alpha) & \text{if } \alpha=b_{r_i+1}\ldots \text{ or } \alpha=c_{r_i}\ldots \\ r(D_j,\alpha) & \text{if } \alpha=b_{r_j+1}\ldots \text{ or } \alpha=c_{r_j}\ldots \\ \emptyset & \text{otherwise.}\end{cases}$$

In the next two theorems, we will show that $(F,\mathscr{L}_F,\mathscr{B}_F)$ is a labeled space.

THEOREM 4.11. *Given a labeled space $(E,\mathscr{L},\mathscr{B})$, let $(F,\mathscr{L}_F)$ be a desingularization of the labeled graph $(E,\mathscr{L})$. If $(E,\mathscr{L},\mathscr{B})$ is weakly left–resolving, then so is $(F,\mathscr{L}_F,\mathscr{B}_F)$.*

Proof. In order to prove that $(F,\mathscr{L}_F,\mathscr{B}_F)$ is weakly left–resolving, we need to show that $r(M\cap N,\alpha)=r(M,\alpha)\cap r(N,\alpha)$ for any $M,N\in\mathscr{B}_F$ and any $\alpha\in\mathscr{L}^*(F)$.

Let $\alpha\in\mathscr{L}^*(F)$, and let $M=A\cup(\cup C_i)\cup(\cup D_j)$, $N=B\cup(\cup C'_k)\cup(\cup D'_l)$, where each union is finite, $C_i=r(A_i,\beta_{r_i})$, $D_j=r(\beta_{s_j})$ and the elements in the collection $\{r_i,s_j\}$ are pairwise unequal. Similarly, $C'_k=r(A'_k,\beta_{d_k}),D'_l=r(\beta_{h_l})$, also the elements in $\{d_k,h_l\}$ are pairwise unequal. To show that $r(M\cap N,\alpha)=r(M,\alpha)\cap r(N,\alpha)$, we will consider following cases:

Case 1: $\alpha=b_1\ldots$, that means α begins in b_1.

Case $1(a)$: $\alpha=b_1\ldots c_{t_d}$, that means α ends in c_{t_d}. Thus $\alpha=\gamma_{t_1}\gamma_{t_2}\ldots\gamma_{t_d}$. Now, $r(M,\alpha)=r(A,\alpha)=r(A,\gamma_{t_1}\gamma_{t_2}\ldots\gamma_{t_d})=r(A,a_{t_1}a_{t_2}\ldots a_{t_d})=r(A,a_{t_1}a_{t_2}\ldots a_{t_d})\in(E,\mathscr{L},\mathscr{B})$.

Similarly, $r(N,\alpha)=r(B,a_{t_1}a_{t_2}\ldots a_{t_d})$

Since $M\cap N=(A\cap B)\cup\big(\cup(C_i\cap C'_k)\big)\cup\big(\cup(C_i\cap D'_l)\big)\cup\big(\cup(D_j\cap C'_k)\big)\cup\big(\cup(D_j\cap D'_l)\big)$, we get

$$\begin{aligned} r(M\cap N,\alpha)&=r(A\cap B,\alpha)\\ &=r(A\cap B,a_{t_1}a_{t_2}\ldots a_{t_d})\\ &=r(A,a_{t_1}a_{t_2}\ldots a_{t_d})\cap r(B,a_{t_1}a_{t_2}\ldots a_{t_d})\\ &=r(M,\alpha)\cap r(N,\alpha).\end{aligned}$$

The third equality is because $(E,\mathscr{L},\mathscr{B})$ is weakly left–resolving.

Case $1(b)$: $\alpha=\gamma_{t_1}\gamma_{t_2}\ldots\gamma_{t_d}b_1b_2\ldots b_r=\gamma_{t_1}\gamma_{t_2}\ldots\gamma_{t_d}\beta_r$.

$$\begin{aligned} r(M,\alpha)=r(A,\alpha)&=r(A,\gamma_{t_1}\gamma_{t_2}\ldots\gamma_{t_d}\beta_r)\\ &=r(r(A,\gamma_{t_1}\gamma_{t_2}\ldots\gamma_{t_d}),\beta_r)\\ &=r(r(A,a_{t_1}a_{t_2}\ldots a_{t_d}),\beta_r).\end{aligned}$$

Similarly, $r(N,\alpha) = r(r(B,a_{t_1}a_{t_2}\dots a_{t_d}),\beta_r)$. Also, $r(M\cap N,\alpha) = r(r(A\cap B,a_{t_1}a_{t_2}\dots a_{t_d}),\beta_r)$. Let $A' = r(A,a_{t_1}a_{t_2}\dots a_{t_d})$, $B' = r(B,a_{t_1}a_{t_2}\dots a_{t_d})$. From Case 1(a), we get that $r(A\cap B,a_{t_1}a_{t_2}\dots a_{t_d}) = A'\cap B'$. Also, we have that $r(M,\alpha) = r(A',\beta_r)$, $r(N,\alpha) = r(B',\beta_r)$, and $r(M\cap N,\alpha) = r(A'\cap B',\beta_r)$. Our goal now is to show that $r(A'\cap B',\beta_r) = r(A',\beta_r)\cap r(B',\beta_r)$. For this, it is enough to show that $r(A',\beta_r)\cap r(B',\beta_r)\subseteq r(A'\cap B',\beta_r)$. Let $t\in r(A',\beta_r)\cap r(B',\beta_r)$. This implies that there exists a unique $x\in A'$ such that $t = v_x^r$. Also, since $t\in r(B',\beta_r)$, we get $x\in B'$. Therefore, $x\in A'\cap B'$ and $t = v_x^r$. This implies that $t\in r(A'\cap B',\beta_r) = r(M\cap N,\alpha)$. Thus, $r(M,\alpha)\cap r(N,\alpha)\subseteq r(M\cap N,\alpha)$. Make a note that the reverse containment is always true. From 1(a) and 1(b), we have that $r(M\cap N) = r(M,\alpha)\cap r(N,\alpha)$ when $\alpha = b_1\dots$

Notice that in particular, $r(A\cap B,\alpha) = r(A,\alpha)\cap r(B,\alpha)$ when $A,B\in\mathscr{B}$ and α begins in b_1.

Case 2: $\alpha = b_n\dots$ or $\alpha = c_{n-1}\dots$ with $n>1$, that means α begins in b_n or c_{n-1} and $n\geqslant 2$.

Then $r(M,\alpha) = \phi$ unless $n-1\in\{r_i,s_j\}$. Similarly, $r(N,\alpha) = \phi$ unless $n-1\in\{d_k,h_l\}$. Also, $r(M\cap N,\alpha) = \phi$ unless $n-1\in\{r_i,s_j\}\cap\{d_k,h_l\}$.

Case $2(a)$: Suppose $n-1\in\{r_i\}\cap\{d_k\}$, say WLOG $n-1 = r_1 = d_1$. Then

$$r(M,\alpha) = r(C_1,\alpha) = r(r(A_1,\beta_{r_1}),\alpha) = r(A_1,\beta_{r_1}\alpha)$$

and

$$r(N,\alpha) = r(C_1',\alpha) = r(r(A_1',\beta_{d_1}),\alpha) = r(r(A_1',\beta_{r_1}),\alpha) = r(A_1',\beta_{r_1}\alpha).$$

Now,

$$\begin{aligned} r(M\cap N,\alpha) = r(C_1\cap C_1',\alpha) &= r(r(A_1\cap A_1',\beta_{r_1}),\alpha)\\ &= r(A_1\cap A_1',\beta_{r_1}\alpha)\\ &= r(A_1,\beta_{r_1}\alpha)\cap r(A_1',\beta_{r_1}\alpha)\\ &= r(M,\alpha)\cap r(N,\alpha). \end{aligned}$$

Case $2(b)$: Suppose $n-1\in\{r_i\}\cap\{h_l\}$, say $n-1 = r_1 = h_1$. Then $r(M,\alpha) = r(r(A_1,\beta_{r_1}),\alpha) = r(A_1,\beta_{r_1}\alpha)$ and $r(N,\alpha) = r(D_1,\alpha) = r(r(\beta_{h_1}),\alpha) = r(\beta_{h_1},\alpha) = r(\beta_{r_1}\alpha)$.

$$\begin{aligned} r(M,\alpha)\cap r(N,\alpha) &= r(A_1,\beta_{r_1}\alpha)\cap r(\beta_{r_1}\alpha) = r(A_1,\beta_{r_1}\alpha).\\ r(M\cap N,\alpha) &= r(C_1\cap D_1',\alpha) = r(C_1,\alpha) = r(A_1,\beta_{r_1}\alpha). \end{aligned}$$

So, $r(M\cap N,\alpha) = r(M,\alpha)\cap r(N,\alpha)$.

Case $2(c)$: The case $n-1\in\{s_j\}\cap\{d_k\}$ is symmetrical to 2(b).

Case $2(d)$: Suppose $n-1\in\{s_j\}\cap\{h_l\}$, say $n-1 = s_1 = h_1$. Then $r(M,\alpha) = r(D_1,\alpha) = r(r(\beta_{r_1}),\alpha) = r(\beta_{r_1}\alpha)$. Similarly, $r(N,\alpha) = r(\beta_{r_1}\alpha)$. Hence, $r(M\cap N,\alpha) = r(M,\alpha)\cap r(N,\alpha)$. $\square$

THEOREM 4.12. *Given a labeled space $(E,\mathscr{L},\mathscr{B})$, let $(F,\mathscr{L}_F)$ be a desingularization of the labeled graph $(E,\mathscr{L})$. $\mathscr{B}_F$ is closed under finite unions, finite intersections, and relative ranges. This makes $(F,\mathscr{L}_F,\mathscr{B}_F)$ a labeled space. Moreover, $\mathscr{B}_F$ contains no singular sets.*

Proof. It is trivial that $\mathscr{B}_F$ is closed under finite unions.

We show that $\mathscr{B}_F$ is closed under finite intersections. For this, let $M = A\cup(\cup C_i)\cup(\cup D_j)$ and $N = B\cup\left(\cup C'_k\right)\cup\left(\cup D'_l\right)$, where each union is finite.

Then $M\cap N = (A\cap B)\cup\left(\cup(C_i\cap C'_k)\right)\cup\left(\cup(C_i\cap D'_l)\right)\cup\left(\cup(D_i\cap D'_j)\right)$ which is in $\mathscr{B}_F$.

Next, we show that $\mathscr{B}_F$ is closed under relative ranges. Since, $r(A\cup B,\alpha) = r(A,\alpha)\cup r(B,\alpha)$, for any sets A, $B\in\mathscr{B}$ and any word α (in any labeled graph), it suffices to show that the set $\mathscr{B}\cup\mathscr{C}\cup\mathscr{D}$ is closed under relative ranges. Let α be a fixed word and we consider sets $B\in\mathscr{B}$, $C\in\mathscr{C}$, and $D\in\mathscr{D}$, and we show that $r(B,\alpha), r(C,\alpha)$, and $r(D,\alpha)\ \in\mathscr{B}\cup\mathscr{C}\cup\mathscr{D}$.

Any fixed word α may have following representations:

- $\alpha_1 = b_n b_{n+1}\dots b_{k_1}c_{k_1}b_1b_2\dots b_{k_2}c_{k_2}\ \dots\ b_1b_2\dots b_{k_j}c_{k_j}b_1b_2\dots b_{r_i}$ $= b_n b_{n+1}\dots b_{k_1}c_{k_1}\gamma_{k_2}\dots\gamma_{k_j}\beta_i$
- $\alpha_2 = c_{k_1}\gamma_{k_2}\gamma_{k_3}\dots\gamma_{k_n}\beta_i$
- $\alpha_3 = b_n b_{n+1}\dots b_{k_1}c_{k_1}\gamma_{k_2}\gamma_{k_3}\dots\gamma_{k_j}$
- $\alpha_4 = c_{k_1}\gamma_{k_2}\gamma_{k_3}\dots\gamma_{k_j}$

We know that, if $n\neq 1$ in α_1, $r(B,\alpha_1)=\phi$. Otherwise,

$$r(B,\alpha_1) = r(r(B,\gamma_{k_1}\gamma_{k_2}\dots\gamma_{k_j}),\beta_{r_i}) = r(A,\beta_{r_i})\in\mathscr{C},$$

where $A = r(B,\gamma_{k_1}\gamma_{k_2}\dots\gamma_{k_j})\in\mathscr{B}$. Also, $r(B,\alpha_2)=\phi$. Moreover, if $n\neq 1$ in α_3, $r(B,\alpha_3)=\phi$. Otherwise, $r(B,\alpha_3) = r(B,a_{k_1}a_{k_2}\dots a_{k_j})\in\mathscr{B}$. And, $r(B,\alpha_4)=\phi$.

Similarly, for any $i\in\{1,2,3,4\}$, $r(C,\alpha_i)$ lies in $\mathscr{B}$ or in $\mathscr{C}$ and $r(D,\alpha_i)$ lies in $\mathscr{B}$, in $\mathscr{C}$ or in $\mathscr{D}$.

That $\mathscr{B}_F$ has no singular sets follows from the fact that the graph F has no sinks and

— For any set $A\in\mathscr{B}$, $\mathscr{L}_F(AF^1)=\{b_1\}$,

— For any set $C = r(A,\beta_i)$, $\emptyset\neq\mathscr{L}_F(CF^1)\subseteq\{b_{i+1},c_i\}$,

— For any set $D = r(b_i)$, $\emptyset\neq\mathscr{L}_F(DF^1)\subseteq\{b_{i+1},c_i\}$, and

— For any set $A\in\mathscr{B}$, $r(A,\gamma_i)\in\mathscr{B}$. $\square$

DEFINITION 4.13. Given a labeled space $(E,\mathscr{L},\mathscr{B})$, let $(F,\mathscr{L}_F)$ be a desingularization of the labeled graph $(E,\mathscr{L})$. We call $(F,\mathscr{L}_F,\mathscr{B}_F)$ a desingularization of the labeled space $(E,\mathscr{L},\mathscr{B})$.

THEOREM 4.14. *Suppose $(E,\mathcal{L},\mathcal{B})$ is a weakly left–resolving labeled space and suppose $(F,\mathcal{L}_F,\mathcal{B}_F)$ is a desingularization of $(E,\mathcal{L},\mathcal{B})$. If a set of projections $\{Q_A : A\in\mathcal{B}_F\}$ and partial isometries $\{T_a : a\in\mathcal{A}_F\}$ forms a representation of $(F,\mathcal{L}_F,\mathcal{B}_F)$, then there is a representation of projections $\{P_A : A\in\mathcal{B}\}$ and partial isometries $\{S_a : a\in\mathcal{A}\}$ of the labeled space $(E,\mathcal{L},\mathcal{B})$ in $C^*(F,\mathcal{L}_F,\mathcal{B}_F)$.*

Proof. For each $A\in\mathcal{B}$, define $P_A := Q_A$, and for each $a_i\in\mathcal{A}$ define $S_{a_i} := T_{\gamma_i}$.

1. Given A and B in $\mathcal{B}$, $P_AP_B = P_{A\cap B}$, and $P_{A\cup B} = P_A+P_B-P_{A\cap B}$ follows trivially.

2. For $a_i,a_j\in\mathcal{A}$, $S^*_{a_i}S_{a_j} = T^*_{\gamma_i}T_{\gamma_j} = Q_{r(\gamma_i)}\delta_{i,j} = P_{r(a_i)}\delta_{i,j}$ follows from [1, Lemma 3.2].

3. For any $a_i\in\mathcal{A}$ and $A\in\mathcal{B}$, $S^*_{a_i}P_A = T^*_{\gamma_i}Q_A = Q_{r(A,\gamma_i)}T^*_{\gamma_i} = P_{r(A,a_i)}S^*_{a_i}$.

4. Suppose $A\in\mathcal{B}$ with $\mathcal{L}(AE^1)$ finite, and $A\cap B=\emptyset$ for all $B\in\mathcal{B}$ satisfying $B\subseteq E^0_{sink}$. Let $\mathcal{L}(AE^1)=\{a_{k_1},a_{k_2},\dots a_{k_n}\}$ with $k_1<k_2<\dots<k_n$. Then

$$
\begin{aligned}
P_A &= Q_A\\
&= T_{\beta_{k_1}}Q_{r(A,\beta_{k_1})}T^*_{\beta_{k_1}} \quad \text{[by repeated use of (CK-5)]}\\
&= T_{\gamma_{k_1}}Q_{r(A,\gamma_{k_1})}T^*_{\gamma_{k_1}} + T_{\beta_{k_1+1}}Q_{r(A,\beta_{k_1+1})}T^*_{\beta_{k_1+1}}\\
&= T_{\gamma_{k_1}}Q_{r(A,\gamma_{k_1})}T^*_{\gamma_{k_1}} + T_{\beta_{k_2}}Q_{r(A,\beta_{k_2})}T^*_{\beta_{k_2}}\\
&= T_{\gamma_{k_1}}Q_{r(A,\gamma_{k_1})}T^*_{\gamma_{k_1}} + T_{\gamma_{k_2}}Q_{r(A,\gamma_{k_2})}T^*_{\gamma_{k_2}} + T_{\beta_{k_2+1}}Q_{r(A,\beta_{k_2+1})}T^*_{\beta_{k_2+1}}\\
&\ \ \vdots\\
&= \sum_{i=1}^{n-1} T_{\gamma_{k_i}}Q_{r(A,\gamma_{k_i})}T^*_{\gamma_{k_i}} + T_{\beta_{k_n}}Q_{r(A,\beta_{k_n})}T^*_{\beta_{k_n}}\\
&= \sum_{i=1}^{n-1} T_{\gamma_{k_i}}Q_{r(A,\gamma_{k_i})}T^*_{\gamma_{k_i}} + T_{\gamma_{k_n}}Q_{r(A,\gamma_{k_n})}T^*_{\gamma_{k_n}}\\
&= \sum_{i=1}^{n} T_{\gamma_{k_i}}Q_{r(A,\gamma_{k_i})}T^*_{\gamma_{k_i}}\\
&= \sum_{i=1}^{n} S_{a_{k_i}}P_{r(A,a_{k_i})}S^*_{a_{k_i}},
\end{aligned}
$$

as desired. □

LEMMA 4.15. *Let $(E,\mathcal{L},\mathcal{B})$ be a labeled space where $\mathcal{A}=\{a_i : i=1,\dots,N\}$. Suppose $\{S_a,P_A : a\in\mathcal{A},\ A\in\mathcal{B}\}$ is a representation of $(E,\mathcal{L},\mathcal{B})$ on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}_n := \left\{h-\sum_{i=1}^{n-1}S_{a_i}S^*_{a_i}(h) : h\in\mathcal{H}\right\}$. If $A\in\mathcal{B}$ and $k\in\mathcal{H}_n$, then $P_A(k)\in\mathcal{H}_n$.*

Proof. If $k = h - \sum_{i=1}^{n-1} S_{a_i} S_{a_i}^*(h)$ then

$$\begin{aligned} P_A(k) &= P_A(h - \sum_{i=1}^{n-1} S_{a_i} S_{a_i}^*(h)) \\ &= P_A(h) - \sum_{i=1}^{n-1} P_A(S_{a_i} S_{a_i}^*(h)) \\ &= P_A(h) - \sum_{i=1}^{n-1} S_{a_i} S_{a_i}^*(P_A(h)) \end{aligned}$$

which is in $\mathscr{H}_n$; this is because $P_A S_a S_a^* = S_a P_{r(A,a)} S_a^* = S_a S_a^* P_A$. Therefore $P_A(k) \in \mathscr{H}_n$. $\square$

The following theorem is analogous to [14, Lemma 6.4 and Lemma 6.5]. However the proofs are different because the desingularization process of an ultragraph is different from the desingularization process of a labeled graph.

THEOREM 4.16. *Let $(E,\mathscr{L},\mathscr{B})$ be a labeled space and let $(F,\mathscr{L}_F,\mathscr{B}_F)$ be a desingularized labeled space of $(E,\mathscr{L},\mathscr{B})$. For every representation $\{S_a, P_A : a \in \mathscr{A}, A \in \mathscr{B}\}$ on a Hilbert space $\mathscr{H}$, there exists a Hilbert space $\mathscr{H}_F = \mathscr{H} \oplus \mathscr{H}_T$ and a representation $\{T_a, Q_M : a \in \mathscr{A}_F, M \in \mathscr{B}_F\}$ on $\mathscr{H}_F$ satisfying:*

1. *$P_A = Q_A$ for every $A \in \mathscr{B}$,*
2. *$S_{a_i} = T_{\gamma_i}$ for every $i = 1, \ldots, N$,*
3. *$P := \sum_i Q_{r(b_i)}$ is the projection of $\mathscr{H}_F$ onto $\mathscr{H}_T$.*

Proof. Let $\mathscr{H}_1 := \mathscr{H}$ and for $n = 2, \ldots$, let $\mathscr{H}_n := \left\{ h - \sum_{i=1}^{n-1} S_{a_i} S_{a_i}^*(h) : h \in \mathscr{H} \right\}$. Define $\mathscr{H}_F := \mathscr{H} \oplus (\oplus_{n=1}^{\infty} \mathscr{H}_n)$. For $A \in \mathscr{B}$, $i = 1, \ldots, N$, and $(h, k_1, k_2, \ldots) \in \mathscr{H}_F$ define the following projections:

$$\begin{aligned} Q_A(h,k_1,k_2,\ldots) &:= (P_A(h),0,0,\ldots) \\ Q_{r(b_i)}(h,k_1,k_2,\ldots) &:= (0,\ldots,0,k_i,0,\ldots) \\ Q_{r(A,\beta_i)}(h,k_1,k_2,\ldots) &:= (0,\ldots,0,P_A(k_i),0,\ldots). \end{aligned}$$

In the last two cases, the nonzero terms appear in the $\mathscr{H}_i$ component. As usual, a projection associated with a disjoint union is defined as the sum of the projections associated with the individual sets.

The projection $Q_{r(A,\beta_i)}$ is valid since $P_A(k_i) \in \mathscr{H}_i$, by Lemma 4.15.

We are using the fact that the elements of $\mathscr{B}_F$ can be written as disjoint unions of finite numbers of elements of $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ when defining projections associated with elements of $\mathscr{B}_F$.

First we define the generating partial isometries.

$$T_{b_i}(h,k_1,k_2,\ldots) := (0,\ldots,0,k_i,0,\ldots),$$

where the nonzero term appears in the $\mathscr{H}_{i-1}$ component. A straight-forward calculation yields that

$$T^*_{b_1}(h,k_1,k_2,\ldots) = (0,h,0,0,\ldots),$$

and

$$T^*_{b_i}(h,k_1,k_2,\ldots) = (0,\ldots,0,k_{i-1} - S_{a_{i-1}}S^*_{a_{i-1}}(k_{i-1}),0,\ldots),$$

where the nonzero term is in the $\mathscr{H}_i$ component.

$$T_{c_i}(h,k_1,k_2,\ldots) := (0,\ldots,0,S_{a_i}(h),0,\ldots),$$

where the nonzero term is the $\mathscr{H}_i$ component; it is easy to see that $S_{a_i}(h) \in \mathscr{H}_i$. This gives us $T^*_{c_i}(h,k_1,k_2,\ldots) = (S^*_{a_i}(k_i),0,0,\ldots)$.

Combining the above definitions, we get

$$T_{\gamma_i}(h,k_1,k_2,\ldots) = (S_{a_i}(h),0,0,\ldots),$$

$$T_{\beta_i}(h,k_1,k_2,\ldots) = (k_i,0,0,\ldots),$$

$$T_{\gamma_i}(h,k_1,k_2,\ldots) = (S_{a_i}(h),0,0,\ldots),$$

and

$$T^*_{\beta_i}(h,k_1,k_2,\ldots) = \left(0,\ldots,0,h - \sum_{k=1}^{i-1} S_{a_k}S^*_{a_k}(h),0,\ldots\right),$$

where the nonzero term is in the $\mathscr{H}_i$ component.

Next we prove that the collection $\{T_a, Q_M : a \in \mathscr{A}_F, M \in \mathscr{B}_F\}$ forms a representation for $(F,\mathscr{L}_F,\mathscr{B}_F)$ by showing that it satisfies (CK-1) – (CK-4) of definition (3.1).

To prove (CK-1), we prove that for all sets $M,N \in \mathscr{B}_F$, $Q_{M\cap N} = Q_M Q_N$ and $Q_{M\cup N} = Q_M + Q_N - Q_{M\cap N}$. To prove that $Q_{M\cap N} = Q_M Q_N$ we will first prove that the property holds for simple sets in $\mathscr{B}\cup\mathscr{C}\cup\mathscr{D}$, then we take $M,N \in \mathscr{B}_F$, written as disjoint unions and show that the property holds.

For $A,B \in \mathscr{B}$, $Q_A Q_B = Q_{A\cap B}$ follows easily.

If $C = r(B,\beta_i)$,

$$\begin{aligned} Q_A Q_C(h,k_1,k_2,\ldots) &= Q_A Q_{r(B,\beta_i)}(h,k_1,k_2,\ldots) = Q_A(0,\ldots,0,P_B(k_i),0,\ldots) \\ &= (P_A(0),0,\ldots) = 0 = Q_{A\cap r(B,\beta_i)}(h,k_1,k_2,\ldots). \end{aligned}$$

That is, $Q_A Q_C = 0 = Q_{A\cap C}$.

If $D = r(b_i)$,

$$\begin{aligned} Q_A Q_D(h,k_1,k_2,\ldots) &= Q_A Q_{r(b_i)}(h,k_1,k_2,\ldots) = Q_A(0,\ldots,0,k_i,0,\ldots) \\ &= (P_A(0),0,\ldots) = 0 = Q_{A\cap r(B,b_i)}(h,k_1,k_2,\ldots). \end{aligned}$$

That is, $Q_A Q_D = 0 = Q_{A\cap D}$.

$$\begin{aligned}
Q_{r(A,\beta_i)}Q_{r(B,\beta_j)}(h,k_1,k_2,\dots) &= Q_{r(A,\beta_i)}(0,\dots,0,P_B(k_j),0,\dots)\\
&= \begin{cases}(0,\dots,P_A(0),0,\dots) & \text{if } \; i\neq j\\ (0,\dots,0,P_AP_B(k_i),0,\dots) & \text{if } \; i=j\end{cases}\\
&= \begin{cases}0 & \text{if } \; i\neq j\\ (0,\dots,0,P_{A\cap B}(k_i),0,\dots) & \text{if } \; i=j\end{cases}\\
&= Q_{r(A,\beta_i)\cap r(B,\beta_j)}(h,k_1,k_2,\dots).
\end{aligned}$$

$$\begin{aligned}
Q_{r(A,\beta_i)}Q_{r(b_j)}(h,k_1,k_2,\dots) &= Q_{r(A,\beta_i)}(0,\dots,0,k_j,0,\dots)\\
&= \begin{cases}(0,\dots,P_A(0),0,\dots) & \text{if } \; i\neq j\\ (0,\dots,0,P_A(k_i),0,\dots) & \text{if } \; i=j\end{cases}\\
&= Q_{r(A,\beta_i)\cap r(b_j)}(h,k_1,k_2,\dots).
\end{aligned}$$

$$\begin{aligned}
Q_{r(b_i)}Q_{r(b_j)}(h,k_1,k_2,\dots) &= Q_{r(b_i)}(0,\dots,0,k_j,0,\dots)\\
&= \begin{cases}0 & \text{if } \; i\neq j\\ (0,\dots,0,k_i,0,\dots) & \text{if } \; i=j\end{cases}\\
&= Q_{r(b_i)\cap r(b_j)}(h,k_1,k_2,\dots).
\end{aligned}$$

Now, let $M = A\cup(\cup_{i=1}^{n_1}C_i)\cup(\cup_{j=1}^{m_1}D_j)$, $N = B\cup(\cup_{k=1}^{n_2}C'_k)\cup(\cup_{l=1}^{m_2}D'_l)$, where each union is finite, and let $C_i = r(A_i,\beta_{r_i})$, $D_j = r(\beta_{s_j})$ where the elements in the collection $\{r_i,s_j\}$ are pairwise unequal. Similarly, $C'_k = r(A'_k,\beta_{d_k})$, $D'_l = r(\beta_{h_l})$, with the elements in $\{d_k,h_l\}$ pairwise unequal. Thus,

$$Q_M = Q_A+\sum_{i=1}^{n_1}Q_{C_i}+\sum_{j=1}^{m_1}Q_{D_j} \quad \text{and} \quad Q_N = Q_B+\sum_{k=1}^{n_2}Q_{C'_k}+\sum_{l=1}^{m_2}Q_{D'_l}.$$

Then

$$\begin{aligned}
Q_MQ_N &= \left[Q_A+\sum_{i=1}^{n_1}Q_{C_i}+\sum_{j=1}^{m_1}Q_{D_j}\right]\cdot\left[Q_B+\sum_{k=1}^{n_2}Q_{C'_k}+\sum_{l=1}^{m_2}Q_{D'_l}\right]\\
&= Q_AQ_B+\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}Q_{C_i}Q_{C'_k}+\sum_{i=1}^{n_1}\sum_{l=1}^{m_2}Q_{C_i}Q_{D'_l}+\sum_{j=1}^{m_1}\sum_{l=1}^{m_2}Q_{D_j}Q_{D'_l}\\
&= Q_{A\cap B}+\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}Q_{C_i\cap C'_k}+\sum_{i=1}^{n_1}\sum_{l=1}^{m_2}Q_{C_i\cap D'_l}+\sum_{j=1}^{m_1}\sum_{l=1}^{m_2}Q_{D_j\cap D'_l}.
\end{aligned}$$

On the other hand,

$$\begin{aligned} M\cap N &= \Big[A\cup(\cup_{i=1}^{n_1}C_i)\cup(\cup_{j=1}^{m_1}D_j)\Big]\cap\big[B\cup(\cup_{k=1}^{n_2}C'_k)\cup(\cup_{l=1}^{m_2}D'_l)\big] \\ &= (A\cap B)\cup\big(\cup_{i=1}^{n_1}\cup_{k=1}^{n_2}C_i\cap C'_k\big)\cup\big(\cup_{i=1}^{n_1}\cup_{l=1}^{m_2}C_i\cap D'_l\big) \\ &\quad\cup\Big(\cup_{j=1}^{m_2}\cup_{l=1}^{m_2}D_j\cap D'_l\Big). \end{aligned}$$

Since this is a disjoint union, we get:

$$Q_{M\cap N}=Q_{A\cap B}+\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}Q_{C_i\cap C'_k}+\sum_{i=1}^{n_1}\sum_{l=1}^{m_2}Q_{C_i\cap D'_l}+\sum_{j=1}^{m_1}\sum_{l=1}^{m_2}Q_{D_j\cap D'_l}.$$

This shows that $Q_MQ_N=Q_{M\cap N}$.

Next, we prove the second part of (CK-1). i.e., $Q_{M\cup N}=Q_M+Q_N-Q_{M\cap N}$. We implement a similar strategy as before; we first prove that the property holds for simple sets in $\mathscr{B}\cup\mathscr{C}\cup\mathscr{D}$ and then prove it for general elements of $\mathscr{B}_F$ written as disjoint unions.

For $A,B\in\mathscr{B}$, $Q_{A\cup B}=Q_A+Q_B-Q_{A\cap B}$ follows easily.

For $A\in\mathscr{B}$ and $C\in\mathscr{C}$, $Q_{A\cup C}=Q_A+Q_C$ follows immediately from the definition since A and C are disjoint.

Similarly, for $A\in\mathscr{B}$ and $D\in\mathscr{D}$, $Q_{A\cup D}=Q_A+Q_D$ follows immediately.

For $C\in\mathscr{C}$ and $D\in\mathscr{D}$,

$$\begin{aligned} Q_{C\cup D} &= \begin{cases} Q_C+Q_D & \text{if } C\cap D=\emptyset \\ Q_D & \text{if } C\subseteq D \end{cases} \\ &= Q_C+Q_D-Q_{C\cap D}. \end{aligned}$$

For $C_1=r(A,\beta_i),C_2=r(B,\beta_j)\in\mathscr{C}$, if $i\neq j$, then $Q_{C_1\cup C_2}=Q_{C_1}+Q_{C_2}$ follows from the fact that $C_1\cap C_2=\emptyset$. If $i=j$ then $C_1\cup C_2=r(A\cup B,\beta_i)$. Therefore $Q_{C_1\cup C_2}(h,k_1,k_2,\ldots)=(0,0,\ldots,P_{A\cup B}(k_i),0,\ldots)=(0,0,\ldots,(P_A+P_B-P_{A\cap B})(k_i),0,\ldots)=(Q_{C_1}+Q_{C_2}-Q_{C_1\cap C_2})(h,k_1,k_2,\ldots)$. Therefore $Q_{C_1\cup C_2}=Q_{C_1}+Q_{C_2}-Q_{C_1\cap C_2}$.

Similarly, if $D_1,D_2\in\mathscr{D}$, we get $Q_{D_1\cup D_2}=Q_{D_1}+Q_{D_2}-Q_{D_1\cap D_2}$.

Now, for the general case, let $M=A\cup(\cup_{i=1}^{n_1}C_{k_i})\cup(\cup_{i=1}^{m_1}D_{l_i})$, where $C_{k_i}=r(A_{k_i},\beta_{k_i})$, $D_{l_i}=r(b_{l_i})$, and M is written as a disjoint union of A, the C_{k_i}'s, $i=1,\ldots,n_1$, and the D_{l_i}'s, $i=1,\ldots,m_1$. Let $N=A'\cup(\cup_{j=1}^{n_2}C'_{r_j})\cup(\cup_{j=1}^{m_2}D'_{s_j})$, also a disjoint union with similar assumptions.

Let $T_1=\{k_i\}\cap\{r_j\}$, $T_2=\{k_i\}\cap\{s_j\}$, $T_3=\{l_i\}\cap\{r_j\}$, $T_4=\{l_i\}\cap\{s_j\}$, and let $T=T_1\cup T_2\cup T_3\cup T_4$. Then $M\cap N=(A\cap A')\cup\Big(\cup_{k_i=r_j\in T_1}(C_{k_i}\cap C'_{r_j})\Big)\cup\Big(\cup_{k_i=s_j\in T_2}(C_{k_i}\cap D'_{s_j})\Big)\cup\Big(\cup_{l_i=r_j\in T_3}(D_{l_i}\cap C'_{r_j})\Big)\cup\Big(\cup_{l_i=s_j\in T_3}(D_{l_i}\cap D'_{s_j})\Big)$, where these are disjoint unions. Also $M\cup N=(A\cup A')\cup\big(\cup_{k_i\notin T}C_{k_i}\big)\cup\big(\cup_{l_i\notin T}D_{l_i}\big)\cup\Big(\cup_{r_j\notin T}C'_{r_j}\Big)\cup\Big(\cup_{r_j\notin T}C'_{r_j}\Big)\cup\Big(\cup_{s_j\notin T}D'_{s_j}\Big)\cup\Big(\cup_{k_i=r_j\in T_1}(C_{k_i}\cup C'_{r_j})\Big)\cup\Big(\cup_{k_i=s_j\in T_2}(C_{k_i}\cup D'_{s_j})\Big)\cup$

$\left(\cup_{l_i=r_j\in T_3}(D_{l_i}\cup C'_{r_j})\right)\cup\left(\cup_{l_i=s_j\in T_3}(D_{l_i}\cup D'_{s_j})\right)$. This is also a disjoint union, except possibly for the inner pairs. Observe that

$$\begin{aligned} Q_{M\cap N} = Q_{A\cap A'} &+ \left(\sum_{k_i=r_j\in T_1} Q_{C_{k_i}\cap C'_{r_j}}\right) + \left(\sum_{k_i=s_j\in T_2} Q_{C_{k_i}\cap D'_{s_j}}\right) \\ &+ \left(\sum_{l_i=r_j\in T_3} Q_{D_{l_i}\cap C'_{r_j}}\right) + \left(\sum_{l_i=s_j\in T_3} Q_{D_{l_i}\cap D'_{s_j}}\right). \end{aligned}$$

Computing $Q_{M\cup N}$ we get:

$$\begin{aligned} Q_{M\cup N} &= Q_{A\cup A'} + \sum_{k_i\notin T} Q_{C_{k_i}} + \sum_{l_i\notin T} Q_{D_{l_i}} + \sum_{r_j\notin T} Q_{C'_{r_j}} + \sum_{s_j\notin T} Q_{D'_{s_j}} \\ &\quad + \sum_{k_i=r_j\in T_1} Q_{C_{k_i}\cup C'_{r_j}} + \sum_{k_i=s_j\in T_2} Q_{C_{k_i}\cup D'_{s_j}} \\ &\quad + \sum_{l_i=r_j\in T_3} Q_{D_{l_i}\cup C'_{r_j}} + \sum_{l_i=s_j\in T_4} Q_{D_{l_i}\cup D'_{s_j}} \\ &= (Q_A + Q_{A'} - Q_{A\cap A'}) + \sum_{k_i\notin T} Q_{C_{k_i}} + \sum_{l_i\notin T} Q_{D_{l_i}} \\ &\quad + \sum_{r_j\notin T} Q_{C'_{r_j}} + \sum_{s_j\notin T} Q_{D'_{s_j}} \\ &\quad + \sum_{k_i=r_j\in T_1} [Q_{C_{k_i}} + Q_{C'_{r_j}} - Q_{C_{k_i}\cap C'_{r_j}}] \\ &\quad + \sum_{k_i=s_j\in T_2} [Q_{C_{k_i}} + Q_{D'_{s_j}} - Q_{C_{k_i}\cap D'_{s_j}}] \\ &\quad + \sum_{l_i=r_j\in T_3} [Q_{D_{l_i}} + Q_{C'_{r_j}} - Q_{D_{l_i}\cap C'_{r_j}}] \\ &\quad + \sum_{l_i=s_j\in T_4} [Q_{D_{l_i}} + Q_{D'_{s_j}} - Q_{D_{l_i}\cap D'_{s_j}}] \\ &= Q_A + Q_{A'} + \sum_{k_i\notin T} Q_{C_{k_i}} + \sum_{l_i\notin T} Q_{D_{l_i}} \\ &\quad + \sum_{r_j\notin T} Q_{C'_{r_j}} + \sum_{s_j\notin T} Q_{D'_{s_j}} \\ &\quad + \sum_{k_i=r_j\in T_1} [Q_{C_{k_i}} + Q_{C'_{r_j}}] \\ &\quad + \sum_{k_i=s_j\in T_2} [Q_{C_{k_i}} + Q_{D'_{s_j}} - Q_{C_{k_i}\cap D'_{s_j}}] \\ &\quad + \sum_{l_i=r_j\in T_3} [Q_{D_{l_i}} + Q_{C'_{r_j}} - Q_{D_{l_i}\cap C'_{r_j}}] \\ &\quad + \sum_{l_i=s_j\in T_4} [Q_{D_{l_i}} + Q_{D'_{s_j}} - Q_{D_{l_i}\cap D'_{s_j}}] \end{aligned}$$

$$
\begin{aligned}
&-\Bigg[Q_{A\cap A'} + \sum_{k_i=r_j\in T_1} Q_{C_{k_i}\cap C'_{r_j}} + \sum_{k_i=s_j\in T_2} Q_{C_{k_i}\cap D'_{s_j}} \\
&\quad + \sum_{l_i=r_j\in T_3} Q_{D_{l_i}\cap C'_{r_j}} + \sum_{l_i=s_j\in T_4} Q_{D_{l_i}\cap D'_{s_j}}\Bigg] \\
&= \Big(Q_A + \sum_{k_i} Q_{C_{k_i}} + \sum_{l_i} Q_{D_{l_i}}\Big) + \Big(Q_{A'} + \sum_{r_j} Q_{C'_{r_j}} + \sum_{s_j} Q_{D'_{s_j}}\Big) \\
&\quad -\Bigg[Q_{A\cap A'} + \sum_{k_i=r_j\in T_1} Q_{C_{k_i}\cap C'_{r_j}} + \sum_{k_i=s_j\in T_2} Q_{C_{k_i}\cap D'_{s_j}} \\
&\quad + \sum_{l_i=r_j\in T_3} Q_{D_{l_i}\cap C'_{r_j}} + \sum_{l_i=s_j\in T_4} Q_{D_{l_i}\cap D'_{s_j}}\Bigg] \\
&= Q_M + Q_N - Q_{M\cap N}.
\end{aligned}
$$

This concludes the proof of (CK-1).

To prove (CK-2), we prove that for any $a,\ b \in \mathscr{A}_F$, $T_a^* T_b = Q_{r(a)}\delta_{a,b}$.

For $b_i, b_j \in \mathscr{A}_F$,

$$
\begin{aligned}
T_{b_i}^* T_{b_j}(h,k_1,\ldots) &= T_{b_i}^*(0,\ldots,0,k_j,0,\ldots) \text{ in the } \mathscr{H}_{j-1} \text{ component} \\
&= \begin{cases} 0 & \text{if } \ i \neq j \\ (0,\ldots,0,k_i,0,\ldots) & \text{if } \ i = j;\ \mathscr{H}_i \text{ component} \end{cases} \\
&= Q_{r(bj)}(h,k_1,\ldots).
\end{aligned}
$$

Therefore, $T_{b_i}^* T_{b_j} = \delta_{i,j} Q_{r(b_j)}$, as expected.

For $b_i, c_j \in \mathscr{A}_F$,

$$
\begin{aligned}
T_{b_i}^* T_{c_j}(h,k_1,\ldots) &= T_{b_i}^*(0,\ldots,0,S_{a_j}(h),0,\ldots) \quad \text{in the } \mathscr{H}_j \text{ component} \\
&= \begin{cases} 0 & \text{if } \ i \neq j \\ (0,\ldots,0,S_{a_j}(h) - S_{a_j}S_{a_j^*}(S_{a_j}(h)),0,\ldots) & \text{if } \ i = j \end{cases} \\
&= 0.
\end{aligned}
$$

Therefore, $T_{b_i}^* T_{c_j} = 0$.

Finally, for $c_i, c_j \in \mathscr{A}_F$,

$$
\begin{aligned}
T_{c_i}^* T_{c_j}(h,k_1,\ldots) &= T_{c_i}^*(0,\ldots,0,S_{a_j}(h),0,\ldots) \text{ in the } \mathscr{H}_j \text{ component} \\
&= \begin{cases} 0 & \text{if } \ i \neq j \\ (S_{a_j}^* S_{a_j}(h),0,\ldots) & \text{if } \ i = j \end{cases} \\
&= \begin{cases} 0 & \text{if } \ i \neq j \\ (P_{r(a_i)}(h),0,\ldots) & \text{if } \ i = j. \end{cases}
\end{aligned}
$$

Therefore, $T_{c_i}^* T_{c_j} = \delta_{i,j} Q_{r(c_j)}$.

These three cases give us (CK-2).

To prove (CK-3), it needs to be shown that $T_a^* Q_M = Q_{r(M,a)} T_a^*$ for all $a \in \mathscr{A}_F$ and all $M \in \mathscr{B}_F$. Recall from the definition of $\mathscr{B}_F$ that any $M \in \mathscr{B}_F$ is of the form $M = A \cup (\cup_{i=1}^n C_i) \cup (\cup_{j=1}^m Dj)$ where each $C_i \in \mathscr{C}$ and each $D_j \in \mathscr{D}$. For the proof, we show the following specific results and then get the general result using the fact that $(F, \mathscr{L}_F, \mathscr{B}_F)$ is weakly left–resolving: First we show that for all i,

I. $T_{b_i}^* Q_A = Q_{r(A,b_i)} T_{b_i}^*$ for $A \in \mathscr{B}$,

II. $T_{b_i}^* Q_C = Q_{r(C,b_i)} T_{b_i}^*$ for $C \in \mathscr{C}$, and

III. $T_{b_i}^* Q_D = Q_{r(D,b_i)} T_{b_i}^*$ for $D \in \mathscr{D}$,

and for all k,

IV. $T_{c_k}^* Q_A = Q_{r(A,c_k)} T_{c_k}^*$ for $A \in \mathscr{B}$,

V. $T_{c_k}^* Q_C = Q_{r(C,c_k)} T_{c_k}^*$ for $C \in \mathscr{C}$, and

VI. $T_{c_k}^* Q_D = Q_{r(D,c_k)} T_{c_k}^*$ for $D \in \mathscr{D}$.

Proof of I: First note that if $A \in \mathscr{B}$ then

$$T_{b_1}^* Q_A(h, k_1, k_2, \ldots) = T_{b_1}^*(P_A(h), 0, 0, \ldots) = (0, P_A(h), 0, \ldots)$$

and

$$Q_{r(A,b_1)} T_{b_1}^*(h, k_1, k_2, \ldots) = Q_{r(A,b_1)}(0, h, 0, \ldots) = (0, P_A(h), 0, \ldots)$$

so that $T_{b_1}^* Q_A = Q_{r(A,b_1)} T_{b_1}^*$. Also, for $i \geqslant 2$,

$$T_{b_i}^* Q_A(h, k_1, k_2, \ldots) = T_{b_i}^*(P_A(h), 0, 0, \ldots) = 0$$

and

$$Q_{r(A,b_i)} = 0 \text{ (since } r(A, b_i) = \emptyset)$$

so that $T_{b_i}^* Q_A = Q_{r(A,b_i)} T_{b_i}^*$.

Proof of II: Let $C \in \mathscr{C}$. Then $C = r(A, \beta_j)$ for some $A \in \mathscr{B}$ and $j \geqslant 1$. If $i \neq j+1$, then $r(C, b_i) = \emptyset$ so that $Q_{r(C,b_i)} = 0$, which implies that $Q_{r(C,b_i)} T_{b_i}^* = 0$. Also,

$$T_{b_i}^*(Q_C(h, k_1, k_2, \ldots)) = T_{b_i}^*(0, 0, \ldots, P_A(k_j), 0, \ldots, 0)$$

where the nonzero term is in the jth coordinate. But since $i \neq j+1$,

$$T_{b_i}^*(0, 0, \ldots, P_A(k_j), 0, \ldots, 0) = (0, 0, \ldots, 0 - S_{a_i} S_{a_i}^*(0), \ldots, 0),$$

which implies that $T_{b_i}^* Q_C = 0$. Thus, if $i \neq j+1$, $Q_{r(C,b_i)} T_{b_i}^* = T_{b_i}^* Q_C$. Now, assume that $i = j+1$. Then

$$\begin{aligned} T_{b_i}^* Q_C(h, k_1, k_2, \ldots) &= T_{b_{j+1}}^*(0, 0, \ldots, P_A(k_j), \ldots, 0) \\ &\quad \text{(nonzero term in the } j\text{th component)} \\ &= (0, \ldots, P_A(k_j) - S_{a_j} S_{a_j}^*(P_A(k_j)), \ldots, 0) \\ &\quad \text{(nonzero term in the } (j+1)\text{st component).} \end{aligned}$$

Also

$$\begin{aligned}
&Q_{r(C,b_i)}T^*_{b_i}(h,k_1,k_2,\dots)\\
&\qquad = Q_{r(C,b_{j+1})}T^*_{b_{j+1}}(h,k_1,k_2,\dots)\\
&\qquad = Q_{r(A,\beta_j)}(0,\dots,k_j - S_{a_j}S^*_{a_j}(k_j),\dots,0)\\
&\qquad \text{(nonzero term in the } (j+1)\text{st component)}\\
&\qquad = (0,\dots,P_A(k_j)-P_A(S_{a_j}S^*_{a_j}(k_j)),\dots,0)\\
&\qquad \text{(nonzero term in the } (j+1)\text{st component).}
\end{aligned}$$

Thus, if $i=j+1$, $Q_{r(C,b_i)}T^*_{b_i} = T^*_{b_i}Q_C$.

Proof of III: Let $D\in\mathscr{D}$. The proof that $T^*_{b_i}Q_D = Q_{r(D,b_i)}T^*_{b_i}$ is similar to the proof of II.

Proof of IV: First note that if $A\in\mathscr{B}$ then

$$T^*_{c_k}Q_A(h,k_1,k_2,\dots) = T^*_{c_k}(P_A(h),0,\dots) = (S^*_{a_k}(0),0,\dots) = 0$$

and

$$Q_{r(A,c_k)} = 0 \text{ since } r(A,c_k)=\emptyset$$

so that

$$T^*_{c_k}Q_A = 0 = Q_{r(A,c_k)}.$$

Proof of V: Let $C\in\mathscr{C}$ so that $C=r(A,\beta_j)$ for some $A\in\mathscr{B}$ and $j\geqslant 1$.

$$\begin{aligned}
T^*_{c_i}Q_C(h,k_1,k_2,\dots) &= T^*_{c_i}(0,\dots,P_A(k_j),\dots,0) \text{ (nonzero term in } j\text{th coordinate)}\\
&= \begin{cases}(S^*_{a_i}P_A(k_i),0,\dots,0) & \text{if } i=j\\ 0 & \text{else.}\end{cases}
\end{aligned}$$

Also

$$\begin{aligned}
Q_{r(C,c_i)}T^*_{c_i}(h,k_1,k_2,\dots) &= Q_{r(r(A,\beta_j),c_i)}T^*_{c_i}(h,k_1,k_2,\dots)\\
&= \begin{cases}Q_{r(A,b_1\dots b_jc_i)}T^*_{c_i}(h,k_1,k_2,\dots) & \text{if } i=j\\ 0 & \text{else}\end{cases}\\
&= \begin{cases}Q_{r(A,\gamma_i)}(S^*_{a_i}(k_i),0,\dots,0) & \text{if } i=j\\ 0 & \text{else}\end{cases}\\
&= \begin{cases}Q_{r(A,a_i)}(S^*_{a_i}(k_i),0,\dots,0) & \text{if } i=j\\ 0 & \text{else}\end{cases}\\
&= \begin{cases}(P_{r(A,a_i)}S^*_{a_i}(k_i),0,\dots,0) & \text{if } i=j\\ 0 & \text{else.}\end{cases}
\end{aligned}$$

Therefore, $T_{c_i}^* Q_C = Q_{r(C,c_i)} T_{c_i}^*$.

Proof of VI: Let $D \in \mathscr{D}$ so that $D = r(b_j)$ for some $j \geqslant 1$.

$$\begin{aligned} T_{c_i}^* Q_D(h,k_1,k_2,\ldots) &= T_{c_i}^* Q_{r(b_j)}(h,k_1,k_2,\ldots) \\ &= T_{c_i}^*(0,\ldots,k_j,\ldots,0) \text{ (in jth coordinate)} \\ &= \begin{cases} (S_{a_i}^*(k_i),0,\ldots,0) & \text{if } i=j \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Also, note that if $i \neq j$, $Q_{r(D,c_i)} = Q_{r(r(b_j),c_i)} = 0$ since $r(r(b_j),c_i) = \emptyset$ when $i \neq j$. Then if $i \neq j$, $Q_{r(D,c_i)} T_{c_i}^* = 0$. Now suppose that $i = j$. Then $r(D,c_i) = r(r(b_i),c_i) = r(r(b_1 \ldots b_i),c_i) = r(b_1 \ldots b_i c_i) = r(\gamma_i) = r(a_i)$. Hence, $Q_{r(D,c_i)} = Q_{r(a_i)}$, which implies that

$$\begin{aligned} Q_{r(D,c_i)} T_{c_i}^*(h,k_1,k_2,\ldots) &= Q_{r(a_i)}(S_{a_i}^*(k_i),0,\ldots,0) \\ &= (P_{r(a_i)} S_{a_i}^*(k_i),0,\ldots,0) \\ &= (S_{a_i}^* S_{a_i} S_{a_i}^*(k_i),0,0,\ldots,0) \\ &= (S_{a_i}^*(k_i),0,0,\ldots,0). \end{aligned}$$

To complete the proof that (CK-3) is satisfied, let $M \in \mathscr{B}_F$ so that, written as a disjoint union,

$$M = A \cup (\cup_{i=1}^n C_i) \cup (\cup_{j=1}^m D_j).$$

Then

$$Q_M = Q_A + \sum_{i=1}^n Q_{C_i} + \sum_{j=1}^m Q_{D_i}.$$

This implies that

$$\begin{aligned} T_a^* Q_M &= T_a^* Q_A + \sum_{i=1}^n T_a^* Q_{C_i} + \sum_{j=1}^m T_a^* Q_{D_i} \\ &= \left[Q_{r(A,a)} + \sum_{i=1}^n Q_{r(C_i,a)} + \sum_{j=1}^m Q_{r(D_i,a)} \right] T_a^* \\ &= Q_{r(M,a) T_a^*}. \end{aligned}$$

To prove (CK-4), it needs to be shown that for all $M \in \mathscr{B}_F$,

$$Q_M = \sum_{a \in \mathscr{L}(MF^1)} T_a Q_{r(M,a)} T_a^*.$$

As in the proof of (CK-3), we prove the following specific cases and then extend to the general result using the fact that $(F, \mathscr{L}_F, \mathscr{B}_F)$ is weakly left–resolving.

I. $M = A$ for some $A \in \mathscr{B}$,

II. $M = C$ for some $C \in \mathscr{C}$ so that $C = r(A, \beta_i)$ for some $i \geqslant 1$, and

III. $M = D$ for some $D \in \mathscr{D}$ so that $D = r(b_j)$ for some $j \geqslant 1$.

Proof of I: If $A \in \mathscr{B}$, then $\mathscr{L}_F(AF^1) = \{b_1\}$. Then $T_{b_1} Q_{r(A,b_1)} T_{b_1}^* = T_{b_1} T_{b_1}^* Q_A = Q_A$.

Proof of II: Suppose $C \in \mathscr{C}$ so that $C = r(A, \beta_i)$ for some $i \geqslant 1$.

If $a_i \in \mathscr{L}(AE^1)$, then $\mathscr{L}_F(r(A,\beta_i)F^1) = \mathscr{L}_F(r(A, b_1 \dots b_i)F^1) = \{c_i, b_{i+1}\}$. We need to show that

$$Q_{r(A,\beta_i)} = T_{c_i} Q_{r(r(A,\beta_i),c_i)} T_{c_i}^* + T_{b_{i+1}} Q_{r(r(A,\beta_i),b_{i+1})} T_{b_{i+1}}^*.$$

Recall that $\gamma_i = b_1 \dots b_i c_i$. Note:

$$\begin{aligned}
T_{c_i} Q_{r(r(A,\beta_i),c_i)} T_{c_i}^* &= T_{c_i} Q_{r(A,\gamma_i)} T_{c_i}^* (h, k_1, k_2, \dots) \\
&= T_{c_i} Q_{r(A,\gamma_i)} (S_{a_i}^*(k_i), 0, 0, \dots) \\
&= T_{c_i} (P_{r(A,a_i)} S_{a_i}^*(k_i), 0, 0, \dots) \\
&= (0, \dots, S_{a_i} P_{r(A,a_i)} S_{a_i}^*(k_i), 0, \dots) \\
&\quad [\text{nonzero term in the } i\text{th component}] \\
&= (0, \dots, P_A S_{a_i} S_{a_i}^*(k_i), 0, \dots) \\
&\quad [\text{nonzero term in the } i\text{th component}].
\end{aligned}$$

Note also that:

$$\begin{aligned}
&T_{b_{i+1}} Q_{r(A,\beta_{i+1})} T_{b_{i+1}}^* (h, k_1, k_2, \dots) \\
&\qquad = T_{b_{i+1}} Q_{r(A,\beta_{i+1})} (0, \dots, k_i - S_{a_i} S_{a_i}^*(k_i), 0, \dots) \\
&\qquad\quad [\text{nonzero term in the } (i+1)\text{st component}] \\
&\qquad = T_{b_{i+1}} (0, \dots, P_A(k_i) - P_A S_{a_i} S_{a_i}^*(k_i), 0, \dots) \\
&\qquad\quad [\text{nonzero term in the } (i+1)\text{st component}] \\
&\qquad = (0, \dots, P_A(k_i) - P_A S_{a_i} S_{a_i}^*(k_i), 0, \dots) \\
&\qquad\quad [\text{nonzero term in the } i\text{th component}].
\end{aligned}$$

Then

$$\begin{aligned}
&[T_{c_i} Q_{r(A,\gamma_i)} T_{c_i}^* + T_{b_{i+1}} Q_{r(A,\beta_{i+1})} T_{b_{i+1}}^*](h, k_1, k_2, \dots) \\
&\qquad\qquad = (0, \dots, P_A(k_i), 0, \dots) \\
&\qquad\qquad = Q_{r(A,\beta_i)}(h, k_1, k_2, \dots)
\end{aligned}$$

so that

$$T_{c_i} Q_{r(r(A,\beta_i),c_i)} T_{c_i}^* + T_{b_{i+1}} Q_{r(r(A,\beta_i),b_{i+1})} T_{b_{i+1}}^* = Q_{r(A,\beta_i)}.$$

Now assume $a_i \notin \mathscr{L}(AE^1)$. Then $\mathscr{L}_F(r(A,b_i)F^1) = \{b_{i+1}\}$.

$$\begin{aligned}
&T_{b_{i+1}} Q_{r(r(A,\beta_i),b_{i+1})} T^*_{b_{i+1}}(h,k_1,k_2,\dots)\\
&\qquad = T_{b_{i+1}} Q_{r(A,\beta_{i+1})} T^*_{b_{i+1}}(h,k_1,k_2,\dots)\\
&\qquad = T_{b_{i+1}} Q_{r(A,\beta_{i+1})}(0,\dots,k_i - S_{a_i}S^*_{a_i}(k_i),0,\dots)\\
&\qquad \text{[nonzero term in the } (i+1)\text{st component]}\\
&\qquad = T_{b_{i+1}}(0,\dots,P_A(k_i) - P_A S_{a_i}S^*_{a_i}(k_i),0,\dots)\\
&\qquad \text{[nonzero term in the } (i+1)st \text{ component]}\\
&\qquad = (0,\dots,P_A(k_i) - P_A S_{a_i}S^*_{a_i}(k_i),0,\dots)\\
&\qquad \text{[nonzero term in the } i\text{th component].}
\end{aligned}$$

But since $a_i \notin \mathscr{L}(AE^1)$, $P_A S_{a_i}S^*_{a_i} = 0$ so that

$$\begin{aligned}
T_{b_{i+1}} Q_{r(r(A,\beta_i),b_{i+1})} T^*_{b_{i+1}}(h,k_1,k_2,\dots) &= (0,\dots,P_A(k_i),0,\dots)\\
&= Q_{r(A,\beta_i)}(h,k_1,k_2,\dots).
\end{aligned}$$

Therefore, we have the desired result.

Proof of III: Suppose $D \in \mathscr{D}$ so that $D = r(b_j)$ for some $j \geqslant 1$. Since we assume that every $a_j \in \mathscr{A}$ is the label for some edge in E^1, it must be the case that for every $j \geqslant 1$, $\mathscr{L}_F(r(b_j)F^1) = \{b_{j+1}, c_j\}$. We need to show that

$$Q_{r(b_j)} = T_{c_j} Q_{r(r(b_j),c_j)} T^*_{c_j} + T_{b_{j+1}} Q_{r(r(b_j),b_{j+1})} T^*_{b_{j+1}}.$$

First note that

$$\begin{aligned}
&T_{c_j} Q_{r(r(b_j),c_j)} T^*_{c_j}(h,k_1,k_2,\dots)\\
&\qquad = T_{c_j} Q_{r(c_j)} T^*_{c_j}(h,k_1,k_2,\dots)\\
&\qquad = T_{c_j} Q_{r(c_j)}(S^*_{a_j}(k_j),0,0,\dots)\\
&\qquad = T_{c_j}(P_{r(a_j)} S^*_{a_j}(k_j),0,0,\dots)\\
&\qquad = (0,\dots,S_{a_j} P_{r(a_j)} S^*_{a_j}(k_j),0,\dots)\\
&\qquad \text{[nonzero term in the } j\text{th component].}
\end{aligned}$$

But by (CK-2), $P_{r(a_j)} = S^*_{a_j} S_{a_j}$ so that $S_{a_j} P_{r(a_j)} S^*_{a_j} = S_{a_j} S^*_{a_j} S_{a_j} S^*_{a_j} = (S_{a_j} S^*_{a_j})^2 = S_{a_j} S^*_{a_j}$. This gives

$$T_{c_j} Q_{r(r(b_j),c_j)} T^*_{c_j}(h,k_1,k_2,\dots) = (0,\dots,S_{a_j} S^*_{a_j}(k_j),0,\dots).$$

Also,

$$\begin{aligned}
&T_{b_{j+1}} Q_{r(r(b_j),b_{j+1})} T^*_{b_{j+1}}(h,k_1,k_2,\dots)\\
&\qquad = T_{b_{j+1}} Q_{r(b_{j+1})} T^*_{b_{j+1}}(h,k_1,k_2,\dots)
\end{aligned}$$

$$= T_{b_{j+1}} Q_{r(b_{j+1})}(0,\dots,k_j - S_{a_j}S_{a_j}^*(k_j),0,\dots)$$

[nonzero entry in the $(j+1)$st component]

$$= (0,\dots,k_j - S_{a_j}S_{a_j}^*(k_j),0,\dots)$$

[nonzero entry in the jth component].

Putting these together, we have that

$$\begin{aligned} T_{c_j} Q_{r(r(b_j),c_j)} T_{c_j}^*(h,k_1,k_2,\dots) + T_{b_{j+1}} Q_{r(r(b_j),b_{j+1})} T_{b_{j+1}}^*(h,k_1,k_2,\dots) \\ = (0,\dots,k_j,0,\dots) \\ = Q_{r(b_j)}(h,k_1,k_2,\dots). \end{aligned}$$

Now assume that $M \in \mathscr{B}_F$. Let

$$M = A \cup (\cup_{i=1}^n C_i) \cup (\cup_{j=1}^m D_j)$$

be written as a disjoint union. For any $a \in \mathscr{A}_f$, we get $r(M,a) = r(A,a) \cup [\cup_{i=n}^n r(C_i,a)] \cup [\cup_{j=1}^m r(D_j,a)]$. Since the collection $\{A, C_i, D_j : i = 1\dots n,\ j = 1\dots m\}$ is pairwise disjoint and $(F, \mathscr{L}_F, \mathscr{B}_F)$ is weakly left–resolving, the collection $\{r(A,a), r(C_i,a), r(D_j,a) : i = 1\dots n,\ j = 1\dots m\}$ is also pairwise disjoint. Therefore

$$\begin{aligned} Q_{r(M,a)} &= Q_{r(A,a)} + \sum_i Q_{r(C_i,a)} + \sum_j Q_{r(D_j,a)} \\ &= Q_{r(A,a)} + Q_{\cup r(C_i,a)} + Q_{\cup r(D_j,a)} \\ &= Q_{r(A,a)} + Q_{r(\cup C_i,a)} + Q_{r(\cup D_j,a)}. \end{aligned}$$

Then

$$\begin{aligned} Q_M &= Q_A + \sum_{i=1}^n Q_{C_i} + \sum_{j=1}^m Q_{D_i} \\ &= \sum_{a\in\mathscr{L}(AF^1)} T_a Q_{r(A,a)} T_a^* + \sum_{b\in\mathscr{L}[(\cup C_i)F^1]} T_b Q_{r(\cup C_i,b)} T_b^* \\ &\quad + \sum_{c\in\mathscr{L}[(\cup D_j)F^1]} T_c Q_{r(\cup D_j,c)} T_c^* \\ &= \sum_{a\in\mathscr{L}(MF^1)} T_a [Q_{r(A,a)} + Q_{r(\cup C_i,a)} + Q_{r(\cup D_j,a)}] T_a^* \\ &= \sum_{a\in\mathscr{L}(MF^1)} T_a Q_{r(M,a)} T_a^*. \end{aligned}$$

This concludes the proof of (CK-4).

Now we prove (1), (2), and (3) of the theorem.

Identifying P_A with Q_A, for each $A \in \mathscr{B}$, and S_{a_i} with T_{γ_i}, for each $i = 1\dots,N$, (1) and (2) of the theorem follow.

To prove (3), notice that the set of projections $\{Q_{r(b_i)}\}$ is pairwise orthogonal and that the sum $\sum_i Q_{r(b_i)}$ converges to a projection, say P, in the multiplier algebra

$M(C^*(F,\mathcal{L}_F,\mathcal{B}_F))$. Moreover, $P(h,k_1,\ldots)=(0,k_1,\ldots)$. Therefore P is the projection of $\mathcal{H}_F$ onto $\mathcal{H}_T$. $\square$

We recall that for any labeled space $(E,\mathcal{L},\mathcal{B})$ and any two words $\mu,\ \nu\in\mathcal{L}^*(E)$, $s_\mu^* s_\nu=0$ unless one of μ, ν extends the other. Thus, in $(F,\mathcal{L}_F,\mathcal{B}_F)$, if $\mu\in\mathcal{L}^*(F)$, $t_{b_1}^* t_\mu$ is zero unless μ begins with b_1, i.e., $s(\mu)=E^0$. In fact,

$$t_{b_1}t_{b_1}^*t_\mu=\begin{cases} t_\mu \text{ if } \mu=b_1\alpha\\ 0 \text{ otherwise.}\end{cases}$$

Similarly

$$t_\nu^*t_{b_1}t_{b_1}^*=\begin{cases} s_\nu^* \text{ if } \nu=b_1\alpha\\ 0 \text{ otherwise.}\end{cases}$$

LEMMA 4.17. *Let $(E,\mathcal{L},\mathcal{B})$ be a labeled space and suppose $(F,\mathcal{L}_F,\mathcal{B}_F)$ is a desingularised labeled space of $(E,\mathcal{L},\mathcal{B})$. Suppose $\{T_a,Q_A : a\in\mathcal{A}_F,\ A\in\mathcal{B}_F\}$ is a representation of $(F,\mathcal{L}_F,\mathcal{B}_F)$. Let $\{S_{a_i}=T_{\gamma_i},\ P_A : a_i\in\mathcal{A},\ A\in\mathcal{A}\}$ be the representation built as in Theorem 4.14. Let $\mathfrak{A}=C^*\{S_{a_i},P_A\}$.*

1. *If $M=A\cup(\cup_{i=1}^{n_1}C_{k_i})\cup(\cup_{j=1}^{m_1}D_{l_j})\in\mathcal{B}_F$, then $Q_MT_{b1}T_{b_1}^*=Q_A$. Consequently, if $r(\mu)\in\mathcal{B}$ then $T_\mu T_{b_1}T_{b_1}^*=T_\mu$.*

2. *For any $i\geqslant 1$ and any $M\in\mathcal{B}_F$, $T_{\gamma_i}Q_M\in\mathfrak{A}$. Consequently, for any $i,j\geqslant 1$ and any $M\in\mathcal{B}_F$, $T_{\gamma_i}Q_MT_{\gamma_j}^*\in\mathfrak{A}$.*

3. *For any $M\in\mathcal{B}_F$ and any $\nu\in\mathcal{L}_F^*(F)$, $T_{b_1}T_{b_1}^*Q_MT_\nu^*T_{b_1}T_{b_1}^*\in\mathfrak{A}$.*

4. *For any $d\in\mathbb{N}$ and any $M\in\mathcal{B}_F$,*

$$T_{\beta_d}Q_MT_{\beta_d}^*=\begin{cases} T_{b_1}T_{b_1}^*+X & \textit{if } M\cap r(b_1)=r(b_1)\\ X & \textit{otherwise}\end{cases}$$

for some $X\in\mathfrak{A}$.

Proof.

1. $Q_AT_{b1}T_{b_1}^*(h,k_1,\ldots)=Q_AT_{b_1}(0,h,0,\ldots)=Q_A(h,0,\ldots)=(P_A(h),0,\ldots)=Q_A(h,k_1,\ldots)$. Therefore $Q_AT_{b1}T_{b_1}^*=Q_A$. Using similar calculations, for $C\in\mathcal{C}$ and $D\in\mathcal{D}$, we get $Q_CT_{b_1}T_{b_1}^*=0$, and $Q_DT_{b1}T_{b_1}^*=0$.

2. Since $T_{\gamma_i}Q_A=T_{\gamma_i}Q_{r(\gamma_i)\cap A}$, and $r(\gamma_i)\in\mathcal{B}$, we get $r(\gamma_i)\cap A\in\mathcal{B}$. Therefore $T_{\gamma_i}Q_A=S_{a_i}P_A\in\mathfrak{A}$.

3. If ν is the empty word, the result follows from (1). Otherwise, $T_\nu^*T_{b_1}T_{b_1}^*$ is zero unless ν starts in b_1, i.e., $s(\nu)\subseteq E^0$. If M is as in (1), then $T_{b_1}T_{b_1}^*Q_MT_\nu^*T_{b_1}T_{b_1}^*=Q_AT_\nu^*T_{b_1}T_{b_1}^*$ this is zero unless $r(\nu)\cap A\neq\emptyset$. The result follows from (1) and (2).

4. First we prove that $T_{b_1}Q_MT_{b_1}^* = T_{b_1}T_{b_1}^*$ whenever $M \cap r(b_1) = r(b_1)$; otherwise $T_{b_1}Q_MT_{b_1}^*$ is in $\mathscr{G}$. Write $M = A \cup (\cup_{i=1}^{n_1} C_{k_i}) \cup (\cup_{j=1}^{m_1} D_{l_j})$, where $C_{k_i} = r(A_{k_i}, \beta_{k_i})$, $D_{l_j} = r(b_{l_j})$, written as a disjoint union. Since $T_{b_1}Q_MT_{b_1}^* = T_{b_1}Q_{r(b_1)\cap M}T_{b_1}^*$, we see that $T_{b_1}Q_MT_{b_1}^*$ is non-zero if one of the k_i's or one of the l_j's is to equal 1. If a $k_i = 1$, say $k_1 = 1$ then $C_1 = r(A_1, b_1)$, for some $A_1 \in \mathscr{B}$. This gives us $T_{b_1}Q_MT_{b_1}^* = T_{b_1}Q_{r(A_1,b_1)}T_{b_1}^* = Q_{A_1}T_{b_1}T_{b_1}^* = Q_{A_1} \in \mathfrak{A}$. If on the other hand an l_j is equal to 1, say $l_1 = 1$, then $T_{b_1}Q_MT_{b_1}^* = T_{b_1}Q_{r(b_1)}T_{b_1}^* = T_{b_1}T_{b_1}^*$. Now, if $d > 1$, recall $T_{\beta_{d-1}} = T_{\gamma_{d-1}} + T_{\beta_d}$. Hence

$$\begin{aligned}
T_{\beta_d} &= T_{\beta_{d-1}} - T_{\gamma_{d-1}} \\
&= T_{\beta_{d-2}} - [T_{\gamma_{d-2}} + T_{\gamma_{d-1}}] \\
&\ \ \vdots \\
&= T_{b_1} - \sum_{i=1}^{d-1} T_{\gamma_i}.
\end{aligned}$$

This gives us

$$\begin{aligned}
T_{\beta_d}Q_MT_{\beta_d}^* &= T_{b_1}Q_MT_{b_1}^* - \sum_{i=1}^{d-1} T_{\gamma_i}Q_MT_{\gamma_i}^* \\
&= T_{b_1}T_{b_1}^* - \sum_{i=1}^{d-1} T_{\gamma_i}Q_MT_{\gamma_i}^*.
\end{aligned}$$

Each $T_{\gamma_i}Q_MT_{\gamma_i}^*$ is in $\mathfrak{A}$. □

The following result is similar to [14, Proposition 6.6]. The results are slightly different due to the fact that the two desingularizations are for different spaces.

THEOREM 4.18. *Let $(E, \mathscr{L}, \mathscr{B})$ be a labeled space and suppose $(F, \mathscr{L}_F, \mathscr{B}_F)$ is a desingularised labeled space of $(E, \mathscr{L}, \mathscr{B})$. Suppose $\{t_a, q_A : a \in \mathscr{A}_F,\ A \in \mathscr{B}_F\}$ is a representation that generates $C^*(F, \mathscr{L}_F, \mathscr{B}_F)$. Let $\{s_{a_i} = t_{\gamma_i},\ p_A : a_i \in \mathscr{A},\ A \in \mathscr{A}\}$ be the representation built as in Theorem 4.14 and let $\mathfrak{A} = C^*\{s_{a_i}, p_A\}$.*

Then

1. $\mathfrak{A} = C^*(E, \mathscr{L}, \mathscr{B})$

2. *If $t_{b_1}t_{b_1}^* \in C^*(E, \mathscr{L}, \mathscr{B})$ then $\mathfrak{A}$ is a full corner of $C^*(F, \mathscr{L}_F, \mathscr{B}_F)$, otherwise, there exists a unital sub-algebra $\mathscr{Z}$ of $C^*(F, \mathscr{L}_F, \mathscr{B}_F)$ such that.*

   - *$\mathscr{Z}$ is isomorphic to a full corner of $C^*(F, \mathscr{L}_F, \mathscr{B}_F)$.*
   - *$C^*(E, \mathscr{L}, \mathscr{B})$ is isomorphic to a sub-algebra of $\mathscr{Z}$ with $\mathscr{Z}/C^*(E, \mathscr{L}, \mathscr{B}) \cong \mathbb{C}$.*

Proof. We will prove that $\mathfrak{A} = C^*(E,\mathcal{L},\mathcal{B})$ by showing that $\mathfrak{A}$ satisfies the universal property for $(E,\mathcal{L},\mathcal{B})$. After that, letting $P = t_{b_1}t_{b_1}^*$, we will prove that $\mathcal{Z}/\mathfrak{A} \cong \mathbb{C}$, where $\mathcal{Z} = P \cdot C^*(F,\mathcal{L}_F,\mathcal{B}_F) \cdot P$.

Suppose $\{S_{a_i}, P_A : a_i \in \mathcal{A},\ A \in \mathcal{B}\}$ is a representation of $(E,\mathcal{L},\mathcal{B})$. Let $\{T_a, Q_M : a \in \mathcal{A}_F,\ M \in \mathcal{B}_F\}$ be the representation of $(F,\mathcal{L}_F,\mathcal{B}_F)$ constructed as in Theorem 4.14. By the universality property, there exists a homomorphism, π from $C^*(F,\mathcal{L}_F,\mathcal{B}_F)$ onto $C^*\{T_a, Q_A\}$. Restricting π to $\mathfrak{A}$ we get $\pi(s_{a_i}) = \pi(t_{\gamma_i}) = T_{\gamma_i} = S_{a_i}$, and $\pi(p_A) = \pi(q_A) = Q_A = P_A$, for each $a_i \in \mathcal{A}$, and each $A \in \mathcal{B}$. Moreover this map is onto. This gives us a homomorphism of $\mathfrak{A}$ onto $C^*\{S_{a_i}, P_A\}$. Therefore $\mathfrak{A} = C^*(E,\mathcal{L},\mathcal{B})$.

Now let $t_\mu q_M t_\nu^* \in C^*(F,\mathcal{L}_F,\mathcal{B}_F)$ be a generating element of $C^*(F,\mathcal{L}_F,\mathcal{B}_F)$. If μ or ν is the empty word then $P \cdot t_\mu q_M t_\nu^* \cdot P = t_{b_1}t_{b_1}^* t_\mu q_M t_\nu^* t_{b_1}t_{b_1}^*$ is in $\mathfrak{A}$, by 4.17 (3) or (1). Otherwise, $P \cdot t_\mu q_M t_\nu^* \cdot P = t_{b_1}t_{b_1}^* t_\mu q_M t_\nu^* t_{b_1}t_{b_1}^*$ is zero unless both μ and ν begin in b_1. In that case, $P \cdot t_\mu q_M t_\nu^* \cdot P = t_\mu q_M t_\nu^*$; also either $\mu = \gamma_{k_1}\gamma_{k_2}\dots\gamma_{k_n}$ or $\mu = \gamma_{k_1}\gamma_{k_2}\dots\gamma_{k_n}\beta_d$, for some $d \geqslant 1$.

In the first case, i.e., if $\mu = \gamma_{k_1}\gamma_{k_2}\dots\gamma_{k_n}$, we have that $r(\mu) = r(\gamma_{k_1}\gamma_{k_2}\dots\gamma_{k_n}) = r(a_{k_1}a_{k_2}\dots a_{k_n}) \in \mathcal{B}$. Therefore $t_\mu q_M t_\nu^* \in \mathfrak{A}$.

In the second case, since $\beta_d = b_1 \dots b_d$, we see that $t_\mu q_M t_\nu^*$ is non zero only when ν ends in b_d, that is, $\nu = \gamma_{l_1}\gamma_{l_2}\dots\gamma_{l_m}\beta_d$. Using $\mu' = \gamma_{k_1}\gamma_{k_2}\dots\gamma_{k_n}$, $\nu' = \gamma_{l_1}\gamma_{l_2}\dots\gamma_{l_m}$, we have $t_\mu q_M t_\nu^* = t_{\mu'}t_{\beta_d} q_M t_{\beta_d}^* t_{\nu'}^* = t_{\mu'}t_{b_1} q_M t_{b_1}^* t_{\nu'}^* - \sum_{i=1}^{d-1} t_{\mu'}t_{\gamma_i} q_M t_{\gamma_i}^* t_{\nu'}^*$. This is in $\mathfrak{A}$ unless $M \cap r(b_1) = r(b_1)$, and both μ' and ν' are empty words. In that case the first term is $t_{b_1}t_{b_1}^*$ plus an element of $\mathfrak{A}$ and each term in the sum is in $\mathfrak{A}$. Therefore $t_\mu q_M t_\nu^*$ is either in $\mathfrak{A}$ or is of the form $t_{b_1}t_{b_1}^* + X$, for some $X \in \mathfrak{A}$. This concludes the proof. □

It is easy to see that $\mathcal{Z}$ is a unital C^*-algebra, where as $C^*(E,\mathcal{L},\mathcal{B})$ may not be. However, if $\mathcal{B}$ contains E^0 (making $C^*(E,\mathcal{L},\mathcal{B})$ unital), we can do better. With a slight modification of the definition of the Hilbert space $\mathcal{H}$, we can show that $\mathfrak{A} \cong P \cdot C^*(F,\mathcal{L}_F,\mathcal{B}_F) \cdot P$.

COROLLARY 4.19. Let $(E,\mathcal{L},\mathcal{B})$ be a labeled space and let $(F,\mathcal{L}_F,\mathcal{B}_F)$ be a desingularized labeled space of $(E,\mathcal{L},\mathcal{B})$. Suppose $E^0 \in \mathcal{B}$. Then $C^*(E,\mathcal{L},\mathcal{B})$ is a full corner of $C^*(F,\mathcal{L}_F,\mathcal{B}_F)$. Therefore $C^*(E,\mathcal{L},\mathcal{B})$ is Morita equivalent to $C^*(F,\mathcal{L}_F,\mathcal{B}_F)$.

Proof. Given a representation $\{S_a, P_A : a \in \mathcal{A}, A \in \mathcal{B}\}$ on a Hilbert space $\mathcal{H}'$, define $\mathcal{H} = P_{E^0}(\mathcal{H}')$. Now use $\mathcal{H}$ in the construction of $C^*(F,\mathcal{L}_F,\mathcal{B}_F)$. The result follows from $T_{b_1}T_{b_1}^* = P_{E^0}$. □

Consider the directed graph and desingularization in Example 4.5.

1. If $\mathcal{B}^1 = \{\emptyset, \{w_1, w_2, w_3, \dots\}\}$ then $C^*(E,\mathcal{L},\mathcal{B}^1) = \mathcal{M}_2(\mathbb{C}) =$ the space of 2×2 matrices of complex numbers, and $C^*(F,\mathcal{L}_F,\mathcal{B}_F^1) = \mathcal{K} =$ the space of compact operators. In this case, it is not difficult to see that $T_{b_1}T_{b_1}^* = Q_{\{w_1,w_2,w_3,\dots\}} + T_{b_1c_1}Q_{\{w_1,w_2,w_3,\dots\}}T_{b_1c_1}^* = P_{\{w_1,w_2,w_3,\dots\}} + S_{a_1}P_{\{w_1,w_2,w_3,\dots\}}S_{a_1}^* \in C^*(E,\mathcal{L},\mathcal{B}^1)$.

2. If $\mathscr{B}^2$ is the set of all finite subsets of $\{w_1, w_2, w_3, \dots\}$ then $C^*(E, \mathscr{L}, \mathscr{B}^2) = \bigoplus_{i=1}^{\infty} \mathscr{M}_2(\mathbb{C})$. Here $T_{b_1} T_{b_1}^*$ is the unit element of the unitization of $C^*(E, \mathscr{L}, \mathscr{B}^2)$.

REMARK 4.20.

1. Given any labeled graph $(E, \mathscr{L})$, the singularization process is at the labeled graph level. As a result, any labeled space $(E, \mathscr{L}, \mathscr{B})$ is desingularized regardless of whether $\mathscr{B}$ contains singular sets or not. It may be possible to modify the process, taking $\mathscr{B}$ into consideration, and get a better result.

2. The process essentially maintains, at the graph level, the loop structure and cofinalities of the original graph. We believe that one may be able to classify the ideals of $C^*(E, \mathscr{L}, \mathscr{B})$ such as gauge–invariant ideals, primitive ideals of $C^*(E, \mathscr{L}, \mathscr{B})$.

3. Given a labeled space $(E, \mathscr{L}, \mathscr{B})$, if $\mathscr{B}$ contains no sets A with $|\mathscr{L}(AE^1)| = \infty$, i.e., each set in $\mathscr{B}$ emits only finitely many (or none) labels, it may be possible to achieve a similar result by only adding an infinite tail to each sink in E and labeling it as $b_1, b_2, \dots$ and leaving the edges of E (and the a_i's) alone. If this conjecture is true, it may be more efficient and practical.

Acknowledgement. The authors would like to thank the reviewers for the thorough and constructive reviews, which improved the overall presentation of this paper significantly.

REFERENCES

[1] D. P. BANJADE, M. EPHREM, *On Labeled Graph C^*-algebras*, Rocky Mountain Journal of Math, vol. 50, no. 3, (2020), 863–870.

[2] T. BATES, T. M. CARLSEN, D. PASK, *C^*-algebras of labelled graphs III – K-theory computations*, Ergodic Theory and Dynam. Systems. **37** (2017), 337–368.

[3] T. BATES, D. PASK, *C^*-algebras of labeled graphs*, J. Operator Theory. **57** (2007), 101–120.

[4] T. BATES, D. PASK, *C^*-algebras of labeled graphs II – simplicity results*, Math. Scand. **104** (2009), no. 2, 249–274.

[5] T. BATES, D. PASK, I. RAEBURN, W. SZYMAŃSKI, *The C^*-algebras of row–finite Graphs*, New York J. Math. **6** (2000), 307–324.

[6] J. CUNTZ, *Simple C^*-algebras Generated by Isometries*, Commun. Math. Phys. bf 57 (1977), 173–185.

[7] D. DRINEN, M. TOMFORDE, *The C^*-algebras of Arbitrary Graphs*, Rocky Mountain J. Math. **35** (2005), no. 1, 105–135.

[8] M. EPHREM, *Characterizing Liminal And Type I Graph C^*-algebras*, J. Operator Theory. **52** (2004) 303–323.

[9] M. EPHREM, *Primitive Ideals of Labeled Graph C^*-algebras*, Houston Journal of Math. **46**, no. 2 (2020), 377–387.

[10] M. ENOMOTO, Y. WATATANI, *A Graph Theory For C^*-algebras*, Mathematica Japonica. **25**, no. 2 (1980) 435–442.

[11] J. JEONG, H. KIM, G. H. PARK, *The structure of gauge-invariant ideals of labeled graph C^*-algebras*, J. Funct. Anal. **262**, (2012), no. 4, 1759–1780.

[12] A. KUMJIAN, D. PASK, I. RAEBURN, *Cuntz-Krieger Algebras of Directed Graphs*, Pacific J. Math. **184**, (1998), 161–174.

[13] A. KUMJIAN, D. PASK, I. RAEBURN, J. RENAULT, *Graphs, Groupoids and Cuntz-Krieger Algebras*, J. Funct. Anal. **144**, (1997), 505–541.

[14] M. TOMFORDE, *A unified approach to Exel-Laca algebras and C^*-algebras associated to graphs*, J. Operator Theory. **50** (2003) no. 2, 345–368.

(Received January 26, 2023)

Debendra P. Banjade
Department of Mathematics and Statistics
Coastal Carolina University
Conway, SC 29528–6054
e-mail: dpbanjade@coastal.edu

Amy Chambers
Department of Mathematics
Tennessee Technological University
Cookeville, TN 38505
e-mail: achambers@tntech.edu

Menassie Ephrem
Department of Mathematics and Statistics
Coastal Carolina University
Conway, SC 29528–6054
e-mail: menassie@coastal.edu