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q–NUMERICAL RADIUS INEQUALITIES FOR PRODUCT OF COMPLEX LINEAR BOUNDED OPERATORS

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Abstract. In this paper, we prove some q-numerical radius power inequalities for a product of operators on a complex Hilbert space. We introduce also the notion of the q-center for bounded operators, and we give the relationship between this q-center, the q-numerical radius and the center of the q-numerical range.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the space of all linear bounded operators on $\mathcal{H}$. Let $\mathbb{S}_{\mathcal{H}} = \{x \in \mathcal{H} : \|x\| = 1\}$. For any $A \in \mathcal{B}(\mathcal{H})$ and $q \in \mathbb{C}$ with $|q| \leqslant 1$, the q-numerical range of A is the set defined by

$$W_q(A) = \{\langle Ax|y\rangle : x, y \in \mathbb{S}_{\mathcal{H}}, \langle x|y\rangle = q\},$$

and the q-numerical radius of A is given by

$$w_q(A) = \sup\{|\lambda| : \lambda \in W_q(A)\}.$$

Note that $W_q(A)$ is a bounded subset of $\mathbb{C}$, because $W_q(A)$ is included in the disk of $\mathbb{C}$ centered at 0 with radius $\|A\|$. If $q = 1$, $W_q(A)$ and $w_q(A)$ are the classical numerical range $W(A)$ and the classical numerical radius $w(A)$, respectively. Then the q-numerical range is a generalization of the classical numerical range.

For operators on a complex Hilbert space we know that the numerical range is a convex subset of $\mathbb{C}$ (see [9]), it is also known that the spectrum is included in the closure of the numerical range. For other properties of the numerical range and numerical radius see [9, 16].

Several authors studied the properties of the q-numerical range, especially to generalize the properties of the classical numerical range. In [20], Tsing proved that the q-numerical range is convex. In [18, 19], the authors gave some basic properties of the q-numerical range, analogous to the classical numerical range. Among these results, they proved that $q\sigma(A) \subset W_q(A)^-$ with $\sigma(A)$ is the spectrum of A and $W_q(A)^-$ is the

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closure of $W_q(A)$. They also studied the properties of the q-numerical radius as a norm (if $q \neq 0$) and as a seminorm (if $q = 0$).

The properties of the q-numerical range for normal operators are introduced in [5], and for reducible and normal matrices in [6, 7, 17]. M. Aleksiejczyk in [1], gave some inequalities related to the diameter of $W_0(A)$ and the diameter of $W_q(A)$, where A is a $n \times n$ complex matrix and $q \in [0,1]$.

The authors in [14, 18] showed that $W_0(A)$ is a circular disk centered at 0 with radius

$$w_0(A) = d(A) = \inf\{\|A - \lambda I\| : \lambda \in \mathbb{C}\} = \|A - c_A I\|,$$

with I is the identity operator on $\mathcal{H}$ and c_A is the Stampfli's center of A.

In [12, 13, 14], M. C. Kaadoud proved that

$$w(AB) \leqslant w(B)w(A) + w_0(B)w_0(A),$$

$$\|A\|^2 \leqslant w^2(A) + w_0^2(A)$$

and

$$w_0(A) \leqslant diamW(A),$$

with $A, B \in \mathcal{B}(\mathcal{H})$ and $diamW(A)$ is the diameter of $W(A)$. In this paper, we give a generalization of these results for the q-numerical range and the q-numerical radius. We generalize also others results in [13, 15].

The paper is organized as follows. In Section 2, the notion of the q-center $c_q(A)$ of A is introduced, and the relationship to $c_q(A)$, $w_q(A)$ and the center $c_{W_q(A)}$ of $W_q(A)$ is established. In section 3, we give a necessary and sufficient condition so that $w_q(A+B) = w_q(A) + w_q(B)$ for $A, B \in \mathcal{B}(\mathcal{H})$, and we show also some power inequalities for the q-numerical radius of a product of operators on $\mathcal{H}$. Section 4, is devoted to prove some inequalities related to $w_0(A)$ and the q-numerical radius of A.

2. The q-center

Let K be a compact subset of $\mathbb{C}$. Denote by R_K and c_K the radius and the center, respectively of the smallest disk $D_K = D(c_K, R_K)$ containing K. Let $|K| = \sup\{|\alpha| : \alpha \in K\}$.

LEMMA 2.1. ([13, Proposition 3]) *Let K be a compact subset of $\mathbb{C}$. Then*

$$R_K = |K - c_K| = \sup_{\alpha \in K} |c_K - \alpha| = \inf_{\lambda \in \mathbb{C}} \sup_{\alpha \in K} |\lambda - \alpha|,$$

where c_K is the unique scalar satisfies $R_K = |K - c_K|$.

COROLLARY 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and $q \in \mathbb{C}$ such that $0 < |q| \leqslant 1$. Then*

$$R_{W_q(A)} = |W_q(A) - c_{W_q(A)}| = w_q\left(A - \frac{1}{q} c_{W_q(A)} I\right).$$

Proof. Since $W_q(A)$ is bounded, $W_q(A)^-$ is compact. Then by Lemma 2.1, we have

$$\begin{aligned} R_{W_q(A)} &= |W_q(A) - c_{W_q(A)}| = \sup_{\alpha \in W_q(A)} |\alpha - c_{W_q(A)}| \\ &= \sup\{|\langle Ax|y\rangle - c_{W_q(A)}| : x,y \in \mathbb{S}_{\mathcal{H}}, \langle x|y\rangle = q\} \\ &= \sup\left\{\left|\left\langle \left(A - \frac{1}{q}c_{W_q(A)}I\right) x|y \right\rangle\right| : x,y \in \mathbb{S}_{\mathcal{H}}, \langle x|y\rangle = q\right\} \\ &= w_q\left(A - \frac{1}{q}c_{W_q(A)}I\right). \quad \square \end{aligned}$$

DEFINITION 2.3. Let $A \in \mathcal{B}(\mathcal{H})$ and $q \in \mathbb{C}$ with $|q| \leqslant 1$. The scalar β that satisfies

$$\inf_{\lambda \in \mathbb{C}} w_q(A - \lambda I) = w_q(A - \beta I),$$

is called a q-center of A, which we indicate by $c_q(A)$.

In this section $q \in \mathbb{C}$ with $0 < |q| \leqslant 1$.

PROPOSITION 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$R_{W_q(A)} = \inf_{\lambda \in \mathbb{C}} w_q(A - \lambda I) = w_q(A - c_q(A)I),$$

and

$$c_q(A) = \frac{1}{q} c_{W_q(A)}.$$

Proof. We have that

$$\begin{aligned} w_q(A - c_q(A)I) &= \inf_{\lambda \in \mathbb{C}} w_q(A - \lambda I) = \inf_{\lambda \in \mathbb{C}} w_q\left(A - \frac{\lambda}{q}I\right) \\ &= \inf_{\lambda \in \mathbb{C}} \sup\{|\langle Ax|y\rangle - \lambda| : x,y \in \mathbb{S}_{\mathcal{H}}, \langle x|y\rangle = q\} \\ &= |W_q(A) - c_{W_q(A)}| = R_{W_q(A)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |W_q(A) - qc_q(A)| &= \sup\{|\langle Ax|y\rangle - qc_q(A)| : x,y \in \mathbb{S}_{\mathcal{H}}, \langle x|y\rangle = q\} \\ &= \sup\{|\langle (A - c_q(A)I)x|y\rangle| : x,y \in \mathbb{S}_{\mathcal{H}}, \langle x|y\rangle = q\} \\ &= w_q(A - c_q(A)I). \end{aligned}$$

Therefore,

$$|W_q(A) - qc_q(A)| = |W_q(A) - c_{W_q(A)}|,$$

and by the unicity of the center $c_{W_q(A)}$ of $W_q(A)$, it follows that $c_{W_q(A)} = qc_q(A)$. $\square$

REMARK 2.5. By Proposition 2.4, we get the unicity of the q-center $c_q(A)$ of $A \in \mathcal{B}(\mathcal{H})$.

PROPOSITION 2.6. *Let $A \in \mathcal{B}(\mathcal{H})$. If a sequence $(A_n) \subset \mathcal{B}(\mathcal{H})$ converges to A then the sequence $(c_q(A_n))$ converges to $c_q(A)$.*

Proof. By [13, Corollary 11], the sequence $(c_{W_q(A_n)})$ converges to $c_{W_q(A)}$ and by Proposition 2.4, we have $c_q(A_n) = \frac{1}{q} c_{W_q(A_n)}$. So, $(c_q(A_n))$ converges to $c_q(A)$. □

LEMMA 2.7. ([13, Corollary 5]) *Let K be a compact subset of $\mathbb{C}$ and $c \in \mathbb{C}$. Then the following assertions are equivalent:*

(i) $c_K = c$.

(ii) $|K - c| < |K - (c + \lambda)|$ *for all* $\lambda \in \mathbb{C}^*$.

(iii) $|K - c|^2 + |\lambda|^2 \leqslant |K - (c + \lambda)|^2$ *for all* $\lambda \in \mathbb{C}$.

COROLLARY 2.8. *Let $A \in \mathcal{B}(\mathcal{H})$. Then the following assertions are equivalent:*

(i) $c = c_q(A)$.

(ii) $w_q(A - cI) < w_q(A - (c + \frac{\lambda}{q})I)|$ *for all* $\lambda \in \mathbb{C}^*$.

(iii) $w_q^2(A - cI) + |\lambda|^2 \leqslant w_q^2(A - (c + \frac{\lambda}{q})I)$ *for all* $\lambda \in \mathbb{C}$.

Proof. Note that $w_q(A - \frac{\lambda}{q}) = |W_q(A) - \lambda|$ for all $\lambda \in \mathbb{C}$ and by Proposition 2.4, $c_{W_q(A)} = qc_q(A)$. Then by Lemma 2.7, (i), (ii) and (iii) are equivalent. □

COROLLARY 2.9. *Let $A \in \mathcal{B}(\mathcal{H})$ such that $0 \in W_q(A)$. Then*

$$|c_q(A)| \leqslant \frac{1}{\sqrt{2}|q|} w_q(A).$$

Proof. By [13, Proposition 25], we have

$$|c_{W_q(A)}| \leqslant \frac{|W_q(A)|}{\sqrt{2}} = \frac{1}{\sqrt{2}} w_q(A),$$

and by Proposition 2.4, it follows that $c_{W_q(A)} = qc_q(A)$. Hence

$$|c_q(A)| \leqslant \frac{1}{\sqrt{2}|q|} w_q(A). \quad \square$$

Let $A \in \mathcal{B}(\mathcal{H})$. In [13], M. C. Kaadoud proved that $R_{W(A)} = w_0(A)$ if and only if $w(A - c_{W(A)}I) = \|A - c_{W(A)}I\|$, in the following proposition we generalize this result for $W_q(A)$.

PROPOSITION 2.10. *Let $A \in \mathcal{B}(\mathcal{H})$. Then $R_{W_q(A)} = w_0(A)$ if and only if*

$$w_q(A - c_q(A)I) = \|A - c_q(A)I\|.$$

Proof. Assume that $R_{W_q(A)} = w_0(A) = \|A - c_A I\|$. Then by Proposition 2.4, it holds:

$$R_{W_q(A)} = w_q(A - c_q(A)I) \leqslant w_q(A - c_A I) \leqslant \|A - c_A I\| = w_0(A) = R_{W_q(A)}.$$

Hence $w_q(A - c_q(A)I) = w_q(A - c_A I)$ and the unicity of the center implies that $c_q(A) = c_A$. Conversely, suppose that $w_q(A - c_q(A)I) = \|A - c_q(A)I\|$. Then

$$w_0(A) = \|A - c_A I\| \leqslant \|A - c_q(A)I\| = w_q(A - c_q(A)I) = R_{W_q(A)}.$$

Since $R_{W_q(A)} \leqslant w_q(A - c_A I)$, it follows that

$$w_0(A) \leqslant R_{W_q(A)} \leqslant \|A - c_A I\| = w_0(A). \quad \square$$

3. q-numerical radius inequalities

In what follows, let $q \in \mathbb{C}$ with $|q| \leqslant 1$.

The next result has been proved for $q = 1$ in [2].

THEOREM 3.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then the following two assertions are equivalents:*

(i) $w_q(A+B) = w_q(A) + w_q(B)$.

(ii) There exists two sequences of unit vectors $(x_n) \subset \mathcal{H}$ and $(y_n) \subset \mathcal{H}$ such that $\langle x_n | y_n \rangle = q$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \langle A^* y_n | x_n \rangle \langle B x_n | y_n \rangle = w_q(A) w_q(B)$.

Proof. Assume that $w_q(A+B) = w_q(A) + w_q(B)$. Since $W_q(A+B)$ is bounded, $W_q(A+B)^-$ is compact. Then there exist two sequences of unit vectors $(x_n) \subset \mathcal{H}$ and $(y_n) \subset \mathcal{H}$ such that $\langle x_n | y_n \rangle = q$ and $w_q(A+B) = \lim_{n\to\infty} |\langle (A+B)x_n | y_n \rangle|$. We have

$$\begin{aligned}
|\langle (A+B)x_n | y_n \rangle|^2 &= |\langle A x_n | y_n \rangle|^2 + |\langle B x_n | y_n \rangle|^2 + 2Re(\langle A^* y_n | x_n \rangle \langle B x_n | y_n \rangle) \\
&\leqslant |\langle A x_n | y_n \rangle|^2 + |\langle B x_n | y_n \rangle|^2 + 2|\langle A^* y_n | x_n \rangle||\langle B x_n | y_n \rangle| \\
&= (|\langle A x_n | y_n \rangle| + |\langle B x_n | y_n \rangle|)^2 \\
&\leqslant (w_q(A) + w_q(B))^2 = w_q^2(A+B).
\end{aligned}$$

Hence $\lim_{n\to\infty} \langle A^* y_n | x_n \rangle \langle B x_n | y_n \rangle = w_q(A) w_q(B)$. Conversely, suppose (ii) fails to hold. Then $\lim_{n\to\infty} Re(\langle A^* y_n | x_n \rangle \langle B x_n | y_n \rangle) = w_q(A) w_q(B)$, $\lim_{n\to\infty} |\langle A^* y_n | x_n \rangle| = w_q(A)$ and $\lim_{n\to\infty} |\langle B x_n | y_n \rangle| = w_q(B)$. So, $\lim_{n\to\infty} |\langle (A+B)x_n | y_n \rangle| = w_q(A) + w_q(B)$ and since $|\langle (A+B)x_n | y_n \rangle| \leqslant w_q(A+B) \leqslant w_q(A) + w_q(B)$, it follows that $w_q(A+B) = w_q(A) + w_q(B)$. $\square$

Let $A,B \in \mathcal{B}(\mathcal{H})$ such that B is self-adjoint and let $q=0$. The authors in [15, Theorem 4] proved that if $w_0(A+B)=w_0(A)+w_0(B)$ and $w_0(A)=\|A\|$ then $w_0(A)w_0(B) \in W^-(B^*A)$. In the next theorem, we generalize this result for any $q\in[0,1]$ and without the hypothesis of B is self-adjoint.

THEOREM 3.2. *Let $A,B \in \mathcal{B}(\mathcal{H})$ such that the dimensional of $\mathcal{H} = dim(\mathcal{H}) \geqslant 2$, and let $q\in[0,1]$. If $w_q(A+B)=w_q(A)+w_q(B)$ and $w_q(A)=\|A\|$ then $w_q(A)w_q(B)\in W^-(B^*A)$.*

Proof. Assume that $w_q(A+B)=w_q(A)+w_q(B)$ and $w_q(A)=\|A\|$. By Theorem 3.1, there exists two sequences of unit vectors $(x_n)\subset\mathcal{H}$ and $(y_n)\subset\mathcal{H}$ such that $\langle x_n|y_n\rangle = q$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}\langle A^*y_n|x_n\rangle\langle Bx_n|y_n\rangle = w_q(A)w_q(B)$. Then for all $n\in\mathbb{N}$, we have $Ax_n=\alpha_n y_n+\beta_n z_n$ with $z_n\in\mathbb{S}_{\mathcal{H}}$, $\langle y_n|z_n\rangle=0$ and we have $\alpha_n=\langle Ax_n|y_n\rangle$, $\beta_n=\langle Ax_n|z_n\rangle$. Note that $\|Ax_n\|^2=|\langle Ax_n|y_n\rangle|^2+|\langle Ax_n|z_n\rangle|^2$ and since

$$\lim_{n\to\infty}|\langle Ax_n|y_n\rangle| = \lim_{n\to\infty}|\langle A^*y_n|x_n\rangle| = w_q(A)=\|A\|,$$

it follows that $\lim_{n\to\infty}\|Ax_n\|^2=\|A\|^2$ and $\lim_{n\to\infty}|\langle Ax_n|z_n\rangle|^2=0$. We have

$$\langle B^*Ax_n|x_n\rangle = \langle Ax_n|y_n\rangle\langle B^*y_n|x_n\rangle + \langle Ax_n|z_n\rangle\langle B^*z_n|x_n\rangle.$$

Since $\lim_{n\to\infty}\langle A^*y_n|x_n\rangle\langle Bx_n|y_n\rangle = w_q(A)w_q(B)$ and $\lim_{n\to\infty}\langle Ax_n|z_n\rangle = 0$, it holds: $\lim_{n\to\infty}\langle B^*Ax_n|x_n\rangle = w_q(A)w_q(B)$. This implies that $w_q(A)w_q(B)\in W^-(B^*A)$. □

In the next example, we show that if $A,B\in\mathcal{B}(\mathcal{H})$ such that $w_q(A+B)=w_q(A)+w_q(B)$ and $w_q(A)\neq\|A\|$ then $w_q(A)w_q(B)$ is not necessarily an scalar in $W^-(B^*A)$.

EXAMPLE 3.3. Let $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $B=\begin{pmatrix}0&0\\1&0\end{pmatrix}=A^*$ and suppose that $q=1$. We have $w_q(A)=w_q(B)=\frac{1}{2}$ and $A+B=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. For $x=\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$, it holds: $\|x\|=1$ and $\langle (A+B)x|x\rangle = 1 = w_q(A)+w_q(B)$. Hence $w_q(A+B)=w_q(A)+w_q(B)$, but $w_q(A)w_q(B)=\frac{1}{4}\notin W^-(B^*A)=W(A^2)=\{0\}$.

LEMMA 3.4. (Buzano inequality, [4]) *Let $a,b,e\in\mathcal{H}$. Then*

$$|\langle a|e\rangle\langle e|b\rangle| \leqslant \frac{\|e\|^2}{2}(\|a\|\|b\|+|\langle a|b\rangle|).$$

COROLLARY 3.5. *Let $A,B\in\mathcal{B}(\mathcal{H})$. If $w_q(A+B)=w_q(A)+w_q(B)$ then*

$$w_q(A)w_q(B)\leqslant\frac{1}{2}(\|A\|\|B\|+w(B^*A)). \tag{3.1}$$

Proof. By Lemma 3.4, we have

$$\frac{1}{2}(\|a\|\|b\|+|\langle a|b\rangle|) \geqslant |\langle a|e\rangle\langle e|b\rangle|, \tag{3.2}$$

for all $a,b,e \in \mathcal{H}$ with $\|e\|=1$. Let $x,y \in \mathbb{S}_{\mathcal{H}}$ such that $\langle x|y\rangle = q$. Let $a = Ax$, $b = Bx$ and $e = y$. Then by (3.2), we infer that

$$\frac{1}{2}(\|Ax\|\|Bx\|+|\langle B^*Ax|x\rangle|) \geqslant |\langle Ax|y\rangle\langle y|Bx\rangle|.$$

If $w_q(A+B) = w_q(A)+w_q(B)$ then by Theorem 3.1, There exist two sequences of unit vectors $(x_n) \subset \mathcal{H}$ and $(y_n) \subset \mathcal{H}$ such that $\langle x_n|y_n\rangle = q$ and $\lim\limits_{n\to\infty}\langle A^*y_n|x_n\rangle\langle Bx_n|y_n\rangle = w_q(A)w_q(B)$. Then

$$\begin{aligned}\frac{1}{2}(\|A\|\|B\|+w(B^*A)) &\geqslant \frac{1}{2}(\|Ax_n\|\|Bx_n\|+|\langle B^*Ax_n|x_n\rangle|)\\ &\geqslant |\langle Ax_n|y_n\rangle\langle y_n|Bx_n\rangle|\\ &= |\langle A^*y_n|x_n\rangle\langle y_n|Bx_n\rangle| \to w_q(A)w_q(B).\end{aligned}$$

This completes the proof. □

REMARK 3.6. Let $A,B \in \mathcal{B}(\mathcal{H})$ such that $w_q(A+B) = w_q(A)+w_q(B)$. If A and B are normal and $AB = BA$, then (3.1) becomes an equality. Indeed, since A and B are normal, $w(A) = \|A\|$ and $w(B) = \|B\|$, and by [11, Theorem 3] we have $w(B^*A) \leqslant w(B)w(A)$. Then (3.1) becomes an equality.

PROPOSITION 3.7. *Let $A \in \mathcal{B}(\mathcal{H})$ be positive and let $x,y \in \mathcal{H}$. Then*

$$|\langle Ax|y\rangle| \leqslant \frac{\|A\|}{2}(\|x\|\|y\|+|\langle x|y\rangle|),$$

and

$$w_q(A) \leqslant \frac{\|A\|}{2}(1+|q|).$$

Proof. Let $x,y \in \mathcal{H}$. By Lemma 3.4, it follows that

$$|\langle x|Ax\rangle\langle Ax|y\rangle| \leqslant \frac{\|Ax\|^2}{2}(\|x\|\|y\|+|\langle x|y\rangle|).$$

If $\langle x|Ax\rangle \neq 0$ then

$$\begin{aligned}|\langle Ax|y\rangle| &\leqslant \frac{\|Ax\|^2}{2\langle x|Ax\rangle}(\|x\|\|y\|+|\langle x|y\rangle|)\\ &= \frac{\|A^{\frac{1}{2}}A^{\frac{1}{2}}x\|^2}{2\|A^{\frac{1}{2}}x\|^2}(\|x\|\|y\|+|\langle x|y\rangle|)\\ &\leqslant \frac{\|A^{\frac{1}{2}}\|^2}{2}(\|x\|\|y\|+|\langle x|y\rangle|)\\ &= \frac{\|A\|}{2}(\|x\|\|y\|+|\langle x|y\rangle|).\end{aligned}$$

If $\langle x|Ax\rangle = 0$ then $A^{\frac{1}{2}}x = 0$, which implies that $Ax = 0$. So, the first inequality is evident. The second inequality follows immediately from the first one. $\square$

REMARK 3.8. If $q = 1$ then the second inequality of Proposition 3.7 becomes equality.

COROLLARY 3.9. *Let $T,S,A \in \mathcal{B}(\mathcal{H})$ with A positive. Then*

$$w_q(SAT) \leqslant \frac{\|A\|}{2}(\|T\|\|S\| + w_q(ST)).$$

Proof. Let $x,y \in \mathbb{S}_{\mathcal{H}}$ such that $\langle x|y\rangle = q$. Then by Proposition 3.7, we have

$$\begin{aligned}|\langle SATx|y\rangle| &= |\langle ATx|S^*y\rangle| \\ &\leqslant \frac{\|A\|}{2}(\|Tx\|\|S^*y\| + |\langle STx|y\rangle|) \\ &\leqslant \frac{\|A\|}{2}(\|T\|\|S\| + w_q(ST)). \quad \square\end{aligned}$$

COROLLARY 3.10. *Let $T,S,A \in \mathcal{B}(\mathcal{H})$ with A positive. Then*

$$w_q^r(SAT) \leqslant \frac{\|A\|^r}{2}(\|T\|^r\|S\|^r + w_q^r(ST)), \textit{ for all} \quad r \geqslant 1.$$

Proof. By Corollary 3.9, we have

$$w_q(SAT) \leqslant \frac{\|A\|}{2}(\|T\|\|S\| + w_q(ST).$$

Since the function $t \mapsto t^r$ is increasing and convex on $[0,+\infty[$, it follows that

$$\begin{aligned}w_q^r(SAT) &\leqslant \|A\|^r\left(\frac{\|T\|\|S\| + w_q(ST)}{2}\right)^r \\ &\leqslant \frac{\|A\|^r}{2}(\|T\|^r\|S\|^r + w_q^r(ST)). \quad \square\end{aligned}$$

PROPOSITION 3.11. *Let $A = U|A|$ be the polar decomposition of $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w_q(A) \leqslant \frac{1}{2}(\|A\| + \|A\|^{\frac{1}{2}} w_q(U|A|^{\frac{1}{2}})).$$

Proof. Let $x,y \in \mathbb{S}_{\mathcal{H}}$ such that $\langle x|y\rangle = q$. We have that

$$\langle Ax|y\rangle = \langle U|A|x \mid y\rangle = \langle |A|^{\frac{1}{2}}x \mid |A|^{\frac{1}{2}}U^*y\rangle.$$

Then by Proposition 3.7, it follows that

$$\begin{aligned}|\langle Ax|y\rangle| &\leqslant \frac{\||A|^{\frac{1}{2}}\|}{2}(\|x\|\,\||A|^{\frac{1}{2}}U^{*}y\| + |\langle x\mid |A|^{\frac{1}{2}}U^{*}y\rangle|)\\ &\leqslant \frac{\|A\|^{\frac{1}{2}}}{2}(\|A\|^{\frac{1}{2}}\|U^{*}\| + |\langle U|A|^{\frac{1}{2}}x|y\rangle|)\\ &\leqslant \frac{\|A\|^{\frac{1}{2}}}{2}(\|A\|^{\frac{1}{2}}\|U^{*}\| + w_q(U|A|^{\frac{1}{2}})).\end{aligned}$$

Since $\|U^{*}\| = \|U\| \leqslant 1$, we infer that $w_q(A) \leqslant \frac{1}{2}(\|A\| + \|A\|^{\frac{1}{2}} w_q(U|A|^{\frac{1}{2}}))$. □

COROLLARY 3.12. *Let $A = U|A|$ be the polar decomposition of $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w_q^r(T) \leqslant \frac{1}{2}(\|A\|^r + \|A\|^{\frac{r}{2}} w_q^r(U|A|^{\frac{1}{2}}), \textit{ for all } \quad r \geqslant 1.$$

Proof. The desired inequality follows from Proposition 3.11 and by using the argument in the proof of Corollary 3.10. □

PROPOSITION 3.13. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$w_q(BA) \geqslant 2(\max(w(A)w_q'(B), w'(A)w_q(B))) - \|A\|\|B\|, \tag{3.3}$$

with

$$w_q'(T) = \inf\{|\lambda| : \lambda \in W_q(T)\} \quad \textit{and} \quad w'(T) = w_1'(T),\ T \in \{A, B\}.$$

Proof. Let $x, y \in \mathbb{S}_{\mathcal{H}}$ such that $\langle x|y\rangle = q$. Let $a = Ax$, $b = B^{*}y$ and $e = x$, in (3.2) we have

$$\frac{1}{2}(\|Ax\|\|B^{*}y\| + |\langle BAx|y\rangle|) \geqslant |\langle Ax|x\rangle\langle Bx|y\rangle|.$$

Hence

$$\frac{1}{2}(\|A\|\|B\| + w_q(BA)) \geqslant \max(w(A)w_q'(B), w'(A)w_q(B)),$$

this completes the proof. □

REMARK 3.14. (i) For $A \in \mathcal{B}(\mathcal{H})$, $w_q(A) \geqslant 2|q|w(A) - \|A\|$. Indeed, we have $W_q(I) = \{q\}$ then if we replace B by I in 3.3, we get

$$w_q(A) \geqslant 2|q|w(A) - \|A\|.$$

(ii) Let $A = I$ and $B \in \mathcal{B}(\mathcal{H})$ such that $w(B) = \|B\|$. Then (3.3) becomes an equality.

Let $A \in \mathcal{B}(\mathcal{H})$ and $q = 0$. The author in [13, Proposition 31] proved that

$$w_0(A) \leqslant diam W(A).$$

In the next proposition, we generalize this result for any $q \in [0,1]$.

PROPOSITION 3.15. *Let $A \in \mathcal{B}(\mathcal{H})$ and $q \in [0,1]$. Then*

$$w_q(A) \leqslant qw(A) + diamW(A). \tag{3.4}$$

Proof. Let $x, y \in \mathbb{S}_{\mathcal{H}}$ such that $\langle x|y\rangle = q$. Then by the polarization identity, it holds:

$$\begin{aligned}
2\langle Ax|y\rangle &= \left\langle A\left(\frac{x+y}{\sqrt{2}}\right)|\frac{x+y}{\sqrt{2}}\right\rangle - \left\langle A\left(\frac{x-y}{\sqrt{2}}\right)|\frac{x-y}{\sqrt{2}}\right\rangle \\
&\quad +i\left\langle A\left(\frac{x+iy}{\sqrt{2}}\right)|\frac{x+iy}{\sqrt{2}}\right\rangle - i\left\langle A\left(\frac{x-iy}{\sqrt{2}}\right)|\frac{x-iy}{\sqrt{2}}\right\rangle \\
&= (1+q)\left\langle A\left(\frac{x+y}{\sqrt{2}\sqrt{1+q}}\right)|\frac{x+y}{\sqrt{2}\sqrt{1+q}}\right\rangle - (1-q) \\
&\quad \left\langle A\left(\frac{x-y}{\sqrt{2}\sqrt{1-q}}\right)|\frac{x-y}{\sqrt{2}\sqrt{1-q}}\right\rangle \\
&\quad +i\left\langle A\left(\frac{x+iy}{\sqrt{2}}\right)|\frac{x+iy}{\sqrt{2}}\right\rangle - i\left\langle A\left(\frac{x-iy}{\sqrt{2}}\right)|\frac{x-iy}{\sqrt{2}}\right\rangle \\
&= \left[\left\langle A\left(\frac{x+y}{\sqrt{2}\sqrt{1+q}}\right)|\frac{x+y}{\sqrt{2}\sqrt{1+q}}\right\rangle - \left\langle A\left(\frac{x-y}{\sqrt{2}\sqrt{1-q}}\right)|\frac{x-y}{\sqrt{2}\sqrt{1-q}}\right\rangle\right] \\
&\quad +i\left[\left\langle A\left(\frac{x+iy}{\sqrt{2}}\right)|\frac{x+iy}{\sqrt{2}}\right\rangle - \left\langle A\left(\frac{x-iy}{\sqrt{2}}\right)|\frac{x-iy}{\sqrt{2}}\right\rangle\right] \\
&\quad +q\left[\left\langle A\left(\frac{x+y}{\sqrt{2}\sqrt{1+q}}\right)|\frac{x+y}{\sqrt{2}\sqrt{1+q}}\right\rangle + \left\langle A\left(\frac{x-y}{\sqrt{2}\sqrt{1-q}}\right)|\frac{x-y}{\sqrt{2}\sqrt{1-q}}\right\rangle\right].
\end{aligned}$$

This implies that

$$2|\langle Ax|y\rangle| \leqslant diamW(A) + diamW(A) + q2w(A).$$

So, $w_q(A) \leqslant diamW(A) + qw(A)$. □

4. Inequalities between the q-numerical radius and the distance to scalar's

PROPOSITION 4.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w_q(A) \leqslant |q|w(A) + \sqrt{1-|q|^2}w_0(A). \tag{4.1}$$

Proof. Let $x, y \in \mathbb{S}_{\mathcal{H}}$ such that $\langle x|y\rangle = q$. Then $y = qx + \sqrt{1-|q|^2}z$ for some $z \in \mathbb{S}_{\mathcal{H}}$ such that $\langle x|z\rangle = 0$. Then

$$\begin{aligned}
|\langle Ax|y\rangle| &= |\langle Ax|x\rangle\overline{q} + \langle Ax|z\rangle\sqrt{1-|q|^2} \\
&\leqslant |q|w(A) + \sqrt{1-|q|^2}w_0(A). \quad \square
\end{aligned}$$

REMARK 4.2. Let $A \in \mathcal{B}(\mathcal{H})$ and $q = 1$, then (4.1) becomes an equality.

Let $A, B \in \mathcal{B}(\mathcal{H})$ and $q = 1$. The author in [12, Corollary 5], proved that

$$w(AB) = w_1(AB) \leqslant w_1(B)w(A) + w_0(B)w_0(A) = w(B)w(A) + w_0(B)w_0(A).$$

In the next proposition, we generalize this result for any $q \in \mathbb{C}$, $|q| \leqslant 1$.

PROPOSITION 4.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$w_q(AB) \leqslant w_q(B)w(A) + d_q(B)w_0(A), \tag{4.2}$$

with

$$d_q^2(B) = \sup\{\|Bx\|^2 - |\langle Bx|y\rangle|^2 : x, y \in \mathbb{S}_{\mathcal{H}}, \langle x|y\rangle = q\}.$$

Proof. Let $x, y \in \mathbb{S}_{\mathcal{H}}$ such that $\langle x|y\rangle = q$. Let $Bx = \alpha y + \beta z$ with $z \in \mathbb{S}_{\mathcal{H}}$ and $\langle y|z\rangle = 0$. Then $\alpha = \langle Bx|y\rangle$, $\beta = \langle Bx|z\rangle$ and $\|Bx\|^2 = |\alpha|^2 + |\beta|^2$. Hence

$$\langle ABx|y\rangle = \langle Bx|y\rangle\langle Ay|y\rangle + \beta\langle Az|y\rangle.$$

This implies that $w_q(AB) \leqslant w_q(B)w(A) + |\beta|w_0(A)$ and since

$$|\beta|^2 = \|Bx\|^2 - |\langle Bx|y\rangle|^2 \leqslant d_q^2(B),$$

it follows that $w_q(AB) \leqslant w_q(B)w(A) + d_q(B)w_0(A)$. $\square$

COROLLARY 4.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $w_0(A) = R_{W(A)}$. Then*

$$w_q(AB) \leqslant w_q(B)w(A) + d_q(B)R_{W(A)}. \tag{4.3}$$

In particular, if A is self-adjoint such that $Co(\sigma(A)) = [\lambda_1, \lambda_2]$ then

$$w_q(AB) \leqslant w_q(B)w(A) + d_q(B)(\|A\| - |\lambda_1 + \lambda_2|/2), \tag{4.4}$$

and if A is positive then

$$w_q(AB) \leqslant \|A\|(w_q(B) + \frac{1}{2}d_q(B)). \tag{4.5}$$

Proof. Since $w_0(A) = R_{W(A)}$, (4.3) follows immediately from Proposition 4.3. If A is self-adjoint and $Co(\sigma(A)) = [\lambda_1, \lambda_2]$ then $R_{W(A)} = \|A\| - |\lambda_1 + \lambda_2|/2$ hence (4.4) follows from (4.3). If A is positive then $w(A) = \|A\|$ and by Proposition 3.7, we have $R_{W(A)} = w_0(A) = w_0(A) \leqslant \frac{\|A\|}{2}$. Thus by (4.3), we infer that

$$\begin{aligned} w_q(AB) &\leqslant w_q(B)\|A\| + d_q(B)R_{W(A)} \\ &\leqslant w_q(B)\|A\| + \frac{1}{2}d_q(B)\|A\|. \quad \square \end{aligned}$$

REMARK 4.5. Let $A \in \mathcal{B}(\mathcal{H})$ and $q = 1$, then we have that

$$\begin{aligned} d_1^2(A) &= \sup\{\|Ax\|^2 - |\langle Ax|y\rangle|^2 : x, y \in \mathbb{S}_{\mathcal{H}}, \langle x|y\rangle = 1\} \\ &= \sup\{\|Ax\|^2 - |\langle Ax|x\rangle|^2 : x \in \mathbb{S}_{\mathcal{H}}\}. \end{aligned}$$

By [3, Theorem 3.2], it holds:

$$\inf\{\|A - \lambda I\| : \lambda \in \mathbb{C}\}^2 = \sup\{\|Ax\|^2 - |\langle Ax|x\rangle|^2 : x \in \mathbb{S}_{\mathcal{H}}\},$$

and since $w_0(A) = \inf\{\|A - \lambda I\| : \lambda \in \mathbb{C}\}$, it follows that $d_1(A) = w_0(A)$.

COROLLARY 4.6. *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *such that* $w_0(A) = R_{W(A)}$. *If* A *is positive then*

$$w(AB) \leqslant \|A\|(w(B) + \frac{1}{2} diamW(B)).$$

Proof. Since $w_0(A) = R_{W(A)}$ and A is positive, by Corollary 4.4 it follows that

$$w(AB) \leqslant \|A\|(w(B) + \frac{1}{2} d_1(B)).$$

We have $d_1(B) = w_0(B)$ (see [1]) and by [13, Proposition 31], we have that $d_1(B) = w_0(B) \leqslant diamW(B)$. This completes the proof. □

In [13, Proposition 34], M. C. Kaadoud proved that, for $A \in \mathcal{B}(\mathcal{H})$ and q=1, it holds:

$$\|A\|^2 \leqslant w_1^2(A) + d_1^2(A) = w^2(A) + w_0^2(A).$$

In the next proposition, we generalize this result for any $q \in \mathbb{C}$, $|q| \leqslant 1$.

PROPOSITION 4.7. *Let* $A \in \mathcal{B}(\mathcal{H})$. *Then*

$$\|A\|^2 \leqslant w_q^2(A) + d_q^2(A). \tag{4.6}$$

Proof. Let $x, y \in \mathbb{S}_{\mathcal{H}}$ such that $\langle x|y\rangle = q$. Let $Ax = \alpha y + \beta z$ with $z \in \mathbb{S}_{\mathcal{H}}$ and $\langle y|z\rangle = 0$ then $\alpha = \langle Ax|y\rangle$ and $\beta = \langle Ax|z\rangle$. Hence

$$\begin{aligned} \|Ax\|^2 &= |\alpha|^2 + |\beta|^2 \\ &\leqslant w_q^2(A) + (\|Ax\|^2 - |\langle Ax|y\rangle|^2) \\ &\leqslant w_q^2(A) + d_q^2(A). \end{aligned}$$

So, $\|A\|^2 \leqslant w_q^2(A) + d_q^2(A)$. □

PROPOSITION 4.8. *Let* $A \in \mathcal{B}(\mathcal{H})$. *Then*

$$d_q(A) = \sup_{\substack{x,y \in \mathbb{S}_{\mathcal{H}} \\ \langle x|y\rangle = q}} \inf_{\lambda \in \mathbb{C}} \|Ax - \lambda y\|.$$

Proof. Note that

$$\inf_{\lambda\in\mathbb{C}} \|a-\lambda b\|^2 = \frac{\|a\|^2\|b\|^2 - |\langle a|b\rangle|^2}{\|b\|^2}, \tag{4.7}$$

for all $a,b \in \mathcal{H}$, $b \neq 0$. This equality is due to Dragomir (see [8]). Let $x,y \in \mathbb{S}_{\mathcal{H}}$ such that $\langle x|y\rangle = q$. Let $a = Ax$ and $b = y$ then by the equality (4.7), it follows that

$$\inf_{\lambda\in\mathbb{C}} \|Ax-\lambda y\|^2 = \|Ax\|^2 - |\langle Ax|y\rangle|^2.$$

Therefore,

$$\sup_{\substack{x,y\in\mathbb{S}_{\mathcal{H}}\\ \langle x|y\rangle=q}} \inf_{\lambda\in\mathbb{C}} \|Ax-\lambda y\|^2 = \sup_{\substack{x,y\in\mathbb{S}_{\mathcal{H}}\\ \langle x|y\rangle=q}} (\|Ax\|^2 - |\langle Ax|y\rangle|^2) = d_q(A)^2. \quad \square$$

COROLLARY 4.9. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$d_q^2(A) + w_q'(A)^2 \leqslant \|A\|^2. \tag{4.8}$$

Proof. Let $x,y \in \mathbb{S}_{\mathcal{H}}$ such that $\langle x|y\rangle = q$. Let $a = Ax$ and $b = y$ then by the equality (4.7), we have

$$\begin{aligned}\inf_{\lambda\in\mathbb{C}} \|Ax-\lambda y\|^2 &= \|Ax\|^2 - |\langle Ax|y\rangle|^2\\ &\leqslant \|A\|^2 - w_q'(A)^2.\end{aligned}$$

Hence Proposition 4.8 implies that $d_q^2(A) + w_q'(A)^2 \leqslant \|A\|^2$. $\square$

REMARK 4.10. (i) Note that the inequalities (4.2), (4.3), (4.4), (4.5), (4.6) and (4.8) are proved in [10] for $q = 1$.

(ii) If $A = \lambda I$ for some $\lambda \in \mathbb{C}$ then $w_0(A) = 0$. So, the inequalities (4.2) and (4.3) becomes equalities for all $B \in \mathcal{B}(\mathcal{H})$.

(iii) The inequality (4.8) becomes equality if $A = I$, since $d_q^2(I) = 1 - |q|^2$ and $w_q'(I)^2 = |q|^2$.

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REFERENCES

[1] M. ALEKSIEJCZYK, *On the diameter of the generalized numerical range*, Demonstratio Mathematica. **30** (1) (1997), 129–136.

[2] A. ABU-OMAR AND F. KITTANEH, *Notes on some spectral radius and numerical radius inequalities*, Studia Mathematica. **227** (2) (2015), 97–109.

[3] M. BARRAA AND M. BOUMAZGOUR, *A note on the orthogonality of bounded linear operators*, Functional Analysis Approximation and Computation. **41** (2012), 65–70.

[4] M. L. BUZANO, *Generalizzazione della diseguaglianza di Cauchy-Schwarz* (Italian), Rend. Sem. Mat. Univ. Politech. Torino. **31** (1974), 405–409.

[5] M. T. CHIEN AND H. NAKAZATO, *The q-numerical ranges of normal operators*, Linear Algebra and its Applications. **53** (6) (2005) 393–416.

[6] M. T. CHIEN AND H. NAKAZATO, *The q-numerical range of a reducible matrix via a normal operator*, Linear Algebra and its Applications. **419** (2006), 440–465.

[7] M. T. CHIEN AND H. NAKAZATO, *The boundary of the q-numerical range of a reducible matrix*, Linear and Multilinear Algebra. **55** (2007), 275–292.

[8] S. S. DRAGOMIR, *Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces*, Tamkang J. Math. **39** (1) (2008), 1–7.

[9] K. E. GUSTAFSON AND D. K. M. RAO, *Numerical range*, Universitext, Springer-Verlag, New York, 1997.

[10] M. S. HOSSEINIA AND B. MOOSAVI, *Some Numerical Radius Inequalities for Products of Hilbert Space Operators*, Filomat. **33** (7) (2019), 2089–2093.

[11] M. C. KAADOUD, *Domaine numérique du produit AB avec A normal*, Serdica Math. J. **32** (2006), 1–6.

[12] M. C. KAADOUD, *Domaine numérique du produit et de la bimultiplication $M_{2,A,B}$*, Proc. Amer. Math. Soc. **132** (2004), 2421–2428.

[13] M. C. KAADOUD, *Géométrie du spectre dans une algèbre de Banach et domaine numérique*, Studia Mathematica. **162** (1) (2004), 1–14.

[14] M. C. KAADOUD, *Domaine numérique de l'opérateur produit $M_{2,A,B}$ de la dérivation généralisée $\delta_{2,A,B}$*, Extracta mathematica. **17** (1) (2002), 59–68.

[15] M. C. KAADOUD, E. H. BENABDI AND M. GUESBA, *Some inequalities related to numerical radius and distance from scalar operators in Hilbert spaces*, Operators and Matrices. **17** (3) (2023), 857–866.

[16] C. S. KUBRUSLY, *Spectral Theory of Operators on Hilbert Spaces*, Birkhäuser, Springer, New York, 2012.

[17] C. K. LI, *q-numerical ranges of normal and convex matrices*, Linear and Multilinear Algebra. **43** (1998), 377–384.

[18] C. K. LI, P. P. MEHTA AND L. RODMAN, *A generalized numerical range: the range of a constrained sesquilinear form*, Linear and Multilinear Algebra. **37** (1994), 25–49.

[19] C. K. LI, AND H. NAKAZATO, *Some Results on the q-Numerical*, Linear and Multilinear Algebra. **43** (1998), 385–409.

[20] N. K. TSING, *The constrained bilinear form and the C-numerical range*, Linear Algebra Appl. **56** (1984), 195–206.

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