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DENSE SUBSET OF MATRICES HAVING EIGENVALUES AND SINGULAR VALUES WITH MINIMUM NUMBER OF REPETITION

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Abstract. In this paper, we introduce a new class of sets namely analytically imaged sets in the space of $m\times n$ matrices. A sufficient condition is obtained for an analytically imaged subset of the set of all $n\times n$ matrices to have a dense subset in terms of algebraic multiplicities of the eigenvalues. Also, the counterparts of this result have been studied for singular values of rectangular matrices and it has been shown that all the results hold for convex subsets of matrices.

1. Introduction and preliminaries

Let Ω be a convex subset of $n\times n$ matrices and Ω' be the set of all matrices in Ω having distinct eigenvalues. In the Corollary 1 of the article [4], the author has shown that Ω' is dense in Ω if Ω' is non-empty. In the article [2], the authors have proved this result for a larger class of subsets instead of convex subset. The authors have also proved some analogous results for singular values of rectangular matrices.

Clearly, these results depend on the matrices whose eigenvalues are non-repeated. Now, two questions arise: Suppose Ω does not contain any matrix with distinct eigenvalues. Is there any dense subset Ω'' of Ω whose elements satisfy certain conditions related to the numbers of repeated eigenvalues? Is there a more larger class of subsets of matrices for which the above stated results hold?

The questions are answered in affirmative in Section 2 and Section 3. We define a class of subsets of matrices called analytically imaged sets and show the existence of dense subset Ω'' of Ω even if Ω does not contain any matrix with distinct eigenvalues. We also prove some results for rectangular matrices related to the singular values.

Before proceeding to the main results, let us introduce some notations which will be used in our results. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the set of all complex, real, rational and natural numbers, respectively. Let $|\mathcal{A}|$ denote the cardinality of the set $\mathcal{A}$. For two sets $\mathcal{A}$ and $\mathcal{B}$, the product $\mathcal{A}\times\mathcal{B}$ represents the set $\{(x,y) : x\in\mathcal{A} \text{ and } y\in\mathcal{B}\}$. We denote the set of all strictly decreasing sequences in $[0,1]$ converging to 0 by $c_0[0,1]$.

Let $\mathbb{C}^{m\times n}$ denote the set of all $m\times n$ complex matrices and I_n denote the $n\times n$ identity matrix. We denote an $n\times n$ diagonal matrix by $\operatorname{diag}(a_1,a_2,\ldots,a_n)$ where a_i is the (i,i)th entry. For an $n\times n$ matrix A, the characteristic polynomial of A is denoted and defined by $\chi_A(t)=\det(tI_n-A)$ (see Definition 8.2.1, [6]). We denote the set of

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all eigenvalues of a given matrix $A \in \mathbb{C}^{n\times n}$ by $\varepsilon(A)$ and for an $a \in \varepsilon(A)$, we denote the algebraic multiplicity of a by $\alpha_A(a)$ (see [6]). For a matrix $A \in \mathbb{C}^{n\times n}$ we define $\mu(A) = \max\{\alpha_A(a) : a \in \varepsilon(A)\}$.

For an $m \times n$ complex matrix A, the Frobenius norm is denoted and defined by $\|A\| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}$, where a_{ij} is the (i,j)th entry of A. In this article, we consider $\mathbb{C}^{m\times n}$ as a topological space induced by the Frobenius norm. Clearly $\mathbb{C}^{m\times n}$ is a metric space. In a metric space $\mathcal{M}$, a point $x \in \mathcal{M}$ is called isolated point if x is not a limit point (see Definition 2.18, [7]).

A function $f(x)$ of the form $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is called an analytic function, where $x \in \mathfrak{u} \subset \mathbb{R}$, $c \in \mathbb{R}$ and $a_n \in \mathbb{C}$ for all n (see Chapter 8, [7]). Let $f(x)$ and $g(x)$ be two analytic functions defined on the open interval $(-r,r) \subset \mathbb{R}$. If the set $\{x : f(x) = g(x)\}$ has a limit point in $(-r,r)$ then $\{x : f(x) = g(x)\} = (-r,r)$. This result is known as the Identity theorem for analytic functions (see Theorem 8.5, [7]).

We call an $m \times n$ matrix valued function $f : \mathfrak{u} \subset \mathbb{R} \to \mathbb{C}^{m\times n}$ an $m \times n$ analytic function if each entry of f is an analytic function defined on $\mathfrak{u}$. We denote the set of all $m \times n$ analytic functions defined on $[0,1]$ by $C^{\omega}_{m\times n}[0,1]$.

Let $f(x) = \sum_{k=0}^{n} a_k x^k$ and $g(x) = \sum_{k=0}^{m} b_k x^k$ be two polynomials of degree n and m, respectively with complex coefficients. The Sylvester matrix of $f(x)$ and $g(x)$ is an $(n+m)\times(n+m)$ matrix, denoted by $\mathrm{Syl}(f,g)$, defined as

$$\mathrm{Syl}(f,g) = \begin{bmatrix} a_n & a_{n-1} & . & . & . & a_0 & 0 & \ldots & . \\ 0 & a_n & a_{n-1} & . & . & . & a_0 & 0 \ldots & . \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ . & . & 0 & a_n & a_{n-1} & . & . & \ldots & a_0 \\ b_m & b_{m-1} & . & . & b_0 & 0 & . & \ldots & . \\ 0 & b_m & b_{m-1} & . & . & b_0 & 0 & \ldots & . \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ . & . & 0 & b_m & b_{m-1} & . & . & \ldots & b_0 \end{bmatrix}.$$

The resultant of two polynomials $f(x)$ and $g(x)$ is the determinant of the $\mathrm{Syl}(f,g)$, and is denoted by $\mathrm{Res}(f,g)$. Also $f(x)$ and $g(x)$ have a common factor if and only if $\mathrm{Res}(f,g) = 0$ (see [5]).

In the next section, we define a new class of subsets of matrices and prove some properties.

2. Analytically imaged set

The motivation to define a new class of sets comes from the questions that arise in the Section 1 and from the proofs of Theorem 1 of article [4] and Theorem 3.5 of article [2]. It is clear from the proofs that the theorems also hold for a subset $\Omega \subset \mathbb{C}^{n\times n}$ if for any two points A and B in Ω, there exists an $n \times n$ analytic function f on $[0,1]$ and a strictly decreasing sequence (x_n) converging to 0 such that $f(0) = A$, $f(1) = B$ and $f(x_n)$ is in Ω for each n. So we define the following class of sets called analytically imaged set.

DEFINITION 2.1. A set $\Omega \subset \mathbb{C}^{m\times n}$ is said to be an analytically imaged set if for A and B $(\neq A)$ in Ω, there is an $m\times n$ analytic function $f_{AB}:[0,1]\to\mathbb{C}^{m\times n}$ and a strictly decreasing sequence (x_n) in $[0,1]$ converging to 0, such that $f_{AB}(0)=A$, $f_{AB}(1)=B$ and $f_{AB}(x_n)$ is in Ω for each n.

Clearly, an analytically imaged set contains at least two elements. Now, we give some examples of analytically imaged set.

EXAMPLE 2.1. Any convex set $\Omega\subset\mathbb{C}^{m\times n}$ is an analytically imaged set. Because for any two points A and B in Ω, we have an $m\times n$ analytic function $f_{AB}(t)=(1-t)A+tB$ defined on $[0,1]$ and a strictly decreasing sequence $(\frac{1}{n})$ in $[0,1]$ converging to 0, such that $f_{AB}(0)=A$, $f_{AB}(1)=B$ and $f_{AB}(\frac{1}{n})$ is in Ω for each $n\in\mathbb{N}$.

EXAMPLE 2.2. The set of all $m\times n$ matrices whose entries are rational numbers is an analytically imaged set because for any two matrices A and B with rational number entries if we consider the same function $f_{AB}(t)$ and the sequence $(\frac{1}{n})$ defined in Example 2.1, then $f_{AB}(0)=A$, $f_{AB}(1)=B$ and each matrix $f_{AB}(\frac{1}{n})$ has rational number entries.

EXAMPLE 2.3. Let $A_0,A_1,\ldots,A_k$ be $m\times n$ complex matrices. Then, the set $\Omega=\{A(t)=\sum_{i=0}^{k}A_it^i : t\in\mathbb{R}\}\subset\mathbb{C}^{m\times n}$ is an analytically imaged set. Because, for any two matrices B and C in Ω, there exist two points x and y in $\mathbb{R}$ such that $B=A(x)$ and $C=A(y)$. So we consider the analytic function $f_{BC}(t)=A((1-t)x+ty)$ defined on $[0,1]$ and the sequence $(\frac{1}{n})$ which give $f_{BC}(0)=B$, $f_{BC}(1)=C$ and $f_{BC}(\frac{1}{n})$ is in Ω for all n in $\mathbb{N}$.

EXAMPLE 2.4. The set of all $n\times n$ invertible complex matrices is an analytically imaged set. For if, we consider two invertible complex matrices A and B, and the function $f_{AB}(t)$ defined in Example 2.1. Now, $\det(f_{AB}(t))$ is a non-constant polynomial in t so it can be 0 for finitely many t. So, there exists a strictly decreasing sequence (x_n) in $[0,1]$ converging to 0, such that $\det(f_{AB}(x_n))\neq 0$ for all n in $\mathbb{N}$. Hence, the result follows.

We call an $n\times n$ complex matrix A, Hermitian if $A=A^*$, where A^* denotes the conjugate transpose of A. The set of all $n\times n$ Hermitian matrices is an analytically imaged set since the set of all $n\times n$ Hermitian matrices forms a convex subset which is an analytically imaged set follows from Example 2.1. Now, we will give another example of analytically imaged set related to Hermitian matrices.

EXAMPLE 2.5. The set of all Hermitian matrices whose eigenvalues lie in the interval $(x,y)\subset\mathbb{R}$ is an analytically imaged set. It is known that for any two Hermitian matrices A and B, whose eigenvalues lie in (x,y), the eigenvalues of any convex combination of A and B lie in (x,y) (see V.1, [1]). So, we consider the function $f_{AB}(t)$ defined in Example 2.1 and the sequence $(\frac{1}{n})$ which give $f_{AB}(0)=A$, $f_{AB}(1)=B$ and eigenvalues of $f_{AB}(\frac{1}{n})$ lie in (x,y) for all n in $\mathbb{N}$. Hence the result is proved.

Next, we prove a property of an analytically imaged set which provides a necessary condition for a set to be analytically imaged.

COROLLARY 2.1. *Let $\Omega \subset \mathbb{C}^{m\times n}$ be an analytically imaged set, then Ω has no isolated points.*

Proof. Let A be in Ω, then for any B in Ω there exists an $m\times n$ analytic function f_{AB} on $[0,1]$ and a strictly decreasing sequence (x_n) in $[0,1]$ converging to 0 such that $f_{AB}(0)=A$, $f_{AB}(1)=B$ and $f_{AB}(x_n)$ is in Ω for all n. Now $f_{AB}(x_n)$ may be equal only for finitely many n, otherwise f_{AB} will be a constant function by the Identity theorem for analytic functions which is not possible. So there is a sub-sequence (x_{n_k}) of (x_n) such that $f_{AB}(x_{n_i}) \neq f_{AB}(x_{n_j})$ if $i\neq j$. Now by continuity of f_{AB} the sequence $f_{AB}(x_{n_k})$ converges to $f_{AB}(0)$, that is to A. Hence A is not an isolated point. As A is arbitrary so Ω has no isolated points. $\square$

Now we prove the following lemmas which can be used to form analytically imaged sets. We also provide some examples after proving the lemmas.

LEMMA 2.1. *Let $\mathfrak{u}$ be an open subset of $\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{C}^{m\times n}$ be a non-constant $m\times n$ analytic function. Then $f(\mathfrak{u})$ is an analytically imaged set.*

Proof. Let A and B be two points in $f(\mathfrak{u})$. So there are x and y in $\mathfrak{u}$ such that $f(x)=A$ and $f(y)=B$. Now we consider a strictly decreasing or increasing sequence (x_n) in $\mathfrak{u}$ when $x<y$ or $x>y$, respectively with $x_1=y$ such that (x_n) converges to x. Such sequences exist as $\mathfrak{u}$ is open. We consider the analytic function $g(t)=x(1-t)+yt$ and let $y_n=\frac{x_n-x}{y-x}$ for all n. So the strictly decreasing sequence (y_n) converges to 0 and $g(y_n)=x_n$ for all n. Hence there is an $m\times n$ analytic function $h(t)=f(g(t))$ defined on $[0,1]$ and a strictly decreasing sequence (y_n) in $[0,1]$ converging to 0 such that $h(0)=f(x)=A$, $h(1)=f(y)=B$ and $h(y_n)=f(x_n)$ is in $f(\mathfrak{u})$ for each n. So $f(\mathfrak{u})$ is an analytically imaged set. $\square$

REMARK 2.1. From the proof of the above Lemma 2.1, it is clear that the lemma is also true for a set $\mathfrak{u}\subset\mathbb{R}$ if $\mathfrak{u}$ has the property that: for each x in $\mathfrak{u}$ there exists a strictly decreasing sequence (x_n) and a strictly increasing sequence (y_n) in $\mathfrak{u}$ converging to x.

EXAMPLE 2.6. Let $A_0, A_1, \ldots, A_k$ be $m\times n$ matrices. Consider the function $f(t)=\sum_{i=0}^{k} A_i t^i$ defined on $\mathbb{R}$. Then for any open interval (x,y), the set $f((x,y))$ is an analytically imaged set. Also, $\{\mathrm{diag}(\sin(t),\sin(2t),\ldots,\sin(nt)) : t\in(x,y)\}$ and $\{\mathrm{diag}(\cos(t),\cos(2t),\ldots,\cos(nt)) : t\in(x,y)\cap\mathbb{Q}\}$ are analytically imaged sets.

LEMMA 2.2. *Let $\mathfrak{u}\subset\mathbb{R}$ has no isolated points and $f:\mathbb{R}\to\mathbb{C}^{m\times n}$ be a non-constant $m\times n$ analytic function. Then $f(\mathfrak{u})$ is an analytically imaged set.*

Proof. Let A and B are in $f(\mathfrak{u})$. So there exist x and y in $\mathfrak{u}$ such that $f(x) = A$ and $f(y) = B$. There must exists a strictly increasing or decreasing sequence (x_n) in $\mathfrak{u}$ which converges to x. First we assume (x_n) is strictly decreasing. We consider the analytic function $g(t) = at^2 + (y - x - a)t + x$ for $a < 0$. So $g(0) = x$ and $g(1) = y$. Now we will show that there exists an $a < 0$ such that $\{x_n : n \in \mathbb{N}\} \subset g([0,1])$. We have $g(t) = at^2 + (y - x - a)t + x$. So for $a < 0$, $g(t)$ has local maxima at $t_{max} = \frac{1}{2} + \frac{x-y}{2a}$ and the value is $x - \frac{(a+x-y)^2}{4a}$. Hence we can choose $a < 0$ such that $0 < t_{max} < 1$ and $g(t_{max})$ is an upper-bound for $\{x_n : n \in \mathbb{N}\}$. Also $g(t)$ is strictly increasing in the interval $[0, t_{max})$. So there exists a strictly decreasing sequence (y_n) in $[0,1]$ converging to 0 such that $y_1 = 1$ and $g(y_n) = x_n$ for $n = 2, 3, \ldots$. Hence there is an $m \times n$ analytic function $h(t) = f(g(t))$ and a strictly decreasing sequence (y_n) such that $h(0) = f(x) = A$, $h(1) = f(y) = B$ and $h(y_n) = f(x_n)$ is in $f(\mathfrak{u})$ for each n.

Now if (x_n) is strictly increasing then we can consider the function $g(t) = at^2 + (y - x - a)t + x$ for $a > 0$. So $g(t)$ has local minima at $t_{min} = \frac{1}{2} + \frac{x-y}{2a}$ and the value is $x - \frac{(a+x-y)^2}{4a}$. Hence we can choose $a > 0$ such that $0 < t_{min} < 1$ and $g(t_{min})$ is a lower bound for $\{x_n : n \in \mathbb{N}\}$. Also $g(t)$ is strictly decreasing in the interval $[0, t_{min})$. Hence by a similar argument as above there is an $m \times n$ analytic function $h(t) = f(g(t))$ and a strictly decreasing sequence (y_n) in $[0,1]$ converging to 0 such that $h(0) = f(x) = A$, $h(1) = f(y) = B$ and $h(y_n) = f(x_n)$ is in $f(\mathfrak{u})$ for each n. Hence $f(\mathfrak{u})$ is an analytically imaged set. $\square$

EXAMPLE 2.7. For a closed interval $[x,y] \subset \mathbb{R}$, the set $\{\mathrm{diag}(\sin(t), \sin(2t), \ldots, \sin(nt)) : t \in [x,y] \cap \mathbb{Q}\}$ is an analytically imaged set.

EXAMPLE 2.8. The set $\{\mathrm{diag}(e^t, e^{2t}, \ldots, e^{nt}) : t \in \mathfrak{c}\}$ is an analytically imaged set where $\mathfrak{c}$ denotes the Cantor set (see [7]).

Equivalently, an analytically imaged set may also be described as follows.

PROPOSITION 2.1. *A set $\Omega \subset \mathbb{C}^{m\times n}$ is analytically imaged if and only if $\Omega = \cup_{(f,(x_n)) \in \mathcal{A}} \{f(x) : x = 0, 1, x_n \text{ for all } n\}$ for some $\mathcal{A} \subset C^{\omega}_{m\times n}[0,1] \times c_0[0,1]$ such that for any two points A and B in Ω there is an element $(f,(x_n))$ in $\mathcal{A}$ which satisfies $f(0) = A$ and $f(1) = B$.*

In the next section, we derive some dense subsets of a given set of $n \times n$ matrices in terms of their eigenvalues. Also, we study the counterpart of these results for singular values of rectangular matrices.

3. Existence of dense subsets

First, we will provide a sufficient condition for an analytically imaged set Ω so that the subset $\Omega' = \{A \in \Omega : \mu(A) = \min_{X \in \Omega} \mu(X)\}$ is dense in Ω. Such subset Ω' exists since for any set $\Omega \subset \mathbb{C}^{n\times n}$, $\mu(A)$ is an element of the finite set $\{1, 2, \ldots, n\}$, where A is in Ω. Hence, $\min_{X \in \Omega} \mu(X)$ always exists and is achieved at least at a matrix

A in Ω which confirms the existence of Ω'. Before proceeding to the result, we give an example of such subset.

EXAMPLE 3.1. Let us consider the following subset of $\mathbb{C}^{3\times 3}$

$$\Omega = \left\{ A(t) = \begin{bmatrix} \cos t^{-1} & -\sin t^{-1} & 0 \\ \sin t^{-1} & \cos t^{-1} & 0 \\ \cos t^{-1} & -\sin t^{-1} & e^{it^{-1}} \end{bmatrix} : t \in (0,1) \right\}.$$

The set Ω is an analytically imaged set which follows from a similar argument given in the Example 2.3. Now, the eigenvalues of $A(t)$ are $e^{it^{-1}}$, $e^{it^{-1}}$ and $e^{-it^{-1}}$. Hence, $\mu(A(t)) = 2$ when t is in $(0,1) \setminus \left\{\frac{1}{n\pi} : n \in \mathbb{N}\right\}$ and $\mu(A(t)) = 3$ otherwise. Hence, Ω' is the following set

$$\Omega' = \left\{ A(t) \in \Omega : t \in (0,1) \setminus \left\{ \frac{1}{n\pi} : n \in \mathbb{N} \right\} \right\}.$$

THEOREM 3.1. *Let Ω be an analytically imaged subset of $\mathbb{C}^{n\times n}$. Then, $\Omega' = \{A \in \Omega : \mu(A) = \min_{X\in\Omega} \mu(X)\}$ is dense in Ω if Ω' contains a matrix whose each eigenvalue has algebraic multiplicity $\min_{X\in\Omega} \mu(X)$.*

Proof. Let $A \in \Omega$ and B be a matrix in Ω' such that each eigenvalue of B has algebraic multiplicity $\min_{X\in\Omega} \mu(X)$. Now, there is an $n \times n$ analytic function $f_{AB} : [0,1] \to \mathbb{C}^{n\times n}$ and a strictly decreasing sequence (x_n) in $[0,1]$ converging to 0 such that $f_{AB}(0) = A$, $f_{AB}(1) = B$ and each $f_{AB}(x_n)$ is in Ω. Set $\chi_x(y) = \det(yI_n - f_{AB}(x)) = y^n + \sum_{i=1}^{n} a_i(x) y^{n-i}$. So $\chi_x(y)$ is a polynomial in y and each $a_i(x)$ is analytic. For each x in $[0,1]$, the eigenvalues of f_{AB} are the roots of the polynomial $\chi_x(y)$.

Let $\min_{X\in\Omega} \mu(X) = k$. Now, $\mathrm{Res}(\chi_x, \chi_x^{(k)})$, the resultant of the polynomials χ_x and $\chi_x^{(k)}$ is an analytic function of x on $[0,1]$ as the coefficients of the polynomials χ_x and $\chi_x^{(k)}$ are analytic functions of x on $[0,1]$, where $\chi_x^{(k)}$ denotes the k-th order derivative of χ_x with respect to y. Also, χ_1 and $\chi_1^{(k)}$ can not have a common root, hence $\mathrm{Res}(\chi_1, \chi_1^{(k)}) \neq 0$. So $\mathrm{Res}(\chi_x, \chi_x^{(k)})$ is 0 for finitely many points in $[0,1]$. Hence, there exists a natural number n_0 such that $\mathrm{Res}(\chi_{x_i}, \chi_{x_i}^{(k)}) \neq 0$ for all $i \geqslant n_0$. So $\mu(f_{AB}(x_i)) \leqslant \min_{X\in\Omega} \mu(X)$ for all $i \geqslant n_0$; also $f_{AB}(x_i) \in \Omega$. Hence $\mu(f_{AB}(x_i)) = \min_{X\in\Omega} \mu(X)$ for all $i \geqslant n_0$ and from this we can conclude that $f_{AB}(x_i) \in \Omega'$ for all $i \geqslant n_0$. Now by the continuity of f_{AB}, it is easy to see that for any $\varepsilon > 0$, the open ball $B(A;\varepsilon)$: centered at A and radius ε has a nonempty intersection with Ω'. As A is arbitrary, hence Ω' is dense in Ω. $\square$

It is clear that Theorem 3.1 makes sense if $\min_{X\in\Omega} \mu(X)$ divides n. Also, we see in Example 3.1 that Ω' does not contain any matrix whose each eigenvalue has algebraic multiplicity 2. Our next theorem shows that we can replace this requirement by a weaker condition. But before proceeding to the next theorem we provide an example for Theorem 3.1.

EXAMPLE 3.2. Let us consider the following subset of $\mathbb{C}^{4\times 4}$

$$\Omega = \left\{ A(a_{ij}) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} : a_{ij} \in \mathbb{C} \quad \text{and} \quad a_{33}a_{44} - a_{34}a_{43} \neq 0 \right\}.$$

By a similar argument given in Example 2.4, we can conclude that Ω is an analytically imaged set. The eigenvalues of $A = A(a_{ij})$ are $1, 1, e_1(A_{34})$ and $e_2(A_{34})$ where $e_1(A_{34})$ and $e_2(A_{34})$ are the eigenvalues of the sub-matrix

$$A_{34} = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}.$$

Now, $\mu(A) = 2$ if $e_1(A_{34}) = e_2(A_{34}) \neq 1$ or $e_1(A_{34}) \neq e_2(A_{34}) \neq 1$, $\mu(A) = 3$ if $e_1(A_{34}) \neq e_2(A_{34}) = 1$ or $e_2(A_{34}) \neq e_1(A_{34}) = 1$ and $\mu(A) = 4$ if $e_1(A_{34}) = e_2(A_{34}) = 1$. Hence,

$$\Omega' = \{A = A(a_{ij}) \in \Omega : e_1(A_{34}) = e_2(A_{34}) \neq 1 \text{ or } e_1(A_{34}) \neq e_2(A_{34}) \neq 1\}.$$

Also, Ω' contains matrices whose each eigenvalue has algebraic multiplicity 2. The following matrix is an example of such matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

From the proof of the Theorem 3.1, It is clear that the Theorem 3.1 is also true if Ω contains a matrix A such that the characteristic polynomial $\chi_A(t)$ and $\min_{X\in\Omega} \mu(X)$-th order derivative of $\chi_A(t)$ have no common root. So, we have the following theorem.

THEOREM 3.2. *Let Ω be an analytically imaged subset of $\mathbb{C}^{n\times n}$. Then, $\Omega' = \{A \in \Omega : \mu(A) = \min_{X\in\Omega} \mu(X)\}$ is dense in Ω if Ω contains a matrix A such that $\chi_A(t)$ and $\min_{X\in\Omega} \mu(X)$-th order derivative of $\chi_A(t)$ have no common root.*

Proof. Let A and B be two matrices in Ω such that $\chi_B(t)$ and $\min_{X\in\Omega} \mu(X)$-th order derivative $\chi_B(t)$ have no common root. So, $\mu(B) \leqslant \min_{X\in\Omega} \mu(X)$ and the minimality condition implies $\mu(B) = \min_{X\in\Omega} \mu(X)$. Hence, B is in Ω'. Rest of the proof is similar to that of Theorem 3.1. $\square$

REMARK 3.1. If Ω has a nonempty interior then $\min_{X\in\Omega} \mu(X) = 1$ as Ω contains matrices with distinct eigenvalues but this is not a necessary condition for $\min_{X\in\Omega} \mu(X) = 1$. It is notable that Ω' is the set of all matrices in Ω having distinct eigenvalues when $\min_{X\in\Omega} \mu(X) = 1$.

We have the following theorem which follows from Theorem 3.1 for $\min_{X\in\Omega}\mu(X) = 1$. This theorem becomes the Corollary 1 of the article [4] if we consider that Ω is a convex set.

THEOREM 3.3. *Let Ω be an analytically imaged subset of $\mathbb{C}^{n\times n}$ and Ω' be the set of all matrices in Ω having distinct eigenvalues. Then, Ω' is dense in Ω if and only if Ω' is nonempty.*

For our next Theorem, we consider a subset Σ of $\mathbb{C}^{n\times n}$ satisfying the following two properties:

1. Each element in Σ has real eigenvalues,
2. For any two elements A and B in Σ there exists an $n\times n$ analytic function f_{AB} : $[0,1]\to\mathbb{C}^{n\times n}$ and a strictly decreasing sequence (x_n) in $[0,1]$ converging to 0 such that $f_{AB}(0)=A$, $f_{AB}(1)=B$, $f_{AB}(x_n)$ is in Σ and each matrix in $f_{AB}([0,1])$ has real eigenvalues.

Example 2.6 (except the first example), 2.7 and 2.8 of Section 2 are some examples of Σ defined above. Now, in the next theorem we show that the set of all matrices in Σ having the maximum number of distinct eigenvalues is dense in Σ. But, before going to the next theorem, we state a result from the article [3] which will be used to prove the theorem.

THEOREM. (Theorem 1.1, [3]) *Let $A(x)$ be an $n\times n$ matrix function with analytic entries on $[a,b]$, where $-\infty\leqslant a<b\leqslant\infty$. Assume that every eigenvalue of $A(x)$ is real on $[a,b]$. Then there exists a unitary matrix $U(x)$ analytic on $[a,b]$ such that $Q(x)=U^{-1}(x)A(x)U(x)$, where $Q(x)$ is a upper triangular matrix whose entries are analytic in x on $[a,b]$.*

THEOREM 3.4. *The subset $\Sigma'=\{A\in\Sigma : |\varepsilon(A)|=\max_{X\in\Sigma}|\varepsilon(X)|\}$ is dense in Σ.*

Proof. The subset Σ' is non-empty since $|\varepsilon(A)|$ is an element of $\{1,2,\dots,n\}$ for any A in Σ. Hence, $\max_{X\in\Sigma}|\varepsilon(X)|$ always exists and is achieved at least at a matrix B in Σ. Let $A\in\Sigma$ and $B\in\Sigma'$. Now there is an $n\times n$ analytic function $f_{AB}:[0,1]\to\mathbb{C}^{n\times n}$ and a strictly decreasing sequence (x_n) in $[0,1]$ converging to 0 such that $f_{AB}(0)=A$, $f_{AB}(1)=B$ and each $f_{AB}(x_n)$ is in Σ. Also $f_{AB}(x)$ has real eigenvalues for each $x\in[0,1]$. Hence there exists an $n\times n$ unitary matrix $g(x)$ whose entries are analytic functions on $[0,1]$ such that $g^{-1}(x)f_{AB}(x)g(x)=h(x)$, where $h(x)$ is an $n\times n$ upper triangular matrix whose entries are analytic functions on $[0,1]$ and $g^{-1}(x)$ is the inverse matrix of $g(x)$. Let $e_1(x),e_2(x),\dots,e_n(x)$ be the eigenvalues of f_{AB}, so each $e_i(x)$ is analytic on $[0,1]$ as these are the diagonal elements of $h(x)$. Let $\max_{X\in\Sigma}|\varepsilon(X)| = k$. Now, $f_{AB}(1)=B$ has k distinct eigenvalues. Hence there are k natural numbers $i_1,i_2,\dots,i_k$ such that $e_{i_1}(1),e_{i_2}(1),\dots,e_{i_k}(1)$ are the k distinct eigenvalues of B. Hence $e_{i_1},e_{i_2},\dots,e_{i_k}$ are k distinct numbers for each $x\in[0,1]\setminus\mathfrak{s}$, where $\mathfrak{s}$ is a finite subset of $[0,1]$. So there exists a natural number n_0 such that $f_{AB}(x_n)$ has at least k distinct eigenvalues for all $n \geqslant n_0$, also $f_{AB}(x_n)$ is in Σ. Hence, $|\varepsilon(f_{AB}(x_n))| = k$ for all $n \geqslant n_0$. So $f_{AB}(x_n) \in \Sigma'$ for all $n \geqslant n_0$. Now it is easily seen that for any $\varepsilon > 0$, the open ball $B(A;\varepsilon)$ has nonempty intersection with Σ' and as A is arbitrary hence Σ' is dense in Σ. $\square$

It is natural to ask whether the counterpart of the above results are true for the singular values. The answer is affirmative.

Let $m \geqslant n$. It is a known result that the singular values of an $m \times n$ matrix A are the positive square root of the eigenvalues of A^*A, where A^* denotes the conjugate transpose of A (see III.7, [1]). We will use this result to prove our next theorems. The notations what we have defined earlier for $\mathbb{C}^{n\times n}$ are not well-defined for $\mathbb{C}^{m\times n}$. So we introduce some new notations. In this section, we restrict ourselves for the case $m \geqslant n$.

The set of all singular values for a given matrix $A \in \mathbb{C}^{m\times n}$ is denoted by $s(A)$ and if $a \in s(A)$ is a repeated singular value of A, then we denote the number of repetitions of a by $\alpha_A^\circ(a)$. For a non-repeated singular value a, $\alpha_A^\circ(a) = 1$. For a matrix $A \in \mathbb{C}^{m\times n}$, let us define $\mu^\circ(A) = \max\{\alpha_A^\circ(a) : a \in s(A)\}$. Clearly $\mu^\circ(A) = \mu(A^*A)$.

The following theorem is a counterpart of Theorem 3.1 for the rectangular matrices.

THEOREM 3.5. *Let Ω be an analytically imaged subset of $\mathbb{C}^{m\times n}$. Then, $\Omega' = \{A \in \Omega : \mu^\circ(A) = \min_{X\in\Omega}\mu^\circ(X)\}$ is dense in Ω if Ω' contains a matrix whose each singular value repeats $\min_{X\in\Omega}\mu^\circ(X)$ times.*

Proof. Let $A \in \Omega$ and $B \in \Omega'$ such that each singular value of B repeats k times, where $k = \min_{X\in\Omega}\mu^\circ(X)$. Then, there is an $m \times n$ analytic function $f_{AB} : [0,1] \to \mathbb{C}^{m\times n}$ and a strictly decreasing sequence (x_n) in $[0,1]$ converging to 0 such that $f_{AB}(0) = A$, $f_{AB}(1) = B$ and each $f_{AB}(x_n)$ is in Ω. Clearly $f_{AB}^* f_{AB}$ is an $n \times n$ matrix function with analytic function entries such that $f_{AB}^*(0) f_{AB}(0) = A^*A$ and $f_{AB}^*(1) f_{AB}(1) = B^*B$. Also each eigenvalue of B^*B has algebraic multiplicity k. Now we can mimic the proof of Theorem 3.1 to show that there exists a natural number n_0 such that $\mu(f_{AB}^*(x_n) f_{AB}(x_n)) = k$ for all $n \geqslant n_0$. Hence $f_{AB}(x_n)$ are in Ω' for all $n \geqslant n_0$. Rest of the proof is similar to that of Theorem 3.1. $\square$

We state our next theorem without proof which is an analogous result of Theorem 3.2 for rectangular matrices. The proof follows from the proof of Theorem 3.2 and Theorem 3.5.

THEOREM 3.6. *Let Ω be an analytically imaged subset of $\mathbb{C}^{m\times n}$. Then, $\Omega' = \{A \in \Omega : \mu^\circ(A) = \min_{X\in\Omega}\mu^\circ(X)\}$ is dense in Ω if Ω contains a matrix A such that $\chi_{A^*A}(t)$ and $\min_{X\in\Omega}\mu^\circ(X)$-th order derivative of $\chi_{A^*A}(t)$ have no common root.*

Existence of the subset Ω' used in Theorem 3.5 and 3.6 can be verified by a similar argument given in Section 3. Also, the above Theorem 3.5 holds if $\min_{x\in\Omega}\mu^\circ(X)$ divides n which is relaxed in Theorem 3.6. Now, we will give an example of a set Ω and its subset Ω' used in Theorem 3.5 and 3.6.

EXAMPLE 3.3. Let us consider the following subset of $\mathbb{C}^{5\times 4}$

$$\Omega = \left\{ A(t,a_{ij}) = \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \\ 0 & 0 & a_{53} & a_{54} \end{bmatrix} : t \in \mathbb{R}, a_{ij} \in \mathbb{C} \text{ and } \operatorname{Rank}(A(t,a_{ij})) = 4 \right\}.$$

Let $B = A(x,b_{ij})$ and $C = A(y,c_{ij})$ be two matrices in Ω. We consider a function $f_{BC}:[0,1] \to \mathbb{C}^{5\times 4}$ defined by $f_{BC}(t) = A((1-t)x+ty,(1-t)b_{ij}+tc_{ij})$. More explicitly,

$$f_{BC}(t) = \begin{bmatrix} \cos((1-t)x+ty) & -\sin((1-t)x+ty) & 0 & 0 \\ \sin((1-t)x+ty) & \cos((1-t)x+ty) & 0 & 0 \\ 0 & 0 & (1-t)b_{33}+tc_{33} & (1-t)b_{34}+tc_{34} \\ 0 & 0 & (1-t)b_{43}+tc_{43} & (1-t)b_{44}+tc_{44} \\ 0 & 0 & (1-t)b_{53}+tc_{53} & (1-t)b_{54}+tc_{54} \end{bmatrix}.$$

Clearly, $f_{BC}(0) = B$ and $f_{BC}(1) = C$. Now, $\det(f_{BC}^*(t)f_{BC}(t))$ is a polynomial in t so it can be 0 only for finitely many values of t since, $\det(B^*B)$ and $\det(C^*C)$ are non-zero. Hence, there exists a decreasing sequence (x_n) in $[0,1]$ converging to 0 such that $\det(f_{BC}^*(x_n)f_{BC}(x_n)) \neq 0$ for all n in $\mathbb{N}$. That is, $\operatorname{Rank}(f_{BC}(x_n))$ is 4 for all n in $\mathbb{N}$. Thus, the set Ω is an analytically imaged set.

Now, the singular values of $A = A(t,a_{ij})$ are $1,1,s_1(A_{34}^5)$ and $s_2(A_{34}^5)$. Here $s_1(A_{34}^5)$ and $s_2(A_{34}^5)$ are the positive square root of the eigenvalues of ${A_{34}^5}^*A_{34}^5$, where A_{34}^5 is the following sub-matrix

$$A_{34}^5 = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \\ a_{53} & a_{54} \end{bmatrix}.$$

So, $\mu^\circ(A) = 2$ if $s_1(A_{34}^5) = s_2(A_{34}^5) \neq 1$ or $s_1(A_{34}^5) \neq s_2(A_{34}^5) \neq 1$, $\mu^\circ(A) = 3$ if $s_1(A_{34}^5) \neq s_2(A_{34}^5) = 1$ or $s_2(A_{34}^5) \neq s_1(A_{34}^5) = 1$ and $\mu^\circ(A) = 4$ if $s_1(A_{34}^5) = s_2(A_{34}^5) = 1$. Hence,

$$\Omega' = \{A = A(t,a_{ij}) \in \Omega : s_1(A_{34}^5) = s_2(A_{34}^5) \neq 1 \text{ or } s_1(A_{34}^5) \neq s_2(A_{34}^5) \neq 1\}.$$

Also, Ω' contains matrices whose each singular value repeats 2 times. The following matrix is an example of such matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The next theorem is a consequence of the Theorem 3.5. This theorem becomes Theorem 3.8 in the article [2] when we consider that Ω is a convex set.

THEOREM 3.7. *Let Ω be an analytically imaged subset of $\mathbb{C}^{m\times n}$. Then Ω', the set of all matrices in Ω having distinct singular values is dense in Ω if and only if Ω' is nonempty.*

Proof. The proof follows from Theorem 3.3 and Theorem 3.5. $\square$

We complete our article with the following theorem which shows that the set of all matrices in Ω having maximum number of distinct singular values is dense in Ω, which is similar to the Theorem 3.4.

THEOREM 3.8. *Let Ω be an analytically imaged subset of $\mathbb{C}^{m\times n}$.Then the set $\Omega' = \{A \in \Omega : |s(A)| = \max_{X\in\Omega} |s(X)|\}$ is dense in Ω.*

Proof. For any two matrices A and B from Ω, we have an $m\times n$ analytic function f_{AB} defined on $[0,1]$ such that $f_{AB}(0) = A$ and $f_{AB}(1) = B$. So the function $f^*_{AB} f_{AB}$ has analytic function entries and for each x in $[0,1]$, the matrix $f^*_{AB}(x) f_{AB}(x)$ has real eigenvalues. So we can prove this theorem with the help of above stated Theorem 1.1 from the article [3] and the proofs of Theorem 3.4 and Theorem 3.5. $\square$

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