Operators
and
Matrices
Volume 18, Number 2 (2024), 441–455
doi:10.7153/oam-2024-18-27

SOME NOVEL INEQUALITIES FOR BEREZIN NUMBER OF OPERATORS

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(*Communicated by F. Kittaneh*)

Abstract. In this paper, some Berezin number inequalities of bounded linear operators defined on a reproducing kernel Hilbert space are developed which generalize and refine the earlier related inequalities. Some applications of the newly obtained inequalities are also provided.

1. Introduction and preliminaries

Let $\mathscr{B}(\mathscr{H})$ denote the C^* - algebra of all bounded linear operators acting on a non trivial complex Hilbert space $\mathscr{H}$ with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$. For $T \in \mathscr{B}(\mathscr{H})$, T^* denotes the adjoint of T and $|T| = \sqrt{T^*T}$. Recall that, the numerical range of $T \in \mathscr{B}(\mathscr{H})$ is defined by

$$W(T) = \{\langle Tx, x \rangle : x \in \mathscr{H}, \ \|x\| = 1\},$$

while the numerical radius is defined as

$$w(T) = \sup\{|\langle Tx, x \rangle| : x \in \mathscr{H}, \ \|x\| = 1\}.$$

It is well known that the norm $\|\cdot\|$ and the numerical radius $\omega(\cdot)$ are equivalent, where one has the two-sided inequality:

$$\frac{1}{2}\|T\| \leqslant w(T) \leqslant \|T\|,$$

for any $T \in \mathscr{B}(\mathscr{H})$.

For some results about the numerical radius inequalities and their applications, we refer to see [1, 7, 13, 22, 23].

Let Θ be a nonempty set. A functional Hilbert space $\mathscr{H}(\Theta)$ is a Hilbert space of complex valued functions, which has the property that point evaluations are continuous i.e., for each $\nu \in \Theta$ the map $f \longmapsto f(\nu)$ is a continuous linear functional on $\mathscr{H}$. The Riesz representation theorem ensues that for each $\nu \in \Theta$ there exists a unique element $k_\nu \in \mathscr{H}$ such that $f(\nu) = \langle f, k_\nu \rangle$ for all $f \in \mathscr{H}$. The set $\{k_\nu : \nu \in \Theta\}$ is called the reproducing kernel of the space $\mathscr{H}$. If $\{e_n\}_{n \geqslant 0}$ is an orthonormal basis for

Mathematics subject classification (2020): Primary 47A63.
Keywords and phrases: Berezin number, Berezin symbol, reproducing kernel Hilbert space, inequalities.

Paper OaM-18-27

a functional Hilbert space $\mathscr{H}$, then the reproducing kernel of $\mathscr{H}$ is given by $k_\nu(z) = \sum_{n=0}^{+\infty} \overline{e_n(\nu)} e_n(z)$ (see [19]). For $\nu \in \Theta$, let $\widehat{k}_\nu = \frac{k_\nu}{\|k_\nu\|}$ be the normalized reproducing kernel of $\mathscr{H}$.

Let T be a bounded linear operator on $\mathscr{H}$, the Berezin symbol of T, which firstly have been introduced by Berezin [8, 9] is the function $\widetilde{T}$ on Θ defined by

$$\widetilde{T}(\nu) := \left\langle T\widehat{k}_\nu, \widehat{k}_\nu \right\rangle.$$

The Berezin set and the Berezin number of the operator T are defined respectively by:

$$\mathbf{Ber}(T) := \left\{ \left\langle T\widehat{k}_\nu, \widehat{k}_\nu \right\rangle : \nu \in \Theta \right\},$$

and

$$\mathbf{ber}(T) := \sup \left\{ \left| \left\langle T\widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right| : \nu \in \Theta \right\}.$$

It is clear that the Berezin symbol $\widetilde{T}$ is the bounded function on Θ whose value lies in the numerical range of the operator T and hence for any $T \in \mathscr{B}(\mathscr{H}(\Theta))$,

$$\mathbf{Ber}(T) \subset W(T) \text{ and } \mathbf{ber}(T) \leqslant \omega(T).$$

Moreover, the Berezin number of an operator T satisfies the following properties:
(i) $\mathbf{ber}(T) = \mathbf{ber}(T^*)$.
(ii) $\mathbf{ber}(T) \leqslant \|T\|$.
(iii) $\mathbf{ber}(\alpha T) = |\alpha| \, \mathbf{ber}(T)$ for all $\alpha \in \mathbb{C}$.
(iv) $\mathbf{ber}(T+K) \leqslant \mathbf{ber}(T) + \mathbf{ber}(K)$ for all $T, K \in \mathscr{B}(\mathscr{H}(\Theta))$.

Notice that, in general, the Berezin number does not define a norm. However, if $\mathscr{H}$ is a reproducing kernel Hilbert space of analytic functions, (for instance on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$), then $\mathbf{ber}(.)$ defines a norm on $\mathscr{B}(\mathscr{H}(\mathbb{D}))$ (see [20, 21]).

The Berezin symbol has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces. A nice property of the Berezin symbol is mentioned next. If $\widetilde{T}(\nu) = \widetilde{K}(\nu)$ for all $\nu \in \Theta$, then $T = K$. Therefore, the Berezin symbol uniquely determines the operator. The Berezin symbol and Berezin number have been studied by many mathematicians over the years, a few of them are [2, 6, 18, 26, 27, 29, 30, 31, 32, 33].

Now, for any operator $T \in \mathscr{B}(\mathscr{H}(\Theta))$, the Berezin norm of T denoted as $\|T\|_{ber}$ is defined by

$$\|T\|_{ber} := \sup_{\nu \in \Theta} \left\| T\widehat{k}_\nu \right\|,$$

where $\widehat{k}_\nu$ is normalized reproducing kernel for $\nu \in \Theta$.

For $T, K \in \mathscr{B}(\mathscr{H}(\Theta))$ it is clear from the definition of the Berezin norm that the following properties hold:
(i) $\|\nu T\|_{ber} = |\nu| \, \|T\|_{ber}$ for all $\nu \in \mathbb{C}$,
(ii) $\|T+K\|_{ber} \leqslant \|T\|_{ber} + \|K\|_{ber}$,
(iii) $\mathbf{ber}(T) \leqslant \|T\|_{ber} \leqslant \|T\|$.

For further results about the Berezin norm inequalities and their applications, we refer to see [3, 4, 5, 10, 11, 17, 25] and references therein.

In this paper, some refinements and generalizations of Berezin norm and Berezin number inequalities of bounded linear operators defined on a reproducing kernel Hilbert space are established. This work is organized as follows: In Section 2, we collect a few lemmas that are required to state and prove the results in the subsequent section. In Section 3, we establish some new refinements and generalizations of Berezin number inequalities.

2. Prerequisites

In this section, we present the following lemmas that will be used to develop our results in this paper.

LEMMA 1. ([24]) *Let* $a,b \geqslant 0$ *and let* $p,q>1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$ab+\min\left\{\frac{1}{p},\frac{1}{q}\right\}\left(a^{\frac{p}{2}}-b^{\frac{q}{2}}\right)^2 \leqslant \frac{a^p}{p}+\frac{b^q}{q}.$$

LEMMA 2. ([12]) *If* a,b,x *are vectors in* $\mathscr{H}$ *with* $\|x\|=1$, *then*

$$|\langle a,x\rangle\langle x,b\rangle| \leqslant \frac{1}{2}\left(\|a\|\,\|b\|+|\langle a,b\rangle|\right).$$

LEMMA 3. ([28]) *Let* $T\in B(\mathscr{H})$ *be a positive operator and let* $x\in\mathscr{H}$ *with* $\|x\|=1$. *Then*
(i) $\langle Tx,x\rangle^r \leqslant \langle T^r x,x\rangle$ *for* $r\geqslant 1$.
(ii) $\langle T^r x,x\rangle \leqslant \langle Tx,x\rangle^r$ *for* $0<r\leqslant 1$.

LEMMA 4. ([22]) *Let* $T\in B(\mathscr{H})$ *and let* f *and* g *be non-negative continuous functions on* $[0,\infty)$ *such that* $f(t)g(t)=t$ *for all* $t\in[0,\infty)$. *Then*

$$|\langle Tx,y\rangle|^2 \leqslant \left\langle f^2(|T|)x,x\right\rangle\left\langle g^2(|T^*|)y,y\right\rangle,$$

for all $x,y\in\mathscr{H}$.

In particular, if $f(t)=g(t)=\sqrt{t}$, then we have

$$|\langle Tx,y\rangle|^2 \leqslant \langle |T|x,x\rangle\langle |T^*|y,y\rangle.$$

LEMMA 5. ([15]) *Let* $x,y,z\in\mathscr{H}$ *with* $z\neq 0$. *Then*

$$\left|\langle x,y\rangle-\frac{\langle x,z\rangle}{\|z\|^2}\langle z,y\rangle\right|^2+\frac{|\langle x,z\rangle|^2}{\|z\|^2}\|y\|^2 \leqslant \|x\|^2\|y\|^2.$$

LEMMA 6. ([14, p. 116]) *Let* $x,y,z\in\mathscr{H}$. *Then*

$$|\langle z,x\rangle|^2+|\langle z,y\rangle|^2 \leqslant \|z\|^2\left(\max\left(\|x\|^2,\|y\|^2\right)+|\langle x,z\rangle|\right).$$

3. Main results

In this section, we present our results. Firstly, we introduce a new refinement of the inequality $\mathbf{ber}(T) \leqslant \|T\|_{ber}$.

THEOREM 1. *Let $T \in B(\mathscr{H}(\Theta))$ be an invertible operator. Then*

$$\mathbf{ber}^2(T) \leqslant \|T\|_{ber}^2 - \inf_{v\in\Theta} \eta^2\left(\widehat{k}_v\right),$$

where $\eta^2\left(\widehat{k}_v\right) = \dfrac{\left|\widetilde{T^2}(v) - \left(\widetilde{T}(v)\right)^2\right|}{\left\|T^*\widehat{k}_v\right\|}$.

Proof. Let $\widehat{k}_v$ be the normalized reproducing kernel of $\mathscr{H}$. We put $x = T\widehat{k}_v$, $y = T^*\widehat{k}_v$ and $z = \widehat{k}_v$ in Lemma 5, we get

$$\left(\frac{\left|\widetilde{T^2}(v) - \left(\widetilde{T}(v)\right)^2\right|}{\left\|T^*\widehat{k}_v\right\|}\right)^2 + \left|\widetilde{T}(v)\right|^2 \leqslant \left\|T\widehat{k}_v\right\|^2.$$

Thus,

$$\left|\widetilde{T}(v)\right|^2 \leqslant \left\|T\widehat{k}_v\right\|^2 - \inf_{v\in\Theta}\left(\frac{\left|\widetilde{T^2}(v) - \left(\widetilde{T}(v)\right)^2\right|}{\left\|T^*\widehat{k}_v\right\|}\right)^2.$$

Now, by taking the supremum over all $v \in \Theta$ in the above inequality, we get the desired result. □

Our next result is stated as follows.

THEOREM 2. *Let $T \in B(\mathscr{H}(\Theta))$ and let $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \geqslant 0)$, then for all $r \geqslant 1$, we have*

$$\mathbf{ber}^{2r}(T) \leqslant \left\|\frac{1}{p} f^{2pr}(|T|) + \frac{1}{q} g^{2qr}(|T^*|)\right\|_{ber} - r_0 \inf_{v\in\Theta} \delta\left(\widehat{k}_v\right),$$

where $\delta\left(\widehat{k}_v\right) = \left(\left\langle f^{2p}(|T|)\widehat{k}_v, \widehat{k}_v\right\rangle^{\frac{r}{2}} - \left\langle g^{2q}(|T^*|)\widehat{k}_v, \widehat{k}_v\right\rangle^{\frac{r}{2}}\right)^2$ *and* $r_0 = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Proof. Let $\widehat{k}_\nu$ be the normalized reproducing kernel of $\mathscr{H}$. Then, we have

$$\begin{aligned}
\left|\widetilde{T}(\nu)\right|^{2r} &\leqslant \left\langle f^2(|T|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^r \left\langle g^2(|T^*|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^r \\
&\quad \text{(by Lemma 4)} \\
&= \left\langle f^{p\frac{2}{p}}(|T|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^r \left\langle g^{q\frac{2}{q}}(|T^*|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^r \\
&\leqslant \left(\left\langle f^{2p}(|T|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^{\frac{1}{p}} \left\langle g^{2q}(|T^*|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^{\frac{1}{q}}\right)^r \\
&\quad \text{(by Lemma 3)} \\
&\leqslant \left(\frac{1}{p}\left\langle f^{2p}(|T|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^r + \frac{1}{q}\left\langle g^{2q}(|T^*|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^r\right) \\
&\quad - r_0\left(\left\langle f^{2p}(|T|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^{\frac{r}{2}} - \left\langle g^{2q}(|T^*|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^{\frac{r}{2}}\right)^2 \\
&\quad \text{(by Lemma 1)} \\
&\leqslant \left(\frac{1}{p}\left\langle f^{2pr}(|T|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle + \frac{1}{q}\left\langle g^{2qr}(|T^*|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right) \\
&\quad - r_0\left(\left\langle f^{2p}(|T|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^{\frac{r}{2}} - \left\langle g^{2q}(|T^*|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^{\frac{r}{2}}\right)^2 \\
&\quad \text{(by Lemma 3)}
\end{aligned}$$

Taking the supremum in over $\nu \in \Theta$, we get

$$\begin{aligned}
\mathbf{ber}^{2r}(T) &\leqslant \mathbf{ber}\left(\frac{1}{p}f^{2pr}(|T|) + \frac{1}{q}g^{2qr}(|T^*|)\right) - r_0 \inf_{\nu\in\Theta}\delta\left(\widehat{k}_\nu\right) \\
&\leqslant \left\|\frac{1}{p}f^{2pr}(|T|) + \frac{1}{q}g^{2qr}(|T^*|)\right\|_{ber} - r_0 \inf_{\nu\in\Theta}\delta\left(\widehat{k}_\nu\right),
\end{aligned}$$

(since $\mathbf{ber}(X) \leqslant \|X\|_{ber}$ for every $X \in B(\mathscr{H}(\Theta))$)

where $\delta\left(\widehat{k}_\nu\right) = \left(\left\langle f^{2p}(|T|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^{\frac{r}{2}} - \left\langle g^{2q}(|T^*|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^{\frac{r}{2}}\right)^2$.

This completes the proof. □

Letting $p = q = 2$ in Theorem 2, we have the following corollary.

COROLLARY 1. *Let $T \in B(\mathscr{H}(\Theta))$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \geqslant 0)$, then*

$$\mathbf{ber}^{2r}(T) \leqslant \frac{1}{2}\left\|f^{4r}(|T|) + g^{4r}(|T^*|)\right\|_{ber} - \frac{1}{2}\inf_{\nu\in\Theta}\delta\left(\widehat{k}_\nu\right),$$

where $\delta\left(\widehat{k}_\nu\right) = \left(\left\langle f^4(|T|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^{\frac{r}{2}} - \left\langle g^4(|T^*|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle^{\frac{r}{2}}\right)^2$.

Considering $f(t) = g(t) = \sqrt{t}$ in Corollary 1, we get the following inequality.

COROLLARY 2. *If* $T \in B(\mathscr{H}(\Theta))$ *and* $r \geqslant 1$, *then*

$$\mathbf{ber}^{2r}(T) \leqslant \frac{1}{2}\left\|(TT^*)^r + (T^*T)^r\right\|_{ber} - \frac{1}{2}\inf_{v\in\Theta} \delta\left(\widehat{k}_v\right),$$

where $\delta\left(\widehat{k}_v\right) = \left(\left\langle |T|^2\widehat{k}_v, \widehat{k}_v\right\rangle^{\frac{r}{2}} - \left\langle |T^*|^2\widehat{k}_v, \widehat{k}_v\right\rangle^{\frac{r}{2}}\right)^2.$

REMARK 1. We note that the inequality in Corollary 2 refines the inequality

$$\mathbf{ber}^r(T) \leqslant \frac{1}{2}\mathbf{ber}\left(|T|^r + |T^*|^r\right), \text{ for all } r \geqslant 1.$$

obtained in [29, Corollary 3.5].

Now, we state the following theorem.

THEOREM 3. *Let* $T \in B(\mathscr{H}(\Theta))$ *and let* f *be a non-negative increasing convex function on* $[0,\infty)$. *Then*

$$f\left(\mathbf{ber}^2(T)\right) \leqslant \frac{1}{4}\left(f\left(\|T^*T + TT^*\|_{ber}\right) + f\left(\|T^*T - TT^*\|_{ber}\right)\right) + \frac{1}{2}f\left(\mathbf{ber}\left(T^2\right)\right).$$

Proof. Let $\widehat{k}_v$ be the normalized reproducing kernel of $\mathscr{H}$. Putting $x = T\widehat{k}_v$, $y = T^*\widehat{k}_v$, and $z = \widehat{k}_v$ in Lemma 6, and using the fact $\max\{a,b\} = (|a+b| + |a-b|)/2$, we obtain

$$\begin{aligned}
&\left|\left\langle \widehat{k}_v, T\widehat{k}_v\right\rangle\right|\left|\left\langle \widehat{k}_v, T^*\widehat{k}_v\right\rangle\right| \\
&\leqslant \frac{1}{2}\left(\left|\left\langle \widehat{k}_v, T\widehat{k}_v\right\rangle\right|^2 + \left|\left\langle \widehat{k}_v, T^*\widehat{k}_v\right\rangle\right|^2\right) \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&\leqslant \frac{1}{2}\left(\max\left\{\left\|T\widehat{k}_v\right\|^2, \left\|T^*\widehat{k}_v\right\|^2\right\} + \left|\left\langle T\widehat{k}_v, T^*\widehat{k}_v\right\rangle\right|\right) \\
&\quad \text{(by Lemma 6)} \\
&= \frac{1}{4}\left(\left|\left\langle (T^*T + TT^*)\widehat{k}_v, \widehat{k}_v\right\rangle\right| + \left|\left\langle (T^*T - TT^*)\widehat{k}_v, \widehat{k}_v\right\rangle\right|\right) \\
&\quad + \frac{1}{2}\left|\left\langle T^2\widehat{k}_v, \widehat{k}_v\right\rangle\right|.
\end{aligned}$$

Whence,

$$
\begin{aligned}
& f\left(\left|\left\langle\widehat{k}_\nu, T\widehat{k}_\nu\right\rangle\right|\left|\left\langle\widehat{k}_\nu, T^*\widehat{k}_\nu\right\rangle\right|\right) \\
& \leqslant f\left(\frac{1}{4}\left(\left|\left\langle(T^*T+TT^*)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|+\left|\left\langle(T^*T-TT^*)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right)+\frac{1}{2}\left|\left\langle T^2\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right) \\
& = f\left(\frac{\frac{1}{2}\left(\left|\left\langle(T^*T+TT^*)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|+\left|\left\langle(T^*T-TT^*)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right)+\left|\left\langle T^2\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|}{2}\right) \\
& \leqslant \frac{1}{2}\left(f\left(\frac{\left|\left\langle(T^*T+TT^*)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|+\left|\left\langle(T^*T-TT^*)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|}{2}\right)+f\left(\left|\left\langle T^2\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right)\right) \\
& \leqslant \frac{1}{4}\left(f\left(\left|\left\langle(T^*T+TT^*)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right)+f\left(\left|\left\langle(T^*T-TT^*)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right)\right) \\
& \quad +\frac{1}{2}f\left(\left|\left\langle T^2\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right).
\end{aligned}
$$

Therefore, we infer that

$$
\begin{aligned}
& f\left(\left|\left\langle\widehat{k}_\nu, T\widehat{k}_\nu\right\rangle\right|\left|\left\langle\widehat{k}_\nu, T^*\widehat{k}_\nu\right\rangle\right|\right) \\
& \leqslant \frac{1}{4}\left(f\left(\left|\left\langle(T^*T+TT^*)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right)+f\left(\left|\left\langle(T^*T-TT^*)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right)\right) \\
& \quad +\frac{1}{2}f\left(\left|\left\langle T^2\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right).
\end{aligned}
$$

Now, by taking supremum over $\nu\in\Theta$ in the above inequality, we get the desired inequality. □

Considering $f(t)=t^r$, $r\geqslant 1$ in Theorem 3, we get the following corollary.

COROLLARY 3. *Let* $T\in B(\mathscr{H}(\Theta))$. *Then*

$$\mathbf{ber}^{2r}(T)\leqslant\frac{1}{4}\left(\|T^*T+TT^*\|_{ber}^r+\|T^*T-TT^*\|_{ber}^r\right)+\frac{1}{2}\mathbf{ber}^r\left(T^2\right),$$

for any $r\geqslant 1$.

In particular, for $r=1$ we have

$$\mathbf{ber}^{2}(T)\leqslant\frac{1}{4}\left(\|T^*T+TT^*\|_{ber}+\|T^*T-TT^*\|_{ber}\right)+\frac{1}{2}\mathbf{ber}\left(T^2\right).$$

REMARK 2. If T is normal operator, we get

$$\mathbf{ber}^{2r}(T)\leqslant 2^{r-2}\|T^*T\|_{ber}^r+\frac{1}{2}\mathbf{ber}^r\left(T^2\right),$$

for any $r\geqslant 1$.

The following theorem is a remarkable generalization and improvement of [10, Theorem 2.15].

THEOREM 4. *Let $T \in B(\mathscr{H}(\Theta))$ and let f, g be nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$, $(t \geqslant 0)$. Then*

$$\mathbf{ber}^{2r}(T) \leqslant \frac{1}{4}\mathbf{ber}\left(f^{4r}(|T|) + g^{4r}(|T^*|)\right) + \frac{1}{4}\mathbf{ber}\left(f^{2r}(|T|)g^{2r}(|T^*|) + g^{2r}(|T^*|)f^{2r}(|T|)\right),$$

for any $r \geqslant 1$.

Proof. Let $\widehat{k}_\nu$ be the normalized reproducing kernel of $\mathscr{H}$. We have

$$\begin{aligned}
&\left|\left\langle T\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right|^{2r} \\
&\leqslant \left\langle f^2(|T|)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle^r \left\langle g^2(|T^*|)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle^r \\
&\quad \text{(by Lemma 4)} \\
&\leqslant \left\langle f^{2r}(|T|)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle \left\langle g^{2r}(|T^*|)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle \\
&\quad \text{(by Lemma 3)} \\
&\leqslant \left(\frac{\left\langle f^{2r}(|T|)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle + \left\langle g^{2r}(|T^*|)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle}{2}\right)^2 \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{4}\left\langle \left(f^{2r}(|T|) + g^{2r}(|T^*|)\right)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle^2 \\
&\leqslant \frac{1}{4}\left\langle \left(f^{2r}(|T|) + g^{2r}(|T^*|)\right)^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle \\
&\quad \text{(by Lemma 3)} \\
&= \frac{1}{4}\left\langle \left(f^{4r}(|T|) + g^{4r}(|T^*|) + f^{2r}(|T|)g^{2r}(|T^*|) + g^{2r}(|T^*|)f^{2r}(|T|)\right)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle \\
&\leqslant \frac{1}{4}\mathbf{ber}\left(f^{4r}(|T|) + g^{4r}(|T^*|) + f^{2r}(|T|)g^{2r}(|T^*|) + g^{2r}(|T^*|)f^{2r}(|T|)\right) \\
&\leqslant \frac{1}{4}\mathbf{ber}\left(f^{4r}(|T|) + g^{4r}(|T^*|)\right) + \frac{1}{4}\mathbf{ber}\left(f^{2r}(|T|)g^{2r}(|T^*|) + g^{2r}(|T^*|)f^{2r}(|T|)\right).
\end{aligned}$$

Thus,

$$\left|\left\langle T\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right|^{2r} \leqslant \frac{1}{4}\mathbf{ber}\left(f^{4r}(|T|) + g^{4r}(|T^*|)\right) + \frac{1}{4}\mathbf{ber}\left(f^{2r}(|T|)g^{2r}(|T^*|) + g^{2r}(|T^*|)f^{2r}(|T|)\right).$$

Now, by taking supremum over $v \in \Theta$ in the above inequality, we get the desired inequality. $\square$

If we take $f(t) = t^{1-p}$ and $g(t) = t^p$ with $0 \leqslant p \leqslant 1$ in Theorem 4, then we get the following corollary.

COROLLARY 4. *Let $T \in B(\mathscr{H}(\Theta))$. Then*

$$\begin{aligned}\mathbf{ber}^{2r}(T) \leqslant & \frac{1}{4}\mathbf{ber}\left(|T|^{4(1-p)r} + |T^*|^{4pr}\right) \\ & + \frac{1}{4}\mathbf{ber}\left(|T|^{2(1-p)r}|T^*|^{2pr} + |T^*|^{2pr}|T|^{2(1-p)r}\right),\end{aligned}$$

for any $p \in [0,1]$ and $r \geqslant 1$.

In particular, for $p = \frac{1}{2}$

$$\mathbf{ber}^{2r}(T) \leqslant \frac{1}{4}\mathbf{ber}\left(|T|^{2r} + |T^*|^{2r}\right) + \frac{1}{4}\mathbf{ber}\left(|T|^r|T^*|^r + |T^*|^r|T|^r\right).$$

REMARK 3. We note that the inequality obtained in Corollary 4 refines the following inequality obtained in [10, Theorem 2.15] as

$$\mathbf{ber}^{2r}(T) \leqslant \frac{1}{4}\mathbf{ber}\left(|T|^{2r} + |T^*|^{2r}\right) + \frac{1}{2}\mathbf{ber}\left(|T|^r|T^*|^r\right),$$

for any $r \geqslant 1$.

Indeed, it can observe that

$$\begin{aligned}\mathbf{ber}^{2r}(T) &\leqslant \frac{1}{4}\mathbf{ber}\left(|T|^{2r} + |T^*|^{2r}\right) + \frac{1}{4}\mathbf{ber}\left(|T|^r|T^*|^r + |T^*|^r|T|^r\right) \\ &\leqslant \frac{1}{4}\mathbf{ber}\left(|T|^{2r} + |T^*|^{2r}\right) + \frac{1}{4}\left(\mathbf{ber}\left(|T|^r|T^*|^r\right) + \mathbf{ber}\left(|T^*|^r|T|^r\right)\right) \\ &= \frac{1}{4}\mathbf{ber}\left(|T|^{2r} + |T^*|^{2r}\right) + \frac{1}{4}\left(\mathbf{ber}\left(|T|^r|T^*|^r\right) + \mathbf{ber}\left(|T|^r|T^*|^r\right)^*\right)\end{aligned}$$

(since $\mathbf{ber}(X) = \mathbf{ber}(X^*)$ for any $X \in B(\mathscr{H}(\Theta))$)

$$= \frac{1}{4}\mathbf{ber}\left(|T|^{2r} + |T^*|^{2r}\right) + \frac{1}{2}\mathbf{ber}\left(|T|^r|T^*|^r\right).$$

The following theorem is an extension of [16, Theorem 2].

THEOREM 5. *Let $T \in B(\mathscr{H}(\Theta))$. Then*

$$\mathbf{ber}^n(T) \leqslant \frac{1}{2^{n-1}}\mathbf{ber}(T^n) + \|T^*\|_{ber}\sum_{p=1}^{n-1}\frac{1}{2^p}\mathbf{ber}^{n-p-1}(T)\,\|T^p\|_{ber},$$

for any $n \geqslant 2$.

Proof. Let $\widehat{k}_v$ be the normalized reproducing kernel of $\mathscr{H}$, we first prove that

$$\left|\widetilde{T}(v)\right|^n \leqslant \frac{1}{2^{n-1}}\left|\widetilde{T^n}(v)\right| + \sum_{p=1}^{n-1}\frac{1}{2^p}\left|\widetilde{T}(v)\right|^{n-p-1}\left\|T^p\widehat{k}_v\right\|\left\|T^*\widehat{k}_v\right\|, \tag{3.1}$$

for any $n \geqslant 2$.

We will use induction on n to establish the required inequality. Substituting $a = T\widehat{k}_v$ and $b = T^*\widehat{k}_v$ in Lemma 2, simply proved that that the inequality (3.1) is true for $n = 2$.

On the other hand, assume that (3.1) is true for n. Applying Lemma 2 with $a = T^n\widehat{k}_v$ and $b = T^*\widehat{k}_v$, we get

$$\left|\widetilde{T^n}(v)\right|\left|\widetilde{T}(v)\right| \leqslant \frac{1}{2}\left(\left\|T^n\widehat{k}_v\right\|\left\|T^*\widehat{k}_v\right\| + \left|\widetilde{T^{n+1}}(v)\right|\right).$$

Under the assumption of induction, we observe that

$$\begin{aligned}\left|\widetilde{T}(v)\right|^{n+1} &= \left|\widetilde{T}(v)\right|^n\left|\widetilde{T}(v)\right| \\ &\leqslant \frac{1}{2^{n-1}}\left|\widetilde{T^n}(v)\right|\left|\widetilde{T}(v)\right| + \sum_{p=1}^{n-1}\frac{1}{2^p}\left|\widetilde{T}(v)\right|^{n-p}\left\|T^p\widehat{k}_v\right\|\left\|T^*\widehat{k}_v\right\| \\ &\leqslant \frac{1}{2^n}\left(\left\|T^n\widehat{k}_v\right\|\left\|T^*\widehat{k}_v\right\| + \left|\widetilde{T^{n+1}}(v)\right|\right) + \sum_{p=1}^{n-1}\frac{1}{2^p}\left|\widetilde{T}(v)\right|^{n-p}\left\|T^p\widehat{k}_v\right\|\left\|T^*\widehat{k}_v\right\| \\ &= \frac{1}{2^n}\left|\widetilde{T^{n+1}}(v)\right| + \sum_{p=1}^{n}\frac{1}{2^p}\left|\widetilde{T}(v)\right|^{n-p}\left\|T^p\widehat{k}_v\right\|\left\|T^*\widehat{k}_v\right\|.\end{aligned}$$

Thus,

$$\left|\widetilde{T}(v)\right|^{n+1} \leqslant \frac{1}{2^n}\left|\widetilde{T^{n+1}}(v)\right| + \sum_{p=1}^{n}\frac{1}{2^p}\left|\widetilde{T}(v)\right|^{n-p}\left\|T^p\widehat{k}_v\right\|\left\|T^*\widehat{k}_v\right\|.$$

Hence, (3.1) is true for $n+1$.
Taking the supremum over all $v \in \Theta$, we obtain

$$\mathbf{ber}^n(T) \leqslant \frac{1}{2^{n-1}}\mathbf{ber}(T^n) + \|T^*\|_{ber}\sum_{p=1}^{n-1}\frac{1}{2^p}\mathbf{ber}^{n-p-1}(T)\|T^p\|_{ber}.$$

Hence, the desired inequality is proved. □

If $n = 2$ in Theorem 5, then we have the following result.

COROLLARY 5. *Let* $T \in B(\mathscr{H}(\Theta))$. *Then*

$$\mathbf{ber}^2(T) \leqslant \frac{1}{2}\left(\|T\|_{ber}\|T^*\|_{ber} + \mathbf{ber}(T^2)\right).$$

REMARK 4. Since $t \longmapsto t^r$, $r \geqslant 1$ is a convex increasing function on $[0,\infty)$ and by using Corollary 5, it is not difficult to see that

$$\mathbf{ber}^{2r}(T) \leqslant \frac{1}{2}\left(\|T\|_{ber}^{r}\|T^*\|_{ber}^{r} + \mathbf{ber}^{r}\left(T^2\right)\right),$$

this inequality proved recently in [16, Theorem 2].

Kittaneh in [22] proved the mixed Schwarz inequality, which asserts

$$|\langle Tx,y\rangle|^2 \leqslant \left\langle |T|^{2(1-t)}x,x\right\rangle \left\langle |T^*|^{2\lambda}y,y\right\rangle, 0 \leqslant t \leqslant 1 \tag{3.2}$$

for $T \in B(H)$ and the vectors $x,y \in H$.

In the following theorem, we give a result about the triangle inequality for the Berezin number.

THEOREM 6. *Let $\mathscr{H} = \mathscr{H}(\Theta)$ be a RKHS on Θ and $T_1,T_2 \in \mathscr{B}(\mathscr{H})$. Then*

$$ber(T_1+T_2) \leqslant \frac{1}{\sqrt{2}}\left(\mathbf{ber}(|T_1|+i|T_2|) + \mathbf{ber}(|T_1^*|+i|T_2^*|)\right)$$

Proof. Let $\widehat{k}_\nu$ be normalized reproducing kernel. Then

$$\begin{aligned}
&\left|\left\langle (T_1+T_2)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right| \\
&\leqslant \left|\left\langle T_1\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\left|\left\langle T_2\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right| \\
&\leqslant \sqrt{\left\langle |T_1|\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\left\langle |T_1^*|\widehat{k}_\nu,\widehat{k}_\nu\right\rangle} + \sqrt{\left\langle |T_2|\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\left\langle |T_2^*|\widehat{k}_\nu,\widehat{k}_\nu\right\rangle} \quad \text{(by (3.2))} \\
&\leqslant \frac{1}{2}\left(\left\langle |T_1|\widehat{k}_\nu,\widehat{k}_\nu\right\rangle + \left\langle |T_1^*|\widehat{k}_\nu,\widehat{k}_\nu\right\rangle + \left\langle |T_2|\widehat{k}_\nu,\widehat{k}_\nu\right\rangle + \left\langle |T_2^*|\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right) \\
&= \frac{\sqrt{2}}{2}\left(\left|\left\langle (|T_1|+i|T_2|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right| + \left|\left\langle (|T_1^*|+i|T_2^*|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right)
\end{aligned}$$

and so

$$\left|\left\langle (T_1+T_2)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right| \leqslant \frac{\sqrt{2}}{2}\left(\left|\left\langle (|T_1|+i|T_2|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right| + \left|\left\langle (|T_1^*|+i|T_2^*|)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right)$$

for all $\nu \in \Theta$. Taking the supremum over $\nu \in \Theta$ above inequality, we have

$$ber(T_1+T_2) \leqslant \frac{1}{\sqrt{2}}\left(\mathbf{ber}(|T_1|+i|T_2|) + \mathbf{ber}(|T_1^*|+i|T_2^*|)\right)$$

which yields the desired result. $\square$

Next result is a refinement of Proposition 3.5 in [32].

THEOREM 7. *Let $T_1, T_2 \in \mathscr{B}(\mathscr{H})$. Then*

$$\mathbf{ber}^2(T_1+T_2) \leqslant \mathbf{ber}\left(T_1^2\right)+\mathbf{ber}\left(T_2^2\right) + \frac{1}{\sqrt{2}}\mathbf{ber}\left(\left(|T_1|^2+|T_2|^2\right)+i\left(|T_1^*|^2+|T_2^*|^2\right)\right).$$

Proof. Putting $T = T_1\widehat{k}_\nu$, $b = T_1^*\widehat{k}_\nu$ and $x = \widehat{k}_\nu$ in (3.2), we have

$$\begin{aligned}\left|\widetilde{T_1}(\nu)\right|^2 &\leqslant \frac{1}{2}\left(\left|\left\langle T_1^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right| + \left\|T_1\widehat{k}_\nu\right\|\left\|T_1^*\widehat{k}_\nu\right\|\right) \\ &= \frac{1}{2}\left(\left|\left\langle T_1^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right| + \sqrt{\left\langle |T_1|^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\left\langle \left|T_1^*\right|^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle}\right) \\ &\leqslant \frac{1}{2}\left(\left|\left\langle T_1^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right| + \frac{1}{4}\left\langle\left(|T_1|^2+|T_1^*|^2\right)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right)\end{aligned}$$

and hence

$$\left|\widetilde{T_1}(\nu)\right| \leqslant \frac{1}{2}\sqrt{2\left|\left\langle T_1^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right| + \left\langle\left(|T_1|^2+\left|T_1^*\right|^2\right)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle}$$

for all $\nu \in \Theta$. From above inequality, we have

$$\begin{aligned}&\left(\left|\widetilde{T_1}(\nu)\right| + \left|\widetilde{T_2}(\nu)\right|\right)^2 \\ &\leqslant \frac{1}{4}\left(\sqrt{2\left|\left\langle T_1^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right| + \left\langle\left(|T_1|^2+\left|T_1^*\right|^2\right)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle}\right. \\ &\quad \left. + \sqrt{2\left|\left\langle T_2^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right| + \left\langle\left(|T_2|^2+\left|T_2^*\right|^2\right)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle}\right)^2 \\ &\leqslant \left|\left\langle T_1^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right| + \left|\left\langle T_2^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right| \\ &\quad + \frac{1}{2}\left(\left\langle\left(|T_1|^2+|T_1^*|^2\right)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle + \left\langle\left(|T_2|^2+|T_2^*|^2\right)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right) \\ &\leqslant \left|\left\langle T_1^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right| + \left|\left\langle T_2^2\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right| \\ &\quad + \frac{1}{\sqrt{2}}\left|\left\langle\left(\left(|T_1|^2+|T_2|^2\right)+i\left(|T_1^*|^2+|T_2^*|^2\right)\right)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right|\end{aligned}$$

(since $|a+b| \leqslant \sqrt{2}\,|a+ib|$ for all $a, b \in \mathbb{R}$) and thus

$$\begin{aligned}\left|\widetilde{(T_1+T_2)}(\nu)\right|^2 &\leqslant \left|\widetilde{T_1^2}(\nu)\right| + \left|\widetilde{T_2^2}(\nu)\right| \\ &\quad + \frac{1}{\sqrt{2}}\left|\left\langle\left(\left(|T_1|^2+|T_2|^2\right)+i\left(|T_1^*|^2+|T_2^*|^2\right)\right)\widehat{k}_\nu, \widehat{k}_\nu\right\rangle\right|\end{aligned}$$

for all $\nu \in \Theta$. So, we get that

$$\begin{aligned}
&\sup_{\nu\in\Theta}\left|\widetilde{(T_1+T_2)}(\nu)\right|^2 \\
&\leqslant \sup_{\nu\in\Theta}\left|\widetilde{T_1^2}(\nu)\right| + \sup_{\nu\in\Theta}\left|\widetilde{T_2^2}(\nu)\right| \\
&\quad + \frac{1}{\sqrt{2}}\sup_{\nu\in\Theta}\left|\left\langle\left(\left(|T_1|^2+|T_2|^2\right)+i\left(|T_1^*|^2+|T_2^*|^2\right)\right)\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|,
\end{aligned}$$

and

$$\begin{aligned}
ber^2(T_1+T_2) &\leqslant ber\left(T_1^2\right)+ber\left(T_2^2\right) \\
&\quad + \frac{1}{\sqrt{2}}ber\left(\left(|T_1|^2+|T_2|^2\right)+i\left(|T_1^*|^2+|T_2^*|^2\right)\right). \quad \square
\end{aligned}$$

THEOREM 8. *Let $T_1, T_2 \in \mathscr{B}(\mathscr{H})$. Then*

$$ber^2(T_1+T_2) \leqslant \frac{3}{2}\left\| |T_1|^2+|T_2|^2\right\|_{ber} + ber\left(T_2^*T_1\right)$$

Proof. Putting $x=\widehat{k}_\nu$, $a=T_1\widehat{k}_\nu$, $b=T_2\widehat{k}_\nu$ in Lemma 2, we have

$$\left|\left\langle T_1\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\left|\left\langle T_2\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right| \leqslant \frac{\left\|T_1\widehat{k}_\nu\right\|\left\|T_2\widehat{k}_\nu\right\|+\left|\left\langle T_1\widehat{k}_\nu,T_2\widehat{k}_\nu\right\rangle\right|}{2}$$

for all $\nu \in \Theta$. Hence, we obtain

$$\begin{aligned}
\left|\widetilde{(T_1+T_2)}(\nu)\right|^2 &\leqslant \left(\left|\left\langle T_1\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|+\left|\left\langle T_2\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\right)^2 \\
&= \left|\left\langle T_1\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|^2+\left|\left\langle T_2\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|^2+2\left|\left\langle T_1\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|\left|\left\langle T_2\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right| \\
&\leqslant \left|\left\langle T_1\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|^2+\left|\left\langle T_2\widehat{k}_\nu,\widehat{k}_\nu\right\rangle\right|^2+\left\|T_1\widehat{k}_\nu\right\|\left\|T_2\widehat{k}_\nu\right\|+\left|\left\langle T_1\widehat{k}_\nu,T_2\widehat{k}_\nu\right\rangle\right| \\
&\leqslant \left\|T_1\widehat{k}_\nu\right\|^2+\left\|T_2\widehat{k}_\nu\right\|^2+\left\|T_1\widehat{k}_\nu\right\|\left\|T_2\widehat{k}_\nu\right\|+\left|\left\langle T_1\widehat{k}_\nu,T_2\widehat{k}_\nu\right\rangle\right| \\
&= \left\|T_1\widehat{k}_\nu\right\|^2+\left\|T_2\widehat{k}_\nu\right\|^2+\sqrt{\left\langle T_1\widehat{k}_\nu,T_1\widehat{k}_\nu\right\rangle\left\langle T_2\widehat{k}_\nu,T_2\widehat{k}_\nu\right\rangle} \\
&\quad +\left|\left\langle T_1\widehat{k}_\nu,T_2\widehat{k}_\nu\right\rangle\right| \\
&\leqslant \widetilde{\left|T_1^2\right|}(\nu)+\widetilde{\left|T_2^2\right|}(\nu)+\frac{1}{2}\left(\widetilde{\left|T_1^2\right|}(\nu)+\widetilde{\left|T_2^2\right|}(\nu)\right)+\left|\widetilde{\left(T_2^*T_1\right)}(\nu)\right| \\
&= \frac{3}{2}\left(\left|T_1^2\right|+\left|T_2^2\right|\right)^{\sim}(\nu)+\left|\widetilde{\left(T_2^*T_1\right)}(\nu)\right|
\end{aligned}$$

and so

$$\sup_{\nu\in\Theta}\left|\widetilde{(T_1+T_2)}(\nu)\right|^2 \leqslant \frac{3}{2}\left\|\left|T_1^2\right|+\left|T_2^2\right|\right\|_{ber}+\sup_{\nu\in\Theta}\left|\widetilde{\left(T_2^*T_1\right)}(\nu)\right|,$$

which yields

$$ber^2(T_1+T_2) \leqslant \frac{3}{2}\left\| |T_1|^2 + |T_2|^2 \right\|_{ber} + ber(T_2^* T_1). \quad \square$$

Acknowledgement. The authors are very grateful to the referee for careful reading and for helpful comments.

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(Received April 6, 2024)

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