Operators
and
Matrices
Volume 18, Number 3 (2024), 559–570 doi:10.7153/oam-2024-18-32

ON LIE TRIPLE CENTRALIZERS OF VON NEUMANN ALGEBRAS

BEHROOZ FADAEE AND HOGER GHAHRAMANI*

(*Communicated by S. McCullough*)

Abstract. Let $\mathscr{U}$ be a von Neumann algebra endowed with the Lie product $[A,B] = AB - BA$ ($A,B \in \mathscr{U}$). In this article, we consider the subsequent condition on an additive mapping ϕ on the von Neumann algebra $\mathscr{U}$ with a suitable projection $P \in \mathscr{U}$:

$$\phi([[A,B],C]) = [[\phi(A),B],C] = [[A,\phi(B)],C]$$

for all $A,B,C \in \mathscr{U}$ with $AB = P$ and we show that $\phi(A) = WA + \xi(A)$ for all $A \in \mathscr{U}$, where $W \in Z(\mathscr{U})$, and $\xi : \mathscr{U} \to Z(\mathscr{U})$ ($Z(\mathscr{U})$ is the center of $\mathscr{U}$) is an additive map in which $\xi([[A,B],C]) = 0$ for any $A,B,C \in \mathscr{U}$ with $AB = P$. We also give some results of the conclusion.

1. Introduction

Let $\mathscr{U}$ be an associative algebra. The additive map $\phi : \mathscr{U} \to \mathscr{U}$ is called a *Lie centralizer* if $\phi([a,b]) = [\phi(a),b]$ for all $a,b \in \mathscr{U}$ and it is called a *Lie triple centralizer* if $\phi([[a,b],c]) = [[\phi(a),b],c]$ for all $a,b,c \in \mathscr{U}$, where $[a,b] = ab - ba$ is the Lie product of a and b in $\mathscr{U}$. It is easily checked that ϕ is a Lie triple centralizer (Lie centralizer) on $\mathscr{U}$ if and only if $\phi([[a,b],c]) = [[a,\phi(b)],c]$ ($\phi([a,b]) = [a,\phi(b)]$) for all $a,b,c \in \mathscr{U}$. Obviously every Lie centralizer is a Lie triple centralizer but the converse is not necessarily true. Lie centralizers and Lie triple centralizers are important classes of mappings related to the Lie structure of algebras that have recently been widely studied from different perspectives on algebras. These studies are aimed at determining the structure of Lie centralizers and Lie triple centralizers or characterizing mappings that act in certain products such as Lie centralizers or Lie triple centralizers. In the following, we will refer to some of the results obtained for the Lie centralizers or Lie triple centralizers. Fošner and Jing in [6] have studied non-additive Lie centralizers on triangular rings, and in [15] non-linear Lie centralizers on generalized matrix algebra have been checked. Jabeen in [13] has described the structure of linear Lie centralizers on a generalized matrix algebra under some conditions. In [5] it has been shown that under some conditions on a unital generalized matrix algebra $\mathscr{U}$, if $\phi : \mathscr{U} \to \mathscr{U}$ is a linear Lie triple centralizer, then $\phi(a) = \lambda a + \xi(a)$ in which $\lambda \in Z(\mathscr{U})$ and ξ is a linear map from $\mathscr{U}$ into $Z(\mathscr{U})$ vanishing at every second commutator $[[a,b],c]$ for all $a,b,c \in \mathscr{U}$, where $Z(\mathscr{U})$ is the center of $\mathscr{U}$. In [17], Lie n-centralizers of generalized matrix algebras have been examined. The authors in [2] have studied the characterization of Lie

Mathematics subject classification (2020): 46L10, 47B47, 47B48.
Keywords and phrases: Von Neumann algebra, Lie triple centralizer, Lie centralizer.
* Corresponding author.

Paper OaM-18-32

centralizers on non-unital triangular algebras through zero products. In [8], linear Lie centralizers at the zero products on some operator algebras are studied, and in [4], linear Lie centralizers through zero products on a 2-torsion free unital generalized matrix algebra under some mild conditions, are described. In [3], the problem of characterizing linear maps behaving like Lie centralizers at idempotent products on triangular algebras is considered, and in [12], additive Lie centralizers through idempotent-products on a 2-torsion free unital prime ring are determined. To find more results in this regard, we refer to [1, 7, 9, 10, 11, 16] and and the references therein. Motivated by these developments, in the present article, we study the additive Lie triple centralizers at idempotent-products on von Neumann algebras. In fact, the following theorem is the main result of the article.

THEOREM 1.1. *Let $\mathscr{U}$ be a von Neumann algebra with unit element I, and $E_1+E_2=I$, where E_1 and E_2 are two orthogonal central projections such that $\mathscr{U}E_1$ is of type I_1 and $\mathscr{U}E_2$ is a von Neumann algebra with no central summands of type I_1. Suppose that $P\in\mathscr{U}E_2$ is a core-free projection with central carrier E_2. Let $\phi:\mathscr{U}\to\mathscr{U}$ be an additive map. Then ϕ satisfies*

$$\phi([[A,B],C]) = [[\phi(A),B],C] = [[A,\phi(B)],C] \quad (\boldsymbol{P})$$

for all $A,B,C\in\mathscr{U}$ with $AB=P$ if and only if $\phi(A)=WA+\xi(A)$ $(A\in\mathscr{U})$, where $W\in Z(\mathscr{U})$, $\xi:\mathscr{U}\to Z(\mathscr{U})$ is an additive map in which $\xi([A,B],C])=0$ for any $A,B,C\in\mathscr{U}$ with $AB=P$.

The following result is obtained from Theorem 1.1 which is a generalization of the obtained result in [5, Remark 4.4] for factors von Neumann algebras.

COROLLARY 1.2. *Let $\mathscr{U}$ be an arbitrary von Neumann algebra, and $\phi:\mathscr{U}\to\mathscr{U}$ be an additive map. Then ϕ is a Lie triple centralizer if and only if there exist an element $W\in Z(\mathscr{U})$ and an additive map $\xi:\mathscr{U}\to Z(\mathscr{U})$ such that $\phi(A)=WA+\xi(A)$ for any $A\in\mathscr{U}$ and $\xi([[A,B],C])=0$ for any $A,B,C\in\mathscr{U}$.*

Also, we obtain the following corollary which is a generalization of [4, Corollary 5.2-(iv)] and [7, Corollary 4.3-(ii)].

COROLLARY 1.3. *Let $\mathscr{U}$ be an arbitrary von Neumann algebra, and $\phi:\mathscr{U}\to\mathscr{U}$ be an additive map. Then ϕ is a Lie centralizer if and only if there exist an element $W\in Z(\mathscr{U})$ and an additive map $\xi:\mathscr{U}\to Z(\mathscr{U})$ such that $\phi(A)=WA+\xi(A)$ for any $A\in\mathscr{U}$ and $\xi([A,B])=0$ for any $A,B\in\mathscr{U}$.*

It should be noted that most of the previous results about von Neumann algebras are on factor von Neumann algebras or von Neumann algebras without central summands of type I_1, but our results are on a wider class of von Neumann algebras, and some of our results are generalizations of some previous results, and it is worth noting that by using the obtained results, it is possible to characterize generalized Lie triple derivations, generalized Lie derivations, Jordan centralizers and Jordan generalized derivations on von Neumann algebras.

Section 2, some preliminaries about von Neumann algebras are provided, and in Section 3, we give the proof of Theorem 1.1 and Corollaries 1.2, 1.3.

2. Preliminaries

A von Neumann algebra $\mathscr{U}$ is a weakly closed, self-adjoint algebra of operators on a complex Hilbert space $\mathscr{H}$ containing the identity operator I. If $P \in \mathscr{U}$ is idempotent (i. e. $P^2 = P$) and self-adjoint ($P^* = P$), then P is called a *projection*. A projection $P \in \mathscr{U}$ is said to be a *central abelian projection* if $P \in \mathrm{Z}(\mathscr{U})$ and $P\mathscr{U}P$ is abelian. The *central carrier* of $T \in \mathscr{U}$ is the smallest central projection P satisfying $PT = T$, and denoted by $\overline{T}$. It is well known that $\overline{T}$ is the projection whose range is the closed linear span of $\{AT(h) : A \in \mathscr{U}, h \in \mathscr{H}\}$. For each self-adjoint operator $S \in \mathscr{U}$, the *core* of S, denoted by $\underline{S}$, is $\sup\{W \in \mathrm{Z}(\mathscr{U}) : W = W^*, W \leqslant S\}$. The projection P is a *core-free projection*, if $P \in \mathscr{U}$ is a projection and $\underline{P} = 0$. A routine verifications shows that $\underline{P} = 0$ if and only if $\overline{I-P} = I$. Note that $\mathscr{U}$ is a von Neumann algebra with no central summands of type I_1 if and only if it has a projection P such that $\underline{P} = 0$ and $\overline{P} = I$. If $\mathscr{U}$ is an arbitrary von Neumann algebra, the unit element I of $\mathscr{U}$ is the sum of two orthogonal central projections E_1 and E_2 such that $\mathscr{U} = \mathscr{U}E_1 \oplus \mathscr{U}E_2$, $\mathscr{U}E_1$ is of type I_1 and $\mathscr{U}E_2$ is a von Neumann algebra with no central summands of type I_1. So $\mathscr{U}E_2$ contains a core-free projection with central carrier E_2. We refer the reader to [14] for the theory of von Neumann algebras.

REMARK 2.1. Let $\mathscr{U}$ be a von Neumann algebra with no central summands of type I_1, and $P \in \mathscr{U}$ be a projection such that $\underline{P} = 0$ and $\overline{P} = I$. We have $\underline{I-P} = 0$ and $\overline{I-P} = I$.

(i) By [14, Corollary 5.5.7] we have

$$\mathrm{Z}(P\mathscr{U}P) = P\mathrm{Z}(\mathscr{U}) \quad \text{and} \quad \mathrm{Z}((I-P)\mathscr{U}(I-P)) = (I-P)\mathrm{Z}(\mathscr{U}).$$

(ii) It follows from the definition of the central carrier that both $span\{AP(h) : A \in \mathscr{U}, h \in \mathscr{H}\}$ and $span\{A(I-P)(h) : A \in \mathscr{U}, h \in \mathscr{H}\}$ are dense in $\mathscr{H}$. So $A \in \mathscr{U}$, $A\mathscr{U}P = \{0\}$ implies $A = 0$ and $A\mathscr{U}(I-P) = \{0\}$ implies $P = 0$.

3. Proving the main results

First, we show the main result for von Neumann algebras with no central summands of type I_1 in the following proposition.

PROPOSITION 3.1. *Let $\mathscr{U}$ be a von Neumann algebra with no central summands of type I_1, and P be a core-free projection with central carrier. Let $\phi : \mathscr{U} \to \mathscr{U}$ be an additive map. Then ϕ satisfies* $(\boldsymbol{P})$ *if and only if $\phi(A) = WA + \xi(A)$ $(A \in \mathscr{U})$, where $W \in \mathrm{Z}(\mathscr{U})$, $\xi : \mathscr{U} \to \mathrm{Z}(\mathscr{U})$ is an additive map in which $\xi([A,B],C]) = 0$ for any $A,B,C \in \mathscr{U}$ with $AB = P$.*

Proof. Assume that ϕ satisfies $(\mathbf{P})$. Set $P_1 := P$ and $P_2 := I - P_1$. By Remark 2.1 P_2 is also core free and $\overline{P_2} = I$. Set $\mathscr{U}_{ij} = P_i \mathscr{U} P_j$ $(i,j = 1,2)$, then $\mathscr{U} = \mathscr{U}_{11} + \mathscr{U}_{12} + \mathscr{U}_{21} + \mathscr{U}_{22}$. So any element A of $\mathscr{U}$ is of the form $A = A_{11} + A_{12} + A_{21} + A_{22}$ for some $A_{ij} \in \mathscr{U}_{ij}$ $(i,j = 1,2)$. The continuation of the proof in this case is done through the following lemmas.

LEMMA 1. $\phi(I), \phi(P_1) \in \mathscr{U}_{11} + \mathscr{U}_{22}$.

Proof. Since $IP_1 = P_1$ and $P_1P_1 = P_1$, we get $0 = \phi([[I,P_1],P_1]) = [[\phi(I),P_1],P_1]$ and $0 = \phi([[P_1,P_1],P_1]) = [[\phi(P_1),P_1],P_1]$, and hence

$$\phi(I)P_1 + P_1\phi(I) - 2P_1\phi(I)P_1 = 0,$$

and

$$\phi(P_1)P_1 + P_1\phi(P_1) - 2P_1\phi(P_1)P_1 = 0.$$

Multiplying the above equation once from left to P_2 and, once from right to P_2, we arrive at $P_2\phi(I)P_1 = P_1\phi(I)P_2 = 0$ and $P_2\phi(P_1)P_1 = P_1\phi(P_1)P_2 = 0$, so $\phi(I), \phi(P_1) \in \mathscr{U}_{11} + \mathscr{U}_{22}$. $\square$

LEMMA 2. $\phi(\mathscr{U}_{ij}) \subseteq \mathscr{U}_{ij}$, *where* $1 \leqslant i \neq j \leqslant 2$.

Proof. For any $A_{12} \in \mathscr{U}_{12}$, since $(I + A_{12})P_1 = P_1$, by assumption, we see that

$$\begin{aligned}
\phi(A_{12}) &= \phi([[I + A_{12}, P_1], P_1]) \\
&= [[I + A_{12}, \phi(P_1)], P_1] \\
&= [[I, \phi(P_1)], P_1] + [[A_{12}, \phi(P_1)], P_1] \\
&= [[A_{12}, \phi(P_1)], P_1] \\
&= [[A_{12}\phi(P_1) - \phi(P_1)A_{12}, P_1] \\
&= A_{12}\phi(P_1)P_1 - A_{12}\phi(P_1) + P_1\phi(P_1)A_{12}
\end{aligned}$$

Multiplying the above equation once from left and right to P_1, once from left and right to P_2, and once from left to P_2 and from right to P_1, we conclude that $P_1\phi(A_{12})P_1 = 0$, $P_2\phi(A_{12})P_2 = 0$ and $P_2\phi(A_{12})P_1 = 0$. Therefore $\phi(A_{12}) = P_1\phi(A_{12})P_2 \in \mathscr{U}_{12}$.

For any $A_{21} \in \mathscr{U}_{21}$, since $P_1(I + A_{21}) = P_1$, we have

$$\begin{aligned}
\phi(A_{21}) &= \phi([[P_1, I + A_{12}], P_2]) \\
&= [[\phi(P_1), I + A_{21})], P_2] \\
&= [[\phi(P_1), I)], P_2] + [[\phi(P_1), A_{21})], P_2] \\
&= [[\phi(P_1), A_{21})], P_2] \\
&= [\phi(P_1)A_{21} - A_{21}\phi(P_1), P_2] \\
&= -A_{21}\phi(P_1)P_2 - P_2\phi(P_1)A_{21} + A_{21}\phi(P_1)
\end{aligned}$$

Multiplying the above equation once from left and right to P_1, once from left and right to P_2, and once from left to P_2 and from right to P_1, and we arrive at $\phi(A_{21}) \in \mathscr{U}_{21}$. $\square$

LEMMA 3. *$\phi(\mathscr{U}_{ii}) \subseteq \mathscr{U}_{11} + \mathscr{U}_{22}$, for $i \in \{1,2\}$.*

Proof. For any invertible $A_{11} \in \mathscr{U}_{11}$, since $A_{11}A_{11}^{-1} = P_1$, we have

$$\begin{aligned}0 &= \phi([[A_{11}, A_{11}^{-1}], P_1]) = [[\phi(A_{11}), A_{11}^{-1}], P_1] \\ &= \phi(A_{11})A_{11}^{-1} - A_{11}^{-1}\phi(A_{11})P_1 - P_1\phi(A_{11})A_{11}^{-1} + A_{11}^{-1}\phi(A_{11}).\end{aligned}$$

Write $\phi(A_{11}) = \sum_{i,j=1}^{2} T_{ij}$. It follows from above equation that $T_{12} = T_{21} = 0$. Consequently, $\phi(A_{11}) \in \mathscr{U}_{11} + \mathscr{U}_{22}$, for any invertible $A_{11} \in \mathscr{U}_{11}$. For any $A_{11} \in \mathscr{U}_{11}$, we may find a sufficiently big number n such that $nP_1 - A_{11}$ is invertible. Thus, by the above and Lemma 1, we have

$$\phi(A_{11}) = n\phi(P_1) - \phi(nP_1 - A_{11}) \in \mathscr{U}_{11} + \mathscr{U}_{22}.$$

For any $B_{22} \in \mathscr{U}_{22}$, write $\phi(B_{22}) = \sum_{i,j=1}^{2} S_{ij}$. By the equation $(P_1 + B_{22})P_1 = P_1$ and Lemma 1, we have

$$\begin{aligned}0 &= \phi([[P_1 + B_{22}, P_1], P_1]) = [[\phi(P_1 + B_{22}), P_1], P_1] \\ &= [[\phi(P_1), P_1], P_1] + [[\phi(B_{22}), P_1], P_1] \\ &= [[\phi(B_{22}), P_1], P_1] \\ &= \phi(B_{22})P_1 + P_1\phi(B_{22}) - 2P_1\phi(B_{22})P_1\end{aligned}$$

It follows that $S_{12} = S_{21} = 0$. Therefore $\phi(B_{22}) \in \mathscr{U}_{11} + \mathscr{U}_{22}$. □

LEMMA 4. *For $i = 1,2$, there exists an additive map $h_i : \mathscr{U}_{ii} \to \mathrm{Z}(\mathscr{U})$ such that $P_j\phi(A_{ii})P_j = h_i(A_{ii})P_j$ for any $A_{ii} \in \mathscr{U}_{ii}$, where $1 \leqslant i \neq j \leqslant 2$.*

Proof. For any invertible element $A_{11} \in \mathscr{U}_{11}$, and each $B_{22} \in \mathscr{U}_{22}$ we have $A_{11}(A_{11}^{-1} + B_{22}) = P_1$, and $(A_{11} + B_{22})A_{11}^{-1} = P_1$. So for each $C_{ij} \in \mathscr{U}_{ij}$, where $(1 \leqslant i \neq j \leqslant 2)$, we get

$$\begin{aligned}0 &= \phi([[A_{11}, A_{11}^{-1} + B_{22}], C_{21}]) \\ &= [[\phi(A_{11}), A_{11}^{-1} + B_{22}], C_{21}] \\ &= [[\phi(A_{11}), A_{11}^{-1}], C_{21}] + [[\phi(A_{11}), B_{22}], C_{21}] \\ &= \phi([[A_{11}, A_{11}^{-1}], C_{21}]) + [[\phi(A_{11}), B_{22}], C_{21}] \\ &= [[\phi(A_{11}), B_{22}], C_{21}]\end{aligned}$$

and

$$\begin{aligned}0 &= \phi([[A_{11}^{-1} + B_{22}, A_{11}], C_{12}]) \\ &= [[\phi(A_{11}^{-1} + B_{22}), A_{11}], C_{12}] \\ &= [[\phi(A_{11}^{-1}) + \phi(B_{22}), A_{11}], C_{12}] \\ &= [[\phi(A_{11}^{-1}), A_{11}], C_{21}] + [[\phi(B_{22}), A_{11}], C_{12}] \\ &= \phi([[A_{11}^{-1}, A_{11}], C_{21}]) + [[\phi(B_{22}), A_{11}], C_{12}] \\ &= [[\phi(B_{22}), A_{11}], C_{21}].\end{aligned}$$

Considering above equations, and using Lemma 3, we arrive at

$$(P_2\phi(A_{11})P_2B_{22} - B_{22}P_2\phi(A_{11})P_2)CP_1 = 0$$

and

$$(P_1\phi(B_{22})P_1A_{11} - A_{11}P_1\phi(B_{22})P_1)CP_2 = 0$$

for any $C \in \mathscr{U}$ and any invertible element $A_{11} \in \mathscr{U}_{11}$. For any $A_{11} \in \mathscr{U}_{11}$, we may find a sufficiently big number n such that $nP_1 - A_{11}$ is invertible. So,

$$(P_2\phi(A_{11})P_2B_{22} - B_{22}P_2\phi(A_{11})P_2)CP_1 = 0$$

and

$$(P_1\phi(B_{22})P_1A_{11} - A_{11}P_1\phi(B_{22})P_1)CP_2 = 0$$

for all $C \in \mathscr{U}$, $A_{11} \in \mathscr{U}_{11}$, and $B_{22} \in \mathscr{U}_{22}$. From Remark 2.1, we conclude that $P_2\phi(A_{11})P_2 \in \mathrm{Z}(\mathscr{U}_{22})$ and $P_1\phi(B_{22})P_1 \in \mathrm{Z}(\mathscr{U}_{11})$, and by fact that $\mathrm{Z}(\mathscr{U}_{22}) = \mathrm{Z}(\mathscr{U})P_2$ and $\mathrm{Z}(\mathscr{U}_{11}) = \mathrm{Z}(\mathscr{U})P_1$ we have $P_2\phi(A_{11})P_2 \in \mathrm{Z}(\mathscr{U})P_2$ and $P_1\phi(B_{22})P_1 \in \mathrm{Z}(\mathscr{U})P_1$. Therefore, for any $A_{11} \in \mathscr{U}_{11}$ and $B_{22} \in \mathscr{U}_{22}$, there are $Z_1, Z_2 \in \mathrm{Z}(\mathscr{U})$ such that

$$P_2\phi(A_{11})P_2 = Z_1P_2 \quad \text{and} \quad P_1\phi(B_{22})P_1 = Z_2P_1.$$

So we can define the maps $h_1 : \mathscr{U}_{11} \to \mathrm{Z}(\mathscr{U})$ by $h_1(A_{11}) = Z_1$ for any $A_{11} \in \mathscr{U}_{11}$ and $h_2 : \mathscr{U}_{22} \to \mathrm{Z}(\mathscr{U})$ by $h_2(B_{22}) = Z_2$ for any $B_{22} \in \mathscr{U}_{22}$. Suppose that $h_1(A_{11}) = Z_1 \in \mathrm{Z}(\mathscr{U})$ and $h_1(A_{11}) = Z_1' \in \mathrm{Z}(\mathscr{U})$. Then we have $\phi(A_{11}) - Z_1 \in \mathscr{U}_{11}$ and $\phi(A_{11}) - Z_1' \in \mathscr{U}_{11}$. It follows that $Z_1' - Z_1 = (\phi(A_{11}) - Z_1) - (\phi(A_{11}) - Z_1') \in \mathscr{U}_{11} \cap \mathrm{Z}(\mathscr{U}) = \{0\}$. So $Z_1 = Z_1'$. In a similar way it is proved that Z_2 is unique. By the uniqueness of Z_1 and Z_2 the maps h_1 and h_2 are well-defined. Moreover, from the uniqueness of Z_1 and Z_2 and additivity of ϕ it follows that h_1 and h_2 are additive. Also,

$$P_2\phi(A_{11})P_2 = h_1(A_{11})P_2 \quad \text{and} \quad P_1\phi(B_{22})P_1 = h_2(B_{22})P_1. \quad \square$$

Now, for any $A = A_{11} + A_{12} + A_{21} + A_{22} \in \mathscr{U}$, we define two additive maps $\xi : \mathscr{U} \to \mathrm{Z}(\mathscr{U})$ and $\psi : \mathscr{U} \to \mathscr{U}$ by

$$\xi(A) = h_1(A_{11}) + h_2(A_{22}) \quad \text{and} \quad \psi(A) = \phi(A) - \xi(A).$$

By Lemmas 1-3, it is clear that $\psi(\mathscr{U}_{ii}) \subseteq \mathscr{U}_{ii}$ for $i = 1, 2$, and $\psi(\mathscr{U}_{ij}) = \phi(\mathscr{U}_{ij}) \subseteq \mathscr{U}_{ij}$ for $1 \leqslant i \neq j \leqslant 2$.

LEMMA 5. *There is an element $W \in \mathrm{Z}(\mathscr{U})$ such that $\psi(A) = WA$ for all $A \in \mathscr{U}$.*

Proof. We divide the proof into the following steps.
Step 1. The following statements hold:

(i) $\psi(A_{ii}B_{ij}) = \psi(A_{ii})B_{ij} = A_{ii}\psi(B_{ij})$ for all $A_{ii} \in \mathscr{U}_{ii}$ and $B_{ij} \in \mathscr{U}_{ij}$, where $1 \leqslant i \neq j \leqslant 2$;

(ii) $\psi(B_{ij}A_{jj}) = \psi(B_{ij})A_{jj} = B_{ij}\psi(A_{jj})$ for all $B_{ij} \in \mathscr{U}_{ij}$ and $A_{jj} \in \mathscr{U}_{jj}$, where $1 \leqslant i \neq j \leqslant 2$

For any invertible element $A_{11} \in \mathscr{U}_{11}$ and any $B_{12} \in \mathscr{U}_{12}$, since $(A_{11}^{-1} + B_{12})A_{11} = P_1$, by Lemma 2, we have

$$\begin{aligned}
\psi(A_{11}B_{12}) &= \phi(A_{11}B_{12})\\
&= \phi([[A_{11}^{-1} + B_{12}, A_{11}], P_1])\\
&= [[\phi(A_{11}^{-1} + B_{12}), A_{11}], P_1]\\
&= [[\phi(A_{11}^{-1}) + \phi(B_{12}), A_{11}], P_1]\\
&= [[\phi(A_{11}^{-1}), A_{11}], P_1] + [[\phi(B_{12}), A_{11}], P_1]\\
&= \phi([[A_{11}^{-1}, A_{11}], P_1]) + [[\phi(B_{12}), A_{11}], P_1]\\
&= [[\phi(B_{12}), A_{11}], P_1]\\
&= [\phi(B_{12})A_{11} - A_{11}\phi(B_{12}), P_1]\\
&= A_{11}\psi(B_{12})
\end{aligned}$$

and

$$\begin{aligned}
\psi(A_{11}B_{12}) &= \phi(A_{11}B_{12})\\
&= \phi([[A_{11}^{-1} + B_{12}, A_{11}], P_1])\\
&= [[A_{11}^{-1} + B_{12}, \phi(A_{11})], P_1]\\
&= [[B_{12}, \phi(A_{11})], P_1]\\
&= P_1\phi(A_{11})P_1B_{12} - B_{12}P_2\phi(A_{11})P_2\\
&= P_1\phi(A_{11})P_1B_{12} - h_1(A_{11})P_1B_{12}\\
&= (P_1\phi(A_{11})P_1 + P_2\phi(A_{11})P_2 - h_1(A_{11})P_1 - h_1(A_{11})P_2)B_{12}\\
&= \phi(A_{11}) - h_1(A_{11})\\
&= \psi(A_{11})B_{12}
\end{aligned}$$

For any $A_{11} \in \mathscr{U}_{11}$, there exists an integer n such that $nP_1 - A_{11}$ is invertible. Note that nP_1 is also invertible. By above results we have $\psi(nP_1B_{12}) = nP_1\psi(B_{12}) = n\psi(P_1)B_{12}$ and $\psi((nP_1 - A_{11})B_{12}) = (nP_1 - A_{11})\psi(B_{12}) = \psi(nP_1 - A_{11})B_{12}$. Thus, $\psi(A_{11}B_{12}) = A_{11}\psi(B_{12}) = \psi(A_{11})B_{12}$, for any $A_{11} \in \mathscr{U}_{11}$ and any $B_{22} \in \mathscr{U}_{22}$.

For any invertible element $A_{11} \in \mathscr{U}_{11}$ and any $B_{21} \in \mathscr{U}_{21}$, since $A_{11}(A_{11}^{-1} + B_{21}) = P_1$ and $[[A_{11}, A_{11}^{-1} + B_{12}], P_1] = -B_{21}A_{11}$, and with the similar arguments as above, it can be checked that

$$\psi(B_{21}A_{11}) = B_{21}\psi(A_{11}) = \psi(B_{21})A_{11}$$

for any $A_{11} \in \mathscr{U}_{11}$ and $B_{21} \in \mathscr{U}_{21}$. For any $A_{22} \in \mathscr{U}_{22}$ and $B_{21} \in \mathscr{U}_{21}$, we have $(P_1 + A_{22} - A_{22}B_{21})(P_1 + B_{21}) = P_1$. By properties of ψ and ξ, we see that

$$\begin{aligned}
-\psi(B_{21}) &= -\phi(B_{21})\\
&= \phi([[P_1 + A_{22} - A_{22}B_{21}, P_1 + B_{21}], P_1])
\end{aligned}$$

$$\begin{aligned}
&= [[\phi(P_1)+\phi(A_{22})-\phi(A_{22}B_{21}),P_1+B_{21}],P_1]\\
&= [[\phi(P_1),P_1+B_{21}],P_1]+[[\phi(A_{22})-\phi(A_{22}B_{21}),P_1+B_{21}],P_1]\\
&= [[\phi(P_1),P_1],P_1]+[[\phi(P_1),B_{21}],P_1]\\
&\quad +[[\phi(A_{22})-\phi(A_{22}B_{21}),P_1],P_1]+[[\phi(A_{22})-\phi(A_{22}B_{21}),B_{21}],P_1]\\
&= [[\psi(P_1)+\xi(P_1),B_{21}],P_1]+[[\psi(A_{22})+\xi(A_{22}),P_1],P_1]\\
&\quad -[[\psi(A_{22}B_{21}),P_1],P_1]+[[\psi(A_{22})+\xi(A_{22}),B_{21}],P_1]\\
&\quad -[[\psi(A_{22}B_{21}),B_{21}],P_1]\\
&= [[\psi(P_1),B_{21}],P_1]-[[\psi(A_{22}B_{21}),P_1],P_1]+[[\psi(A_{22}),B_{21}],P_1]\\
&= -B_{21}\psi(P_1)-\psi(A_{22}B_{21})+\psi(A_{22})B_{21})
\end{aligned}$$

So we have

$$\psi(A_{22}B_{21})=\psi(A_{22})B_{21}$$

for all $A_{22}\in\mathscr{U}_{22}$ and $B_{21}\in\mathscr{U}_{21}$, because $\psi(B_{21})=\psi(B_{21}P_1)=B_{21}\psi(P_1)$. Also,

$$\psi(A_{22}B_{21})=\psi(A_{22}B_{21}P_1)=A_{22}B_{21}\psi(P_1)=A_{22}\psi(B_{21}),$$

for any $A_{22}\in\mathscr{U}_{22}$ and $B_{21}\in\mathscr{U}_{21}$.

For any $A_{22}\in\mathscr{U}_{22}$ and $B_{12}\in\mathscr{U}_{12}$, since $(P_1+B_{12})(P_1+A_{22}-B_{12}A_{22})=P_1$ and $[[(P_1+B_{12}),(P_1+A_{22}-B_{12}A_{22})],P_2]=-B_{12}$, and with the similar arguments as above, it can be checked that

$$\psi(B_{12}A_{22})=B_{12}\psi(A_{22})=\psi(B_{12})A_{22}.$$

Step 2. $\psi(A_{ii}B_{ii})=\psi(A_{ii})B_{ii}=A_{ii}\psi(B_{ii})$ for all $A_{ii},B_{ii}\in\mathscr{U}_{ii}$, where $i\in\{1,2\}$.

For any $A_{ii},B_{ii}\in\mathscr{U}_{ii}$ and any $S_{ij}\in\mathscr{U}_{ij}$ $(1\leqslant i\neq j\leqslant 2)$, by Step 1, we have

$$\psi(A_{ii}B_{ii}S_{ij})=\psi(A_{ii}B_{ii})S_{ij},$$

and on other hand

$$\psi(A_{ii}B_{ii}S_{ij})=A_{ii}\psi(B_{ii}S_{ij})=A_{ii}\psi(B_{ii})S_{ij},$$

Comparing the above two equations, we see that $\psi(A_{ii}B_{ii})S_{ij}=A_{ii}\psi(B_{ii})S_{ij}$ holds for all $S_{ij}\in\mathscr{U}_{ij}$. From Remark 2.1, it follows that $\psi(A_{ii}B_{ii})=A_{ii}\psi(B_{ii})$ for any $A_{ii},B_{ii}\in\mathscr{U}_{ii}$, where $i=1,2$. Also, for any $A_{ii},B_{ii}\in\mathscr{U}_{ii}$ and any $S_{ji}\in\mathscr{U}_{ji}$ $(1\leqslant i\neq j\leqslant 2)$, by Step 1, we get

$$\psi(S_{ji}A_{ii}B_{ii})=S_{ji}\psi(A_{ii}B_{ii}),$$

and on other hand

$$\psi(S_{ji}A_{ii}B_{ii})=\psi(S_{ji}A_{ii})B_{ii}=S_{ji}\psi(A_{ii})B_{ii},$$

Comparing the above two equations and by Remark 2.1, we see that $\psi(A_{ii}B_{ii})=\psi(A_{ii})B_{ii}$ for any $A_{ii},B_{ii}\in\mathscr{U}_{ii}$, where $i\in\{1,2\}$.

Step 3. $\psi(A_{ij}B_{ji})=\psi(A_{ij})B_{ji}=A_{ij}\psi(B_{ji})$ for all $A_{ij}\in\mathscr{U}_{ij}$ and $B_{ji}\in\mathscr{U}_{ji}$, where $1\leqslant i\neq j\leqslant 2$.

Assume that $A_{ij} \in \mathscr{U}_{ij}$ and $B_{ji} \in \mathscr{U}_{ji}$, $1 \leqslant i \neq j \leqslant 2$. It follows from Steps 1 and 2 that

$$\psi(A_{ij}B_{ji}) = \psi(P_iA_{ij}B_{ji}) = \psi(P_i)A_{ij}B_{ji} = \psi(A_{ij})B_{ji},$$

and

$$\psi(A_{ij}B_{ji}) = \psi(A_{ij}B_{ji}P_i) = A_{ij}B_{ji}\psi(P_i) = A_{ij}\psi(B_{ji}).$$

Step 4. The desired result in Lemma 5 is valid.

From Steps 1-3 and the fact that each $\mathscr{U}_{ij}$ is a invariant subspace for ψ, it follows that

$$\psi(AB) = A\psi(B) = \psi(A)B$$

for all $A, B \in \mathscr{U}$. Set $W := \psi(I)$. So

$$\psi(A) = \psi(AI) = A\psi(I) = AW \quad \text{and} \quad \psi(A) = \psi(IA) = \psi(I)A = WA$$

for all $A, B \in \mathscr{U}$, and $W \in \mathrm{Z}(\mathscr{U})$. □

LEMMA 6. $\xi([[A,B],C]) = 0$ *for all* $A, B, C \in \mathscr{U}$ *with* $AB = P$.

Proof. For any $A, B, C \in \mathscr{U}$ with $AB = P$, by Lemma 5 we have

$$\begin{aligned}
\xi([[A,B],C]) &= \phi([[A,B],C]) - \psi([[A,B],C]) \\
&= [[\phi(A),B],C] - \psi([[A,B],C]) \\
&= [[\psi(A) + \xi(A),B],C] - \psi([[A,B],C]) \\
&= [[\psi(A),B],C] - \psi([[A,B],C]) \\
&= W[[A,B],C] - W[[A,B],C] \\
&= 0,
\end{aligned}$$

since $W \in \mathrm{Z}(\mathscr{U})$. □

Now, by the definition of ψ and Lemma 5, we get $\phi(A) = WA + \xi(A)$ for any $A \in \mathscr{U}$, where $W \in \mathrm{Z}(\mathscr{U})$. From Lemma 6, it follows that $\xi : \mathscr{U} \to \mathrm{Z}(\mathscr{U})$ is an additive mapping in which $\xi([[A,B],C]) = 0$ for any $A, B, C \in \mathscr{U}$ with $AB = P$. So the desired result is valid.

The converse is clear. □

Now we are ready to present the proof of the main theorem.

Proof of Theorem 1.1. Let ϕ satisfies $(\mathbf{P})$. Suppose that $A, B, C, Y \in \mathscr{U}$ such that $BYE_2 = P$. Put $X := AE_1 + BE_2$. Since $\mathscr{U}E_1 \subseteq \mathrm{Z}(\mathscr{U})$, it follows that $[X, YE_2] = [BE_2, YE_2] = [B,Y]E_2$. From our assumption $XYE_2 = P$, and we have $\phi([[X,YE_2],C]) = [[\phi(X),YE_2],C] = [[X,\phi(YE_2)],C]$. So

$$\begin{aligned}
\phi([[BE_2,YE_2],C]) &= [[\phi(X),YE_2],C] \\
&= [[\phi(X)E_2,YE_2],C] \\
&= [[\phi(AE_1)E_2,YE_2],C] + [[\phi(BE_2)E_2,YE_2],C].
\end{aligned}$$

We multiply the sides of the above identity by E_2. Consequently

$$\phi([[BE_2,YE_2],C])E_2 = [[\phi(AE_1)E_2,YE_2],C]+[[\phi(BE_2)E_2,YE_2],C]. \tag{1}$$

By setting $A=0$ in (1) we see that

$$\phi([[BE_2,YE_2],C])E_2 = [[\phi(BE_2)E_2,YE_2],C]. \tag{2}$$

for all $B,Y\in \mathscr{U}E_2$ with $BYE_2=P$. Also, we have

$$\begin{aligned}\phi([[BE_2,YE_2],C])E_2 &= [[X,\phi(YE_2)],C]\\ &= [[AE_1,\phi(YE_2)],C]+[[BE_2,\phi(YE_2)],C].\end{aligned}$$

So by the fact that $\mathscr{U}E_1\subseteq \mathrm{Z}(\mathscr{U})$ we arrive at

$$\phi([[BE_2,YE_2],C])E_2 = [[BE_2,\phi(YE_2)],C] \tag{3}$$

for all $B,Y\in \mathscr{U}E_2$ with $BYE_2=P$. Equations (2) and (3) show that the additive mapping $\varphi:\mathscr{U}E_2\to\mathscr{U}E_2$ defined by $\varphi(AE_2)=\phi(AE_2)E_2$, on $\mathscr{U}E_2$ satisfies $(\mathbf{P})$. By our assumption $\mathscr{U}E_2$ is a von Neumann algebra with no central summands of type I_1, and $P\in\mathscr{U}E_2$ is a projection such that $\underline{P}=0$ and $\overline{P}=E_2$. So by Proposition 3.1, there are $W_1\in \mathrm{Z}(\mathscr{U}E_2)\subseteq \mathrm{Z}(\mathscr{U})$ and an additive mapping $\xi_1:\mathscr{U}E_2\to \mathrm{Z}(\mathscr{U}E_2)\subseteq \mathrm{Z}(\mathscr{U})$ such that

$$\phi(AE_2)E_2=\varphi(AE_2)=W_1AE_2+\xi_1(AE_2) \tag{4}$$

for all $A\in\mathscr{U}$ and $\xi_1([[AE_2,BE_2],CE_2])=0$ for all $A,B,C\in\mathscr{U}$ with $ABE_2=P$. Assume that for $Y\in\mathscr{U}$ the element YE_2 is invertible in $\mathscr{U}E_2$ (i.e., $YE_2\in Inv(\mathscr{U}E_2)$). Taking $B:=P(YE_2)^{-1}$ and $X=AE_1+BE_2$ for $A\in\mathscr{U}$. So $BYE_2=P$, and from (1) and (2), for any $C\in\mathscr{U}$ it follows that $[[\phi(AE_1)E_2,YE_2],C]=0$. Since each element of $\mathscr{U}E_2$ is a sum of two invertible elements of $\mathscr{U}E_2$, it results that $[[\phi(AE_1)E_2,YE_2],C]=0$ for all $A,C,Y\in\mathscr{U}$. However, by Kleinecke-Shirokov and the fact that the spectral radius is submultiplicative on commuting elements, it results that $[\phi(AE_1)E_2,YE_2]=0$ for all $A,Y\in\mathscr{U}$. So

$$\phi(AE_1)E_2\in \mathrm{Z}(\mathscr{U}E_2)\subseteq \mathrm{Z}(\mathscr{U})$$

for all $A\in\mathscr{U}$. Also

$$\varphi(A)E_1\in\mathscr{U}E_1\subseteq \mathrm{Z}(\mathscr{U})$$

for all $A\in\mathscr{U}$. Now by (4) we have

$$\begin{aligned}\phi(A) &= \phi(A)E_1+\phi(AE_1)E_2+\phi(AE_2)E_2\\ &= \phi(A)E_1+\phi(AE_1)E_2+W_1AE_2+\xi_1(AE_2)\\ &= WA+\xi(A),\end{aligned}$$

for all $A\in\mathscr{U}$, where $W:=W_1E_2\in \mathrm{Z}(\mathscr{U})$ and $\xi:\mathscr{U}\to\mathscr{U}$ is an additive map defined by $\xi(A)=\phi(A)E_1+\phi(AE_1)E_2+\xi_1(AE_2)$. By above all three summands lie in $\mathrm{Z}(\mathscr{U})$, thus ξ maps $\mathscr{U}$ into $Z(\mathscr{U})$. Finally according to these results for $A,B,C \in \mathscr{U}$ where $AB = P$ we have

$$\begin{aligned}\xi([[A,B],C]) &= \phi([[A,B],C]) - W[[A,B],C]\\ &= [[\phi(A),B],C] - [[WA,B],C]\\ &= [[\xi(A),B],C] = 0.\end{aligned}$$

The converse is clear. □

Proof of Corollary 1.2. Let ϕ be a Lie triple centralizer. If $\mathscr{U}$ is an abelian von Neumann algebra, then ϕ maps $\mathscr{U}$ into $Z(\mathscr{U}) = \mathscr{U}$. Also, from the fact that ϕ is a Lie triple centralizer, it follows that $\phi([[A,B],C]) = 0$ for any $A,B,C \in \mathscr{U}$. So in this case the result is valid. Now let's assume that $\mathscr{U}$ is non-abelian. In this case the unit element I of $\mathscr{U}$ is the sum of two orthogonal central projections E_1 and E_2 such that $\mathscr{U} = \mathscr{U}E_1 \oplus \mathscr{U}E_2$, $\mathscr{U}E_1$ is of type I_1 and $\mathscr{U}E_2$ is a von Neumann algebra with no central summands of type I_1. So there exist a core-free projection $P \in \mathscr{U}E_2$ with central carrier E_2. Because ϕ is a Lie triple centralizer, it satisfies the condition $(\mathbf{P})$ on $\mathscr{U}$. Hence by Theorem 1.1, $\phi(A) = WA + \xi(A)$ for all $A \in \mathscr{U}$, where $W \in Z(\mathscr{U})$, $\xi : \mathscr{U} \to Z(\mathscr{U})$ is an additive map. It is sufficient to prove that for any $A,B,C \in \mathscr{U}$ we have $\xi([[A,B],C]) = 0$. This part can be proved in similar manner with the proof of Theorem 1.1.

The converse is clear. □

Proof of Corollary 1.3. Let ϕ be a Lie centralizer. It is easily checked that ϕ is a Lie triple centralizer. According to this point, the result is obtained from Corollary 1.2. It should be noted that for $A,B \in \mathscr{U}$ we have

$$\begin{aligned}\xi([A,B]) &= \phi([A,B]) - W[A,B]\\ &= [\phi(A),B] - [WA,B]\\ &= [\xi(A),B] = 0,\end{aligned}$$

The converse is clear. □

Declarations

- All authors contributed to the study conception and design and approved the final manuscript.
- The authors have no relevant financial or non-financial interests to disclose.
- Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Acknowledgement. The work of Behrooz Fadaee has been supported financially by Vice Chancellorship of Research and Technology at University of Kurdistan under research Project No. 01/9/11/17885.

The authors thank the referee(s) for careful reading of the manuscript and for helpful suggestions.

REFERENCES

[1] M. ASHRAF AND M. A. ANSARI, *Multiplicative generalized Lie n-derivations of unital rings with idempotents*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **116**, 92 (2022), https://doi.org/10.1007/s13398-022-01233-5.

[2] R. BEHFAR AND H. GHAHRAMANI, *Lie maps on triangular algebras without assuming unity*, Mediterranean J. Math. **18** (2021), 1–28.

[3] B. FADAEE, A. FOŠNER AND H. GHAHRAMANI, *Centralizers of Lie Structure of triangular algebras*, Results Math. **77** (2022), https://doi.org/10.1007/s00025-022-01756-8.

[4] B. FADAEE AND H. GHAHRAMANI, *Lie centralizers at the zero products on generalized matrix algebras*, J. Algebra Appl. **21** (2022), https://doi.org/10.1142/S0219498822501651.

[5] B. FADAEE, H. GHAHRAMANI AND W. JING, *Lie triple centralizers on generalized matrix algebras*, Quaestiones Math. **46** (2023), 281–300.

[6] A. FOŠNER AND W. JING, *Lie centralizers on triangular rings and nest algebras*, Adv. Oper. Theory, **4** (2019), 342–350.

[7] A. FOŠNER, H. GHAHRAMANI AND F. WEI, *Lie centralizers and generalized Lie derivations at zero products*, Rocky Mountain J. Math, to appear.

[8] H. GHAHRAMANI AND W. JING, *Lie centralizers at zero products on a class of operator algebras*, Ann. Funct. Anal. **12** (2021), 1–12.

[9] H. GHAHRAMANI, A. H. MOKHTARI AND F. WEI, *Lie centralizers and commutant preserving maps on generalized matrix algebras*, J. Algebra Appl. (2023), https://doi.org/10.1142/S0219498824501068.

[10] P. GHIMIRE, *Linear Lie centralizers of the algebra of strictly block upper triangular matrices*, Oper. Matrices, **15** (2021), 303–311.

[11] P. GHIMIRE, *Linear Lie centralizers of the algebra of dominant block upper triangular matrices*, Linear and Multilinear Algebra, **70** (2022), 5040–5051.

[12] SH. GOODARZI AND A. KHOTANLOO, *Lie Centralizers and generalized Lie derivations on prime rings by local actions*, Commun. Algebra, (2023), https://doi.org/10.1080/00927872.2023.2228429.

[13] A. JABEEN, *Lie (Jordan) centralizers on generalized matrix algebras*, Commun. Algebra, (2020), https://doi.org/10.1080/00927872.2020.1797759.

[14] R. V. KADISON AND J. R. RINGROSE, *Fundamentals of the theory of operator algebras*, Vol. I, Academic Press, New York, 1983, Vol. II, Academic Press, New York, 1986.

[15] L. LIU, *On nonlinear Lie centralizers of generalized matrix algebras*, Linear and Multilinear Algebra, (2020), https://doi.org/10.1080/03081087.2020.1810605.

[16] L. LIU AND K. GAO, *Characterizations of Lie centralizers of triangular algebras*, Linear and Multilinear Algebra, (2022), https://doi.org/10.1080/03081087.2022.2104788.

[17] H. YUAN AND Z. LIU, *Lie n-centralizers of generalized matrix algebras*, AIMS Math. **8** (2023), 14609–14622.

(Received August 11, 2023)

Behrooz Fadaee
Department of Mathematics
Faculty of Science, University of Kurdistan
P.O. Box 416, Sanandaj, Kurdistan, Iran
e-mail: behroozfadaee@yahoo.com
b.fadaee@uok.ac.ir

Hoger Ghahramani
Department of Mathematics
Faculty of Science, University of Kurdistan
P.O. Box 416, Sanandaj, Kurdistan, Iran
e-mail: h.ghahramani@uok.ac.ir
hoger.ghahramani@yahoo.com