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MULTIPLICATIVELY NUMERICAL RANGE–PRESERVING MAPS

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Abstract. Given a complex Hilbert space $\mathscr{H}$, we denote by $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$, and $\mathscr{A}$, $\mathscr{B}$ two subsets of $\mathscr{B}(\mathscr{H})$ containing all operators of rank at most one. Let $W(A)$ the numerical range of $A \in \mathscr{B}(\mathscr{H})$. For an infinite-dimensional space $\mathscr{H}$, we prove that surjective maps $\phi_1, \phi_2 : \mathscr{A} \longrightarrow \mathscr{B}$ satisfy

$$W(\phi_1(A)\phi_2(B)) = W(AB), \quad (A, B \in \mathscr{A})$$

if and only if there exist $\mu, \nu \in \mathbb{C}$ with $\mu\nu = 1$, a bounded invertible linear operator U on $\mathscr{H}$ and a unitary operator V on $\mathscr{H}$ such that $\phi_1(A) = \mu VAU^{-1}$ and $\phi_2(A) = \nu UAV^*$ for all $A \in \mathscr{A}$. We also obtain an analogue result for the finite-dimensional case. Furthermore, some known results are obtained as immediate consequences of our main results.

1. Introduction

Throughout this paper, let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle\ ,\ \rangle)$. Let $\mathscr{F}_1(\mathscr{H})$ be the set of all operators of rank at most one in $\mathscr{B}(\mathscr{H})$. This means that $\mathscr{F}_1(\mathscr{H}) := \{x \otimes y \ : \ x \in \mathscr{H} \text{ and } y \in \mathscr{H}\}$, where $(x \otimes y)(z) := \langle z, y\rangle x$ for all $x, y, z \in \mathscr{H}$. If $\mathscr{H}$ has dimension $n < \infty$, we identify $\mathscr{H}$ with the Hilbert space $\mathbb{C}^n$ and $\mathscr{B}(\mathscr{H})$ with the algebra $\mathscr{M}_n(\mathbb{C})$ of $n \times n$-complex matrices. According to the above identifications, we have $\langle x, y\rangle = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$ and $x \otimes y = x^t\overline{y}$, where x^t stands for the transpose of x, for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{C}^n$. For a ring automorphism τ of $\mathbb{C}$, we set $x^\tau = (\tau(x_1), \tau(x_2), \ldots, \tau(x_n))$, $x^{\tau^*} = (\overline{\tau(\overline{x_1})}, \overline{\tau(\overline{x_2})}, \ldots, \overline{\tau(\overline{x_n})})$, and $M^\tau = (\tau(t_{ij}))$ for all $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $M = (t_{ij}) \in \mathscr{M}_n(\mathbb{C})$. The identity operator of $\mathscr{B}(\mathscr{H})$ will be denoted by I, and A^* will stand for the adjoint of operator $A \in \mathscr{B}(\mathscr{H})$. Note that if $A : \mathscr{H} \to \mathscr{H}$ is a bounded conjugate linear operator on $\mathscr{H}$ (i.e., A is additive and $A(\lambda x) = \overline{\lambda}A(x)$ for all $\lambda \in \mathbb{C}$ and $x \in \mathscr{H}$), then its adjoint A^* is the bounded conjugate linear operator on $\mathscr{H}$ defined by $\langle A(x), y\rangle = \overline{\langle x, A^*(y)\rangle}$ for all $x, y \in \mathscr{H}$. The numerical range of an operator $A \in \mathscr{B}(\mathscr{H})$ is

$$W(A) := \{\langle A(x), x\rangle : \ x \in \mathscr{H}, \ \|x\| = 1\}.$$

The numerical radius of A is given by

$$w(A) := \sup\{|z| : z \in W(A)\}.$$

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Numerical range of operators is a very important concept and is extensively studied in both theory and applications. Particularly, several researchers have studied numerical range preserving maps on various operator algebras; see [8, 9, 10, 11, 13, 18, 24, 19, 23, 25].

Recently, there has been interest in studying maps ϕ on matrices or operators satisfying $F(\phi(A)\bullet\phi(B)) = F(A\bullet B)$. Here , $F(\cdot)$ is a spectral function or a spectral set such as the spectrum, the local spectrum, the numerical and generalized numerical range, the numerical radius and generalized numerical radius, the ε-pseudo spectral radius, the ε-pseudo spectrum, the ε-condition spectral radius, and the ε-condition spectrum. On the other hand, $A\bullet B$ stands for different kinds of products such as the usual product AB, the triple product ABA, the Jordan product $AB+BA$, the skew product A^*B, the skew triple product AB^*A, the skew-Jordan product AB^*+B^*A, and the Lie product $AB-BA$; see for instance [1, 2, 3, 6, 7, 14, 15, 16, 17, 18, 19, 25] and the references therein.

In [21], Molnàr characterized maps preserving the spectrum of operator or matrix products. His result has been extended in several directions. In particular, Abdelali and Aharmim characterized in [4] couple of maps $\phi_1 : \mathscr{M}_{m,n}(\mathbb{C}) \longrightarrow \mathscr{M}_{p,q}(\mathbb{C})$ and $\phi_2 : \mathscr{M}_{n,m}(\mathbb{C}) \longrightarrow \mathscr{M}_{q,p}(\mathbb{C})$ satisfying

$$\sigma(\phi_1(A)\phi_2(B)) = \sigma(AB), \quad ((A,B) \in \mathscr{M}_{m,n}(\mathbb{C}) \times \mathscr{M}_{n,m}(\mathbb{C})),$$

where $\sigma(A)$ is the spectrum of A and $\mathscr{M}_{m,n}(\mathbb{C})$ the vector space of matrices of m rows and n columns with complex entries. Recently, Bourhim and Lee studied in [5] surjective maps between algebras of operators $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ on Banach spaces X and Y respectively, such that

$$\sigma_{\phi_1(A)\phi_2(B)}(x_0) = \sigma_{AB}(y_0), \quad (A,B \in \mathscr{B}(X)),$$

where $x_0 \in X$ and $y_0 \in Y$ are fixed nonzero vectors and $\sigma_A(x_0)$ is the local spectrum of A at x_0. In [2] Nkhaylia and Abdelali characterized surjective maps $\phi_1, \phi_2 : \mathscr{A} \longrightarrow \mathscr{B}$ such that the ε-pseudo spectrum of $\phi_1(A)\phi_2(B)$ coincides with that of AB for all $A, B \in \mathscr{A}$, where $\mathscr{A}$ and $\mathscr{B}$ are subsets of $\mathscr{B}(\mathscr{H})$ containing all operators of rank at most one.

In the same spirit, we investigate in this paper maps $\phi_1, \phi_2 : \mathscr{A} \longrightarrow \mathscr{B}$ that preserve multiplicatively the numerical range in this way

$$W(\phi_1(A)\phi_2(B)) = W(AB), \quad (A,B \in \mathscr{A}), \tag{1.1}$$

where $\mathscr{A}$ and $\mathscr{B}$ are two subsets of $\mathscr{B}(\mathscr{H})$ containing all operators of rank at most one.

The paper is divided into four sections. After the foregoing one, we state in section 2 the main results. In section 3, we give some auxiliary results that are needed for the proof of our main results. Sections 4 and 5 will be devoted to the proof of the main results.

2. Main results

Here, we deal with two maps rather than a single one as studied in [19].

THEOREM 2.1. *Let $\mathscr{H}$ be an infinite-dimensional complex Hilbert space, and $\mathscr{A}$, $\mathscr{B}$ two subsets of $\mathscr{B}(\mathscr{H})$ containing $\mathscr{F}_1(\mathscr{H})$. Suppose that $\phi_1, \phi_2 : \mathscr{A} \longrightarrow \mathscr{B}$ are two surjective maps. Then, ϕ_1 and ϕ_2 satisfy*

$$W(\phi_1(A)\phi_2(B)) = W(AB), \quad (A, B \in \mathscr{A}) \tag{2.1}$$

if and only if there exist $\mu, \nu \in \mathbb{C}$ with $\mu\nu = 1$, a bounded invertible linear operator U on $\mathscr{H}$ and a unitary operator V on $\mathscr{H}$ such that $\phi_1(A) = \mu VAU^{-1}$ and $\phi_2(A) = \nu UAV^*$ for all $A \in \mathscr{A}$.

The following result refines the above theorem in the finite-dimensional case.

THEOREM 2.2. *Let $n \geqslant 3$, and $\mathscr{A}$, $\mathscr{B}$ two subsets of $\mathscr{M}_n(\mathbb{C})$ containing $\mathscr{F}_1(\mathbb{C})$. Suppose that $\phi_1, \phi_2 : \mathscr{A} \longrightarrow \mathscr{B}$ are two surjective maps. Then ϕ_1 and ϕ_2 satisfy*

$$W(\phi_1(A)\phi_2(B)) = W(AB), \quad (A, B \in \mathscr{A}) \tag{2.2}$$

if and only if there exist $\mu, \nu \in \mathbb{C}$ with $\mu\nu = 1$, a nonsingular matrix $U \in \mathscr{M}_n(\mathbb{C})$ and a unitary matrix $V \in \mathscr{M}_n(\mathbb{C})$ such that $\phi_1(A) = \mu VAU^{-1}$ and $\phi_2(A) = \nu UAV^*$ for all $A \in \mathscr{A}$.

We end this section by the two following corollaries. First, we characterize maps preserving the numerical range of the product AB. Note that, let $\mathscr{H}$ and $\mathscr{K}$ be two complex Hilbert spaces, surjective maps $\phi : \mathscr{B}(\mathscr{H}) \longrightarrow \mathscr{B}(\mathscr{K})$ satisfying

$$W(\phi(A)\phi(B)) = W(AB), \quad (A, B \in \mathscr{B}(\mathscr{H})),$$

were described in [19, Theorem 1.2].

COROLLARY 2.3. *Let $\mathscr{H}$ be a complex Hilbert space with $\dim(\mathscr{H}) \geqslant 3$, and $\mathscr{A}$, $\mathscr{B}$ two subsets of $\mathscr{B}(\mathscr{H})$ containing all operators of rank at most one. Then, a surjective map $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ satisfies*

$$W(\phi(A)\phi(B)) = W(AB), \quad (A, B \in \mathscr{A})$$

if and only if there exist a unitary operator V on $\mathscr{H}$, and $\lambda \in \{1, -1\}$ such that $\phi(A) = \lambda VAV^*$ for all $A \in \mathscr{A}$.

Next, let $\mathscr{A}$ and $\mathscr{B}$ be two subsets of $\mathscr{B}(\mathscr{H})$ which containing all operators of rank at most one. Surjective maps $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ satisfying

$$W(\phi(A)^*\phi(B)) = W(A^*B), \quad (A, B \in \mathscr{B}(\mathscr{H})),$$

were described in [19, Corollary 4.3]. Such characterization is now a consequence of our main results.

COROLLARY 2.4. *Let $\mathscr{H}$ be a complex Hilbert spaces with $\dim(\mathscr{H}) \geqslant 3$, and $\mathscr{A}$, $\mathscr{B}$ two subsets of $\mathscr{B}(\mathscr{H})$ containing all operators of rank at most one. Then, a surjective map $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ satisfies*

$$W(\phi(A)^*\phi(B)) = W(A^*B), \quad (A,B \in \mathscr{A})$$

if and only if there exist a complex unit λ and two unitary operators $U,V \in \mathscr{B}(\mathscr{H})$ such that $\phi(A) = \lambda UAV^*$ for all $A \in \mathscr{A}$.

The rest of this paper is organized as follows. In section 2, we present some useful results on the numerical range and the numerical radius. These results are needed in sections 3 and 4 which will be devoted to the proofs of our results.

3. Preliminaries

In this section, we state some results that we will use in the proof of the main results. The first result collects some useful known properties of the numerical range and the numerical radius.

PROPOSITION 3.1. *For an operator $A \in \mathscr{B}(\mathscr{H})$, the following statements hold.*

i) For every nonzero scalars $\alpha, \beta \in \mathbb{C}$, we have $W(\alpha I + \beta A) = \{\alpha\} + \beta \cdot W(A)$.

ii) For every unitary operator $U \in \mathscr{B}(\mathscr{H})$, we have $W(UAU^*) = W(A)$.

iii) For every conjugate unitary operator U, we have $W(UAU^*) = W(A^*) = \overline{W(A)}$. Consequently, we have:

iv) For every nonzero scalar $\beta \in \mathbb{C}$, we have $w(\beta A) = |\beta| w(A)$.

v) For every bijective isometry $U \in \mathscr{B}(\mathscr{H})$, we have $w(UAU^{-1}) = w(A)$.

vi) For every conjugate isometry operator U, we have $w(UAU^{-1}) = w(A)$.

The second result, quoted from [24, Lemma 2.4], will be the backbone of the proof of our main results. It identifies the numerical radius of rank one operators.

LEMMA 3.2. *Let $(\mathscr{H}, \langle\ ,\ \rangle)$ be a complex Hilbert space. If x and y are two vectors in $\mathscr{H}$, then*

$$w(x \otimes y) = \frac{1}{2}\left(|\langle x, y\rangle| + \|x\|\,\|y\|\right).$$

The following lemma, established in [2, Theorem 2.2], determines the structure of mappings that preserve the double product divisors of zero.

LEMMA 3.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be two subsets of $\mathscr{B}(\mathscr{H})$ containing $\mathscr{F}_1(\mathscr{H})$. Suppose that $\phi_1, \phi_2 : \mathscr{A} \longrightarrow \mathscr{B}$ are two surjective maps satisfying*

$$\phi_1(A)\phi_2(B) = 0 \Longleftrightarrow AB = 0, \quad (A,B \in \mathscr{A}).$$

Then the following statements hold.

1. *If* $\dim(\mathscr{H}) = \infty$, *then there exist two maps*

$$h,k : \mathscr{H} \times \mathscr{H} \to \mathscr{H},\ (x,y) \mapsto k_y(x), (x,y) \mapsto h_x(y)$$

and a bounded invertible linear or conjugate linear operator U *on* $\mathscr{H}$ *such that*

$$\phi_1(x \otimes y) = k_y(x) \otimes (U^{-1})^*(y) \ \text{ and } \ \phi_2(x \otimes y) = U(x) \otimes h_x(y), \quad (x,y \in \mathscr{H}). \tag{3.1}$$

2. *If* $\mathscr{H} = \mathbb{C}^n$, *with* $n \geqslant 3$, *then there exist two maps* $k,h : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $(x,y) \mapsto k_y(x), (x,y) \mapsto h_x(y)$, *a nonsingular matrix* $U \in \mathscr{M}_n(\mathbb{C})$, *and a ring automorphism* τ *of* $\mathbb{C}$ *such that*

$$\phi_1(x \otimes y) = k_y(x) \otimes (U^{-1})^*(y^{\tau^*}) \ \text{ and } \ \phi_2(x \otimes y) = U(x^\tau) \otimes h_x(y), \quad (x,y \in \mathbb{C}^n). \tag{3.2}$$

Finally, as a consequence of Lemma 3.3, we shall present our Uhlhorn-type version of Wigner's theorem [22].

LEMMA 3.4. *Let* $\mathscr{H}$ *be a complex Hilbert space with* $\dim(\mathscr{H}) \geqslant 3$. *Suppose that* $h,k : \mathscr{H} \longrightarrow \mathscr{H}$ *are two surjective maps satisfying*

$$\langle h(x), k(y) \rangle = 0 \iff \langle x,y \rangle = 0, \quad (x,y \in \mathscr{H}). \tag{3.3}$$

Then the following statements hold.

1. *If* $\dim(\mathscr{H}) = \infty$, *then there exist there exist two functionals* $\mu, \nu : \mathscr{H} \longrightarrow \mathbb{C} \setminus \{0\}$ *and a bounded invertible linear or conjugate linear operator* U *on* $\mathscr{H}$ *such that*

$$h(x) = \mu(x)U(x) \ \text{ and } \ k(x) = \nu(x)(U^{-1})^*(x) \ \text{ for all } x \in \mathscr{H}.$$

2. *If* $\mathscr{H} = \mathbb{C}^n$, *with* $n \geqslant 3$, *then there exist there exist two functionals* $\mu, \nu : \mathbb{C}^n \longrightarrow \mathbb{C} \setminus \{0\}$ *and a nonsingular matrix* $U \in \mathscr{M}_n(\mathbb{C})$, *and a ring automorphism* τ *of* $\mathbb{C}$ *such that*

$$h(x) = \mu(x)U(x^\tau) \ \text{ and } \ k(x) = \nu(x)(U^{-1})^*(x^{\tau^*}) \ \text{ for all } x \in \mathbb{C}^n.$$

Proof. Applying Lemma 3.3 in the case when $\mathscr{A} = \mathscr{B} = \mathscr{F}_1(\mathscr{H})$ and $\phi_1(x \otimes y) = x \otimes k(y)$, $\phi_2(x \otimes y) = h(x) \otimes y$ for all $x \otimes y \in \mathscr{F}_1(\mathscr{H})$. Since $h,k : \mathscr{H} \longrightarrow \mathscr{H}$ are surjective maps satisfying

$$\langle h(x), k(y) \rangle = 0 \iff \langle x,y \rangle = 0 \text{ for all } x,y \in \mathscr{H},$$

it follows that $\phi_1, \phi_2 : \mathscr{F}_1(\mathscr{H}) \longrightarrow \mathscr{F}_1(\mathscr{H})$ are two surjective maps satisfying

$$\phi_1(A)\phi_2(B) = 0 \iff AB = 0 \text{ for all } A,B \in \mathscr{F}_1(\mathscr{H}).$$

Therefore, by Lemma 3.3 there exist two maps $\alpha, \beta : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ such that one of the following results:

1. If $\dim(\mathscr{H}) = \infty$, then there exists a bounded invertible linear or conjugate linear operator U on $\mathscr{H}$ such that

   $\phi_1(x\otimes y) = \alpha(x,y)\otimes(U^{-1})^*(y)$ and $\phi_2(x\otimes y) = U(x)\otimes\beta(x,y)$ for all $x,y\in\mathscr{H}$.

2. If $\mathscr{H} = \mathbb{C}^n$, with $n \geqslant 3$, then there exist a nonsingular matrix $U \in \mathscr{M}_n(\mathbb{C})$, and a ring automorphism τ of $\mathbb{C}$ such that

   $\phi_1(x\otimes y) = \alpha(x,y)\otimes(U^{-1})^*(y^{\tau^*})$ and $\phi_2(x\otimes y) = U(x^\tau)\otimes\beta(x,y)$ for all $x\in\mathbb{C}^n$.

On the other hand, since $\phi_1(x\otimes y) = x\otimes k(y)$, $\phi_2(x\otimes y) = h(x)\otimes y$ for all $x\otimes y \in \mathscr{F}_1(\mathscr{H})$, then we conclude that

1. if $\dim(\mathscr{H}) = \infty$, then there exist there exist two functionals $\mu,\nu : \mathscr{H} \longrightarrow \mathbb{C}\setminus\{0\}$ such that

$$h(x) = \mu(x)U(x) \text{ and } k(x) = \nu(x)(U^{-1})^*(x) \text{ for all } x\in\mathscr{H},$$

2. if $\mathscr{H} = \mathbb{C}^n$, with $n \geqslant 3$, then there exist there exist two functionals $\mu,\nu : \mathbb{C}^n \longrightarrow \mathbb{C}\setminus\{0\}$ such that

$$h(x) = \mu(x)U(x^\tau) \text{ and } k(x) = \nu(x)(U^{-1})^*(x^{\tau^*}) \text{ for all } x\in\mathbb{C}^n. \quad \square$$

4. Proof of Theorems 2.1 and 2.2

From Proposition 3.1 we immediately deduce the "*if*" part of Theorems 2.1 and 2.2. Therefore, we only need to prove the "*only if*" part of Theorems 2.1 and 2.2. Then assume that $\phi_1,\phi_2 : \mathscr{A} \longrightarrow \mathscr{B}$ are two surjective maps satisfying

$$W(\phi_1(A)\phi_2(B)) = W(AB) \tag{4.1}$$

for all $A,B\in\mathscr{A}$. This implies that

$$\phi_1(A)\phi_2(B) = 0 \Longleftrightarrow AB = 0$$

for all $A,B\in\mathscr{A}$. Thus we can assume that ϕ_1 and ϕ_2 have the forms given in Lemma 3.3. Therefore,

$$\phi_1(x\otimes y) = k_y(x)\otimes(U^{-1})^*(y^{\tau_1^*}) \text{ and } \phi_2(x\otimes y) = U(x^{\tau_1})\otimes h_x(y) \tag{4.2}$$

for all $x,y\in\mathscr{H}$, where

$$x^{\tau_1} = \begin{cases} x^\tau & \text{if } \mathscr{H} = \mathbb{C}^n \\ x & \text{if } \dim(\mathscr{H}) = \infty \end{cases}$$

for all $x\in\mathscr{H}$. We also denote

$$\mu(z) := \begin{cases} \tau(z) & \text{if } \mathscr{H} = \mathbb{C}^n \\ \overline{z} & \text{if } \dim(\mathscr{H}) = \infty \text{ and } U \text{ is conjugate linear} \\ z & \text{if } \dim(\mathscr{H}) = \infty \text{ and } U \text{ is linear} \end{cases}$$

for all $z \in \mathbb{C}$.

The rest of the proof will be provided by checking four assertions.

ASSERTION 1. For every non orthogonal vectors x and y, we have $h_x(\cdot)$ and $k_y(\cdot)$ are surjective maps satisfying

$$\langle k_y(u), h_x(v)\rangle = \frac{\langle x,y\rangle}{\mu(\langle x,y\rangle)}\langle u,v\rangle \quad \text{for all} \quad u,v \in \mathscr{H}. \tag{4.3}$$

Moreover,

$$\langle k_y(u), h_x(v)\rangle = 0 \Longleftrightarrow \langle u,v\rangle = 0 \quad \text{for all} \quad u,v \in \mathscr{H}. \tag{4.4}$$

Proof. Let $x,y \in \mathscr{H}$ such that $\langle x,y\rangle \neq 0$. First, let us show that $k_y(\cdot)$ and $h_x(\cdot)$ are surjective. To do it, for $k_y(\cdot)$, it suffice to show that the folowing equality holds

$$\phi_1\left(\{u \otimes y : \ u \in \mathscr{H} \setminus \{0\}\}\right) = \left\{u \otimes (U^{-1})^*(y^{\tau_1^*}) : \ u \in \mathscr{H} \setminus \{0\}\right\}. \tag{4.5}$$

Since $\phi_1\left(\{u \otimes y : \ u \in \mathscr{H} \setminus \{0\}\}\right) \subseteq \left\{u \otimes (U^{-1})^*(y^{\tau_1^*}) : \ u \in \mathscr{H} \setminus \{0\}\right\}$ is obvious, we only need to establish the other inclusion. Let u be a nonzero vector in $\mathscr{H}$, since ϕ_1 preserves rank one operators in both directions, then there exists $v \otimes w \in \mathscr{F}_1(\mathscr{H})$ such that

$$u \otimes (U^{-1})^*(y^{\tau_1^*}) = \phi_1(v \otimes w) = k_w(v) \otimes (U^{-1})^*(w^{\tau_1^*}).$$

It follows that y and w are linearly dependent, and so $w = \alpha y$ for some nonzero scalar α. Thus,

$$\phi(\overline{\alpha} v \otimes y) = \phi(v \otimes w) = u \otimes (U^{-1})^*(y^{\tau_1^*}),$$

and the equality (4.5) is proved. Thus, the surjectivity of $k_y(\cdot)$ follows immediately. By a similar argument as above, we conclude that $h_x(\cdot)$ is surjective.

Second, let us show that $k_y(\cdot)$ and $h_x(\cdot)$ satisfy the conclusion of the Assertion 1. Let $u,v \in \mathscr{H}$, by the equalities (4.1) and (4.2), we have

$$\begin{aligned} W\left(\mu(\langle x,y\rangle)k_y(u) \otimes h_x(v)\right) &= W\left(\phi_1(u \otimes y)\phi_2(x \otimes v)\right) \\ &= W\left((u \otimes y)(x \otimes v)\right) \\ &= W(\langle x,y\rangle u \otimes v). \end{aligned}$$

Hence,

$$W\left(\mu(\langle x,y\rangle)k_y(u) \otimes h_x(v)\right) = W(\langle x,y\rangle u \otimes v).$$

Thus, by taking trace,

$$\mu(\langle x,y\rangle)\langle k_y(u), h_x(v)\rangle = \langle x,y\rangle\langle u,v\rangle.$$

So,

$$\langle k_y(u), h_x(v)\rangle = \frac{\langle x,y\rangle}{\mu(\langle x,y\rangle)}\langle u,v\rangle$$

for all $u,v \in \mathscr{H}$, and we have

$$\langle k_y(u), h_x(v)\rangle = 0 \Longleftrightarrow \langle u,v\rangle = 0$$

for all $u,v\in\mathscr{H}$. $\square$

ASSERTION 2. There exist there exist two functionals $d,l:\mathscr{H}\setminus\{0\}\times\mathscr{H}\setminus\{0\}\to\mathbb{C}\setminus\{0\}$, $(x,y)\mapsto d_y(x)$; $(x,y)\mapsto l_x(y)$ such that one of the following results:

1. If $\dim(\mathscr{H})=\infty$, then there exists a bounded invertible linear or conjugate linear operator V such that
$$k_y(x)=d_y(x)V(x) \text{ and } h_x(y)=l_x(y)(V^{-1})^*(y) \text{ for all } x,y\in\mathscr{H}\setminus\{0\}. \tag{4.6}$$
2. If $\mathscr{H}=\mathbb{C}^n$, with $n\geqslant 3$, then there exist a unitary matrix $V\in\mathscr{M}_n(\mathbb{C})$ and a ring automorphism τ' of $\mathbb{C}$ such that
$$k_y(x)=d_y(x)V(x^{\tau'}) \text{ and } h_x(y)=l_x(y)(V^{-1})^*(y^{\tau'^*}) \text{ for all } x,y\in\mathbb{C}^n\setminus\{0\}. \tag{4.7}$$

Proof. Let $x,y\in\mathscr{H}$ such that $\langle x,y\rangle\neq 0$. By Assertion 1, we see that $h_x(\cdot)$ and $k_y(\cdot)$ are surjective maps satisfying (4.4). Then in light of Lemma 3.4, we have the following two possibilities:

1. If $\dim(\mathscr{H})=\infty$, then there exist there exist two functionals $\mu_y,\nu_x:\mathscr{H}\longrightarrow\mathbb{C}\setminus\{0\}$ and two bounded invertible linear or conjugate linear operators V_x,V_y on $\mathscr{H}$ with V_x and V_y are linearly dependent such that
$$k_y(u)=\mu_y(u)V_y(u) \text{ and } h_x(u)=\nu_x(u)(V_x^{-1})^*(u) \text{ for all } u\in\mathscr{H}. \tag{4.8}$$
2. If $\mathscr{H}=\mathbb{C}^n$, with $n\geqslant 3$, then there exist there exist two functionals $\mu_y,\nu_x:\mathbb{C}^n\longrightarrow\mathbb{C}\setminus\{0\}$ and two nonsingular matrices $V_x,V_y\in\mathscr{M}_n(\mathbb{C})$ with V_x and V_y are linearly dependent and two ring automorphisms τ_x, τ_y of $\mathbb{C}$ such that
$$k_y(u)=\mu_y(u)V_y(u^{\tau_y}) \text{ and } h_x(u)=\nu_x(u)(V_x^{-1})^*\left(u^{\tau_x^*}\right) \text{ for all } u\in\mathbb{C}^n. \tag{4.9}$$

Now, let us show that $\tau_y=\tau_x^*$ for all $x,y\in\mathbb{C}^n\setminus\{0\}$. Let $x,y\in\mathbb{C}^n$ such that $\langle x,y\rangle\neq 0$. Let $\mu\in\mathbb{C}$ and $w,z\in\mathbb{C}^n$ be two unit vectors such that $\langle w,z\rangle=0$. Clearly, $\langle \mu w-z,w+\overline{\mu}z\rangle=0$ and by (4.4) and (4.9), we have
$$\begin{aligned}
0&=\left\langle k_y(\mu w-z),h_x(w+\overline{\mu}z)\right\rangle\\
&=\left\langle V_y\left((\mu w-z)^{\tau_y}\right),(V_x^{-1})^*\left((w+\overline{\mu}z)^{\tau_x^*}\right)\right\rangle\\
&=\tau_y(\mu)\left\langle V_y(w^{\tau_y}),(V_x^{-1})^*(w^{\tau_x^*})\right\rangle-\overline{\tau_x^*(\overline{\mu})}\left\langle V_y(z^{\tau_y}),(V_x^{-1})^*(z^{\tau_x^*})\right\rangle\\
&\quad+\overline{\tau_x^*(\overline{\mu})}\tau_y(\mu)\left\langle V_y(w^{\tau_y}),(V_x^{-1})^*(z^{\tau_x^*})\right\rangle-\left\langle V_y(z^{\tau_y}),(V_x^{-1})^*(w^{\tau_x^*})\right\rangle\\
&=\tau_y(\mu)\left\langle V_y(w^{\tau_y}),(V_x^{-1})^*(w^{\tau_x^*})\right\rangle-\overline{\tau_x^*(\overline{\mu})}\left\langle V_y(z^{\tau_y}),(V_x^{-1})^*(z^{\tau_x^*})\right\rangle.
\end{aligned}$$

Therefore,
$$\tau_y(\mu)\left\langle V_y(w^{\tau_y}),(V_x^{-1})^*(w^{\tau_x^*})\right\rangle-\overline{\tau_x^*(\overline{\mu})}\left\langle V_y(z^{\tau_y}),(V_x^{-1})^*(z^{\tau_x^*})\right\rangle=0 \tag{4.10}$$

for all $\mu \in \mathbb{C}$. Choosing $\mu = 1$ in the above equality, we obtain that

$$\left\langle V_y(w^{\tau_y}), (V_x^{-1})^*(w^{\tau_x^*}) \right\rangle = \left\langle V_y(z^{\tau_y}), (V_x^{-1})^*(z^{\tau_x^*}) \right\rangle.$$

This, together with Equation (4.10) imply that

$$\left(\tau_y(\mu) - \overline{\tau_x^*(\overline{\mu})}\right) \left\langle V_y(z^{\tau_y}), (V_x^{-1})^*(z^{\tau_x^*}) \right\rangle = 0$$

for all $\mu \in \mathbb{C}$ and all unit vectors $x \in \mathbb{C}^n$. Again by (4.4) and the fact that $\langle z, z \rangle \neq 0$, we have $\left\langle V_y(z^{\tau_y}), (V_x^{-1})^*(z^{\tau_x^*}) \right\rangle \neq 0$. Then

$$\tau_y(\mu) = \overline{\tau_x^*(\overline{\mu})} = \tau_x(\mu)$$

for all $\mu \in \mathbb{C}$. So, $\tau_y = \tau_x$ for all $x, y \in \mathbb{C}^n$ such that $\langle x, y \rangle \neq 0$. In the other case, if $x, y \in \mathscr{H} \setminus \{0\}$ such that $\langle x, y \rangle = 0$, then there exists $z \in \mathscr{H}$ such that $\langle x, z \rangle \neq 0$ and $\langle y, z \rangle \neq 0$. Then $\tau_x = \tau_z$ and $\tau_y = \tau_z$. Therefore, $\tau_y = \tau_x$ for all $x, y \in \mathscr{H} \setminus \{0\}$. Hence, there exists a ring automorphism τ' of $\mathbb{C}$ such that $\tau_x = \tau'$ for all $x \in \mathscr{H} \setminus \{0\}$.

Finally, let us show that there exists a nonzero scalar β_x dependent on x and if $\dim(\mathscr{H}) = \infty$ (resp, $\mathscr{H} = \mathbb{C}^n$) there exists a bounded invertible linear or conjugate linear operator (resp, nonsingular matrix) V independent on x such that $V_x = \beta_x V$. Let x, y be two nonzero vectors in $\mathscr{H}$. If $x, y \in \mathscr{H}$ be two non orthogonal vectors. By (4.4), we have

$$\begin{aligned}
\left\langle z^{\tau'}, w^{\tau_1'^*} \right\rangle = 0 &\iff \tau_1'(\langle z, w \rangle) = 0 \\
&\iff \langle z, w \rangle = 0 \\
&\iff \langle k_y(z), h_x(w) \rangle = 0 \\
&\iff \left\langle V_y(z^{\tau_1'}), (V_x^{-1})^*(w^{\tau_1'^*}) \right\rangle = 0 \\
&\iff \left\langle V_x^{-1} V_y(z^{\tau_1'}), w^{\tau_1'^*} \right\rangle = 0
\end{aligned}$$

for all $w, z \in \mathbb{C}^n$, where

$$x^{\tau_1'} = \begin{cases} x^{\tau'} & \text{if } \mathscr{H} = \mathbb{C}^n \\ x & \text{if } \dim(\mathscr{H}) = \infty \end{cases}$$

for all $x \in \mathscr{H}$. It then follows that

$$\langle z, w \rangle = 0 \iff \left\langle V_x^{-1} V_y(z), w \right\rangle = 0$$

for all $w, z \in \mathbb{C}^n$. Therefore, $V_x^{-1} V_y(z)$ and z are linearly independent for all $z \in \mathbb{C}^n$, and thus $V_y = \lambda_{x,y} V_x$ for some nonzero scalar $\lambda_{x,y} \in \mathbb{C}$. In the other case, if $x, y \in \mathscr{H} \setminus \{0\}$ such that $\langle x, y \rangle = 0$, then there exists $z \in \mathscr{H}$ such that $\langle x, z \rangle \neq 0$ and $\langle y, z \rangle \neq 0$. Then V_x and V_z are linearly dependent and V_y and V_z are also linearly dependent. Therefore, V_x and V_y are linearly dependent for all $x, y \in \mathscr{H} \setminus \{0\}$. So, there exists a functional

$\beta : \mathscr{H} \setminus \{0\} \to \mathbb{C} \setminus \{0\}; x \mapsto \beta_x$ and if $\dim(\mathscr{H}) = \infty$ (resp, $\mathscr{H} = \mathbb{C}^n$) there exists a bounded invertible linear or conjugate linear operator (resp, nonsingular matrix) V on $\mathscr{H}$ such that

$$V_x = \beta_x V \quad \text{for all} \quad x \in \mathscr{H} \setminus \{0\}. \tag{4.11}$$

Denote $d_x(u) = \beta_x \mu_x(u)$ and $l_x(u) = \frac{\nu_x(u)}{\overline{\beta_x}}$ for all $x, u \in \mathscr{H} \setminus \{0\}$. This, together with (4.8), (4.9) and (4.11) tell us that

$$k_x(u) = d_x(u) V(u^{\tau_1'}) \quad \text{and} \quad h_x(u) = l_x(u)(V^{-1})^*(u^{\tau_1'^*}) \tag{4.12}$$

for all $x, u \in \mathscr{H} \setminus \{0\}$. $\square$

ASSERTION 3. There exist two nonzero scalars μ and ν satisfying $\mu\nu = 1$ and

$$\phi_1(A) = \mu V A U^{-1} \quad \text{and} \quad \phi_2(A) = \nu U A V^{-1} \quad (A \in \mathscr{A}).$$

Proof. From Lemma 3.4 and Assertion 2, it follows that

$$\phi_1(x \otimes y) = d_y(x) V(x^{\tau_1'}) \otimes (U^{-1})^*(y^{\tau_1^*}) \quad \text{and} \quad \phi_2(x \otimes y) = \overline{l_x(y)} U(x^{\tau_1}) \otimes (V^{-1})^*(y^{\tau_1'^*})$$

for all $x, y \in \mathscr{H}$. We define the maps $h_1, h_2 : \mathscr{F}_1(\mathscr{H}) \mapsto \mathbb{C} \setminus \{0\}$ by

$$h_1(x \otimes y) = d_y(x) \quad \text{if} \quad x \otimes y \neq 0, \quad h_1(0) = 1, \tag{4.13}$$

and

$$h_2(x \otimes y) = \overline{l_x(y)} \quad \text{if} \quad x \otimes y \neq 0, \quad h_2(0) = 1. \tag{4.14}$$

Hence,

$$\phi_1(x \otimes y) = h_1(x \otimes y) V(x^{\tau_1'}) \otimes (U^{-1})^*(y^{\tau_1^*})$$

and

$$\phi_2(x \otimes y) = h_2(x \otimes y) U(x^{\tau_1}) \otimes (V^{-1})^*(y^{\tau_1'^*}) \tag{4.15}$$

for all $x, y \in \mathscr{H}$.

To complete the proof of Assertion 3, we will proceed in two steps.

Step 1. There exist two nonzero scalars μ and ν satisfying $\mu\nu = 1$ and

$$h_1(x \otimes y) = \mu \text{ and } h_2(x \otimes y) = \nu \text{ for all nonzero } x \otimes y \in \mathscr{F}_1(\mathscr{H}). \tag{4.16}$$

By equations (4.3), (4.12), (4.13) and (4.14), we have

$$\begin{aligned}
\frac{\langle x, y \rangle}{\eta(\langle x, y \rangle)} \langle u, v \rangle &= \left\langle k_y(u), h_x(v) \right\rangle \\
&= \left\langle l_y(u) V(u^{\tau_1'}), d_x(v)(V^{-1})^*(v^{\tau_1'^*}) \right\rangle \\
&= \left\langle h_1(u \otimes y) V(u^{\tau_1'}), \overline{h_2(x \otimes v)}(V^{-1})^*(v^{\tau_1'^*}) \right\rangle \\
&= h_1(u \otimes y) h_2(x \otimes v) \eta'(\langle u, v \rangle)
\end{aligned}$$

for all $x\otimes u,\ y\otimes v\in\mathscr{F}_1(\mathscr{H})$ with $\langle y,u\rangle\neq 0$, where

$$\eta'(z):=\begin{cases}\tau'(z) & \text{if } \mathscr{H}=\mathbb{C}^n\\ \overline{z} & \text{if } \dim(\mathscr{H})=\infty \text{ and } V \text{ is conjugate linear}\\ z & \text{if } \dim(\mathscr{H})=\infty \text{ and } V \text{ is linear}\end{cases}$$

for all $z\in\mathbb{C}$. Therefore,

$$h_1(u\otimes y)h_2(x\otimes v)=\frac{\langle x,y\rangle\,\langle u,v\rangle}{\eta\left(\langle x,y\rangle\right)\eta'\left(\langle u,v\rangle\right)}\tag{4.17}$$

for all $u\otimes y, x\otimes v\in\mathscr{F}_1(\mathscr{H})$ with $\langle x,y\rangle\neq 0$ and $\langle u,v\rangle\neq 0$. We have two cases to discuss:

Case 1. If $\dim(\mathscr{H})=\infty$, we have

$$h_1(u\otimes y)h_2(x\otimes v)=\frac{\langle x,y\rangle\,\langle u,v\rangle}{\eta\left(\langle x,y\rangle\right)\eta'\left(\langle u,v\rangle\right)}=1\tag{4.18}$$

for all $u\otimes y, x\otimes v\in\mathscr{F}_1(\mathscr{H})$ with $\langle x,y\rangle\neq 0$ and $\langle u,v\rangle\neq 0$. Now, let $x,y,u,v\in\mathscr{H}$ with $\langle u,v\rangle\neq 0$ and $\langle x,y\rangle=0$, choose $w_1,w_2\in\mathscr{H}$ with $\langle w_1,w_2\rangle\neq 0$, $\langle w_1,y\rangle\neq 0$ and $\langle x,w_2\rangle\neq 0$. Therefore, the equality (4.18) implies that

$$h_1(u\otimes y)h_2(w_1\otimes v)=1\ ,\ h_1(u\otimes w_2)h_2(w_1\otimes v)=1 \text{ and } h_1(u\otimes w_2)h_2(x\otimes v)=1,$$

and thus

$$\begin{aligned}h_1(u\otimes y)h_2(x\otimes v)&=h_1(u\otimes y)h_1(u\otimes w_2)h_2(w_1\otimes v)h_2(x\otimes v)\\&=h_1(u\otimes y)h_2(w_1\otimes v)h_1(u\otimes w_2)h_2(x\otimes v)\\&=1.\end{aligned}$$

Next, assume that $\langle u,v\rangle=0$, choose $z,w\in\mathscr{H}$ with $\langle z,w\rangle\neq 0$, $\langle z,v\rangle\neq 0$ and $\langle u,w\rangle\neq 0$. By (4.18), we have

$$h_1(u\otimes y)h_2(z\otimes w)=1\ ,\ h_1(z\otimes y)h_2(z\otimes w)=1 \text{ and } h_1(z\otimes y)h_2(x\otimes v)=1.$$

Then

$$\begin{aligned}h_1(u\otimes y)h_2(x\otimes v)&=h_1(u\otimes y)h_1(z\otimes y)h_2(z\otimes w)h_2(x\otimes v)\\&=h_1(u\otimes y)h_2(z\otimes w)h_1(z\otimes y)h_2(x\otimes v)\\&=1.\end{aligned}$$

Consequently,

$$h_1(u\otimes y)h_2(x\otimes v)=1$$

for all $u\otimes y, x\otimes v\in\mathscr{F}_1(\mathscr{H})$.

Case 2. If $\mathscr{H}=\mathbb{C}^n$, obviously, from (4.17), we have

$$h_1(x\otimes u)h_2(y\otimes v)=\frac{\langle y,u\rangle\,\langle x,v\rangle}{\eta\left(\langle y,u\rangle\right)\eta'_x\left(\langle x,v\rangle\right)}=1\tag{4.19}$$

for all $x,y,u,v \in \mathbb{C}^n$ with $\langle y,u\rangle$ and $\langle x,v\rangle$ are nonzero integers. Next, assume that $x,y,u,v \in \mathbb{C}^n$ such that $\langle y,u\rangle \neq 0$ and $\langle x,v\rangle$ is nonzero integer, considering y and u are linearly dependent or independent, we discuss the two cases respectively.

If y and u are linearly independent, then

$$\{z \in \mathbb{C}^n \ : \ \langle z,y\rangle = 1\} \text{ and } \{z \in \mathbb{C}^n \ : \ \langle z,u\rangle = 1\}$$

are two non-parallel affine hyperplanes of $\mathbb{C}^n$. So

$$\{z \in \mathbb{C}^n \ : \ \langle z,y\rangle = 1\} \cap \{z \in \mathbb{C}^n \ : \ \langle z,u\rangle = 1\}$$

is a nonempty affine hyperplane, hence there exists $z \in \mathbb{C}^n$ such that $\langle z,z\rangle$ is a positive integer and

$$\langle z,y\rangle = 1 \text{ and } \langle z,u\rangle = 1.$$

Therefore, the equality (4.19) implies that

$$h_1(x\otimes u)h_2(z\otimes v) = 1\ ,\ h_1(x\otimes z)h_2(z\otimes v) = 1 \text{ and } h_1(x\otimes z)h_2(y\otimes v) = 1.$$

Hence

$$\begin{aligned} h_1(x\otimes u)h_2(y\otimes v) &= h_1(x\otimes u)h_1(x\otimes z)h_2(z\otimes v)h_2(y\otimes v) \\ &= h_1(x\otimes u)h_2(z\otimes v)h_1(x\otimes z)h_2(y\otimes v) \\ &= 1. \end{aligned}$$

If y and u are linearly dependent. Choose $z \in \mathbb{C}^n$ such $\langle z,z\rangle$ is a positive integer and y,z are linearly independent. By the similar discussion above we conclude that

$$h_1(x\otimes u)h_2(z\otimes v) = 1\ ,\ h_1(x\otimes z)h_2(z\otimes v) = 1 \text{ and } h_1(x\otimes z)h_2(y\otimes v) = 1,$$

hence

$$\begin{aligned} h_1(x\otimes u)h_2(y\otimes v) &= h_1(x\otimes u)h_1(x\otimes z)h_2(z\otimes v)h_2(y\otimes v) \\ &= h_1(x\otimes u)h_2(z\otimes v)h_1(x\otimes z)h_2(y\otimes v) \\ &= 1. \end{aligned}$$

We conclude that

$$h_1(x\otimes u)h_2(y\otimes v) = 1$$

for all $x,y,u,v \in \mathbb{C}^n$ with $\langle y,u\rangle \neq 0$ and $\langle x,v\rangle$ is nonzero integer. Finally, if $x,y,u,v \in \mathbb{C}^n$ with $\langle y,u\rangle = 0$ and $\langle x,v\rangle$ is nonzero integer. One can choose $z \in \mathbb{C}^n$ such that $\langle z,u\rangle\,\langle z,y\rangle \neq 0$, then

$$h_1(x\otimes u)h_2(z\otimes v) = 1\ ,\ h_1(x\otimes z)h_2(z\otimes v) = 1 \text{ and } h_1(x\otimes z)h_2(y\otimes v) = 1,$$

hence

$$\begin{aligned} h_1(x\otimes u)h_2(y\otimes v) &= h_1(x\otimes u)h_1(x\otimes z)h_2(z\otimes v)h_2(y\otimes v) \\ &= h_1(x\otimes u)h_2(z\otimes v)h_1(x\otimes z)h_2(y\otimes v) \\ &= 1. \end{aligned}$$

We conclude that

$$h_1(x\otimes u)h_2(y\otimes v)=1$$

for all $x,y,u,v\in\mathbb{C}^n$ with $\langle x,v\rangle$ is nonzero integer. Now, let $x,y,u,v\in\mathbb{C}^n$ with $\langle x,v\rangle$ is not integer. By a similar way as in the above discussion, we get

$$h_1(x\otimes u)h_2(y\otimes v)=1$$

for all $x,y,u,v\in\mathbb{C}^n$ with $\langle x,v\rangle$ is not integer. Hence,

$$h_1(x\otimes u)h_2(y\otimes v)=1 \tag{4.20}$$

for all $x,y,u,v\in\mathbb{C}^n$. We conclude that if $\mathscr{H}=\mathbb{C}^n$ or $\dim(\mathscr{H})=\infty$, we have

$$h_1(x\otimes u)h_2(y\otimes v)=1$$

for all $x\otimes u, y\otimes v\in\mathscr{F}_1(\mathscr{H})$. Consequently, there exist two nonzero scalars μ and ν satisfying $\mu\nu=1$ and

$$h_1(x\otimes y)=\mu \text{ and } h_2(x\otimes y)=\nu \tag{4.21}$$

for all $x\otimes y\in\mathscr{F}_1(\mathscr{H})$.

Step 2. The statement of Assertion 3 holds.

By equations (4.17), (4.20) we conclude that τ and τ' are continuous, and this implies that τ and τ' are either the identity or the conjugation.

In the rest of this section U denotes the bounded invertible linear or conjugate linear operator on $\mathscr{H}$ given in Lemma 3.3, and V is also the bounded invertible linear or conjugate linear operator on $\mathscr{H}$ given in the previous assertion.

By equations (4.1), (4.15), we have

$$\begin{aligned} W\left(\nu\phi_1(A)U(x)\otimes(V^{-1})^*(y)\right) &= W\left(\phi_1(A)h_2(x\otimes y)U(x)\otimes(V^{-1})^*(y)\right)\\ &= W\left(\phi_1(A)\phi_2(x\otimes y)\right)\\ &= W\left(A(x)\otimes y\right)\end{aligned}$$

for all $A\in\mathscr{A}\setminus\{0\}$, $x,y\in\mathscr{H}$. Hence

$$W\left(\nu\phi_1(A)U(x)\otimes(V^{-1})^*(y)\right)=W\left(A(x)\otimes y\right) \tag{4.22}$$

for all $A\in\mathscr{A}\setminus\{0\}$, $x,y\in\mathscr{H}$. This implies that

$$\left\langle\nu\phi_1(A)U(x),(V^{-1})^*(y)\right\rangle=\langle A(x),y\rangle \tag{4.23}$$

for all $A\in\mathscr{A}\setminus\{0\}$, $x,y\in\mathscr{H}$. By Equation (4.22), we have $\ker(\Phi(A))=\ker(A)$ for all $A\in\mathscr{A}\setminus\{0\}$ where $\Phi(A):=V^{-1}\phi_1(A)U$. Let $A\in\mathscr{A}\setminus\{0\}$, then according to the space decomposition, we have

$$\mathscr{H}=\mathscr{H}_1\oplus\ker(A),$$

where $\mathscr{H}_1$ is an algebraic complement of $\ker(A)$. Now, we show that $\Phi(A)$ and A are linearly dependent. Let $x \in \mathscr{H}_1 \setminus \{0\}$, and note there exists a unique nonzero scalar λ_x such that $\Phi(A)(x) = \lambda_x A(x)$. Now, we claim that the scalar-valued function $x \mapsto \lambda_x$ is constant on $\mathscr{H}_1 \setminus \{0\}$. Indeed, let $x, y \in \mathscr{H}_1 \setminus \{0\}$, and observe first that if x and y are linearly dependent, then obviously $\lambda_x = \lambda_y$. Thus we may assume that x and y are linearly independent. Then,

$$\Phi(A)(x+y) = \Phi(A)(x) + \Phi(A)(y) = \lambda_x A(x) + \lambda_y A(y).$$

On the other hand,

$$\Phi(A)(x+y) = \lambda_{x+y} A(x+y) = \lambda_{x+y} A(x) + \lambda_{x+y} A(y).$$

Therefore,

$$(\lambda_{x+y} - \lambda_x) A(x) = (\lambda_y - \lambda_{x+y}) A(y).$$

Thus

$$A\left((\lambda_{x+y} - \lambda_x)x - (\lambda_y - \lambda_{x+y})y\right) = 0.$$

So, $(\lambda_{x+y} - \lambda_x)x - (\lambda_y - \lambda_{x+y})y = 0$ and thus $\lambda_x = \lambda_{x+y} = \lambda_y$ since x and y are linearly independent vectors in $\mathscr{H}_1$. Thus, there exists a nonzero scalar λ such that $\Phi(A)(x) = \lambda A(x)$ for all $x \in \mathscr{H}_1$, and hence $\Phi(A)(x) = \lambda A(x)$ for all $x \in \mathscr{H}$. Consequently $\Phi(A)$ and A are linearly dependent, and there exists a functional $\alpha : \mathscr{A} \mapsto \mathbb{C} \setminus \{0\}$ such that

$$\phi_1(A) = \alpha(A) VAU^{-1} \quad (A \in \mathscr{A}). \tag{4.24}$$

Equation (4.23) implies that $\alpha(A) = \frac{1}{\nu} = \mu$ for all $A \in \mathscr{A}$. Clearly, we have $\alpha(A) = h_1(A)$ if A is of rank one. In the same manner, there exists a functional $\beta : \mathscr{A} \mapsto \mathbb{C} \setminus \{0\}$ such that

$$\phi_2(A) = \beta(A) UAV^{-1} \quad (A \in \mathscr{A}), \tag{4.25}$$

with $\beta(A) = h_2(A)$ if $A \in \mathscr{F}_1(\mathscr{H})$ and $\beta(A) = \nu$ for all $A \in \mathscr{A}$. $\square$

ASSERTION 4. V is a unitary operator, U is a linear operator.

Proof. Let $x \in \mathscr{H}$ be a nonzero vector and $\alpha \in \mathbb{C} \setminus \{0\}$. From the equalities (4.1), (4.15) and (4.21), we have

$$\begin{aligned}
\alpha W(x \otimes x) &= W((\alpha x \otimes x)(x \otimes x)) \\
&= W(\phi_1(\alpha x \otimes x)\phi_2(x \otimes x)) \\
&= W\left(\left(\mu V(\alpha x) \otimes (U^{-1})^*(x)\right)(\nu U(x) \otimes V(x))\right) \\
&= W(\mu \nu V \mu(\langle x, x\rangle) V(\alpha x) \otimes V(x)) \\
&= \alpha^V W(V(x) \otimes V(x))
\end{aligned}$$

and hence

$$\alpha W(x \otimes x) = \alpha^V W(V(x) \otimes V(x)). \tag{4.26}$$

We know that

$$\begin{cases} \alpha W(x\otimes x) = \alpha W\left(\|x\|^2 \frac{x}{\|x\|} \otimes \frac{x}{\|x\|}\right) = \|x\|^2[0,\alpha] = [0,\|x\|^2\alpha] \\ \alpha^V W\left(V(x)\otimes V(x)\right) = \alpha^V W\left(\|V(x)\|^2 \frac{V(x)}{\|V(x)\|} \otimes \frac{V(x)}{\|V(x)\|}\right) \\ \qquad\qquad = \|V(x)\|^2\alpha^V[0,1] = [0,\|V(x)\|^2\alpha^V] \end{cases}$$

From this together with equation (4.26), we infer that $\|V(x)\|^2\alpha^V = \|x\|^2\alpha$ for all $x \in \mathscr{H}\setminus\{0\}$ and $\alpha \in \mathbb{C}\setminus\{0\}$. For $\alpha = 1$ we obtain $\|V(x)\|^2 = \|x\|^2$, hence $\alpha^V = \alpha$ for all $\alpha \in \mathbb{C}\setminus\{0\}$. Thus, V is unitary operator.

Next, let $x \in \mathscr{H}$ be a unit vector. By the equalities (4.1), (4.15) and (4.21), we have

$$\begin{aligned} \alpha W(x\otimes x) &= W((x\otimes\overline{\alpha}x)(x\otimes x)) \\ &= W(\phi_1(x\otimes\overline{\alpha}x)\phi_2(x\otimes x)) \\ &= W\left(\left(\mu V(x)\otimes(U^{-1})^*(\overline{\alpha}x)\right)(\nu U(x)\otimes V(x))\right) \\ &= \mu(\alpha)W\left(V(x)\otimes V(x)\right) \end{aligned}$$

for all $\alpha \in \mathbb{C}\setminus\{0\}$. Since

$$\begin{cases} \alpha W(x\otimes x) = \alpha[0,1] = [0,\alpha] \\ \mu(\alpha)W\left(V(x)\otimes V(x)\right) = \mu(\alpha)[0,1] = [0,\mu(\alpha)] \end{cases}$$

we conclude that $\mu(\alpha) = \alpha$ for all $\alpha \in \mathbb{C}$ and so U is a bounded linear operator. Thus, we obtain the desired forms of ϕ_1 and ϕ_2. $\square$

5. Proof of Corollaries 2.3 and 2.4

We start with the proof of Corollary 2.3. The "if" part is straightforward. So, assume that

$$W(AB) = W(\phi(A)\phi(B)) \tag{5.1}$$

for all $A, B \in \mathscr{A}$. By Theorems 2.1 and 2.2, there exist $\mu, \nu \in \mathbb{C}$ with $\mu\nu = 1$, a bounded invertible linear operator U on $\mathscr{H}$ and a unitary operator V on $\mathscr{H}$ such that

$$\phi(A) = \mu VAU^{-1}$$

and

$$\phi(A) = \nu UAV^*$$

for all $A \in \mathscr{A}$. It then follows that

$$\mu VAU^{-1} = \phi(A) = \nu UAV^*$$

for all $A \in \mathscr{A}$. Therefore, for every $x\otimes u \in \mathscr{F}_1(\mathscr{H})$, we have $\mu V(x)\otimes(U^{-1})^*(u) = \nu U(x)\otimes V(u)$. This implies that $U(x)$ and $V(x)$ are linearly dependent for all $x \in \mathscr{H}$.

Thus there exists $\beta \in \mathbb{C}$ such that $U(x) = \beta V(x)$ for all $x \in \mathscr{H}$, and then, $U = \beta V$, so, $\phi(A) = \lambda VAV^*$, with $\lambda = \nu\beta$. Finally, we claim that $\lambda^2 = 1$. Indeed, let us fix a unit vector x in $\mathscr{H}$, by equation (5.1), we have

$$\begin{aligned}
[0,\lambda^2] &= W(\lambda^2 x\otimes x) \\
&= W(\lambda^2(x\otimes x)(x\otimes x)) \\
&= W\left((\lambda V(x\otimes x)V^*)(\lambda V(x\otimes x)V^*)\right) \\
&= W\left(\phi(x\otimes x)\phi(x\otimes x)\right) \\
&= W((x\otimes x)(x\otimes x)) \\
&= W(x\otimes x) \\
&= [0,1],
\end{aligned}$$

and then, $\lambda^2 = 1$. The proof of Corollary 2.3 is complete.

Now, we prove Corollary 2.4. Checking the "if" part is straightforward. For the "only if" part assume that

$$W(\phi(A^*)^*\phi(B)) = W((A^*)^*B) = W(AB)$$

for all $A, B \in \mathscr{A}$. By Theorems 2.1 and 2.2, there exist $\mu, \nu \in \mathbb{C}$ with $\mu\nu = 1$, a bounded invertible linear operator U on $\mathscr{H}$ and a unitary operator V on $\mathscr{H}$ such that

$$\phi(A) = \mu VAU^{-1}$$

and

$$(\phi(A^*))^* = \nu UAV^*$$

for all $A \in \mathscr{A}$. In particular, we have

$$\mu VA^*U^{-1} = \phi(A^*) = \left((\phi(A^*))^*\right)^* = (\nu UAV^*)^* = \overline{\nu}VA^*U^*. \tag{5.2}$$

For $x \in \mathscr{H}$, if $A = x\otimes x$, then, equation (5.2) implies that $\mu V(x)\otimes(U^{-1})^*(x) = \overline{\nu}V(x)\otimes U(x)$. It then follows that

$$\left(\overline{\mu}(U^{-1})^* - \nu U\right)(x) = 0$$

for all $x \in \mathscr{H}$. Since $\mu\nu = 1$ we have, $(U^{-1})^*(x) = |\nu|^2 U(x)$ for all $x \in \mathscr{H}$. Thus, U is a λ multiple of a unitary operator R, where λ is a complex scalar. So, we conclude that $\phi(A) = \lambda RAV^*$ for all $A \in \mathscr{A}$. Now, we show that $|\lambda| = 1$. By Equation (5.1), for a unit vector x, we have

$$\begin{aligned}
\left[0,|\lambda|^2\right] &= W\left(|\lambda|^2 x\otimes x\right) \\
&= W\left(\overline{\lambda}V(x\otimes x)R^*\lambda R(x\otimes x)V^*\right) \\
&= W\left((\lambda R(x\otimes x)V^*)^*(\lambda R(x\otimes x)V^*)\right) \\
&= W\left(\phi(x\otimes x)^*\phi(x\otimes x)\right) \\
&= w((x\otimes x)^*(x\otimes x)) \\
&= W(x\otimes x) \\
&= [0,1],
\end{aligned}$$

and then, $|\lambda| = 1$ and this completes the proof.

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REFERENCES

[1] Z. E. A. ABDELALI AND H. NKHAYLIA, *Maps preserving the pseudo spectrum of skew triple product of operators*, Linear and Multilinear Algebra. 2019, vol. **67**, no. 11, p. 2297–2306.

[2] Z. E. A. ABDELALI AND H. NKHAYLIA, *Multiplicatively pseudo spectrum-preserving maps*, Linear and Multilinear Algebra and Function Spaces. Contemporary Mathematics. 2020, vol. **750**, p. 43–69.

[3] Z. E. A. ABDELALI AND H. NKHAYLIA, *Condition spectrum of rank one operators and preservers of the condition spectrum of skew product of operators*, Complex Analysis and Operator Theory. 2020, vol. **14**, no. 7, p. 69.

[4] Z. E. A. ABDELALI AND B. AHARMIM, *Multiplicatively spectrum preserving maps on rectangular matrices*, Linear and Multilinear Algebra. 2021, vol. **69**, no. 16, p. 3099–3111.

[5] A. BOURHIM AND J. E. LEE, *Multiplicatively local spectrum-preserving maps*, Linear Algebra and its Applications. 2018, vol. **549**, p. 291–308.

[6] A. BOURHIM AND J. MASHREGHI, *Maps preserving the local spectrum of triple product of operators*, Linear and Multilinear Algebra. 2015, vol. **63**, no. 4, p. 765–773.

[7] A. BOURHIM AND J. MASHREGHI, *Maps preserving the local spectrum of product of operators*, Glasgow Mathematical Journal. 2015, vol. **57**, no. 3, p. 709–718.

[8] Z. BAI, J. HOU AND Z. XU, *Maps preserving numerical radius on C^*-algebras*, Studia Mathematica. 2004, vol. **2**, no. 162, p. 97–104.

[9] Z. BAI AND J. HOU, *Numerical radius distance preserving maps on $B(\mathscr{H})$*, Proceedings of the American Mathematical Society. 2004, vol. **132**, no. 5, p. 1453–1461.

[10] J. CUI AND J. HOU, *Linear maps preserving the closure of numerical range on nest algebras with maximal atomic nest*, Integral Equations and Operator Theory. 2003, vol. **46**, no. 3, p. 253–266.

[11] J. CUI AND J. HOU, *Non-linear numerical radius isometries on atomic nest algebras and diagonal algebras*, Journal of Functional Analysis. 2004, vol. **206**, no. 2, p. 414–448.

[12] J. T. CHAN, *Numerical radius preserving operators on $B(\mathscr{H})$*, Proceedings of the American Mathematical Society. 1995, vol. **123**, no. 5, p. 1437–1439.

[13] J. T. CHAN, *Numerical radius preserving operators on C^*-algebras*, Archiv der Mathematik. 1998, vol. **123**, no. 6, p. 486–488.

[14] J. CUI, C. K. LI AND N. S. SZE, *Unitary similarity invariant function preservers of skew products of operators*, Journal of Mathematical Analysis and Applications. 2017, vol. **454**, no. 2, p. 716–729.

[15] J. CUI, C. K. LI AND Y. T. POON, *Pseudospectra of special operators and pseudo spectrum preservers*, Journal of Mathematical Analysis and Applications. 2014, vol. **419**, no. 2, p. 1261–1273.

[16] J. CUI, V. FORSTALL, C. K. LI AND V. YANNELLO, *Properties and Preservers of the Pseudospectrum*, Linear Algebra and its Applications. 2012, vol. **436**, no. 2, p. 316–325.

[17] J. CUI, C. K. LI AND Y. T. POON, *Preservers of unitary similarity functions on lie products of matrices*, Linear Algebra and its Applications. 2016, vol. **498**, p. 160–180.

[18] J. HOU, K. HE AND X. ZHANG, *Nonlinear maps preserving numerical radius of indefinite skew products of operators*, Linear Algebra and its Applications. 2009, vol. **430**, no. 8–9, p. 2240–2253.

[19] J. HOU AND Q. DI, *Maps preserving numerical range of operator products*, Proceedings of the American Mathematical Society. 2006, vol. **134**, no. 5, p. 1435–1446.

[20] L. MOLNÀR, *Selected preserver problems on algebraic structures of linear operators and on function spaces*, Lecture Notes in Mathematics, 1895, Springer-Verlag, Berlin, 2007.

[21] L. MOLNÀR, *Some characterizations of the automorphisms of $B(\mathscr{H})$ and $C(X)$*, Proceedings of the American Mathematical Society. 2002, vol. **130**, no. 1, p. 111–120.

[22] L. MOLNÀR, *Orthogonality preserving transformations on indefinite inner product spaces: generalization of Uhlhorn's version of Wigner's theorem*, Journal of Functional Analysis. 2002, vol. **194**, no. 2, p. 248–262.

[23] M. OMLADIČ, *On operators preserving the numerical range*, Linear Algebra and its Applications. 1990, vol. **134**, p. 31–51.

[24] X. ZHANG, J. HOU AND K. HE, *Maps preserving numerical radius and cross norms of operator products*, Linear and Multilinear Algebra. 2009, vol. **57**, no. 5, p. 523–534.

[25] Y. ZHANG AND X. FANG, *Preserves of the c-numerical radius of operator Jordan semi-triple products*, Operators and Matrices. 2021, vol. **15**, no. 2, p. 471–483.

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