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ON ANDO–HIAI TYPE INEQUALITIES FOR SECTORIAL MATRICES

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Abstract. In a recent paper of the same journal, Zhao, Zheng and Jiang generalized a norm inequality of Ando and Hiai for sectorial matrices. We first improve their main result by reducing the coefficient to a smaller one. We also present an analogous inequality involving the logarithmic mean.

1. Introduction

Let $\mathbb{M}_n$ be the set of all $n \times n$ complex matrices. The conjugate transpose of $A \in \mathbb{M}_n$ is denoted by A^*. We know that every $A \in \mathbb{M}_n$ admits the Cartesian decomposition

$$A = \Re A + i\Im A,$$

where $\Re A := (A+A^*)/2$ and $\Im A := (A-A^*)/2i$ are called the Hermitian part of A and the skew-Hermitian part of A, respectively. The matrix A is accretive if $\Re A$ is positive definite.

In [5], the geometric mean for two accretive matrices $A, B \in \mathbb{M}_n$ is defined as

$$A\sharp B = \left(\frac{2}{\pi}\int_0^\infty (tA + t^{-1}B^{-1})^{-1}\frac{dt}{t}\right)^{-1}.$$

A weighted version (see [17]) is given by

$$A\sharp_\alpha B = \frac{\sin\alpha\pi}{\pi}\int_0^\infty t^{\alpha-1}(A^{-1} + tB^{-1})^{-1}dt. \tag{1}$$

It could be easily verified that $A\sharp_{1/2}B = A\sharp B$. And when A, B are positive definite, $A\sharp_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$, which coincides with the usual definition of the weighted geometric mean for two positive definite matrices; see [17] for more details.

Recall that the numerical range of $A \in \mathbb{M}_n$ is defined as

$$W(A) = \{v^*Av : v \in \mathbb{C}^n,\ v^*v = 1\}.$$

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In the past few years, there have been many studies devoted to the class of matrices whose numerical range is contained in a sector on the complex plane

$$S_\theta = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leqslant (\Re z)\tan\theta\},$$

where $\theta \in [0, \pi/2)$. We refer the interested reader to [4, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18] and references therein. Sector matrices are intimately connected with positive definite matrices, one observes that if $W(A) \subset S_0$, then A is necessarily positive definite. Indeed, if $W(A) \subset S_\theta$, then $\Re A$ is positive definite. For two Hermitian matrices $A, B \in \mathbb{M}_n$, we write $A \geqslant B$ (or $B \leqslant A$) to mean that $A - B$ is positive semidefinite.

A norm on $\mathbb{M}_n$ is unitarily invariant if $\|UAV\| = \|A\|$ for any $A, U, V \in \mathbb{M}_n$ with U, V being unitary. The usual operator norm, Hilbert-Schmidt norm, trace norm are unitarily invariant.

Recently, Zhao, Zheng and Jiang [22] established the following norm inequality for sectorial matrices.

THEOREM 1.1. [22, Theorem 3.1] *Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\theta$ and let $0 \leqslant \alpha \leqslant 1$. Then for any unitarily invariant norm*

$$\|(A\sharp_\alpha B)^r\| \leqslant (\sec\theta)^{4+2r}\sec(r\theta)\|A^r\sharp_\alpha B^r\|,\ 0 \leqslant r \leqslant 1.$$

When $\theta = 0$, Theorem 1.1 reduces to the following remarkable result of Ando and Hiai proved in 1994.

THEOREM 1.2. [1] *Let $A, B \in \mathbb{M}_n$ be positive definite and let $0 \leqslant \alpha \leqslant 1$. Then for any unitarily invariant norm*

$$\|(A\sharp_\alpha B)^r\| \leqslant \|A^r\sharp_\alpha B^r\|, \quad 0 \leqslant r \leqslant 1.$$

The goal of this paper is twofold. Firstly, we improve Theorem 1.1 by reducing the coefficient $(\sec\theta)^{4+2r}\sec(r\theta)$ to a smaller one. Secondly, we present an analogue of Theorem 1.1 involving the logarithmic mean, though we conjecture that the analogous result is also true for all unitarily invariant norms.

Recall that the logarithmic mean of two positive numbers a and b, which is of interest in geometry, statistics, and thermodynamics, is defined as

$$L(a,b) = \frac{a-b}{\log a - \log b}.$$

The logarithmic mean has different expressions, among which two frequently used formulas are as follows (e.g. [3])

$$L(a,b) = \int_0^1 a^{1-t}b^t\,\mathrm{d}t \tag{2}$$

and

$$L(a,b) = \left[\int_0^1 \frac{dt}{(1-t)a+tb}\right]^{-1}. \tag{3}$$

Apparently inspired by (2), the authors of [20] defined the logarithmic mean for two accretive matrices $A,B \in \mathbb{M}_n$ as

$$L(A,B) = \int_0^1 A\sharp_t B \,\mathrm{d}t$$

and studied many of its properties. Later, it was shown that (3) could serve the same purpose, that is,

$$\int_0^1 A\sharp_t B \,\mathrm{d}t = \left[\int_0^1 ((1-t)A+tB)^{-1}dt\right]^{-1};$$

see [13, Proposition 2.1] for more details.

2. Main results

Before presenting the main results, we give some auxiliary tools.

LEMMA 2.1. *Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\theta$. Then*

$$\Re A^{-1} \leqslant (\Re A)^{-1} \leqslant (\sec\theta)^2 \Re A^{-1}.$$

Proof. The first inequality is from [9, Lemma 2.4] while the second one is from [10, Lemma 3]. □

LEMMA 2.2. [5, Corollary 2.4] *Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\theta$. Then $W(A^r) \subset S_{r\theta}$ for any $0 \leqslant r \leqslant 1$.*

LEMMA 2.3. *Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\theta$. Then*

$$\|\Re A\| \leqslant \|A\| \leqslant \sec\theta \|\Re A\|.$$

Proof. The first inequality is from [21, Lemma 3.1] while the second one is from [2, p. 74]. □

If $A \in \mathbb{M}_n$ is accretive, it follows from (1) immediately that for any $0 \leqslant r \leqslant 1$,

$$A^r = I\sharp_r A = \frac{\sin r\pi}{\pi}\int_0^\infty t^{r-1}(I+tA^{-1})^{-1}dt. \tag{4}$$

LEMMA 2.4. *Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\theta$ and let $0 \leqslant r \leqslant 1$. Then*

$$(\Re A)^r \leqslant \Re A^r \leqslant (\sec\theta)^{2r}(\Re A)^r.$$

Proof. The first inequality is from [22, Proposition 2.7]. Now we proceed to the proof of the second inequality. By the first inequality in Lemma 2.1,

$$\Re(I+tA^{-1})^{-1} \leqslant (I+t\Re A^{-1})^{-1}.$$

By the second inequality in Lemma 2.1,

$$\begin{aligned} I+t\Re A^{-1} &\geqslant I+t(\cos\theta)^2(\Re A)^{-1} \\ &= I+t((\sec\theta)^2\Re A)^{-1}, \end{aligned}$$

and so

$$\Re(I+tA^{-1})^{-1} \leqslant (I+t((\sec\theta)^2\Re A)^{-1})^{-1}.$$

Now by (4),

$$\begin{aligned} \Re A^r &= \frac{\sin r\pi}{\pi}\int_0^\infty \Re(I+tA^{-1})^{-1}t^{r-1}\,dt \\ &\leqslant \frac{\sin r\pi}{\pi}\int_0^\infty (I+t((\sec\theta)^2\Re A)^{-1})^{-1}t^{r-1}\,dt \\ &= ((\sec\theta)^2\Re A)^r = (\sec\theta)^{2r}(\Re A)^r. \quad \square \end{aligned}$$

LEMMA 2.5. *Let $A,B\in\mathbb{M}_n$ with $W(A),W(B)\subset S_\theta$. Then*

$$(\Re A)\sharp_\alpha(\Re B) \leqslant \Re(A\sharp_\alpha B) \leqslant (\sec\theta)^2((\Re A)\sharp_\alpha(\Re B)).$$

The first inequality is from [17, Theorem 2.4] while the second inequality is from [22, Proposition 2.8], see also[19, Lemma 5].

LEMMA 2.6. [20] *Let $A,B\in\mathbb{M}_n$ with $W(A),W(B)\subset S_\theta$. Then*

$$L(\Re A,\Re B) \leqslant \Re L(A,B) \leqslant (\sec\theta)^2 L(\Re A,\Re B).$$

We are in a position to improve Theorem 1.1. Our first result reads as follows.

THEOREM 2.7. *Let $A,B\in\mathbb{M}_n$ with $W(A),W(B)\subset S_\theta$ and let $0\leqslant\alpha\leqslant 1$. Then for any unitarily invariant norm*

$$\|(A\sharp_\alpha B)^r\| \leqslant (\sec\theta)^{4r}\sec(r\theta)\|A^r\sharp_\alpha B^r\|,\ 0\leqslant r\leqslant 1.$$

Proof. It is easy to see that $W(A\sharp_\alpha B) \subset S_\theta$. Therefore, by Lemma 2.2, $W((A\sharp_\alpha B)^r)\subset S_{r\theta}$. By the second inequality in Lemma 2.3,

$$\|(A\sharp_\alpha B)^r\| \leqslant \sec(r\theta)\|\Re(A\sharp_\alpha B)^r\|. \tag{5}$$

Now we estimate

$$\begin{aligned} \|\Re(A\sharp_\alpha B)^r\| &\leqslant (\sec\theta)^{2r}\|(\Re(A\sharp_\alpha B))^r\| && \text{by Lemma 2.4} \\ &\leqslant (\sec\theta)^{2r}\left\|\Big((\sec\theta)^2((\Re A)\sharp_\alpha(\Re B))\Big)^r\right\| && \text{by Lemma 2.5} \\ &= (\sec\theta)^{4r}\left\|\Big((\Re A)\sharp_\alpha(\Re B)\Big)^r\right\| \\ &\leqslant (\sec\theta)^{4r}\|(\Re A)^r\sharp_\alpha(\Re B)^r\| && \text{by Theorem 1.2} \\ &\leqslant (\sec\theta)^{4r}\|(\Re A^r)\sharp_\alpha(\Re B^r)\| && \text{by Lemma 2.4} \\ &\leqslant (\sec\theta)^{4r}\|\Re(A^r\sharp_\alpha B^r)\| && \text{by Lemma 2.5} \\ &\leqslant (\sec\theta)^{4r}\|A^r\sharp_\alpha B^r\|. \end{aligned}$$

That is,

$$\|\Re(A\sharp_\alpha B)^r\| \leqslant (\sec\theta)^{4r}\|A^r\sharp_\alpha B^r\|. \tag{6}$$

The desired result follows from (5) and (6). □

The following result is a new inequality involving the logarithmic mean.

THEOREM 2.8. *Let $A,B\in\mathbb{M}_n$ be positive definite with some positive numbers m,M such that $0<m\leqslant A,B\leqslant M$. Then for any $0\leqslant r\leqslant 1$*

$$L(A,B)^r \leqslant \kappa^r L(A^r,B^r),$$

where $\kappa=\frac{M}{m}$.

Proof. Since it is clear that for any $0\leqslant t\leqslant 1$,

$$((1-t)A+tB)^r \leqslant M^r$$

and

$$(1-t)A^r+tB^r \geqslant m^r,$$

we have

$$((1-t)A+tB)^r \leqslant \kappa^r\Big((1-t)A^r+tB^r\Big).$$

Taking inverses on both sides yields

$$\kappa^r((1-t)A+tB)^{-r} \geqslant ((1-t)A^r+tB^r)^{-1},$$

and so

$$\kappa^r\int_0^1((1-t)A+tB)^{-r}dt \geqslant \int_0^1((1-t)A^r+tB^r)^{-1}dt.$$

Taking inverses on both sides again yields

$$\begin{aligned}\left(\int_0^1((1-t)A+tB)^{-r}dt\right)^{-1} &\leqslant \kappa^r\left(\int_0^1((1-t)A^r+tB^r)^{-1}dt\right)^{-1}\\ &= \kappa^r L(A^r,B^r).\end{aligned} \tag{7}$$

On the other hand, let $X(t)=((1-t)A+tB)^{-1}$, $t\in[0,1]$. Since $f(x)=x^r$ is operator concave for $0\leqslant r\leqslant 1$, we have

$$\frac{\sum_{k=1}^n X^r(k/n)}{n} \leqslant \left(\frac{\sum_{k=1}^n X(k/n)}{n}\right)^r,$$

which is true for all positive integer n. Letting $n\to\infty$ yields

$$\int_0^1 X^r(t)dt \leqslant \left(\int_0^1 X(t)dt\right)^r,$$

that is,

$$\int_0^1 ((1-t)A+tB)^{-r}dt \leqslant \left(\int_0^1 ((1-t)A+tB)^{-1}dt\right)^r.$$

Taking inverses on both sides again yields

$$\left(\int_0^1 ((1-t)A+tB)^{-r}dt\right)^{-1} \geqslant \left(\int_0^1 ((1-t)A+tB)^{-1}dt\right)^{-r} = L(A,B)^r. \tag{8}$$

Combining (7) and (8) gives

$$L(A,B)^r \leqslant \kappa^r L(A^r,B^r).$$

This completes the proof. $\square$

The previous theorem immediately yields the following norm inequality.

COROLLARY 2.9. *Let $A,B \in \mathbb{M}_n$ be positive definite with some positive numbers m,M such that $0 < m \leqslant A,B \leqslant M$. Then for any unitarily invariant norm and $0 \leqslant r \leqslant 1$,*

$$\|L(A,B)^r\| \leqslant \kappa^r \|L(A^r,B^r)\|,$$

equivalently,

$$\left\|\left(\int_0^1 A\sharp_\alpha B d\alpha\right)^r\right\| \leqslant \kappa^r \left\|\int_0^1 A^r \sharp_\alpha B^r d\alpha\right\|,$$

where $\kappa = \frac{M}{m}$.

REMARK 2.10. Under the same condition as Corollary 2.9, we are not sure whether it is true that

$$\left\|\left(\int_0^1 A\sharp_\alpha B d\alpha\right)^r\right\| \leqslant \left\|\int_0^1 A^r \sharp_\alpha B^r d\alpha\right\|.$$

If this were true, then it would be a nice complement of Theorem 1.2.

The last result of the paper is an extension of Corollary 2.9 to sectorial matrices.

THEOREM 2.11. *Let $A,B \in \mathbb{M}_n$ with $W(A),W(B) \subset S_\theta$ and with some positive numbers m,M such that $0 < m \leqslant \Re A, \Re B \leqslant M$. Then for any unitarily invariant norm and $0 \leqslant r \leqslant 1$,*

$$\|L(A,B)^r\| \leqslant (\sec r\theta)(\sec\theta)^{4r}\kappa^r \|L(A^r,B^r)\|,$$

where $\kappa = \frac{M}{m}$.

Proof. It is easy to see that $W(L(A,B)) \subset S_\theta$ and so by Lemma 2.2, $W(L(A,B)^r) \subset S_{r\theta}$. By the second inequality of Lemma 2.3, we have

$$\|L(A,B)^r\| \leqslant \sec r\theta \|\Re L(A,B)^r\|. \tag{9}$$

We proceed to estimate

$$\begin{aligned}
\|\Re L(A,B)^r\| &\leqslant (\sec\theta)^{2r} \left\| \left(\Re L(A,B)\right)^r \right\| && \text{by Lemma 2.4} \\
&\leqslant (\sec\theta)^{2r} \left\| \left((\sec\theta)^2 L(\Re A, \Re B)\right)^r \right\| && \text{by Lemma 2.6} \\
&= (\sec\theta)^{4r} \left\| \left(L(\Re A, \Re B)\right)^r \right\| \\
&\leqslant (\sec\theta)^{4r} \kappa^r \left\| L\left((\Re A)^r, (\Re B)^r\right) \right\| && \text{by Corollary 2.9} \\
&\leqslant (\sec\theta)^{4r} \kappa^r \left\| L\left(\Re A^r, \Re B^r\right) \right\| && \text{by Lemma 2.4} \\
&\leqslant (\sec\theta)^{4r} \kappa^r \left\| \Re L\left(A^r, B^r\right) \right\| && \text{by Lemma 2.6} \\
&\leqslant (\sec\theta)^{4r} \kappa^r \left\| L\left(A^r, B^r\right) \right\| && \text{by Lemma 2.3.}
\end{aligned}$$

That is,

$$\|\Re L(A,B)^r\| \leqslant (\sec\theta)^{4r} \kappa^r \left\| L\left(A^r, B^r\right) \right\|. \tag{10}$$

The desired result follows from (9) and (10). This completes the proof. □

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