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ON MINIMAL SMALLEST SINGULAR VALUE OF SUBFRAMES FOR SIGNAL RECOVERY

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Abstract. In this paper, we mainly study the smallest singular value of submatrices consisting of row vectors bounded by 1, and we establish that the minimal smallest singular value of submatrices of matrices of size $n+1$ times n consisting of row vectors bounded by 1 is equal to $\frac{1}{\sqrt{n}}$ if and only if the rows of $\operatorname{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1})A$ are the coordinates of the $n+1$ vertices of a regular n-simplex on the unit $(n-1)$-sphere S^{n-1} in $\mathbb{R}^n$ for some $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1}) \in \{-1,1\}^{n+1}$. Moreover, we establish that the minimal smallest singular value of submatrices of matrices of size n times 2 consisting of row vectors bounded by 1 is sharply bounded above by $\sqrt{2}\sin\frac{\pi}{2n}$, and furthermore, this bound is achieved if and only if the rows of $\operatorname{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)A$ are the coordinates of n adjacent vertices of a regular $2n$-gon on the unit circle S^1 in $\mathbb{R}^2$ for some $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \{-1,1\}^n$. Additionally, we show that the equiangular frames in the projective spaces do not form the matrices in the general dimensions with the optimal smallest singular value of the submatrices, contrary to the case of matrices of dimension $n+1$ by n or negative to the conjectures based on the phenomena in the low dimensions.

1. Introduction

In signal processing, measurements, frames, transformations, and dictionaries (see, for example, [3], [33], [11], [14] and [8]), all of which are essentially matrices, have been studied. As the main features or characteristics of a matrix or linear transformation, the singular values and their generalized forms have been studied, for example, in [26], [23], [15], [42], and [27]. It is well known that the singular values of a matrix are determined by both the magnitudes and the angles of the row vectors of the matrix in Euclidean space, as the singular values of a matrix are the square roots of the eigenvalues of the matrix multiplied by its transpose, and the eigenvalues of a matrix measure how much the matrix stretches or shrinks the vectors along different directions (see, for example, [9], [16] and [34]).

Recent research has focused on the study of rectangular matrices, as seen in works such as [42], [23], [37], [28], and [43]. In [30], we showed that the matrix problem can be reduced to a problem of combinatorial geometry by considering the columns of a "portrait" matrix (number of rows greater than that of columns) as points in a bounded region in a plane. When the magnitudes of all rows of a rectangular matrix are bounded, we can estimate the smallest singular values of submatrices with respect to the size of

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the matrix, since there are configurations of matrices whose minimal smallest singular values are of the order of a power of the size with a negative exponent. In [30], we also established some estimates of the distances between points in a set or the distances from points to lines connecting two other points in a set of points in a bounded region, and the decay rate of these distances determines in some sense the decay rate of the smallest singular values of submatrices with bounded row vectors. It is worth noting that the combinatorial geometry problem we considered is related to the Heilbronn triangle problem (see, for example, [19], [49], and [6]). There have been efforts to develop algorithms for finding counterexamples to the original Heilbronn conjecture (see, for example [24] and [19]). However, to the best of our knowledge, there does not appear to be an implementable algorithm that finds explicit or concrete point sets. It would be interesting to determine the optimal arrangements of n points in a square or unit disk for the Heilbronn triangle problem or for our problem of interest.

In [5], Lipschitz bounds were obtained for the nonlinear analysis map and theoretical performance bounds were established for each reconstruction algorithm. Additionally, [30]discovered a relationship between the smallest singular value of submatrices and the minimum distance of points to connecting lines in a bounded set. Through the use of integral geometry and combinatorial geometry, the decay rate of the minimal distance for sets of points was established when the number of points on the boundary of the convex hull of any subset is not too large relative to the set's cardinality. Numerical experiments were conducted in [30]to analyze the minimum distance for various point sets, including extreme configurations. However, the sharpness of the bound for the minimal smallest singular value frames of low dimensions for signal processing was not demonstrated in[30].

As for the applications of the theoretical work on the smallest singular value of submatrices, it is related to numerical erasure robustness of frames and robustness in signal reconstruction, and recently, there have been works on this, on which one can see for example [13], [12], and [29]. The stability of frames of low dimensions for signal recovery, as well as their applicability to signal processing, relies to some extent on the smallest singular values. These values play a crucial role in compressed sensing, matrix recovery (as discussed in [47]), phase retrieval (as explored in [2] and [18]), and various other fields. Consequently, this paper's findings can be utilized to construct robust frames that are resilient to errors in signal processing, as demonstrated in works such as [39] and [21].

The minimal smallest singular value of submatrices provides a useful measure for assessing the stability of frames for signal processing (see, for example, [17] and [48]). By considering this value, we can construct stable frames that provide robust representations of signals against noise and errors (see, for example, [36] and [32]). Stable frames with large minimal smallest singular values have many applications in signal processing (see, for example, [41] and [25]). By using these frames in tasks such as signal reconstruction and denoising, we can improve the performance of signal processing algorithms and provide more accurate and robust representations of signals (see, for example, [46] and [10]). Overall, the minimal smallest singular value of submatrices is an important measure for assessing the stability of frames for signal processing (see, for example, [1]). By using various techniques to construct stable frames with large minimal smallest singular values, we can improve the performance of signal processing algorithms and provide more accurate and robust representations of signals (see, for example, [20]).

The main contribution of this paper is to establish that the minimal smallest singular value of submatrices of matrices of size $n+1$ times n consisting of row vectors bounded by 1 is equal to $\frac{1}{\sqrt{n}}$ if and only if the rows of $\operatorname{diag}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{n+1})A$ are the coordinates of the $n+1$ vertices of a regular n-simplex on the unit $(n-1)$-sphere S^{n-1} in $\mathbb{R}^n$ for some $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{n+1})\in\{-1,1\}^{n+1}$. Moreover, we establish that the minimal smallest singular value of submatrices of matrices of size n times 2 consisting of row vectors bounded by 1 is sharply bounded above by $\sqrt{2}\sin\frac{\pi}{2n}$, and furthermore, this bound is achieved if and only if the rows of $\operatorname{diag}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)A$ are the coordinates of n adjacent vertices of a regular $2n$-gon on the unit circle S^1 in $\mathbb{R}^2$ for some $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\in\{-1,1\}^n$. Additionally, we show that the equiangular frames in the projective spaces do not form the matrices in the general dimensions with the optimal smallest singular value of the submatrices, contrary to the case of matrices of dimension $n+1$ by n or negative to the conjectures based on the phenomena in the low dimensions. we show that the equiangular frames in the projective spaces do not form the matrices in the general dimensions with the optimal smallest singular value of the submatrices, contrary to the case of matrices of dimension $n+1$ by n or negative to the conjectures based on the phenomena in the low dimensions. The applications of this work may include, but are not limited to, improving the numerical stability of compressed sensing, constructing error-free frames in noisy settings, constructing frames needed for a given level of stability, and proving further or more general results about numerically erasure-robust frame (NERF) bounds or frame erasure robustness.

This paper is structured as follows: In Section 2, we prove the theorem on the minimal smallest singular value of slim matrices, and, in particular, we show the optimal decay rate for the base case; in Section 3, we show that the equiangular frames in projective spaces do not form the matrices in general dimensions with the optimal smallest singular value of submatrices, contrary to the case of matrices of dimension $n+1$ by n, that is, negative to conjectures based on low dimensions.

2. On the minimal smallest singular value

First, the following basic lemma about submatrices can be easily established, and in this paper, we denote the k-singular value of a matrix A by $\sigma_k(A)$ and the submatrix of A consisting of rows with indices in the set $S\subseteq\{1,\ldots,N\}$ by A_S. There are probably many ways to prove it (see, for example, [45] and [44]), but we provide here a direct proof from the perspective of vector extension.

LEMMA 2.1. *For any real matrix A of size N by n with $N\geqslant n$, one has*

$$\sigma_n(A)\geqslant\sigma_n(A_S) \tag{2.1}$$

for all $S\subseteq\{1,\ldots,N\}$ with $|S|=n$, and

$$\sigma_1(A)\geqslant\sigma_1(A_S) \tag{2.2}$$

for all $S \subseteq \{1,\ldots,N\}$ *with* $|S| = n$.

Proof. For any $S \subseteq \{1,\ldots,N\}$ with $|S| = n$,

$$\sigma_n(A_S) = \inf_{v\in\mathbb{R}^n,\|v\|=1} \|A_S v\|; \tag{2.3}$$

and on the other hand,

$$\sigma_n(A) = \inf_{V\subseteq\mathbb{R}^n,\dim(V)=1} \|A|_V\| = \inf_{v\in\mathbb{R}^n,\|v\|=1} \|Av\|. \tag{2.4}$$

Since Av is basically a vector extension of $A_S v$ for every $v \in \mathbb{R}^n, \|v\| = 1$, we have

$$\|A_S v\| \leqslant \|Av\| \tag{2.5}$$

for every $v \in \mathbb{R}^n, \|v\| = 1$. Thus, it follows from (2.3) and (2.4) that

$$\sigma_n(A_S) \leqslant \sigma_n(A) \tag{2.6}$$

for any $S \subseteq \{1,\ldots,n+1\}$ with $|S| = n$. Therefore, we obtain (2.1), and similarly, we also obtain (2.2). □

More generally, the following lemma can also be easily established. There are probably many ways to prove it (see, for example, [45] and [44]), but we provide here a direct proof from the perspective of vector extension.

LEMMA 2.2. *For any real matrix* A *of size* N *by* n *with* $N \geqslant n$*, one has*

$$\sigma_k(A) \geqslant \sigma_k(A_S) \tag{2.7}$$

for $k = 1,2,\cdots n$ *for all* $S \subseteq \{1,\ldots,N\}$ with $|S| = n$.

Proof. For any $S \subseteq \{1,\ldots,N\}$ with $|S| = n$,

$$\sigma_k(A_S) = \inf_{V\subseteq\mathbb{R}^n,\dim(V)=n-k+1} \|A_S|_V\|; \tag{2.8}$$

and on the other hand,

$$\sigma_k(A) = \inf_{V\subseteq\mathbb{R}^n,\dim(V)=n-k+1} \|A|_V\|. \tag{2.9}$$

Since Av is basically a vector extension of $A_S v$ for every $v \in \mathbb{R}^n, \|v\| = 1$, we have

$$\|A_S v\| \leqslant \|Av\| \tag{2.10}$$

for every $v \in \mathbb{R}^n, \|v\| = 1$ and therefore

$$\|A_S|_V\| \leqslant \|A|_V\| \tag{2.11}$$

for every $V \subseteq \mathbb{R}^n$ with $\dim(V) = n-k+1$. Thus, it follows from (2.8) and (2.9) that

$$\sigma_k(A) \geqslant \sigma_k(A_S) \tag{2.12}$$

for $k = 1,2,\cdots,n$ for all $S \subseteq \{1,\ldots,N\}$ with $|S| = n$. $\square$

From the growth rate of the smallest singular value of random matrices established in [4], one can obtain that

$$\sigma_n(A) \to \left(2-\sqrt{2}\right)\sqrt{n} \tag{2.13}$$

for $N = 2n$. On the other hand,

$$\sigma_n(A_S) \leqslant O\left(\frac{1}{\sqrt{n}}\right). \tag{2.14}$$

In [30], it was proved that

$$\min_{S\subseteq\{1,\ldots,n+1\},|S|=n} \sigma_n(A_S) \leqslant \frac{1}{\sqrt{n}} \tag{2.15}$$

for any $n+1$ by n matrix $A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{n+1} \end{bmatrix}$ with $\|\mathbf{a}_i\| \leqslant 1$, $i = 1,\ldots,n+1$. Here we further establish the following theorem:

THEOREM 2.3. *For any $n+1$ by n matrix $A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{n+1} \end{bmatrix}$ with $\|\mathbf{a}_i\| \leqslant 1$, $i = 1,\ldots,n+1$, the inequality*

$$\min_{S\subseteq\{1,\ldots,n+1\},|S|=n} \sigma_n(A_S) \leqslant \frac{1}{\sqrt{n}} \tag{2.16}$$

is sharp, and the equality holds if and only if the rows of

$$diag(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{n+1})A \tag{2.17}$$

are the coordinates of the $n+1$ vertices of a regular n-simplex on the unit $(n-1)$-sphere S^{n-1} in $\mathbb{R}^n$ for some $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{n+1}) \in \{-1,1\}^{n+1}$.

Proof. Since $\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}$ are linear dependent, there are $c_1, \ldots, c_{n+1}$, such that

$$\sum_{i=1}^{n+1} c_i\mathbf{a}_i = 0 \tag{2.18}$$

with

$$\sum_{i=1}^{n+1} c_i^2 = 1. \tag{2.19}$$

Without loss of generality, assume $|c_{n+1}| \leqslant \min(|c_1|,\ldots,|c_n|)$. If $c_{n+1}=0$, (2.16) is trivial, because there exists an S such that A_S is singular. It suffices to consider the case of $c_{n+1}\neq 0$. Therefore,

$$c_{n+1}\mathbf{a}_{n+1} = -\sum_{i=1}^{n} c_i\mathbf{a}_i. \tag{2.20}$$

By (2.19),

$$(n+1)c_{n+1}^2 \leqslant \sum_{i=1}^{n+1} c_i^2 = 1. \tag{2.21}$$

It follows that

$$|c_{n+1}| \leqslant \frac{1}{\sqrt{n+1}} \tag{2.22}$$

and furthermore

$$\frac{\|c_{n+1}\mathbf{a}_{n+1}\|}{\sqrt{1-c_{n+1}^2}} \leqslant \frac{1}{\sqrt{n+1}}\cdot\frac{\sqrt{n+1}}{\sqrt{n}} = \frac{1}{\sqrt{n}}. \tag{2.23}$$

Since

$$\min_{S\subseteq\{1,\ldots,n+1\},|S|=n} \sigma_n(A_S) \leqslant \frac{\|\sum_{i=1}^{n} c_i\mathbf{a}_i\|}{\sqrt{\sum_{i=1}^{n} c_i^2}} = \frac{\|c_{n+1}\mathbf{a}_{n+1}\|}{\sqrt{1-c_{n+1}^2}}, \tag{2.24}$$

therefore (2.16) follows from (2.23).

Obviously, when the rows of A are the coordinates of the $n+1$ vertices of a regular n-simplex on the unit $(n-1)$-sphere S^{n-1} in $\mathbb{R}^n$, we have

$$\min_{S\subseteq\{1,\ldots,n+1\},|S|=n} \sigma_n(A_S) = \frac{1}{\sqrt{n}}. \tag{2.25}$$

Thus it also shows the sharpness of the inequality (2.16).

Coversely, if any $n+1$ by n matrix $A=\begin{bmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_{n+1}\end{bmatrix}$ with $\|\mathbf{a}_i\|\leqslant 1$, $i=1,\ldots,n+1$, has

$$\min_{S\subseteq\{1,\ldots,n+1\},|S|=n} \sigma_n(A_S) = \frac{1}{\sqrt{n}}, \tag{2.26}$$

then A has full rank and therefore there are c_1, $\ldots$, c_{n+1}, none of which is zero, with

$$\sum_{i=1}^{n+1} c_i^2 = 1, \tag{2.27}$$

such that

$$\sum_{i=1}^{n+1} c_i\mathbf{a}_i = 0. \tag{2.28}$$

For every

$$i_0 \in \mathscr{I} := \{1\leqslant i\leqslant n+1 : |c_i| \leqslant \min(|c_1|,\ldots,|c_{n+1}|)\}, \tag{2.29}$$

it follows from (2.25) and (2.28) that

$$\frac{1}{\sqrt{n}} = \min_{S\subseteq\{1,\dots,n+1\},|S|=n} \sigma_n(A_S) \leqslant \frac{\left\|\sum_{1\leqslant i\leqslant n+1, i\neq i_0} c_i\mathbf{a}_i\right\|}{\sqrt{\sum_{1\leqslant i\leqslant n+1, i\neq i_0} c_i^2}} = \frac{\left|c_{i_0}\right|\left\|\mathbf{a}_{i_0}\right\|}{\sqrt{1-c_{i_0}^2}}, \tag{2.30}$$

and furthermore, it follows from (2.27) and (2.29) that

$$\frac{\left|c_{i_0}\right|\left\|\mathbf{a}_{i_0}\right\|}{\sqrt{1-c_{i_0}^2}} \leqslant \frac{\left\|\mathbf{a}_{i_0}\right\|}{\sqrt{n}}. \tag{2.31}$$

Combining (2.30) and (2.31) yields that $\left\|\mathbf{a}_{i_0}\right\| \geqslant 1$. Furthermore, since $\|\mathbf{a}_i\| \leqslant 1$, $i = 1,\dots,n+1$, hence we obtain that $\left\|\mathbf{a}_{i_0}\right\| = 1$ for all $i_0 \in \mathscr{I}$ and therefore $\left|c_{i_0}\right| = \frac{1}{\sqrt{n}}$ for all $i_0 \in \mathscr{I}$. Moreover, it follows from (2.23) and (2.31) that $|c_i| = \frac{1}{\sqrt{n}}$ for all $i = 1,\dots,n+1$. Therefore, we obtain that

$$\frac{1}{\sqrt{n}} = \min_{S\subseteq\{1,\dots,n+1\},|S|=n} \sigma_n(A_S) \leqslant \frac{\left\|\sum_{1\leqslant j\leqslant n+1, j\neq i} c_j\mathbf{a}_j\right\|}{\sqrt{\sum_{1\leqslant j\leqslant n+1, j\neq i} c_j^2}} = \frac{|c_i|\,\|\mathbf{a}_i\|}{\sqrt{1-c_i^2}} = \frac{\|\mathbf{a}_i\|}{\sqrt{n}}, \tag{2.32}$$

which yields that $\|\mathbf{a}_i\| \geqslant 1$ for all $i = 1,\dots,n+1$. Furthermore, since $\|\mathbf{a}_i\| \leqslant 1$, $i = 1,\dots,n+1$, hence we obtain that $\|\mathbf{a}_i\| = 1$ for all $i = 1,\dots,n+1$. Therefore, there exists some $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{n+1}) \in \{-1,1\}^{n+1}$, such that

$$\sum_{i=1}^{n+1} \varepsilon_i\mathbf{a}_i = 0 \tag{2.33}$$

and

$$\|\varepsilon_i\mathbf{a}_i\| = 1. \tag{2.34}$$

Now, let M_i be symmetric matrices of size $n+1$ by $n+1$ with diagonal entries

$$(M_i)_{j,j} = \begin{cases} \frac{1}{n} & \text{for} j \neq i \\ 0 & \text{for} j = i \end{cases} \tag{2.35}$$

and non-diagonal entries

$$(M_i)_{k,l} = \begin{cases} \frac{(AA^T)_{k,l}}{n-1} & \text{for } k \neq i \text{ and } l \neq i \\ 0 & \text{for } k = i \text{ or } l = i \end{cases} \tag{2.36}$$

for $k \neq l$, then we decompose AA^T as

$$AA^T = \sum_{i=1}^{n+1} M_i, \tag{2.37}$$

because

$$\min_{S\subseteq\{1,\dots,n+1\},|S|=n} \sigma_n(A_S) = \frac{1}{\sqrt{n}}. \tag{2.38}$$

Let $\left[AA^T\right]_i$ and $[M_i]_i$ be the submatrices of AA^T and M_i respectively obtained by removing the i-the row and the i-th column of AA^T and M_i respectively, then it is obvious that

$$n\left[AA^T\right]_i - I_{n\times n} = n\,(n-1)\,[M_i]_i \tag{2.39}$$

and it follows that the n-th eigenvalue

$$\lambda_n\left([M_i]_i\right) = \frac{\lambda_n\left(n\left[AA^T\right]_i - I_{n\times n}\right)}{n\,(n-1)} \geqslant 0 \tag{2.40}$$

since

$$\min_{S\subseteq\{1,\ldots,n+1\},|S|=n} \sigma_n\left(A_S\right) = \frac{1}{\sqrt{n}}. \tag{2.41}$$

Therefore, M_i is positive semi-definite for every i from 1 through $n+1$. Since AA^T is positive semi-definite and singular, then there exists a non-zero vector $v\in\mathbb{R}^{n+1}$ such that $A^T v^T = 0$. Therefore, by (2.37), it follows that

$$\sum_{i=1}^{n+1} vM_iv^T = 0. \tag{2.42}$$

and since M_i for $i=1,\cdots,n+1$ are all positive semi-definite, hence

$$vM_iv^T = 0 \tag{2.43}$$

for every i from 1 through $n+1$. Furthermore, we obtain that

$$vM_i = \mathbf{0} \tag{2.44}$$

for $i=1,\cdots,n+1$. Now by grouping the equations in the system (2.44), for each fixed j between 1 and $n+1$, we have

$$v\left(M_i\right)_j = 0 \tag{2.45}$$

for $1\leqslant i\leqslant n+1$ and $i\neq j$, in which $\left(M_i\right)_j$ denotes the j-th column of M_i. Without loss of generality, we assume

$$v = \left(x_1,\cdots,x_n,1\right). \tag{2.46}$$

In particular, it follows from (2.45) that

$$n\left(AA^T\right)_{k,n+1}x_k - n\sum_{i=1}^{n}\left(AA^T\right)_{i,n+1}x_i = n-1 \tag{2.47}$$

for $k=1,\cdots,n$, and by solving this simple linear system, one obtains that

$$n\left(AA^T\right)_{i,n+1}x_i = -1 \tag{2.48}$$

for $i=1,\cdots,n$. Moreover, in particular, for any fixed i between 1 and n, it follows from (2.45) that

$$n\left(AA^T\right)_{i,k}x_k - n\sum_{j=1}^{n}\left(AA^T\right)_{i,j}x_j - n\left(AA^T\right)_{i,n+1} = (n-1)x_i \tag{2.49}$$

for $k = 1, \cdots, n$, and by solving this simple linear system, one obtains that

$$n\left(AA^T\right)_{i,j} x_j = n\left(AA^T\right)_{i,n+1} = -x_i \tag{2.50}$$

for $j = 1, \cdots, n$. Combining (2.48) and (2.50), we have

$$n\left|\left(AA^T\right)_{i,n+1}\right| = |x_i| = 1 \tag{2.51}$$

for $i = 1, \cdots, n$, and furthermore, it follows from (2.50) that

$$n\left|\left(AA^T\right)_{i,j}\right| = 1 \tag{2.52}$$

for $1 \leqslant i < j \leqslant n$. Thus, the diagonal entries of AA^T are all 1 and the absolute value of its off-diagonal entries of AA^T are all $\frac{1}{n}$. Therefore, each row vector of A lies on one of $n+1$ equiangular lines in $\mathbb{R}^n$ that intersect at the origin and have an angle of $\arccos\frac{1}{n}$ between every pair. Hence, the rows of $\operatorname{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1})A$ are the coordinates of the $n+1$ vertices of a regular n-simplex on the unit $(n-1)$-sphere S^{n-1} in $\mathbb{R}^n$ for some $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1}) \in \{-1, 1\}^{n+1}$. $\square$

REMARK 2.4. For an example of regular n-simplex on the unit $(n-1)$-sphere S^{n-1} in $\mathbb{R}^n$ for $n = 3$, see Figure 2.1 on page 281.

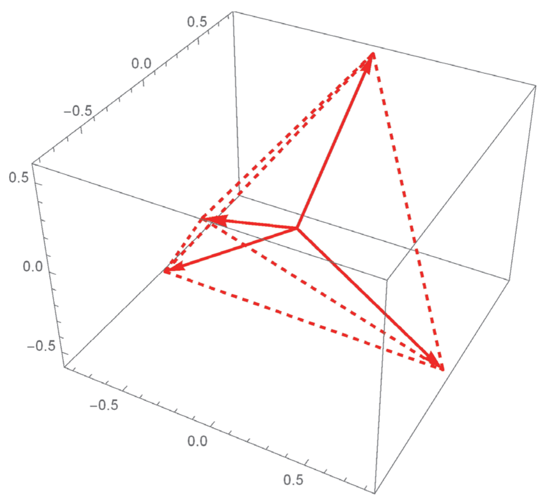

Figure 2.1: *An example of regular n-simplex on the unit $(n-1)$-sphere S^{n-1} for $n = 3$*

For matrices of size $n+2$ by n, one can have the following lemma for the case of $n = 2$.

LEMMA 2.5. *For any 4 by 2 matrix* $A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{bmatrix}$ *with* $\|\mathbf{a}_i\| \leqslant 1,\ i = 1,2,3,4$, *one has*

$$\min_{S\subseteq\{1,\dots,n+2\},|S|=2} \sigma_n(A_S) \leqslant \sqrt{2}\sin\frac{\pi}{8} \tag{2.53}$$

for $n = 2$ *and the inequality is sharp.*

Proof. Without loss of generality, we let

$$A = \begin{bmatrix} 1 & 0 \\ \cos x & \sin x \\ \cos y & \sin y \\ \cos z & \sin z \end{bmatrix} \tag{2.54}$$

and consider the Grammian matrix $A_S A_S^T$ and its eigenvalues. Suppose that

$$\min_{S\subseteq\{1,\dots,n+2\},|S|=2} \sigma_n(A_S) > \sqrt{2}\sin\frac{\pi}{8} \tag{2.55}$$

for $n = 2$. Since the square of the minimum of the smallest singular values of the submatrices of A, $\min_{S\subseteq\{1,\dots,n+2\},|S|=2} \sigma_n(A_S)$, for $n = 2$ is equal to

$$\begin{gathered}\min(1-\cos x, \cos x + 1, 1-\cos(x-y), \cos(x-y)+1,\\ 1-\cos y, \cos y + 1, 1-\cos z, \cos z + 1,\\ 1-\cos(x-z), \cos(x-z)+1, 1-\cos(z-y), \cos(z-y)+1),\end{gathered} \tag{2.56}$$

then because of the periodicity of the absolute value of sine and cosine functions, there exist x_1, y_1 and z_1 such that

$$\tfrac{x_1-2x_0}{\pi},\ \tfrac{y_1-2y_0}{\pi},\ \tfrac{z_1-2z_0}{\pi} \in \mathbb{Z} \tag{2.57}$$

which implies that

$$\frac{x_1-y_1-2x_0+2y_0}{\pi},\ \frac{y_1-z_1-2y_0+2z_0}{\pi} \in \mathbb{Z}, \tag{2.58}$$

for some

$$\tfrac{\pi}{8} < x_0, y_0, z_0, y_0 - x_0, z_0 - y_0 < \tfrac{3\pi}{8}, \tag{2.59}$$

as a consequence of (2.55).

Without loss of generality, we can assume that

$$\frac{\pi}{8} < x_0 \leqslant y_0 \leqslant z_0 < \frac{3\pi}{8}. \tag{2.60}$$

Therefore, it follows from (2.59) that

$$\frac{3\pi}{8} > z_0 > y_0 + \frac{\pi}{8} > x_0 + \frac{\pi}{4} > \frac{3\pi}{8} \tag{2.61}$$

which is a contradiction.

Regarding the sharpness of the inequality (2.53), the equality in (2.53) can be achieved by taking

$$A=\begin{bmatrix} 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{2.62}$$

Thus, we have completed the proof. □

Regarding the possible 4 by 2 matrice for which the equality in (2.53) holds, we have the following lemma.

LEMMA 2.6. *Let* $A=\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{bmatrix}$ *be a* 4 *by* 2 *matrix with* $\|\mathbf{a}_i\| \leqslant 1,\ i=1,2,3,4$. *Then*

$$\min_{S\subseteq\{1,\ldots,n+2\},|S|=2} \sigma_n(A_S)=\sqrt{2}\sin\frac{\pi}{8} \tag{2.63}$$

for $n=2$ *if and only if the rows of* $\operatorname{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)A$ *are the coordinates of 4 adjacent vertices of a regular octagon on the unit circle* S^1 *in* $\mathbb{R}^2$ *for some* $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\in\{-1,1\}^4$.

Proof. Because of the last part of the proof for Lemma 2.5, we only need to prove that the row vectors of matrix A or their negative vectors must be the coordinates of the 4 adjacent vertices of a regular octagon, given that matrix A has the equality (2.63).

If matrix A has the equality (2.63), then

$$\sigma_2(A_S)\geqslant\sqrt{2}\sin\frac{\pi}{8} \tag{2.64}$$

for all subsets S of indices with $|S|=2$ and there exists a subset S_0 of indices with $|S_0|=2$ such that

$$\sigma_2\left(A_{S_0}\right)=\sqrt{2}\sin\frac{\pi}{8}. \tag{2.65}$$

Now, consider the row vectors of A and their negative vectors, and reorder them counterclockwise as $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_8$. The sum of the angles between adjacent vectors of these 8 vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_8$ is 2π.

Suppose that there exists an angle between adjacent vectors of the 8 vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_8$ that is strictly less than $\frac{\pi}{4}$. Then, there exists a subset S_0 of indices $\{1,2,3,4\}$ with $|S_0|=2$, such that $\sigma_2(A_{S_0})<\sqrt{2}\sin\frac{\pi}{8}$, which contradicts equation (2.64). Therefore, the angles between adjacent vectors of the 8 vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_8$ are all no less than $\frac{\pi}{4}$.

Furthermore, suppose that there exists an angle between adjacent vectors of the 8 vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_8$ that is strictly greater than $\frac{\pi}{4}$. Then, their sum would be greater than 2π, which is a contradiction. Therefore, the angles between adjacent vectors of the 8 vectors $\{\mathbf{v}_i\}_{i=1}^{8}$ are all equal to $\frac{\pi}{4}$. It follows that the rows of $\operatorname{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)A$ are the coordinates of 4 adjacent vertices of an octagon in $\mathbb{R}^2$ for some $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\in\{-1,1\}^4$.

Suppose that there exists one of the 8 vectors $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_8$ whose length is strictly less than 1. Then, there would exist, there exists a subset S_0 of indices $\{1,2,3,4\}$ with $|S_0|=2$, such that $\sigma_2(A_{S_0})<\sqrt{2}\sin\frac{\pi}{8}$, which contradicts equation (2.64). Therefore, $\|\mathbf{v}_i\|=1$ for $i=1,2,\dots,8$, which implies $\|\mathbf{a}_i\|=1$ for $i=1,2,\dots,8$.

Hence, the rows of $\operatorname{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)A$ are the coordinates of 4 adjacent vertices of a regular octagon on the unit circle S^1 in $\mathbb{R}^2$ for some $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\in\{-1,1\}^4$. □

More generally, we have the following theorem.

THEOREM 2.7. *For any n by 2 matrix A with $n\geqslant 4$ and $\|\mathbf{a}_i\|\leqslant 1$, $i=1,2,\cdots,n$, one has*

$$\min_{S\subseteq\{1,\dots,n\},|S|=2}\sigma_2(A_S)\leqslant\sqrt{2}\sin\frac{\pi}{2n} \tag{2.66}$$

and the inequality is sharp. Furthermore, the equality in (2.66) holds if and only if the rows of $\operatorname{diag}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)A$ are the coordinates of n adjacent vertices of a regular $2n$-gon on the unit circle S^1 in $\mathbb{R}^2$ for some $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\in\{-1,1\}^n$.

Proof. Let

$$A=\begin{bmatrix}\cos x_1 & \cos x_1\\ \cos x_2 & \sin x_2\\ \vdots & \vdots\\ \cos x_n & \sin x_n\end{bmatrix} \tag{2.67}$$

and consider the Grammian matrix $A_SA_S^T$ and its eigenvalues. Suppose that

$$\min_{S\subseteq\{1,\dots,n\},|S|=2}\sigma_2(A_S)>\sqrt{2}\sin\frac{\pi}{2n}. \tag{2.68}$$

Since the square of the minimum of the smallest singular values of the submatrices of A,

$$\min_{S\subseteq\{1,\dots,n\},|S|=2}\sigma_n(A_S)=\min_{1\leqslant i<j\leqslant n}\left(1-\cos(x_i-x_j),\cos(x_i-x_j)+1\right), \tag{2.69}$$

then because of the periodicity of the absolute value of sine and cosine functions, there exist $\tilde{x}_i$, $i=1,2,\cdots,n$, such that

$$\tfrac{\tilde{x}_i-2\hat{x}_i}{\pi}\in\mathbb{Z} \tag{2.70}$$

for $i=1,2,\cdots,n$, which implies that

$$\frac{\tilde{x}_j-\tilde{x}_i-2\hat{x}_j+2\hat{x}_i}{\pi}\in\mathbb{Z} \tag{2.71}$$

for $1 \leqslant i < j \leqslant n$ for some

$$\tfrac{\pi}{2n} < \hat{x}_i < \tfrac{(n-1)\pi}{2n} \tag{2.72}$$

and

$$\tfrac{\pi}{2n} < \hat{x}_j - \hat{x}_i < \tfrac{(n-1)\pi}{2n} \tag{2.73}$$

for $1 \leqslant i < j \leqslant n$ as a consequence of (2.68).

Without loss of generality, we can assume that

$$\frac{\pi}{2n} < \hat{x}_1 \leqslant \hat{x}_2 \cdots \leqslant \hat{x}_n < \frac{(n-1)\pi}{2n}. \tag{2.74}$$

Therefore, it follows from (2.72) and (2.73) that

$$\begin{aligned} \tfrac{(n-1)\pi}{2n} > \hat{x}_n > \hat{x}_{n-1} + \tfrac{\pi}{2n} > \hat{x}_{n-2} + \tfrac{\pi}{n} > \cdots \\ \cdots > \hat{x}_1 + \tfrac{(n-2)\pi}{2n} > \tfrac{(n-1)\pi}{2n} \end{aligned} \tag{2.75}$$

which is a contradiction.

Regarding the sharpness of the inequality (2.66), the equality in (2.66) can be achieved by taking

$$A = \begin{bmatrix} 1 & 0 \\ \cos\frac{\pi}{n} & \sin\frac{\pi}{n} \\ \cos\frac{2\pi}{n} & \cos\frac{2\pi}{n} \\ \vdots & \vdots \\ \cos\frac{(n-1)\pi}{n} & \sin\frac{(n-1)\pi}{n} \end{bmatrix}. \tag{2.76}$$

If the equality in (2.66) holds, namely,

$$\min_{S\subseteq\{1,\ldots,n\},|S|=2} \sigma_2(A_S) = \sqrt{2}\sin\frac{\pi}{2n}, \tag{2.77}$$

let us consider the row vectors of A and their negative vectors, and reorder them counterclockwise as $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n}$. The sum of the angles between adjacent vectors of these $2n$ vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n}$ is 2π.

Suppose that there exists an angle between adjacent vectors of the $2n$ vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n}$ that is strictly less than $\frac{\pi}{n}$. Then, there exists a subset S_0 of indices $\{1,2,\cdots,n\}$ with $|S_0| = 2$ such that $\sigma_2(A_{S_0}) < \sqrt{2}\sin\frac{\pi}{2n}$, which contradicts with (2.77). Therefore, the angles between adjacent vectors of the $2n$ vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n}$ are all no less than $\frac{\pi}{n}$.

Furthermore, suppose that there exists an angle between adjacent vectors of the $2n$ vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n}$ that is strictly greater than $\frac{\pi}{n}$. Then, their sum would be greater than 2π, which is a contradiction. Therefore, the angles between adjacent vectors of the $2n$ vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n}$ are all equal to $\frac{\pi}{n}$. It follows that the rows of $\operatorname{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)A$ are the coordinates of n adjacent vertices of an $2n$-gon in $\mathbb{R}^2$ for some $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \{-1,1\}^n$.

Suppose that there exists one of the $2n$ vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n}$ whose length is strictly less than 1. Then, there would exist subset S_0 of indices $\{1,2,\cdots,n\}$ with

$|S_0| = 2$ such that $\sigma_2(A_{S_0}) < \sqrt{2}\sin\frac{\pi}{2n}$, which contradicts with (2.77). Therefore, $\|\mathbf{v}_i\| = 1$ for $i = 1,2,\ldots,2n$, which implies $\|\mathbf{a}_i\| = 1$ for $i = 1,2,\ldots,n$.

Hence, the rows of $\operatorname{diag}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)A$ are the coordinates of n adjacent vertices of a regular $2n$-gon on the unit circle S^1 in $\mathbb{R}^2$ for some $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n) \in \{-1,1\}^n$. Thus, we have completed the proof. $\square$

REMARK 2.8. For an example of row vectors of n by 2 matrix A with $\operatorname{diag}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)A$ being the coordinates of the n adjacent vertices of a regular $2n$-gon on the unit circle S^1 in $\mathbb{R}^2$ for some $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n) \in \{-1,1\}^n$ for $n = 4$, see Figure 2.2 on page 286.

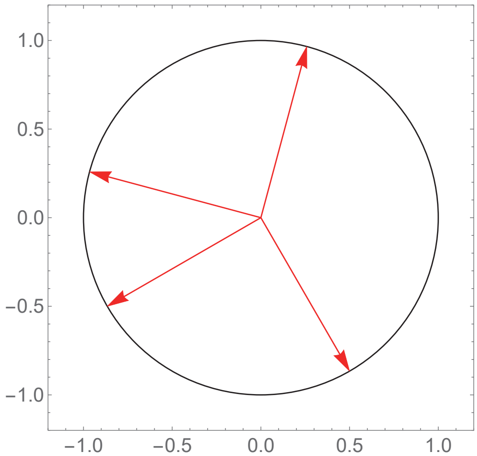

Figure 2.2: *An example of row vectors in a regular $2n$-gon on the unit circle for $n = 4$*

3. Equiangular frames in the projective spaces

In the previous section, we have shown that equiangular frames form matrices in some dimensions with the optimal smallest singular value of submatrices. In this section, however, we show that equiangular frames in projective spaces do not form matrices in general dimensions with the optimal smallest singular value of the submatrices, contrary to the case of matrices of dimension $n+1$ times n and dimension n times 2 for $n \geqslant 4$, in other words, negative to the conjectures based on the low dimensions.

In [7], optimal frames in projective spheres were found for some sequences of dimensions. In particular, the optimal frames of 6 vectors in projective spheres in $\mathbb{R}^4$ can be found as follows.

LEMMA 3.1. *An optimal equiangular frame of 6 vectors in projective spheres in $\mathbb{R}^4$ is*

$$F = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 \end{bmatrix}, \tag{3.1}$$

whose Grammian matrix

$$G = \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix}. \tag{3.2}$$

However, the smallest minimal singular value of the submatrix of F is 0.

Proof. Since

$$G = F^t F, \tag{3.3}$$

the angle between every two row-vectors of F is $\arccos \frac{1}{3}$. Therefore, the row vectors of F form an equiangular frame of 6 vectors in projective spheres in $\mathbb{R}^4$. On the other hand, one can quickly check that the smallest minimal singular value of the submatrix of F is indeed 0. $\square$

REMARK 3.2. In matrix theory and operator theory, the image of an operator can be viewed as the dual of its kernel or null space. This duality essentially creates corresponding relations between the restricted isometry property, the Johnson-Lindenstrauss embedding, and the null space property in signal processing, including compressed sensing, phase retrieval, and others (see, for example, [40], [22], and [38]). Therefore, the stability of frames can be directly applied to these signal processing techniques.

4. On applications

4.1. On stability of frames

Frames are sets of vectors in a Hilbert space that provide stable and redundant representations of signals. In signal processing, the stability of a frame is an important property that ensures the robustness of the signal representation against noise and errors.

One way to measure the stability of a frame is by considering the minimal smallest singular value of its submatrices.

The minimal smallest singular value of a matrix is the smallest singular value among all its submatrices. For a frame, this value provides a measure of how well-conditioned the frame is with respect to perturbations. A frame with a large minimal smallest singular value is considered to be stable, as small perturbations in the signal representation will not result in large errors in the reconstructed signal.

First, let us consider a frame

$$\mathcal{F} = \{f_1, f_2, \dots, f_n\} \tag{4.1}$$

for a finite-dimensional Hilbert space $\mathcal{H}$ with dimension d. The analysis operator A associated with this frame is defined as $A : \mathcal{H} \to \mathbb{C}^n$ such that

$$Af = (\langle f, f_1\rangle, \langle f, f_2\rangle, \dots, \langle f, f_n\rangle)^T \tag{4.2}$$

for any $f \in \mathcal{H}$. The frame operator S is defined as $S = A^*A$, where A^* denotes the adjoint of A.

The singular values of a matrix M are the square roots of the eigenvalues of M^*M. The minimal smallest singular value $\sigma_{\min}^{(d)}(\mathcal{F})$ of a frame $\mathcal{F}$ is defined as

$$\sigma_{\min}^{(d)}(\mathcal{F}) = \min_{J \subseteq [n], |J| = d} \sigma_{\min}(A_J), \tag{4.3}$$

where A_J denotes the submatrix of A consisting of columns indexed by J and $\sigma_{\min}(A_J)$ denotes the smallest singular value of A_J. This value provides a measure of how well-conditioned the frame is with respect to perturbations.

A frame with a large minimal smallest singular value is considered to be stable, as small perturbations in the signal representation will not result in large errors in the reconstructed signal. In contrast, a frame with a small minimal smallest singular value is considered to be unstable, as small perturbations can result in large errors.

In conclusion, the minimal smallest singular value of submatrices provides a useful measure for assessing the stability of frames for signal processing. By considering this value, we can construct stable frames that provide robust representations of signals against noise and errors.

4.2. On robustness of frames in signal processing

Stable frames with large minimal smallest singular values have many applications in signal processing.

One application of stable frames is in signal reconstruction. In many signal processing tasks, we need to reconstruct a signal from a set of measurements. By using a stable frame as a measurement matrix, we can improve the accuracy and robustness of the reconstruction process. For example, in compressed sensing, stable frames can be used to recover sparse signals from a small number of measurements.

Another application of stable frames is in denoising. In many signal processing tasks, we need to remove noise from a signal to improve its quality. By using a stable frame as a basis for representing the signal, we can improve the performance of denoising algorithms. For example, in wavelet denoising, stable frames can be used to provide a sparse representation of the signal that is robust against noise.

In addition to signal reconstruction and denoising, there are many other applications of stable frames in signal processing. For example, stable frames can be used in image processing, speech processing, and data compression. By using stable frames with large minimal smallest singular values, we can improve the performance of these algorithms and provide more accurate and robust representations of signals.

In conclusion, stable frames with large minimal smallest singular values have many applications in signal processing. By using these frames in tasks such as signal reconstruction and denoising, we can improve the performance of signal processing algorithms and provide more accurate and robust representations of signals.

4.3. Applications of minimal smallest singular value

The minimal smallest singular value of submatrices of a matrix or frame has numerous applications in various fields. This section provides a comprehensive list of these applications, along with relevant references to support the information.

4.3.1. Condition number

One of the applications of the minimal smallest singular value of submatrices is to compute the condition number of a matrix. The condition number provides information about the sensitivity of the matrix to changes in the input and is widely used in numerical analysis and scientific computing. The condition number of a matrix is the quotient of the largest singular values of the matrix over the smallest singular values of the matrix, and likewise for each of its submatrices. A high condition number indicates that the matrix is sensitive to changes in the input, while a low condition number indicates that the matrix is relatively insensitive to changes in the input.

4.3.2. Robustness analysis

Another application of the minimal smallest singular value of submatrices is to assess the robustness of a system. In fields such as control theory and optimization, it is important to be able to assess how a system will behave under uncertainties or perturbations. The minimal smallest singular value of submatrices can be used to quantify the robustness of a system by measuring its sensitivity to changes in the input.

4.3.3. Signal and image processing

Submatrices of a matrix or frame can also be used to represent signals or images. The minimal smallest singular value is utilized to analyze the quality of the representation. It helps in quantifying the accuracy of signal reconstruction and identifying the presence of noise or artifacts.

For example, in signal processing, the minimal smallest singular value of submatrices can be used to estimate the SNR (signal-to-noise ratio) of a signal. The SNR is a measure of the quality of a signal and is inversely proportional to the noise power. A high SNR indicates that the signal is of high quality, while a low SNR indicates that the signal is noisy.

In image processing, the minimal smallest singular value of submatrices can be used to estimate the quality of an image. The quality of an image is affected by a number of factors, including the amount of noise, the amount of compression, and the quality of the camera. The minimal smallest singular value can be used to quantify the impact of these factors on the quality of an image.

4.3.4. Data compression

The minimal smallest singular value of submatrices plays a crucial role in data compression techniques such as Singular Value Decomposition (SVD) and Principal Component Analysis (PCA) (see, for example, [50]). SVD and PCA are both linear dimensionality reduction techniques that can be used to reduce the size of a data set while preserving its essential features.

SVD decomposes a matrix into three matrices: a left singular matrix, a diagonal matrix of singular values, and a right singular matrix. The singular values of a matrix are arranged in descending order, with the largest singular value being the most important. PCA is a special case of SVD where the left and right singular matrices are the same.

In both SVD and PCA, the minimal smallest singular value is used to determine the number of principal components to retain. The principal components with the smallest singular values are the least important and can be discarded without significantly affecting the quality of the representation.

4.3.5. Image denoising

By analyzing the minimal smallest singular value of submatrices, one can also estimate the level of noise present in an image. This information is used in denoising algorithms to enhance the visual quality of images by reducing unwanted noise while preserving important details.

For example, in image denoising, the minimal smallest singular value of submatrices can be used to estimate the noise power. The noise power is a measure of the strength of the noise in an image. A high noise power indicates that the image is noisy, while a low noise power indicates that the image is relatively noise-free.

The minimal smallest singular value of submatrices can also be used to identify the type of noise present in an image. Different types of noise have different statistical properties, and the minimal smallest singular value can be used to identify the statistical properties of the noise in an image.

4.3.6. Optimization algorithms

The minimal smallest singular value of submatrices is employed in optimization algorithms to assess the convergence properties and efficiency. In optimization, the goal is to find the minimum or maximum of a function. Optimization algorithms are used to find the optimal solution to an optimization problem.

The minimal smallest singular value of submatrices can be used to assess the convergence properties of an optimization algorithm.

4.3.7. Machine learning

In machine learning tasks, such as feature selection, clustering, and classification, the minimal smallest singular value of submatrices can be used to evaluate the separability and discriminative power of features (see, for example, [35]). It aids in identifying informative and relevant features for accurate model training and prediction.

4.3.8. Network analysis

The minimal smallest singular value of submatrices is utilized in network analysis to analyze the connectivity and robustness of complex networks (see, for example, [31]). It helps identify critical nodes or components that are crucial for maintaining network functionality and resilience against failures or attacks.

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