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ON THE ESTIMATION OF q–NUMERICAL RADIUS OF HILBERT SPACE OPERATORS

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Abstract. The objective of this article is to estimate the q-numerical radius of bounded linear operators on complex Hilbert spaces. One of our main results states that for a bounded linear operator T in a Hilbert space $\mathcal{H}$ and $q \in [0,1]$, the relation

$$\omega_q^2(T) \leqslant q^2\omega^2(T) + (1 - q^2 + q\sqrt{1-q^2})\|T\|^2$$

holds where $\omega(T)$, $\omega_q(T)$ are the numerical radius and q-numerical radius of T respectively. Several refined new upper bounds follow from this result. Finally, the q-numerical radius of 2×2 operator matrices is explored and several new results are established.

1. Introduction

Let $\mathcal{H}$ denotes a complex Hilbert space with the inner product $\langle .,. \rangle$ and $\mathcal{B}(\mathcal{H})$ denotes the C^*-algebra of bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, the operator norm of T can be defined as

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

Another expression of $\|T\|$ in terms of the inner product is as follows

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|.$$

A norm $|||.|||$ on $\mathcal{B}(\mathcal{H})$ is said to be a unitarily invariant norm if it satisfies $|||UTV||| = |||T|||$ for all $T \in \mathcal{B}(\mathcal{H})$ and for all unitary operators U and V in $\mathcal{B}(\mathcal{H})$. A norm $|||.|||$ on $\mathcal{B}(\mathcal{H})$ is said to be a weakly unitarily invariant norm if it satisfies $|||UTU^*||| = |||T|||$ for all $T \in \mathcal{B}(\mathcal{H})$ and for all unitary operators U in $\mathcal{B}(\mathcal{H})$.

The numerical range of T is denoted and defined by

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H},\ \|x\| = 1\}.$$

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The most important properties of $W(T)$ are that it always forms a convex set and its closure contains the spectrum of T. The numerical radius $\omega(T)$ and the Crawford number $c(T)$ of $T \in \mathcal{B}(\mathcal{H})$ are defined by

$$\begin{aligned}\omega(T) &= \sup_{\|x\|=1} |\langle Tx, x\rangle|, \\ c(T) &= \inf_{\|x\|=1} |\langle Tx, x\rangle|.\end{aligned}$$

The numerical radius $\omega(T)$ defines a weakly unitarily invariant norm in $\mathcal{B}(\mathcal{H})$. The operator norm and numerical radius are both equivalent which follows from the following well-known inequality

$$\frac{\|T\|}{2} \leqslant \omega(T) \leqslant \|T\|. \tag{1}$$

The above inequalities are sharp. Equality holds in the first inequality if $T^2 = 0$ and in the second inequality if T is a normal operator. For the past several years researchers have attempted to refine the above inequality. Kittaneh [14, 15] proved, respectively, that, if $T \in \mathcal{B}(\mathcal{H})$, then

$$\omega(T) \leqslant \frac{1}{2}\||T| + |T^*|\| \leqslant \frac{1}{2}\left(\|T\| + \sqrt{\|T^2\|}\right), \tag{2}$$

$$\frac{1}{4}\|T^*T + TT^*\| \leqslant \omega^2(T) \leqslant \frac{1}{2}\|T^*T + TT^*\|, \tag{3}$$

and if $A, B, C, D, S, T \in \mathcal{B}(\mathcal{H})$ then

$$\omega(ATB + CSD) \leqslant \frac{1}{2}\|A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2\alpha}B + C|S^*|^{2(1-\alpha)}C^* + D^*|S|^{2\alpha}D\|, \tag{4}$$

for all $\alpha \in [0,1]$ where $|T| = (T^*T)^{\frac{1}{2}}$, the absolute value of T. Later on these bounds were refined extensively. In recent years, there has been a major advancement in the estimation of the numerical range. For a detailed review of the numerical radius inequalities, we refer to the articles [1, 4, 5, 16, 21, 23], their references, and the book [3].

There are several generalizations of the classical numerical range in the literature. Our focus will be on the q-numerical range and its radius of an operator. Let $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. The q-numerical range $W_q(T)$ and q-numerical radius $\omega_q(T)$ of T are defined respectively as

$$\begin{aligned}W_q(T) &= \{\langle Tx, y\rangle : x, y \in \mathcal{H},\ \|x\| = \|y\| = 1,\ \langle x, y\rangle = q\}, \\ \omega_q(T) &= \sup_{z \in W_q(T)} |z|.\end{aligned}$$

It is easy to verify that if $q = 1$ then $W_q(T)$ reduces to the classical numerical range $W(T)$. The set $W_q(T)$ was first introduced by Marcus and Andresen [19] in 1977 for a linear transformation T defined over an n-dimensional unitary space. Nam-Kiu Tsing [25] established the convexity of the q-numerical range. Several properties of $W_q(T)$ are discussed by Li et al. [17] and Li and Nakazato [18]. Chien and Nakazato [7] described the boundary of the q-numerical range of a square matrix using the concept of the Davis-Wieldant shell. The q-numerical range of shift operators is also studied [6,8]. Duan [9] draws attention to the vital significance that the idea of q-numerical range plays in characterizing the perfect distinguishability of quantum operations. Recently, Moghaddam et al. [10] have studied several q-numerical radius bounds. The following are a few of the inequalities they have derived:

$$\frac{q}{2(2-q^2)}\|T\| \leqslant \omega_q(T) \leqslant \|T\|, \tag{5}$$

$$\omega_q^2(T) \leqslant \frac{q^2}{4}\left(\|T\|+\sqrt{\|T^2\|}\right)^2 + (1-q^2+2q\sqrt{1-q^2})\||T|\|^2, \tag{6}$$

$$\frac{q^2}{4(2-q^2)^2}\|T^*T+TT^*\| \leqslant \omega_q^2(T) \leqslant \frac{(q+2\sqrt{1-q^2})^2}{2}\|T^*T+TT^*\|. \tag{7}$$

These results are in fact generalizations of the corresponding inequalities in (1), (2), and (3) respectively for numerical radius.

Our interest in this paper lies in the direction of obtaining refined q-numerical radius inequalities. In section 2, we establish upper bounds for q-numerical radius that generalize the results of [10]. In addition, the q-numerical radius bounds for 2×2 operator matrices are also discussed in section 3. Several examples with figures are provided to supplement the results.

2. q-numerical radius of $T \in \mathcal{B}(\mathcal{H})$

First, we recall a few important properties of the q-numerical radius in the following lemmas.

LEMMA 2.1. [12, p. 380] *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $q \in [0,1]$. *Then*

(i) *if* $\dim\mathcal{H} = 1$ *then* $W_q(T)$ *is non-empty if and only if* $q = 1$ *and for* $\dim\mathcal{H} \geqslant 2$, $W_q(T)$ *is always non-empty,*

(ii) $W_q(T)$ *is a bounded subset of* $\mathbb{C}$ *and it is compact if* $\mathcal{H}$ *is finite-dimensional,*

(iii) $W_q(U^*TU) = W_q(T)$ *for any unitary operator* $U \in \mathcal{B}(\mathcal{H})$,

(iv) $W_q(aT+bI) = aW_q(T)+bq$ *for complex numbers* a *and* b,

(v) $W_{\lambda q}(T) = \lambda W_q(T)$ *for any complex numeber* λ *with* $|\lambda| = 1$,

(vi) $W_q(T)^* = W_q(T)^* = \{\bar{z} : z \in W_q(T)\}$,

(vii) $q\sigma(T) \subseteq \overline{W_q(T)}$.

LEMMA 2.2. [10] *The q-numerical radius defines a semi-norm on $\mathcal{B}(\mathcal{H})$.*

LEMMA 2.3. [17, Proposition 2.11] *Let $T \in \mathcal{B}(\mathcal{H})$ and $m(T) = \min\{\|T-\lambda I\| : \lambda \in \mathbb{C}\}$. Then*

$$\text{either } W_0(T) = \{z : |z| < m(T)\} \quad \text{or, } \quad W_0(T) = \{z : |z| \leqslant m(T)\}.$$

The number $m(T)$ is known as the transcendental radius of T. Stampfli [24] proved that there exists a unique complex number $\mu \in \overline{W(T)}$ such that

$$m(T) = \min\{\|T-\lambda I\| : \lambda \in \mathbb{C}\} = \|T-\mu I\|.$$

Prasanna [22] derived another expression for $m(T)$ which is

$$m^2(T) = \sup_{\|x\|=1} (\|Tx\|^2 - |\langle Tx, x\rangle|^2).$$

For our study, the following results are crucial.

LEMMA 2.4. (Bessel's Inequality) *Let $\mathcal{E}$ be a orthonormal set in $\mathcal{H}$ and $h \in \mathcal{H}$. Then*

$$\sum_{e \in \mathcal{E}} |\langle h, e\rangle|^2 \leqslant \|h\|^2.$$

LEMMA 2.5. [11] *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$|\langle Tx, y\rangle|^2 \leqslant \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle,$$

for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$.

Now we are ready to prove the q-numerical radius inequalities.

THEOREM 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Then*

$$\omega_q^2(T) \leqslant q^2 \omega^2(T) + (1 - q^2 + q\sqrt{1-q^2})\|T\|^2. \tag{8}$$

Proof. For $q = 1$ the relation holds trivially. Let $q \in [0,1)$ and $x, y \in \mathcal{H}$ such that $\|x\| = 1 = \|y\|$ with $\langle x, y\rangle = q$. Then y can be expressed as $y = qx + \sqrt{1-q^2}z$, where $\|z\| = 1$ and $\langle x, z\rangle = 0$. In this setting we have

$$|\langle Tx, y\rangle| \leqslant q|\langle Tx, x\rangle| + \sqrt{1-q^2}|\langle Tx, z\rangle|. \tag{9}$$

Let $\mathcal{E}$ be any orthonormal set in $\mathcal{H}$ containing x and z. From Bessel's Inequality it follows

$$\sum_{e \in \mathcal{E}\setminus\{x\}} |\langle Tx, e\rangle|^2 + |\langle Tx, x\rangle|^2 \leqslant \|Tx\|^2$$
$$\Rightarrow |\langle Tx, z\rangle|^2 \leqslant \sum_{e \in \mathcal{E}\setminus\{x\}} |\langle Tx, e\rangle|^2 \leqslant \|Tx\|^2 - |\langle Tx, x\rangle|^2.$$

Using the above relation in equation (9), we get

$$\begin{aligned}
|\langle Tx,y\rangle| &\leqslant q|\langle Tx,x\rangle| + \sqrt{1-q^2}(\|Tx\|^2 - |\langle Tx,x\rangle|^2)^{\frac{1}{2}} \qquad (10)\\
\Rightarrow |\langle Tx,y\rangle|^2 &\leqslant \left(q|\langle Tx,x\rangle| + \sqrt{1-q^2}(\|Tx\|^2 - |\langle Tx,x\rangle|^2)^{\frac{1}{2}}\right)^2\\
&= q^2|\langle Tx,x\rangle|^2 + 2q\sqrt{1-q^2}|\langle Tx,x\rangle|(\|Tx\|^2 - |\langle Tx,x\rangle|^2)^{\frac{1}{2}}\\
&\quad +(1-q^2)(\|Tx\|^2 - |\langle Tx,x\rangle|^2)\\
&\leqslant q^2|\langle Tx,x\rangle|^2 + 2q\sqrt{1-q^2}|\langle Tx,x\rangle|(\|Tx\|^2 - |\langle Tx,x\rangle|^2)^{\frac{1}{2}}\\
&\quad +(1-q^2)\|Tx\|^2\\
&\leqslant q^2|\langle Tx,x\rangle|^2 + q\sqrt{1-q^2}(|\langle Tx,x\rangle|^2 + \|Tx\|^2 - |\langle Tx,x\rangle|^2)\\
&\quad +(1-q^2)\|Tx\|^2 \quad \text{(applying AM-GM inequality)}\\
&= q^2|\langle Tx,x\rangle|^2 + (1-q^2+q\sqrt{1-q^2})\|Tx\|^2\\
&\leqslant q^2\omega^2(T) + (1-q^2+q\sqrt{1-q^2})\|T\|^2.
\end{aligned}$$

Taking supremum for all $x,y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$ and $\langle x,y\rangle = q$ we get

$$\omega_q^2(T) \leqslant q^2\omega^2(T) + (1-q^2+q\sqrt{1-q^2})\|T\|^2. \quad \square$$

Now we mention a few observations and derive several corollaries based on the above Theorem.

1. From the result $\omega(T) \leqslant \frac{1}{2}(\|T\| + \|T^2\|^{\frac{1}{2}})$, mentioned in Theorem 1 of [14], we have the following corollary from Theorem 2.6.

COROLLARY 2.7. *For $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$ we have*

$$\omega_q^2(T) \leqslant \frac{q^2}{4}(\|T\| + \|T^2\|^{\frac{1}{2}})^2 + (1-q^2+q\sqrt{1-q^2})\|T\|^2. \tag{11}$$

Using the fact $\||T|\| = \|T\|$, we have the following relations

$$\begin{aligned}
\omega_q^2(T) &\leqslant \frac{q^2}{4}(\|T\| + \|T^2\|^{\frac{1}{2}})^2 + (1-q^2+q\sqrt{1-q^2})\|T\|^2\\
&\leqslant \frac{q^2}{4}(\|T\| + \|T^2\|^{\frac{1}{2}})^2 + (1-q^2+2q\sqrt{1-q^2})\||T|\|^2.
\end{aligned}$$

This shows that the inequality mentioned in (11) provides an improvement on the result (6), proved in Theorem 2.10 in [10].

2. From the relation $\omega^2(T) \leqslant \frac{1}{2}\|T^*T + TT^*\|$ as obtained in Theorem 1 in [15] we have the following corollary which follows from the Theorem 2.6.

COROLLARY 2.8. *For $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$, the following relation holds*

$$\omega_q^2(T) \leqslant \frac{q^2}{2}\|T^*T+TT^*\| + (1-q^2+q\sqrt{1-q^2})\|T\|^2. \tag{12}$$

Here we prove the refinement of the above result in comparison to the existing upper bound (7) of $\omega^2(T)$ mentioned in Theorem 3.1 in [10]. For this, we use the fact that

$$\|T^*T+TT^*\| \geqslant \|T\|^2.$$

From Corollary 2.8 it follows

$$\begin{aligned}\omega_q^2(T) &\leqslant \frac{q^2}{2}\|T^*T+TT^*\| + (1-q^2+q\sqrt{1-q^2})\|T\|^2 \\ &\leqslant \left(1-\frac{q^2}{2}+q\sqrt{1-q^2})\right)\|T^*T+TT^*\| \\ &\leqslant \frac{(q+2\sqrt{1-q^2})^2}{2}\|T^*T+TT^*\|.\end{aligned}$$

The last inequality follows from the fact that

$$\begin{aligned}&\frac{(q+2\sqrt{1-q^2})^2}{2} - \left(1-\frac{q^2}{2}+q\sqrt{1-q^2})\right) \\ &= \frac{2q^2+4(1-q^2)-2+2q\sqrt{1-q^2}}{2} \geqslant 0.\end{aligned}$$

REMARK 1. It is crucial to note that the upper bound of $\omega_q^2(T)$ given in equation (7) reduces to $2\|T^*T+TT^*\|$ when q approaches zero whereas the upper bound stated in Corollary 2.8 reduces to $\|T\|^2$ as $q=0$.

Now we provide an example to demonstrate our results. For this, the following lemma on the q-numerical radius of 2×2 matrices is required.

LEMMA 2.9. ([20]) *Suppose $0 \leqslant q \leqslant 1$ and $T \in M_2(\mathbb{C})$. Then T is unitarily similar to $e^{it}\begin{pmatrix}\gamma & a\\ b & \gamma\end{pmatrix}$ for some $0 \leqslant t \leqslant 2\pi$ and $0 \leqslant b \leqslant a$. Also,*

$$W_q(T) = e^{it}\{\gamma q + r((c+pd)\cos s + i(d+pc)\sin s) : 0 \leqslant r \leqslant 1, 0 \leqslant s \leqslant 2\pi\},$$

with $c = \frac{a+b}{2}, d = \frac{a-b}{2}$ and $p = \sqrt{1-q^2}$.

EXAMPLE 2.10. Consider the matrix $T = \begin{pmatrix}\frac{1}{200} & \frac{1}{25}\\ \frac{1}{36} & \frac{1}{200}\end{pmatrix}$. From Lemma 2.9,

$$\begin{aligned}W_q(T) = \Big\{&\frac{q}{200} + \frac{r}{1800}\Big((61+11\sqrt{1-q^2})\cos s + i(11+61\sqrt{1-q^2})\sin s\Big) : \\ &0 \leqslant r \leqslant 1, 0 \leqslant s \leqslant 2\pi)\Big\}.\end{aligned}$$

Therefore, $\omega_q(T) = \frac{q}{200} + \frac{1}{1800}(61 + 11\sqrt{1-q^2})$. Also $\|T\| = 0.0418$ and $\|T^2\| = 0.0015$, computed correct upto four decimal places using MATLAB command `norm`. In this example we will compare the upper bounds (6) and (7) of $\omega_q(T)$ obtained in [10] with our results Corollary 2.7 and Corollary 2.8. From the relations (6) and (11) we respectively get

$$\omega_q(T) \leqslant \sqrt{0.0017 - 0.0001q^2 + 0.0035q\sqrt{1-q^2}} = f_1(q) \tag{13}$$

and

$$\omega_q(T) \leqslant \sqrt{0.0017 - 0.0001q^2 + 0.0017q\sqrt{1-q^2}} = f_2(q). \tag{14}$$

Figure 1 presents the graphical representation of upper bounds (13), (14) and $\omega_q(T)$.

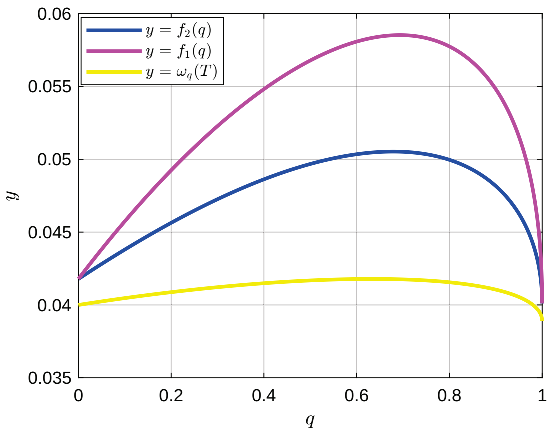

Figure 1: *Comparision of $\omega_q(T)$ with upper bounds $f_1(q)$ and $f_2(q)$ for Example 2.10*

Again using MATLAB, $\|T^*T + TT^*\| = 0.0031$. Hence from inequalities (7) and (12) we get the upper bounds

$$\omega_q(T) \leqslant 0.0394(q + 2\sqrt{1-q^2}) = f_3(q) \tag{15}$$

and

$$\omega_q(T) \leqslant \sqrt{0.0017 - 0.0002q^2 + 0.0017q\sqrt{1-q^2}} = f_4(q). \tag{16}$$

Figure 2 presents the graphical representation of upper bounds (15), (16) and $\omega_q(T)$.

It is clear from Figure 1 and 2 that upper bounds obtained using Corollary 2.7 and Corollary 2.8 are more refined than those of [10].

Using the concept of Crawford number $c(T)$, we can further refine Theorem 2.6 and also the corresponding corollaries 2.7 and 2.8 as follows.

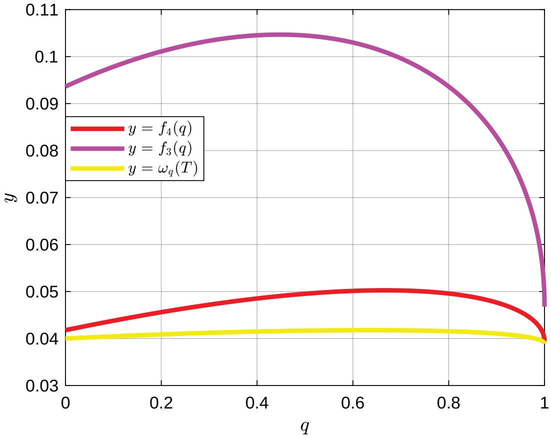

Figure 2: *Comparision of $\omega_q(T)$ with upper bounds $f_3(q)$ and $f_4(q)$ for Example 2.10*

THEOREM 2.11. *Let $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Then*

$$\omega_q^2(T) \leqslant q^2\omega^2(T) + (1-q^2+q\sqrt{1-q^2})\|T\|^2 - (1-q^2)c(T). \tag{17}$$

Proof. For $q = 1$ the inequality holds trivially. Let $q \in [0,1)$. In accordance with the proof of Theorem 2.6, it follows from the relation (10) that

$$\begin{aligned}
|\langle Tx,y\rangle| &\leqslant q|\langle Tx,x\rangle| + \sqrt{1-q^2}(\|Tx\|^2 - |\langle Tx,x\rangle|^2)^{\frac{1}{2}} \\
\Rightarrow |\langle Tx,y\rangle|^2 &\leqslant \left(q|\langle Tx,x\rangle| + \sqrt{1-q^2}(\|Tx\|^2 - |\langle Tx,x\rangle|^2)^{\frac{1}{2}}\right)^2 \\
&= q^2|\langle Tx,x\rangle|^2 + 2q\sqrt{1-q^2}|\langle Tx,x\rangle|(\|Tx\|^2 - |\langle Tx,x\rangle|^2)^{\frac{1}{2}} \\
&\quad + (1-q^2)(\|Tx\|^2 - |\langle Tx,x\rangle|^2) \\
&\leqslant q^2|\langle Tx,x\rangle|^2 + (1-q^2+q\sqrt{1-q^2})\|Tx\|^2 - (1-q^2)|\langle Tx,x\rangle|^2 \\
&\leqslant q^2\omega^2(T) + (1-q^2+q\sqrt{1-q^2})\|T\|^2 - (1-q^2)c^2(T),
\end{aligned}$$

where the second to last inequality follows from the AM-GM inequality. This proves the desired result. □

REMARK 2. Clearly the relation (17) improves the relation (8) of Theorem 2.6 when $c(T) > 0$. If $c(T) = 0$, then the the upper bound in (17) reduces to the upper bound of (8). In this regard it is worth mentioning that $c(T) > 0$ if and only if $0 \notin \overline{W(T)}$.

The Corollary 2.7 and Corollary 2.8 can be improved by using the Theorem 2.11 as follows.

COROLLARY 2.12. *For $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$ we have*

$$\omega_q^2(T) \leqslant \frac{q^2}{4}(\|T\| + \|T^2\|^{\frac{1}{2}})^2 + (1-q^2+q\sqrt{1-q^2})\|T\|^2 - (1-q^2)c^2(T).$$

COROLLARY 2.13. *For $T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$, the following relation holds*

$$\omega_q^2(T) \leqslant \frac{q^2}{2}\|T^*T + TT^*\| + (1-q^2+q\sqrt{1-q^2})\|T\|^2 - (1-q^2)c^2(T).$$

Now we provide a more general q-numerical radius inequality from which several other inequalities related to products and commutators of operators follows. This result is q-numerical radius version of the inequality (4).

THEOREM 2.14. *If $A,B,C,D,S,T \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$, then*

$$\begin{aligned}\omega_q(ATB+CSD) \leqslant\ & \frac{q}{2}\|B^*|T|^{2\alpha}B + A|T^*|^{2(1-\alpha)}A^* + D^*|S|^{2\alpha}D + C|S^*|^{2(1-\alpha)}C^*\| \\ & +\left(\sqrt{1-q^2} + \sqrt{2q\sqrt{1-q^2}}\right)\left(\sqrt{\|B^*|T|^{2\alpha}B\|\|A|T^*|^{2(1-\alpha)}A^*\|}\right. \\ & \left. + \sqrt{\|D^*|S|^{2\alpha}D^*\|\|C|S^*|^{2(1-\alpha)}C^*\|}\right).\end{aligned}$$

Proof. The case $q=1$ follows directly from the relation (4) which was derived in [15]. Let $x,y \in \mathcal{H}$ such that $\|x\| = \|y\| = 1$ with $\langle x,y\rangle = q$. Then we have $y = qx + \sqrt{1-q^2}z$ where $\|z\| = 1$ and $\langle x,z\rangle = 0$. Then from Lemma 2.5 we have

$$\begin{aligned}& |\langle (ATB+CSD)x,y\rangle| \\ \leqslant\ & |\langle TBx, A^*y\rangle| + |\langle SDx, C^*y\rangle| \\ \leqslant\ & \langle B^*|T|^{2\alpha}Bx,x\rangle^{\frac{1}{2}}\left\langle A|T^*|^{2(1-\alpha)}A^*y,y\right\rangle^{\frac{1}{2}} + \langle D^*|S|^{2\alpha}Dx,x\rangle^{\frac{1}{2}}\left\langle C|S^*|^{2(1-\alpha)}C^*y,y\right\rangle^{\frac{1}{2}} \\ \leqslant\ & \langle B^*|T|^{2\alpha}Bx,x\rangle^{\frac{1}{2}}\left(q^2\left\langle A|T^*|^{2(1-\alpha)}A^*x,x\right\rangle + (1-q^2)\left\langle A|T^*|^{2(1-\alpha)}A^*z,z\right\rangle\right. \\ & \left. +2q\sqrt{1-q^2}\left|\left\langle A|T^*|^{2(1-\alpha)}A^*x,z\right\rangle\right|\right)^{\frac{1}{2}} \\ & + \langle D^*|S|^{2\alpha}Dx,x\rangle^{\frac{1}{2}}\left(q^2\left\langle C|S^*|^{2(1-\alpha)}C^*x,x\right\rangle + (1-q^2)\left\langle C|S^*|^{2(1-\alpha)}C^*z,z\right\rangle\right. \\ & \left. +2q\sqrt{1-q^2}\left|\left\langle C|S^*|^{2(1-\alpha)}C^*x,z\right\rangle\right|\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left\langle B^*|T|^{2\alpha}Bx,x\right\rangle^{\frac{1}{2}}\left(q\left\langle A|T^*|^{2(1-\alpha)}A^*x,x\right\rangle^{\frac{1}{2}}+\sqrt{1-q^2}\left\langle A|T^*|^{2(1-\alpha)}A^*z,z\right\rangle^{\frac{1}{2}}\right.\\
&\quad\left.+\sqrt{2q\sqrt{1-q^2}}\left|\left\langle A|T^*|^{2(1-\alpha)}A^*x,z\right\rangle\right|^{\frac{1}{2}}\right)\\
&\quad+\left\langle D^*|S|^{2\alpha}Dx,x\right\rangle^{\frac{1}{2}}\left(q\left\langle C|S^*|^{2(1-\alpha)}C^*x,x\right\rangle^{\frac{1}{2}}+\sqrt{1-q^2}\left\langle C|S^*|^{2(1-\alpha)}C^*z,z\right\rangle^{\frac{1}{2}}\right.\\
&\quad\left.+\sqrt{2q\sqrt{1-q^2}}\left|\left\langle C|S^*|^{2(1-\alpha)}C^*x,z\right\rangle\right|^{\frac{1}{2}}\right)\\
&= q\left(\left\langle B^*|T|^{2\alpha}Bx,x\right\rangle^{\frac{1}{2}}\left\langle A|T^*|^{2(1-\alpha)}A^*x,x\right\rangle^{\frac{1}{2}}+\left\langle D^*|S|^{2\alpha}Dx,x\right\rangle^{\frac{1}{2}}\right.\\
&\quad\left.\times\left\langle C|S^*|^{2(1-\alpha)}C^*x,x\right\rangle^{\frac{1}{2}}\right)+\sqrt{1-q^2}\left(\left\langle B^*|T|^{2\alpha}Bx,x\right\rangle^{\frac{1}{2}}\left\langle A|T^*|^{2(1-\alpha)}A^*z,z\right\rangle^{\frac{1}{2}}\right.\\
&\quad\left.+\left\langle D^*|S|^{2\alpha}Dx,x\right\rangle^{\frac{1}{2}}\left\langle C|S^*|^{2(1-\alpha)}C^*z,z\right\rangle^{\frac{1}{2}}\right)\\
&\quad+\sqrt{2q\sqrt{1-q^2}}\left(\left\langle B^*|T|^{2\alpha}Bx,x\right\rangle^{\frac{1}{2}}\left|\left\langle A|T^*|^{2(1-\alpha)}A^*x,z\right\rangle\right|^{\frac{1}{2}}\right.\\
&\quad\left.+\left\langle D^*|S|^{2\alpha}Dx,x\right\rangle^{\frac{1}{2}}\left|\left\langle C|S^*|^{2(1-\alpha)}C^*x,z\right\rangle\right|^{\frac{1}{2}}\right)\\
&\leqslant \frac{q}{2}\left\langle\left(B^*|T|^{2\alpha}B+A|T^*|^{2(1-\alpha)}A^*+D^*|S|^{2\alpha}D+C|S^*|^{2(1-\alpha)}C^*\right)x,x\right\rangle\\
&\quad+\sqrt{1-q^2}\left(\left\langle B^*|T|^{2\alpha}Bx,x\right\rangle^{\frac{1}{2}}\left\langle A|T^*|^{2(1-\alpha)}A^*z,z\right\rangle^{\frac{1}{2}}+\left\langle D^*|S|^{2\alpha}Dx,x\right\rangle^{\frac{1}{2}}\right.\\
&\quad\left.\times\left\langle C|S^*|^{2(1-\alpha)}C^*z,z\right\rangle^{\frac{1}{2}}\right)+\sqrt{2q\sqrt{1-q^2}}\left(\left\langle B^*|T|^{2\alpha}Bx,x\right\rangle^{\frac{1}{2}}\right.\\
&\quad\left.\times\left|\left\langle A|T^*|^{2(1-\alpha)}A^*x,z\right\rangle\right|^{\frac{1}{2}}+\left\langle D^*|S|^{2\alpha}Dx,x\right\rangle^{\frac{1}{2}}\left|\left\langle C|S^*|^{2(1-\alpha)}C^*x,z\right\rangle\right|^{\frac{1}{2}}\right)\\
&\leqslant \frac{q}{2}\|B^*|T|^{2\alpha}B+A|T^*|^{2(1-\alpha)}A^*+D^*|S|^{2\alpha}D+C|S^*|^{2(1-\alpha)}C^*\|\\
&\quad+\left(\sqrt{1-q^2}+\sqrt{2q\sqrt{1-q^2}}\right)\left(\sqrt{\|B^*|T|^{2\alpha}B\|\|A|T^*|^{2(1-\alpha)}A^*\|}\right.\\
&\quad\left.+\sqrt{\|D^*|S|^{2\alpha}D\|\|C|S^*|^{2(1-\alpha)}C^*\|}\right).
\end{aligned}
$$

The result follows by taking supremum over all such $x,y\in\mathcal{H}$ with $\|x\|=\|y\|=1$, and $\langle x,y\rangle=q$ on the left-hand side. $\square$

Here we mention a few particular cases of the above theorem.

REMARK 3. Consider a few particular cases of the above result:

(i) If $T=B=I$ and $S=0$ then

$$\omega_q(A) \leqslant \frac{q}{2}\|AA^*+I\| + \left(\sqrt{1-q^2}+\sqrt{2q\sqrt{1-q^2}}\right)\|A\|.$$

Also, if $A=B=I$, $S=0$ and $\alpha=\frac{1}{2}$ then we have

$$\begin{aligned}\omega_q(T) &\leqslant \frac{q}{2}\||T|+|T^*|\| + \left(\sqrt{1-q^2}+\sqrt{2q\sqrt{1-q^2}}\right)\sqrt{\||T|\|\||T^*|\|}\\ &= \frac{q}{2}\||T|+|T^*|\| + \left(\sqrt{1-q^2}+\sqrt{2q\sqrt{1-q^2}}\right)\|T\|.\end{aligned}$$

(ii) If $T=I$ and $S=0$ then

$$\omega_q(AB) \leqslant \frac{q}{2}\|AA^*+BB^*\| + \left(\sqrt{1-q^2}+\sqrt{2q\sqrt{1-q^2}}\right)\|A\|\|B\|.$$

(iii) If $T=I$, $C=B$ and $D=A$ then

$$\omega_q(AB+BA) \leqslant \frac{q}{2}\|AA^*+B^*B+A^*A+BB^*\| + \left(\sqrt{1-q^2}+\sqrt{2q\sqrt{1-q^2}}\right)\|A\|^2\|B\|^2.$$

3. q-numerical radius of 2×2 operator matrices

In this section, we derive a few bounds for the q-numerical range of 2×2 operator matrices. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ are two complex Hilbert spaces with the inner product $\langle .,. \rangle$. Then $\mathcal{H}_1 \oplus \mathcal{H}_2$ forms a Hilbert space and any operator $T \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ has an 2×2 matrix representation of the form

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)$, $C \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$, and $D \in \mathcal{B}(\mathcal{H}_2)$.

As the q-numerical radius forms a weakly unitarily invariant norm, from the result mentioned in (p. 107 [2]), we can deduce the following inequalities

$$\omega_q\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \leqslant \omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \tag{18}$$

and

$$\omega_q\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leqslant \omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right). \tag{19}$$

The numerical radius of diagonal operator matrices enjoy the following equality ([13])

$$\omega\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) = \max\{\omega(A),\omega(D)\}. \tag{20}$$

A similar conclusion is not true for q-numerical radius. This is seen in the Example 3.2 that follows. The following lemma provides the shape of the q-numerical radius of Hermitian matrices for $\{q \in \mathbb{C} : |q| \leqslant 1\}$.

LEMMA 3.1. (Theorem 3.5 [12]) *If T is an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $|q| \leqslant 1$, then the numerical range $W_q(T)$ equals the (closed) elliptic disc with foci $q\lambda_1$ and $q\lambda_n$ and minor axis of length $\sqrt{1-|q|^2}(\lambda_1 - \lambda_n)$.*

EXAMPLE 3.2. Let $q \in [0,1]$. For any 2×2 matrix $T = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $a, d > 0$ and $a \neq d$, the q-numerical range of T is given by

$$W_q(T) = \left\{ (x,y) : \frac{\left(x - \frac{q}{2}(a+d)\right)^2}{\frac{1}{4}(a-d)^2} + \frac{y^2}{\frac{1}{4}(1-q^2)(a-d)^2} \leqslant 1 \right\}. \tag{21}$$

The q-numerical radius of $T = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ can be obtained via the following maximization problem

$$\max \sqrt{x^2+y^2}, \text{ subject to } \frac{\left(x - \frac{q}{2}(a+d)\right)^2}{\frac{1}{4}(a-d)^2} + \frac{y^2}{\frac{1}{4}(1-q^2)(a-d)^2} = 1.$$

Solving this we get

$$\max \sqrt{x^2+y^2} = \left(\frac{q}{2}(a+d) + \frac{1}{2}|a-d|\right) \text{ occurs at } \left(\frac{q}{2}(a+d) + \frac{1}{2}|a-d|, 0\right).$$

Hence

$$\omega_q\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = \left(\frac{q}{2}(a+d) + \frac{1}{2}|a-d|\right) > \max\{qa, qd\} \text{ when } q \neq 1.$$

The above equation implies that when $q \neq 1$, the q-numerical range does not satisfy the relation

$$\omega_q\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) = \max\{\omega(A)_q, \omega_q(D)\}$$

similar to the relation mentioned in (20).

In this regard, the following result provides upper and lower bound of the q-numerical radius.

THEOREM 3.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $D \in \mathcal{B}(\mathcal{H}_2)$ and $q \in [0,1]$. Then the following inequalities hold* *(i)* $\max\left\{\omega_q(A), \omega_q(D), \omega_q\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\} \leqslant \omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$,

(ii) $\omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \max\{\|A\|, \|D\|\} + \left(1 - \frac{3q^2}{4} + q\sqrt{1-q^2}\right)^{\frac{1}{2}} (\|B\| + \|C\|)$,

(iii) $\omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \sqrt{1-q^2}\left(\|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2\right)^{\frac{1}{2}}$
$+q\left(\max\{\omega(A), \omega(D)\} + \frac{\|B\|+\|C\|}{2}\right)$.

Proof.

(i) The case $q = 1$ follows directly from relation (20). Let $q \in [0,1)$ and $x_1, y_1 \in \mathcal{H}_1$ such that $\|x_1\| = \|y_1\| = 1$ with $\langle x_1, y_1 \rangle = q$. Then

$$\begin{aligned}\omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &\geqslant \left|\left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right\rangle\right| \\ &= |\langle Ax_1, y_1 \rangle|.\end{aligned}$$

Taking supremum over all such x_1 and y_1 with $\|x_1\| = \|y_1\| = 1$ and $\langle x_1, y_1 \rangle = q$, it follows that

$$\omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \geqslant \omega_q(A). \tag{22}$$

In a similar way, it can be proved that

$$\omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \geqslant \omega_q(D). \tag{23}$$

Finally from the relations (19), (22), and (23) we get

$$\max\left\{\omega_q(A), \omega_q(D), \omega_q\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\} \leqslant \omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right).$$

(ii) Note that the q-numeical range forms a semi-norm and the following relation holds

$$\omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \omega_q\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + \omega_q\left(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}\right) + \omega_q\left(\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}\right). \tag{24}$$

Since $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, Theorem 2.5 of [10] and (24) imply that

$$\omega_q\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \omega_q\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + \left(1 - \frac{3q^2}{4} + q\sqrt{1-q^2}\right)^{\frac{1}{2}} (\|B\| + \|C\|). \tag{25}$$

Let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$, with

$$\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\| = 1, \quad \text{and} \quad \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = q.$$

Observe that

$$\begin{aligned} \left| \left\langle \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle \right| &\leqslant |\langle Ax_1, y_1 \rangle| + |\langle Dx_2, y_2 \rangle| \\ &\leqslant \|A\| \|x_1\| \|y_1\| + \|D\| \|x_2\| \|y_2\| \\ &\leqslant \max\{\|A\|, \|D\|\}. \end{aligned}$$

Hence we have the following result:

$$\omega_q \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) \leqslant \max\{\|A\|, \|D\|\}. \tag{26}$$

The desired upper bound of $\omega_q \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right)$ follows from the inequalities (25) and (26).

(iii) To prove the last inequality let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$, with

$$\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\| = 1, \quad \text{and} \quad \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = q.$$

In this setting, we can take

$$y_1 = qx_1 + \sqrt{1-q^2} z_1, \quad \text{and} \quad y_2 = qx_2 + \sqrt{1-q^2} z_2$$

where $z_1 \in \mathcal{H}_1$, $z_2 \in \mathcal{H}_2$ with

$$\left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\| = 1, \quad \text{and} \quad \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle = 0.$$

We assume that $\|x_1\| = \cos\theta$, $\|x_2\| = \sin\theta$, $\|z_1\| = \cos\phi$, and $\|z_2\| = \sin\phi$ where $\theta, \phi \in [0, \frac{\pi}{2}]$. Also, we use the fact that for any $a, b \in \mathbb{R}$,

(1) $\max\limits_{\theta}(a\cos\theta + b\sin\theta) = \sqrt{a^2+b^2}$,

(2) $\max\limits_{\theta}(a\cos^2\theta + b\sin^2\theta) = \max\{a, b\}$.

In this setting we have

$$\left|\left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle\right| \leqslant |\langle Ax_1, y_1\rangle| + |\langle Bx_2, y_1\rangle| + |\langle Cx_1, y_2\rangle| + |\langle Dx_2, y_2\rangle|. \tag{27}$$

The subsequent computations are mentioned below:

$$\begin{aligned}
&|\langle Ax_1, y_1\rangle| + |\langle Bx_2, y_1\rangle| + |\langle Cx_1, y_2\rangle| + |\langle Dx_2, y_2\rangle| \\
&= \left|\left\langle Ax_1, qx_1 + \sqrt{1-q^2}z_1 \right\rangle\right| + \left|\left\langle Bx_2, qx_1 + \sqrt{1-q^2}z_1 \right\rangle\right| \\
&\quad + \left|\left\langle Cx_1, qx_2 + \sqrt{1-q^2}z_2 \right\rangle\right| + \left|\left\langle Dx_2, qx_2 + \sqrt{1-q^2}z_2 \right\rangle\right| \\
&\leqslant q(|\langle Ax_1, x_1\rangle| + |\langle Bx_2, x_1\rangle| + |\langle Cx_1, x_2\rangle| + |\langle Dx_2, x_2\rangle|) \\
&\quad + \sqrt{1-q^2}\,(|\langle Ax_1, z_1\rangle| + |\langle Bx_2, z_1\rangle| + |\langle Cx_1, z_2\rangle| + |\langle Dx_2, z_2\rangle|) \\
&\leqslant q\left(\omega(A)\cos^2\theta + \omega(D)\sin^2\theta + \frac{1}{2}(\|B\| + \|C\|)\sin 2\theta\right) \\
&\quad + \sqrt{1-q^2}(\|A\|\cos\theta\cos\phi + \|B\|\sin\theta\cos\phi \\
&\quad + \|C\|\cos\theta\sin\phi + \|D\|\sin\theta\sin\phi) \\
&\leqslant q\left(\max\{\omega(A), \omega(D)\} + \frac{\|B\| + \|C\|}{2}\right) \\
&\quad + \sqrt{1-q^2}\left(\|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

The desired result follows from the above inequality and relation (27). □

REMARK 4. Some highlights of the aforementioned theorem are given below:

(i) If $B = C = 0$ in Theorem 3.3(i) and (ii), it follows

$$\max\left\{\omega_q(A), \omega_q(D)\right\} \leqslant \omega_q\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \leqslant \max\{\|A\|, \|D\|\}.$$

The above relation implies that $\max\left\{\omega_q(A), \omega_q(D)\right\}$ actually provides a lower bound of $\omega_q\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right)$ where it is proved that q-numerical radius fails to satisfy an analogous relation as of equation (20).

(ii) If $q = 1$ then the first and the third relations of Theorem 3.3 reduce to existing lower and upper bounds of $\omega\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$ mentioned in [13].

(iii) The upper bounds obtained in (ii) and (iii) of Theorem 3.3 are noncomparable. Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1.5442 + 1.4193i & 0.0859 + 0.2916i \\ -1.4916 + 0.1978i & -0.7423 + 1.5877i \end{bmatrix}$, randomly generated by MATLAB command `randn`. Figure 3, demonstrates the comparison of the upper bounds (ii) and (iii) of Theorem 3.3 for the matrix T.

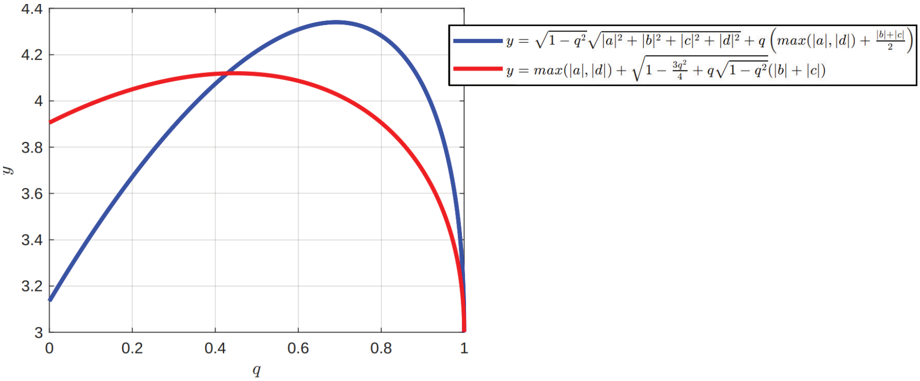

Figure 3: *Comparision of upper bounds (ii) and (iii) of Theorem* 3.3 *for* $\omega_q(T)$

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